Appendix S1 - Mathematical dissertation on the proposed algorithms

From zero to infinity: minimum to maximum diversity of the planet by spatio-parametric Rao's quadratic entropy

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1 Hill's numbers and generalized entropy

Hill (1973) expressed parametric diversity as the "numbers equivalent" of Rényi's generalized entropy, as:

$$K_\alpha = \frac{1}{\left(\sum_{i=1}^{N} p_i \times p_i^{\alpha-1}\right)^{\frac{1}{\alpha-1}}} \tag{1}$$

where the numbers equivalent K_α is the theoretical number of equally-abundant DNs (i.e. all those with $p_i = \frac{1}{K_\alpha}$) that are needed in order that its diversity be H_α (Patil & Taillie, 1982).

Hill's K_α has the form of the reciprocal of a generalized mean of order $\alpha - 1$. Jost (2006) further showed that, like for H_α, the numbers equivalents of all parametric and non-parametric measures of diversity that can be expressed as monotonic functions of $\sum p_i^\alpha$ have the form of the reciprocal of a generalized mean of order $\alpha - 1$ (for details, Jost , 2006).

2 Mathematical proof: for $\alpha \to 0$ Q_0 is the geometric mean among the generalized means, for $\alpha \to \infty$ Q_∞ is the maximum distance between pixel values pairs

We want to compute

$$\lim_{\alpha \to 0} Q_\alpha \quad \text{where} \quad Q_\alpha = \left(\sum_{i,j=1}^{N} \frac{1}{N^2} d_{ij}^\alpha\right)^{\frac{1}{\alpha}}. \tag{2}$$

By $\exp(\log(x)) = x$ we can rewrite Q_α as

$$Q_\alpha = \left(\sum_{i,j=1}^{N} \frac{1}{N^2} d_{ij}^\alpha\right)^{\frac{1}{\alpha}} = \exp\left(\log\left(\sum_{i,j=1}^{N} \frac{1}{N^2} d_{ij}^\alpha\right)^{\frac{1}{\alpha}}\right) = \exp\left(\frac{1}{\alpha}\log\left(\sum_{i,j=1}^{N} \frac{1}{N^2} d_{ij}^\alpha\right)\right)$$

reminding that if $N > 1$, there is at least one distance $d_{ij} > 0$. We use this last expression to calculate (2). We use the following two well known results.

Theorem 1 (De l'Hôpital)**.** *Let $f_1, g_1 : (a,b) \mapsto \mathbb{R}$ be two functions such that*

- $\lim_{x\to a} f_1(x) = \lim_{x\to a} g_1(x) = 0$
- *f_1 and g_1 are differentiable in (a,b) with $g_1'(x) \neq 0$ for every $x \in (a,b)$*
- *the limit $\lim_{x\to a} \frac{f_1'(x)}{g_1'(x)} = L$ with $L \in \mathbb{R}$*

then

$$\lim_{x\to a} \frac{f_1(x)}{g_1(x)} = L.$$

Theorem 2 (Limit composition)**.** *Let $f_2 : (a,b) \mapsto \mathbb{R}$ and let $g_2 : (c,d) \mapsto \mathbb{R}$ be two functions such that the image set of g_2 is contained in the domain of f_2, i.e. $\mathcal{I}mg(g_2) \subseteq (a,b)$. Let $x_0 \in (c,d)$, if it holds that*

- *$\lim_{x\to x_0} g_2(x) = y_0$ with $g_2(x) \neq y_0$ definitely for $x \to x_0$*
- $\lim_{y\to y_0} f_2(y) = l$

with $a, b, c, d, x_0, y_0, l \in \mathbb{R} \cup \pm\infty$ then

$$\lim_{x\to x_0} (f_2 \circ g_2)(x) = l.$$

We apply Theorem (2) to calculate the limit (2) with $f_2(x) = \exp(x)$ and

$$g_2(\alpha) = \frac{1}{\alpha}\log\left(\sum_{i,j=1}^{N} \frac{1}{N^2} d_{ij}^\alpha\right).$$

(all assumptions of the theorem hold). Setting $x_0 = 0$, we have to compute

$$\lim_{\alpha \to 0} g_2(\alpha). \tag{3}$$

which will be accomplished using Theorem (1) by setting $f_1 : (0, +\infty) \mapsto \mathbb{R}$

$$f_1(\alpha) = \log \left(\sum_{i,j=1}^{N} \frac{1}{N^2} d_{ij}^{\alpha} \right)$$

and $g_2 : (0, +\infty) \mapsto \mathbb{R}$, $g_2(\alpha) = \alpha$. Then we have

$$f_1(0) = \lim_{\alpha \to 0} f_1(\alpha) = \log(\frac{1}{N^2} \sum_{i,j=1}^{N} 1) = \log(1) = 0$$

as the limit exists and

$$g_1(0) = \lim_{\alpha \to 0} g_1(\alpha) = 0.$$

Both functions f_1 and g_1 are differentiable. Lastly we observe that $g_1'(\alpha) \equiv 1$. Since all the assumptions of Theorem 1 hold then

$$\begin{aligned} \lim_{\alpha \to 0} \frac{f_1(\alpha)}{g_1(\alpha)} &= \lim_{\alpha \to 0} \frac{f_1'(\alpha)}{g_1'(\alpha)} = \lim_{\alpha \to 0} \frac{(\frac{1}{N^2} \sum_{i,j=1}^{N} d_{ij}^{\alpha})^{-1} (\frac{1}{N^2} \sum_{i,j=1}^{N} d_{ij}^{\alpha} \log d_{ij})}{1} \\ &= \frac{1}{N^2} \sum_{i,j=1}^{N} \log d_{ij} = \sum_{i,j=1}^{N} \log(d_{ij})^{\frac{1}{N^2}} = \prod_{i,j=1}^{N} \log(d_{ij}^{\frac{1}{N^2}}) \end{aligned} \tag{4}$$

By Equation (4) we have the expression of Equation 3. Let

$$y_0 = \prod_{i,j=1}^{N} \log(d_{ij}^{\frac{1}{N^2}})$$

and we conclude by observing

$$\lim_{y \to y_0} \exp(y) = \exp \left(\prod_{i,j=1}^{N} \log(d_{ij}^{\frac{1}{N^2}}) \right) = \prod_{i,j=1}^{N} \exp(\log(d_{ij}^{\frac{1}{N^2}})) = \prod_{i,j=1}^{N} d_{ij}^{\frac{1}{N^2}} = \sqrt[N^2]{\prod_{i,j=1}^{N} d_{ij}}.$$

Now we want to compute

$$\lim_{\alpha \to +\infty} Q_\alpha \quad \text{where} \quad Q_\alpha = \left(\sum_{i,j=1}^{N} \frac{1}{N^2} d_{ij}^{\alpha} \right)^{\frac{1}{\alpha}}$$

We define $d = \max\{d_{ij} | i, j \in \{1, \ldots, N\}\}$ and we rewrite Q_α as

$$Q_\alpha = \left(\sum_{i,j=1}^{N} \frac{1}{N^2} d_{ij}^\alpha \right)^{\frac{1}{\alpha}} = \left(\sum_{i,j=1}^{N} \frac{1}{N^2} d^\alpha \left(\frac{d_{ij}}{d} \right)^\alpha \right)^{\frac{1}{\alpha}} = d \left(\sum_{i,j=1}^{N} \frac{1}{N^2} \left(\frac{d_{ij}}{d} \right)^\alpha \right)^{\frac{1}{\alpha}}$$

Next we observe that

$$\frac{d_{ij}}{d} \leq 1$$

by construction and there exist a pair $(\bar{i}, \bar{j})$ such that $\frac{d_{\bar{i},\bar{j}}}{d} = 1$. Therefore it follows that

$$\sum_{i,j=1}^{N} \frac{1}{N^2} \left(\frac{d_{ij}}{d} \right)^\alpha = \frac{1}{N^2} \sum_{i,j=1}^{N} \left(\frac{d_{ij}}{d} \right)^\alpha = \frac{1}{N^2} \left(1 + \sum_{\substack{i,j=1 \\ (i,j) \neq (\bar{i},\bar{j})}}^{N} \left(\frac{d_{ij}}{d} \right)^\alpha \right) \leq 1$$

for every $\alpha > 1$. And the limit in (4) is

$$\lim_{\alpha \to +\infty} d \left(\sum_{i,j=1}^{N} \frac{1}{N^2} \left(\frac{d_{ij}}{d} \right)^\alpha \right)^{\frac{1}{\alpha}} = d = \max_{i,j} d_{ij}.$$

References

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