The Weak Objectivity of Mathematics and Its Reasonable Effectiveness in Science

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Abstract

Philosophical analysis of mathematical knowledge are commonly conducted within the realist/antirealist dichotomy. Nevertheless, philosophers working within this dichotomy pay little attention to the way in which mathematics evolves and structures itself. Focusing on mathematical practice, I propose a weak notion of objectivity of mathematical knowledge that preserves the intersubjective character of mathematical knowledge but does not bear on a view of mathematics as a body of mind-independent necessary truths. Furthermore, I show how that the successful application of mathematics in science is an important trigger for the objectivity of mathematical knowledge.

Keywords Mathematical knowledge · Objectivity · Platonism · Nominalism · Applicability of mathematics

1 Introduction

Starting a paper with a long quotation is probably not the most elegant way to set things moving. Nevertheless, the following words from Hilary Putnam beautifully serve as an introduction to the subject of the present article. This is why, at the risk of being inelegant, I report them here:

> I urge first that mathematics should be interpreted realistically and objectively. But unfortunately, belief in the objectivity of mathematics has generally gone along with belief that "mathematical objects" have an unconditional and superphysical reality, and with the idea that mathematical knowledge is strictly *a priori*. But actually, the criterion of truth in mathematics is the success of its

ideas in practice; mathematical knowledge is corrigible and not absolute; thus it resembles *empirical* knowledge in many respects (Putnam 1975, p. 529)

Putnam's words, echoing Kreisel's point that the interesting question is not about the existence of mathematical objects but about the objectivity of mathematical discourse, sound menacing to the hears of the platonist who claims for the existence of mathematical objects. And the intimidation is even more strong if we accept the idea that mathematical knowledge is 'corrigible and not absolute'. How can Putnam say that? And, more importantly, how can we account for the objectivity of mathematics if we abandon the idea of an infallible and absolute mathematics? At first glance, it seems that behind Putnam's words there are good news for the nominalist philosopher ("mathematical objects do not exist... I told you, Plato!"). Nevertheless, the happy nominalist has no reason to be so happy. Although Putnam believes that the question over the existence of mathematical objects is not an interesting one and we have to avoid commitment to mathematical objects, a standpoint that is certainly very attractive to the anti-realist, he is a realist of some sort (for Putnam mathematical statements are true about possibilities, not about objects, and these truths are true independently of the human mind). Moreover, and this is a lesson that I think the nominalist has to gulp down when reading the passage above, an analysis of the nature of mathematical knowledge should be conducted in tandem with an analysis of the success of mathematics 'in practice', considering mathematical knowledge as 'corrigible and not absolute'. Engaged in her philosophical battle with the platonist, a battle completely focused on ontology, the nominalist seems not to pay the sufficient attention to the practice of mathematics, its evolution and its constitution.[1] This is, I think, a big mistake. Accepting the ontological arena as the proper and unique location for the dispute on the nature of mathematical knowledge has led both the nominalist and the platonist to undervalue, or even bypass, some 'evolutionary' features of mathematics that are extremely interesting from a philosophical point of view.

I take Putnam's words as trigger for the following question: how can we account for the objectivity of mathematics and make this objectivity compatible with the corrigible and not absolute character of mathematics? In the first two sections of this paper I shall provide an answer to this question. Elaborating on some ideas that have been proposed in Friend (2014) and Ferreirós (2015), I shall propose a weak notion of objectivity of mathematical knowledge that does not arise from some aprioristic view of mathematical knowledge but it is the result of a practice (or better a sequence of practices) in mathematics. In the development of these practices the (weak) objectivity of mathematical results is weakened or reinforced through

[1] Platonist and nominalist philosophies of mathematics come in degree and I am over simplifying here. Nevertheless, even acknowledging the subtle nuances that characterize these philosophies, my point still holds: the contemporary battle in philosophy of mathematics is mainly focused on ontology and other aspects of mathematics, as for instance the way in which mathematical knowledge evolves, are totally omitted from this battleground. This is particularly evident from how the recent discussion over Indispensability Argument(s) has been carried out. Nonetheless, there are exceptions to this attitude and I shall say more on these later in the paper.

crosschecking within mathematics or even through the interaction with the empirical sciences. Thus in my account objectivity comes in degree, and it is exactly this objectivity that confers mathematical knowledge epistemic stability within the mathematical community. Nevertheless, I do not take a higher degree of weak objectivity as a mark of truth. In order to illustrate my position, I shall briefly report some examples of mathematical results that have acquired their objectivity and the relative stability as the result of a sequence of practices (in mathematics and in science) and the crosschecking process that came with these practices. Next, in the third section of my paper, I will address a different (although strictly related) question: how can we make the effectiveness of mathematics in science compatible with a notion of objectivity that makes mathematics 'corrigible and not absolute'? This question is particularly important because, as it is clear from the debate around the Indispensability Arguments for mathematical realism, it is precisely the 'good' applicability of mathematics, namely the possibility to get successful inferences about the world through mathematics, that some philosophers consider as an indicator of the absolute and necessary character of mathematics. Thus an account of objectivity of mathematics should say something on how this objectivity is supposed to work in practice, namely when we use mathematics in science.

2 The Fallible and Corrigible Character of Mathematics

Is mathematics fallible? Yes. The Euler's sum of like powers conjecture, which was formulated by Euler and states that at least n nth powers are required to sum to an nth power for $n > 2$, was shown to be false by Lander and Parkin (1966). Is mathematical knowledge corrigible? Yes. As an example of the corrigible character of mathematics take the famous four color conjecture, proposed by Francis Guthrie in 1852.[2] The conjecture was proved by Alfred Kempe in 1879 and Peter Guthrie Tait in 1880. The problem was considered 'solved' and the two proofs of the relative theorem remained unchallenged for about eleven years. Nevertheless, Kempe's proof was shown to be false by Percy Heawood in 1890, while a gap in Tait's argument was found by the Danish mathematician Julius Petersen in 1891 (Thomas 1998). Many attempts to prove the theorem followed. In some cases these attempts did not survive, while in other cases the proofs were corrected and improved. More than a century of developing the necessary theoretical machinery would have pass before it was established that Francis Guthrie's conjecture was true. The complete proof of the Four-Color theorem was finally achieved in 1976 by Kenneth Appel and Wolfgang Haken, with the help of an IBM 360 in Urbana (Appel and Haken 1977).[3] The mathematical tools developed during this century-long story strongly contributed to the development of new stems and branches of mathematics, one being graph

[2] Guthrie's conjecture that four colors are sufficient to color the world map so that adjacent countries receive distinct colors is equivalent to the mathematical statement that any plane graph is 4-face-colorable.

[3] The authors published a revised version of their proof in Appel and Haken (1989).

theory.[4] For the unconvinced reader who is thinking that, after all, a conjecture is not a well respected piece of mathematical knowledge until it is confirmed or disconfirmed, here is another example of the fallible and corrigible character of mathematics.[5] Lagrange's multiplier rule, originally introduced by Lagrange in his *Mécanique Analytique*, was subjected to several allegedly correct proofs during the 19th century. Nevertheless, all these proofs were later seen as flawed and it was only in 1906 that Hilbert gave a proof filling all the gaps of the previous demonstrations (Goldstine 1980).

But the unconvinced reader is still unconvinced. Why? First, even if we acknowledge that there are proofs that are not correct, we can always say that these are not part of our best mathematical knowledge (indeed, they are rectified at a later stage). Second, in mathematics we usually assess statements (theorems) deductively, from an established set of axioms. Therefore, once the axioms have been correctly set in place, no failure will follow. If this is what we assume mathematics to be about, then Appel and Haken's computer proof does not count as a proof (not in the traditional sense, at least). Third, and here the platonist jumps into the discussion, the fact that some mathematical results are subject to correction can be interpreted as a manifestation of an absolute, not fallible, mathematical knowledge. After all, we only have to get to it. And we can make errors in our search for the truth. That is the mathematical knowledge we are talking about. And that mathematical knowledge is not fallible at all.

I have various replies to these observations, and I address these in turn. First, as it happened in the case of the two false proofs of the four color theorem, a false mathematical proof can be seen as a *bona fide* part of our mathematical knowledge. This happens at *a* particular stage of mathematical development, before that particular piece of mathematics is subject to a potential revision (similarly to what happens in empirical science). Consider, for instance, the book *Proofs from THE BOOK* by Martin Aigner and Günter M. Ziegler, now in its 6th edition (Aigner and Ziegler 2018). The proofs in that book are generally considered correct, however the fact that such book is now in its 6th edition well supports the idea that some proofs may be subject to improvements, and even corrections, in the future. Thus, if proofs *are* part of what we consider as mathematical knowledge, and if proofs are corrigible, then mathematical knowledge should be seen as corrigible. Second, even if within an axiomatic system we assess the truth or falsity of mathematical statements, what about the axioms themselves? What about the axiom of choice and its justification? Don't we accept it as a self-evident, intrinsically necessary, fundamental principle? What about the truth value of the continuum hypothesis? Do we have reasons to discard the possibility that, at a later stage of development of mathematics, the continuum hypothesis will come out as false? Furthermore, although some mathematicians do not consider a computer-based proof as a mathematical

[4] The history of the four-color theorem and its import for the development of various areas of modern mathematics are examined in detail in Fritsch and Fritsch (1998) and Wilson (2013).

[5] An analysis of the important and guiding role that conjectures play in mathematics is offered in Mazur (1997).

proof (and, indeed, the question of a computer-free proof of the four-color theorem still remains), this does not mean that the same mathematicians question the validity of results such as that obtained by Appel and Haken. In fact, Appel and Haken's result is generally accepted as valid by mathematicians.[6]The point is that within the development of mathematics several 'BOOKS' have been written, and some of them contained mathematical results that were considered as *bona fide* pieces of mathematical knowledge according to some standards (e.g., diagram-based reasoning, or even physical principles used in the context of geometry, as shown by Archimedes' works). But some of these results were proven to be false in a later stage. And therefore what was considered as indubitable and infallible according to some standards, at a precise historical moment, was indeed fallible. Finally, concerning the third remark put forward in the previous paragraph, I agree that the existence of errors and corrections in mathematics does not threaten a platonist point of view (indeed, it is not my intention here to propose such a criticism). Nevertheless, the fact that mathematical results are improved and subject to correction at a later stage does not support the platonist position either. And a parallel with the pessimistic meta-induction argument in philosophy of science well illustrates the point: mathematical results which were (considered as) successful were found to be flawed in some respect, so we have no reason to believe that our currently mathematical results are faultless.[7]

There are many other examples that may be added here and that show the fallible and corrigible character of mathematical knowledge. All these examples, although rarely displayed in a contemporary textbook of mathematics, suggest that the building of that solid edifice we call mathematical knowledge is the result of a complex and not error-free process. In this process some bricks are thrown away, modified, added or even substituted with new blocks that show a better fit with the edifice itself. This is not new. But if mathematics has such a fallible and corrigible character, any specific philosophical position on the nature of mathematical knowledge should account for it. Nevertheless, the contemporary discussion between platonists and nominalists seems to have no account (and interest) about this 'imperfect' aspect of mathematics (except for the platonistic observation that such fallibility is ascribable to mathematicians and not to mathematics, which must be considered a source of absolute and indubitable truth). On the other hand, even if we acknowledge the fallible and corrigible character of mathematics, we are still missing an important part of the story. Mathematics shows an impressive form of inter-subjectivity and we want to grant its results a form of objectivity that makes justice of such inter-subjective character.

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[6] The philosophical significance of Appel and Haken's proof of the four-color theorem and its impact on the notion of proof are analyzed in Tymoczko (1979). The influence of computer science on contemporary mathematics and the challenges that philosophy has to meet when addressing these developments are discussed in Avigad (2008).

[7] It is important to clarify here that with the expression 'mathematical results' I am considering not only statements of theorems but also proofs, which should be included in what we consider 'mathematical knowledge'.

3 The Objectivity of Mathematics

Objectivity is not a univocal notion and it is used to express a complex variety of metaphysical and epistemological views. Typically, what is objective is taken to be independent (or to exist independently) of human thought, as opposed to something that depends (or whose existence is not independent) of human intellectual activity. This notion of objectivity, which is based on the idea of 'ontological independence' (Kölbel 2002), is commonly used to identify mathematics as a body of mind-independent absolute and necessary truths. On this characterization, to consider the content of mathematical knowledge as objective is to accept a realist position (realism in truth value if we consider that the truth value of sentences of a mathematical theory is independent to our ability to establish them; realism in ontology if we consider that the ontology of mathematical discourse is independent of our knowledge of mathematics). Call this 'strong objectivity'. Strong objectivity is frequently adopted among philosophers of mathematics, and it is easy to see how it can accommodate the intersubjective character of mathematics. Nevertheless, there is no *prima facie* reason to prefer this strong notion of objectivity to a notion of objectivity that does not bear on absolute truth and ontological dependence. What I have in mind is a weaker sense of objectivity, which although not dependent on the existence of mathematical objects and on the truth of mathematical theories is nonetheless capable to account for the intersubjective character of mathematics and its success in application.

Before passing to the notion of weak objectivity, let me spend some words on truth evaluability and ontological dependence. What is 'truth' in mathematics? I take truth in mathematics to be 'truth *in* a theory', and not as a synonymous for 'absolute truth' or 'truth *of* a theory' (the first being the mark of realism in ontology while the second of realism in truth value). The sentence 'the sum of the angles of a triangle is 180°' is true in Euclidean geometry, while it is false in Bolyai-Lobachevsky geometry, in which Euclid's fifth postulate is replaced with a different postulate. '3+5=8' is true in Peano arithmetics but it is false in arithmetic mod 5, where 3+5=3. I am aware that this attitude towards truth in mathematics will not satisfy some, but I also consider that it better renders the use of 'truth' that we find in mathematical practice.[8] What about ontological dependence? Although the discussion that I offer below will make clear how my considerations are not framed within a particular ontological attitude, I think that is important to clarify my position here. I consider questions of existence (or non existence) of mathematical objects as independent from the objective and intersubjective character of mathematics. After all, this opinion seems to be largely shared by mathematicians. Philosophers of mathematics interested in ontology usually quote famous mathematicians to support their philosophical standpoints and argue for or against the strongly objective character of mathematics. Nevertheless, the majority of the mathematical community seems to

[8] Ferreirós adopts the same stance toward 'truth': "My use of the word 'truth' at this point must be relativized by implicit or explicit reference to a mathematical theory. This agrees with the practice of most mathematicians; hence it should not be perceived as a shortcoming" (Ferreirós 2015, p. 8).

be neutral to these ontological issues, although regarding mathematics as objective and intersubjective.[9] Thus the question: if it is not about absolute truth and ontology, in what sense mathematical knowledge is objective?

Mathematics can be seen as the result of a sequence of different and interconnected practices (Ferreirós 2015).[10] These practices are based on different activities and therefore they should be considered as distinct. Examples of practices include the practice of counting, measuring, drawing, manipulating objects (these practices are called by Ferreirós ‘technical practices’ and are considered as more elementary, or proto-mathematical, because rooted in our particular cognitive abilities),[11] the practice of calculating and that of using a symbolic framework. Practices are agent-based because are performed by human agents, nonetheless they are not subjective and relative only to the individual because they are adopted and shared within the mathematical community. Furthermore, they are interconnected and they mutually interact. It is precisely in this interaction that symbolic and theoretical mathematical frameworks, as for instance the framework of Euclidean geometry, are defined. And it is this interaction that, according to Ferreirós, constraints mathematical knowledge and makes it objective:

> we have working knowledge of several different practices and strata of knowledge, together with their systematic interconnections. This causes links that restrict the admissible–for instance, when a new framework is being developed–and that are responsible for much of the objectivity of mathematical results and developments. The interplay of practices acts as a constraint and a guide (Ferreirós 2015, p. 39)

I agree with Ferreirós that practices and their interplay *constrain* mathematical knowledge and therefore contribute to its objectivity (its strong inter-subjective character). Nevertheless, I think that within this interplay of practices what confers mathematics its objectivity is the process of crosschecking, a process that can be

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[9] Let me note that it is not my intention here to give an argument against platonism. Moreover, the fact that I consider the notion of objectivity in mathematics as independent from the notion of existence does not mean that I am excluding a possible connection between the two. In this respect, I adopt a skeptic position and I leave to the realist the task to show that the objectivity of mathematics is the manifestation of the existence of a realm of abstract and timeless entities.

[10] The philosophical analysis of history of mathematics and practices of working mathematicians has become an important concern for many philosophers of mathematics since the emergence of anti-foundational works such as Lakatos’ *Proofs and Refutations* (Mancosu 2008; cf. also Tymoczko 1985). Although the term ‘mathematical practice’ is generally used to indicate the way in which mathematicians do mathematics, its use may vary depending on the author. In Kitcher (1984) an analysis of the growth of mathematical knowledge is given in terms of practices, and every practice is peculiar of a particular historical period in the development of mathematics. Differently from Kitcher, Ferreirós considers that different levels of practices can coexist during the same period (Ferreirós 2015, pp. 4–5). In what follows I adopt Ferreirós’ view on practices, which I think offers a better rendering of how mathematics is practiced and develops.

[11] The opinion that some parts of mathematics, as for instance elementary geometry, are grounded in basic cognitive skills is shared by many philosophers of mathematics (cf. Giaquinto 2007). Giuseppe Longo calls “cognitive foundation of mathematics” the project of accounting for the intersubjective and conceptually-stable character of mathematics in terms of early cognitive processes (Longo 2003).

internal or external to mathematics. It is therefore the notion of crosschecking that I see as central to the objectivity of mathematical discourse.

The notion of crosschecking in mathematics, and more particularly the kind of objectivity that can be recovered through an analysis in terms of this notion, is extensively discussed in Friend (2014). Friend considers various kinds of crosscheckings in mathematics, as for instance embeddings or reductions, but in general the process of crosschecking consists in applying one mathematical theory, or even one mathematical result, to check another theory or other mathematical results. In order for the crosschecking be possible, there should be something in common between the checking mathematics and the checked mathematics. These common notions are called by friend 'fixtures' and they are taken as preconditions for crosschecking in mathematics (Friend 2014, p. 151).[12] But what about objectivity? Like Friend, I consider that crosschecking reinforces mathematical results and theories, thus contributing to their inter-subjective character within the mathematical community. It is not an instrument of absolute-truth evaluability but it "supplants the need for absolute truth, absolute and independent ontology, a foundation or for a single orientation" (Friend 2014, p. 152). And it is exactly the failure or success of crosscheckings that confers mathematical results less or more objectivity within the community of mathematicians.[13] As Friend points out:

> Moreover, the crosschecking is robust since it is rigorous. There are plenty of contexts where attempts at cross application do not work. It is not the case that everything in mathematics fits together in any way we choose, and it is the failure of cross-application which is evidence for the objectivity and non-triviality of mathematics. This sort of objectivity is not grounded in an ontology. Rather, some successful instances of fit, or convergence, are evidence for some successful instances of fit and convergence, nothing more (Friend 2014, p. 171)

Thus my thesis is: mathematical practices constrain mathematical knowledge but it is through the crosschecking process that this knowledge acquires more or less objectivity. This kind of objectivity is a weak form of objectivity (as compared to the strong form of objectivity which is dependent on ontology and absolute truth). It is dependent on the agents because practices are made by agents. And therefore not subject-independent in an absolute sense. Nevertheless, practices are shared by mathematicians and thus they should be regarded as partly autonomous from the individual agent. The same autonomy holds for crosscheckings: crosscheckings are made internally (within mathematics) or externally (through the application of

[12] The intuition behind this requirement is easy to catch: if we want to check A using B, there should be a way to 'see' the information and the objects of B (or at least that piece of information which is relevant) from the perspective of A (and viceversa), namely a constant mathematical idea that serves as a basis for comparison between theories.

[13] Friend considers Wright's criteria for objectivity (Wright 1992), and particularly his notions of cognitive command and width of cosmological role, as components of her account of objectivity. Similarly, Shapiro (2011) applies Wright's criteria to the notion of objectivity in mathematics. Although I agree with Friend and Shapiro in considering Wright's criteria as useful in shaping a notion of objectivity in mathematics, I won't discuss this issue here and I will leave it for future work.

mathematics in science), but they are oriented by practices. Thus, on this (weak) notion of objectivity, mathematical knowledge is neither arbitrary nor 'subjective' (e.g., relative only to the individual).

Take, for instance, the case of the Pythagorean theorem. In Greek geometry this result was constrained by different practices: measuring, counting, symbolic manipulation and diagram-based practices. Many geometrical proofs of this theorem were proposed but it is with the development of number theory and particularly algebra that it acquired more 'objectivity'. Through algebraic methods mathematicians were able to 'read' the Pythagorean theorem and its objects from a fresh perspective, thus making the result more stable. Moreover, with the advent of non-Euclidean geometries and the emergence of new ways of practicing mathematics, the theorem was subject to an even stronger crosschecking. Indeed, non-Euclidean geometries did not falsify Euclidean geometry but showed that the scope of its theorems only cover those systems in which the parallel postulate is assumed (although the Pythagorean theorem lost its status of absolute truth; but, again, truth is truth in a theory and this particular theorem holds in Euclidean geometry, where the parallel postulate is assumed). Therefore the objectivity of the Pythagorean theorem was reinforced further. This is an example of what I call 'internal crosschecking', namely a crosschecking that comes from the application of mathematics to mathematics. Nevertheless, I also propose another form of crosschecking that contributes to the objectivity of mathematical knowledge. I call this second form of crosschecking 'external' because it comes from the application of mathematics in science and from the use of empirical principles to justify mathematical theorems. In the case of the Pythagorean theorem, several cases of application in science can be mentioned since the prominent use of Euclidean geometry in physics. Furthermore, several mechanical proof of the Pythagorean theorem can be given (Levi 2009; Kogan 1974). And although these 'physical proofs' have not influenced so much the history of this particular theorem, they define another type of crosschecking that contributes to the objectivity of a mathematical result.[14]

The are some observations that I think are worth adding at this point. First, this notion of weak objectivity well fits with various aspects of mathematical practice, as for instance that of giving different proofs of the same theorem. Why do mathematicians look for different proofs of the same theorem? Why do we have so many proofs for the Pythagorean theorem? One reply might be that mathematics is funny, and mathematicians look for different proofs because they are curious. A

[14] The crosscheckings that come from the application of mathematics in science and the use of physical principles to justify theorems are external because in both cases the interactions do not fall within the boundaries of mathematics. Nevertheless, it is important to stress the difference between the two forms of interactions. We apply mathematics in science when we use mathematics to represent some features of an empirical (physical, biological, etc.) setting and infer informations about it. This sense of applicability has many philosophical facets and has received extensive attention among philosophers (Steiner 2005). What is less known, at least to those philosophers of mathematics with no interest in history of science, is that physical principles can led to establish mathematical results. This second sense of applicability (of physics to mathematics) appears prominently in Archimedes' works, and particularly in his treatise *Geometrical Solutions Derived From Mechanics* (Archimedes 2009). A philosophical discussion of the use of physical principles in mathematics is offered in Urquhart (2008a) and Skow (2013).

more interesting answer would be that they look for different proofs because they are interested in particular aesthetic, pragmatic and epistemic virtues, or because they want to extend a theorem's range of validity, or even because they want to explore a topic through a distinct mathematical technique and search for explanations in mathematics.[15] All these are reasonable answers. But it is without doubt that proving several times the same result from different perspectives makes that result more (or less) stable within the mathematical community. And this is especially true when that result is crosschecked with a different piece of mathematics. Take, for instance, the case the purely analytic proof of the Intermediate Value theorem as provided by Bolzano. Bolzano's guiding ideal in searching for a solely analytic (non-geometrical) proof of this theorem was purity of methods (Detlefsen and Arana 2011). Nevertheless, Bolzano himself considered the geometrical proofs of the theorem as showing *that* the result was true. In this sense, Bolzano's analytical treatment can be seen as a crosschecking that reinforced the objectivity of the result.

The case of Bolzano is particularly interesting because it shows how two different ways of practicing mathematics (one associated with the geometrical treatment and the other with the analytical one) constrained a particular mathematical result (the Intermediate Value theorem). Nevertheless, it is with the crosschecking made by Bolzano that the result acquired more objectivity. Practices define the 'environment' in which the crosscheckings hold, however what I see as really central to the notion of objectivity is the crosschecking process itself. To draw a parallel between the case of Bolzano and that of the Four-Color theorem, it is true that the computer proof achieved in by Kenneth Appel and Wolfgang Haken is accepted by mathematicians as showing that the theorem holds. Nevertheless, many mathematicians are attempting to construct a 'manual proof' of the Four-Color theorem, independent of the use of computers. Here we have, again, two different ways of practicing mathematics (one very recent coming from the use of computers in proving theorems and the other more traditional). It is the interaction of these practices, and more particularly the possibility to crosscheck the result from the standpoint of one practice to that of another, that is responsible for the objectivity of the result.

Secondly, I want to connect the discussion of the present section with the fallible and corrigible character of mathematics discussed above. I argue that the sense of objectivity that I am sketching here makes justice of the fallibility and corrigibility of mathematics. How? Consider the case of the four-color theorem. It is reasonable to suppose that, before spotting the errors made by Kempe and Peter Guthrie Tait in their proofs, mathematicians considered the Four-Color theorem and the relative proofs as authentic and objective pieces of mathematical knowledge. Indeed, this was the case and even influential mathematicians and logicians such as Charles Sanders Peirce regarded Kempe's proof as such (Fritsch and Fritsch 1998, pp. 15–16). It is therefore useful and reasonable to adopt a weak notion of objectiv- ity of mathematical knowledge that does not confer to mathematical knowledge an absolute and infallible character, but that is capable to account for the 'objectivity'

[15] John W. Dawson has recently explored these motivations for re-proving theorems in Chapter 2 of his book *Why Prove it Again? Alternative Proofs in Mathematical Practice* (Dawson 2015).

that mathematicians attributed to Kempe's results. With the discovery of errors the original proofs proposed by Kempe and Peter Guthrie Tait were reconsidered and subject to a process of revision and crosschecking from different areas of mathematics, where different practices and ways of tackling a problem were in use. For instance Percy John Heawood, who discovered the fallacy in Kempe's proof, tackled the Four Color Problem using methods from elementary number theory. And other important steps were made by mathematicians working in different areas, as for example topology and the more recent graph theory. In this fascinating and complex process, which gave birth to new mathematical methods and variants of the original four color theorem, the objectivity of a theorem was constrained by different practices and secured by the many crosscheckings that came with these practices. The crosscheckings conferred the result (and its proofs) more objectivity, but this does not mean that this objectivity is absolute and cannot be weakened by further developments of mathematics. Geometrical knowledge obtained through diagrammatic reasoning was considered by Greek geometers as a source of strong objectivity (Netz 1999), however the same knowledge got a reinforced objectivity (not diagram-based) with the development of mathematics and the possibility to crosscheck geometrical theorems from different areas of mathematics, as for instance in graph theory. The notion of objectivity that I am proposing here poses no problems in accounting for this historical process, and therefore has the merit of providing a genuine mirroring of *how* mathematical knowledge evolves.

Before passing to discuss the implications of the notion of weak objectivity in the context of the applicability of mathematics, let me add a comment on the relations between the weak and the strong sense of objectivity. Strong objectivity builds on the insight that mathematical knowledge is true and absolute. It is therefore easy to see that, on this conception, the weak sense of objectivity poses no problems. For instance, we may claim that a mathematical result is objective in the strong sense and in the weak sense proposed above. This is probably not very interesting for those who regard mathematical knowledge as having a strong objective character. Nevertheless, and this is relevant to the point I want to make in this article, strong objectivity collapses into weak objectivity when we have cases of mathematical knowledge that is subject to revision (e.g the proofs of the four color theorem). Therefore the notion of weak objectivity has the benefit to make us consider 'objective' even those pieces of mathematical knowledge that may be subject to a revision at a further historical moment. On the other hand, the adoption of a notion of strong objectivity is not flexible enough to accomodate such potential revisions without embracing a very optimistic (platonistic) standpoint.

4 The Reasonable Effectiveness of Mathematics in Science

The fact that mathematics successfully applies in the empirical sciences has led philosophers of mathematics to elaborate different accounts of how this is possible. For most platonists, the successful application of mathematics supports a realist commitment to the mathematical entities used in application.[16] Philosophers who adopt this standpoint typically assume a strong reading of objectivity and regard mathematics as a body of absolute truths. Thus the success of mathematics in application is a sign of its objectivity, and the objectivity of mathematics is what makes its effectiveness in science reasonable. Call this a realist-account of application. Nevertheless, the platonist position is not the only option to go and other anti-realist accounts of application have been proposed. Many of these attempts have been proposed by nominalists, but not all of them are nominalistic (cf. Bueno 2016). Furthermore, there are also accounts of application that are neutral on the realism/anti-realism issue in the philosophy of mathematics. Bueno and Colyvan's inferential conception of application, for instance, can be adopted by both realist and anti-realist parties to account for the successful application of mathematics in science (Bueno and Colyvan 2011). In this last section my proposal is not to provide an account of applicability, nor to discuss applicability in connection with ontological issues. What I want to show is that the (weak) sense of objectivity that I have sketched in the previous section is compatible with successful cases of applicability. And this without making the applicability of mathematics an unreasonable, or even arbitrary and accidental, practice. The moral is therefore that successful application of mathematics does not necessarily require strong objectivity of mathematical knowledge.

There are many different ways in which mathematics successfully applies in science, but in general we can distinguish two different types of application: when mathematics (or better a piece of mathematical knowledge) is introduced for application; when mathematics is not specifically introduced for application but later, at a further historical moment, we discover that we can apply it with success. An example of the former is given from the introduction of some pieces of basic geometrical knowledge, for instance when the concept of straight line is introduced to measure the minimum length between two points in real space. Another example is given by the Dirichlet principle, which was introduced in physics (more precisely, in potential theory) in the middle of the 19th century (Monna 1975). In these cases the origins of a mathematical concept and the associated result can be sometimes traced in a non-mathematical idea, and mathematics is specifically introduced in the context of a physical application. The key role of physics is to "produce an intuitive 'natural' context for various abstract mathematical constructions" (Atiyah et al. 2010, p. 915). It is no mystery, then, that the same result will reapplied successfully again in a similar setting (e.g., obtaining the minimal distance between two points through a straight line). And it is plausible to think that, in these cases, it is the successful introduction of mathematics for application (and the possibility to repeat application) than confers mathematical knowledge some objectivity (through what

16 This attitude is reflected in the role that applied mathematics plays, according to the platonist, in the enhanced (or explanatory) indispensability argument (Baker 2009).

I call external external crosschecking), and not the objectivity of mathematics that makes the application successfully.[17]

As for the second type of application, a famous example is that of group theory in quantum mechanics. Group theory was developed at the beginning of the twentieth century, however it was applied with success in quantum mechanics only later. Another example is given by non-Euclidean geometries, which were developed long before Einstein used them in the context of spacetime theories. Philosophers have provided accounts of this successful applicability (see, for instance, French 2000). Whatever the account, the brute fact is that some pieces of mathematical knowledge (not all mathematical knowledge) find an application later their introduction into the corpus of mathematics. This application is successful for some reason and represents a form of external crosschecking that contributes to the objectivity of the mathematical knowledge in question. This is particularly true in the context of modern applications (such as the case of group theory and quantum mechanics), where the 'physical' content of the scientific theory is highly mathematized, and therefore to have a successful application often amounts to have a sort of internal crosschecking with other mathematical results that model the physical theory.

5 Conclusions

Four years after the publication of Putnam's article cited in the introduction, Reuben Hersh made a famous assertion that strongly resonates with Putnam's closing lines: "It is reasonable to propose a new task for mathematical philosophy: not to seek indubitable truth, but to give an account of mathematical knowledge as it really is—fallible, corrigible, tentative and evolving, as is every other kind of human knowledge" (Hersh 1979, p. 43). The conception of objectivity I am proposing here goes in the direction suggested by Hersh. I argued that the objective, intra-subjective character of mathematical knowledge is the result of crosscheckings that come internally (within mathematics) or externally (from the application of mathematics in science or even from the application of physics to mathematics).

Does this mean we should give up the idea of strong objectivity in mathematics? No, I think we don't. Although the conception of objectivity sketched here is better suited to fit within an anti-realist account of mathematical knowledge and application, and I suppose that various forms of structuralism are potentially compatible with the sense of weak objectivity that I propose, I didn't rule out the possibility

[17] There also also more tricky, though similar, cases. One of them is the case of the delta function, introduced in Dirac's formulation of quantum mechanics to represent the mass density function of a point particle of mass 1 situated at the origin. Mathematically speaking, the delta function was defined on the real line so that it was zero everywhere except at the origin, with integral equals to 1. However, this function can't be defined on the classical real line. The Dirac function was therefore introduced for application, but it was accepted as a legitimate piece of mathematical knowledge only after its interpretation as a distribution (Urquhart 2008b). In this case, the objectivity of the mathematical result was not provided by the application itself but rather from the internal crosschecking that came later within mathematics (embedding of the delta function in the theory of distributions).

that a particular form of realism (in ontology or in truth value) may be on the right track. And therefore the doors (of Plato's heaven) are still open to those who want to embrace a strong notion of objectivity for mathematical knowledge.

Finding a positive characterization of what makes mathematics objective is hard. My modest proposal here has been to outline a notion of objectivity that, being separated from the notions of truth and existence, well renders the way mathematical knowledge structures and evolves. Further analysis and case studies are needed to make this notion more precise and convincing.

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