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Qualitative properties of solutions to parabolic anisotropic equations, part II — The anisotropic Trudinger's equation

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Abstract

We study the local regularity properties of weak solutions to a special class of anisotropic doubly nonlinear parabolic operators, whose prototype is the anisotropic Trudinger's equation

$$u_t - \sum_{i=1}^{N} D_i\left(u^{2-p_i}|D_i u|^{p_i-2}D_i u\right) = 0, \qquad u \geqslant 0.$$

We prove a parabolic Harnack inequality, valid without any restrictions on the exponents p_is. When the range of diffusion exponents is restricted, solutions are Hölder continuous.

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Contents

1 | INTRODUCTION AND MAIN RESULTS

1.1 | Introduction

In the framework of our program on the qualitative properties of local weak solutions to parabolic anisotropic operators, here our motivation stems from the classic research in [49] and the relatively recent studies in [24, 32, 36, 37, 40], where the regularity results such as the local boundedness, Harnack's inequality, and Hölder continuity of solutions to doubly nonlinear parabolic equations

$$u_t - div\left(u^{2-p}|Du|^{p-2}Du\right) = 0, \quad u \geqslant 0,$$

were proved. Such type of equations in the literature are known as Trudinger's equations, following his pioneering work in [49]. In this paper, we are concerned with nonnegative solutions to anisotropic Trudinger's equations with measurable and bounded coefficients, whose prototype is

$$u_t - \sum_{i=1}^{N} D_i\left(u^{2-p_i}|D_i u|^{p_i-2}D_i u\right) = 0, \quad u \geqslant 0. \tag{1.1}$$

We observe that when $p_i \equiv p$, Equation (1.1) enjoys a similar structure to Trudinger's equation, with the only difference that the modulus of ellipticity depends on each directional derivatives. In this case, the change of variables $v = u^{\frac{1}{p-1}}$ turns Equation (1.1) into

$$\left(v^{p-1}\right)_t + \sum_{i=1}^{N} \gamma_i D_i \left(|D_i v|^{p-2} D_i v\right) = 0\,, \qquad \gamma_i > 0\,, \tag{1.2}$$

whose parabolic and elliptic energy terms are homogeneous with respect to the multiplication by a constant. The original idea of [49] was that this homogeneity makes the low-order regularity theory simpler, but on the other hand, the treatment of the time derivative is encumbered, at the point that adding or subtracting a constant to a solution does not produce a solution to the same kind of equation. For this reason, the study of the Harnack inequality and of the Hölder continuity seems to be dinstinct, see, for instance, [2] and [24]. Here, below we address both studies, for operators whose prototype is (1.1). More precisely, let $\Omega \subset \mathbb{R}^N$ be open and bounded, $T > 0$ and let us consider numbers

$$1 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_N < \infty\,.$$

We study the local regularity properties of nonnegative solutions to the equation

$$u_t - \sum_{i=1}^{N} D_i A_i(x,t,u,Du) = 0, \quad \text{locally weakly in} \quad \Omega_T := \Omega \times (0,T), \tag{1.3}$$

where $A_i(x,t,s,\xi) : \Omega_T \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ are Carathéodory functions, satisfying the following structure conditions:

$$\begin{cases} A_i(x,t,s,\xi)\,\xi_i \geqslant K_1\, s^{2-p_i}\,|\xi_i|^{p_i}, & \forall (s,\xi) \in \mathbb{R}_+ \times \mathbb{R}^N, \\ |A_i(x,t,s,\xi)| \leqslant K_2\, s^{2-p_i}\,|\xi_i|^{p_i-1}, & \forall\, i = 1, \dots, N, \end{cases} \tag{1.4}$$

for positive constants K_1, K_2. The formulation of the structure conditions (1.4) can be rephrased under the assumption of the chain rule, see Section 8.1 for the details.

1.2 | Main results

Our first result is the Harnack inequality. The special feature of the following theorem being its validity regardless of any restrictions on the gap between p_N and p_1.

The parameters $N, p_1, \dots, p_N, K_1, K_2$ will be referred to as *the data*.

Theorem 1.1. *Let u be a nonnegative, local weak solution to (1.3)–(1.4) in Ω_T. There exist positive constants $C, \bar{C}$ depending only on the data such that, for all cylinders*

$$\mathcal{K}_{8\rho}(x_0) \times (t_0 - \bar{C}(8\rho)^p, t_0 + \bar{C}(8\rho)^p) \subset \Omega_T\,,$$

there holds

$$\frac{1}{C} \sup_{\mathcal{K}_\rho(x_0)} u(\cdot, t_0 - \bar{C}\,\rho^p) \leqslant u(x_0,t_0) \leqslant C \inf_{\mathcal{K}_\rho(x_0)} u(\cdot, t_0 + \bar{C}\,\rho^p), \tag{1.5}$$

where

$$\mathcal{K}_\rho(x_0) := \prod_{i=1}^N \left\{ |x_i - x_{0i}| < \rho^{\frac{p}{p_i}} \right\} \quad \text{and} \quad \frac{1}{p} := \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i}.$$

Our second result is the local Hölder continuity of solutions. The proof distinguishes between the cases when u is large or close to zero. In the former case, the equation can be regarded as the anisotropic p-Laplacian (cf. [13, 15]), whose regularity theory is known only under the additional assumption that the gap between p_N and p_1 is sufficiently small (see Lemma 8.2). For this reason, a constraint on the spareness of the p_is appears. See also [21] for this technique in the case of porous medium equations.

Theorem 1.2. *Let u be a nonnegative, local weak solution to (1.3)–(1.4). There exists a number $\epsilon_* \in (0,1)$ depending only on the data such that, if*

$$p_N - p_1 \leqslant \epsilon_*, \tag{1.6}$$

then u has a locally Hölder continuous representative.

1.3 | Novelty and significance

The issues touched in here have been considered in the standard case $p_i \equiv p$ starting with Caffarelli and Friedman [6], Caffarelli and Evans [7], DiBenedetto and Friedman [17], and DiBenedetto [22] and continued in [1, 18, 19, 21, 29, 30, 45, 51, 52]. Since Trudinger's equation seems to be one of the few nonlinear parabolic equation that admits a Harnack inequality just as for the heat equation,† it is the natural candidate for generalizing these results to nonlinear equations with nonstandard growth, particularly to anisotropic equations. The study of regularity properties of solutions to parabolic equations with nonstandard growth goes back to the 1970's. Kolodii [33, 34] gave his seminal contribution on the local boundedness after which several other came along, see, for example, [43, 46, 47]. At the same time, the basic qualitative properties such as continuity and the Harnack inequality of solutions to anisotropic elliptic and parabolic equations are still poorly understood. Regarding the continuity of solutions, surprisingly, the most extensively studied case is the anisotropic porous medium equation [5, 25–28]. To the best of our knowledge, partial results for anisotropic elliptic equations can be found in [12, 14, 41, 42] or in [20] under the additional condition that the gap between p_N and p_1 is small enough. For the elliptic prototype, the authors in [4] show that locally bounded weak solutions are Lipschitz continuous, and hence continuous, without any restrictions on the spread of the p_is. As for the anisotropic parabolic equations, some Harnack-type inequalities were obtained [8, 9, 13, 15]. Questions on boundedness in a wider anisotropic regime were addressed in [10], while in [11], the authors proved the expansion of positivity. In this paper, we continue the study of the regularity properties for anisotropic parabolic equations started in [10, 11], namely, we prove Harnack's inequality and Hölder continuity of weak solutions to a class of anisotropic parabolic equations having as prototype the Trudinger's equation (1.3). In this context, it is a striking fact that the boundedness (proved in [10]) and Harnack's inequality for solutions of such equations can be proved without any restrictions on the numbers p_i and have the same form as for the heat equation (cf. Theorem 1.1).

1.3.1 | The method

In [16], Ennio De Giorgi developed an original geometric method for the boundedness and regularity of solutions to elliptic equations with discontinuous, but measurable and bounded coefficients. The fundamental ideas of this technique have been successfully applied to get regularity for local minima of calculus of variations and/or solutions to the corresponding elliptic and parabolic equations with standard growth (for an exhaustive overview on the subject, see, for example, [21, 22, 38, 39]). The core ideas of our proofs are based on this method and also on Krylov and Safonov [35] procedure, when proving the Harnack inequality, and on the intrinsic scaling method (developed by DiBenedetto [22]) when proving the Hölder continuity of solutions. Although the strategy for the proof of our main results goes as in the standard p-growth, we had to overcome the difficulties inherent to the presence of the anisotropy. The proof of the Hölder continuity follows the lines of [36] and [37], with the complication that the case where u is away from zero does not allow us to transform (1.1) into the p-Laplacian operator, for which there would have been already a complete picture. Instead, it turns (1.1) into the anisotropic p_i-Laplacian, for which the regularity theory is still under development.

† Meaning with this that it does not require to consider an intrinsic geometry (see for this [22]).

1.3.2 | Structure of the paper

The paper is organized as follows. In Section 2, we collect some preliminary and technical properties, being Section 3 devoted to the proof of a De Giorgi type lemma. Expansion of positivity is proved in Section 4. Our first main result, the Harnack inequality, is proved in Section 5. Finally, in Section 6, we prove Hölder continuity of weak solutions to (1.3). Motivation for the functional framework, the definition of solution with the exponential mollification, and other tools of the trade are collected in the Appendix.

2 | NOTATION

2.1 | Indexes

From now on, we assume that the numbers $\{p_1, p_2, \dots, p_N\}$ are fixed. In order to ease notation, we further assume, without loss of generality, that

$$1 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_N < \infty$$

and we set

$$\frac{1}{p} := \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} = \frac{1}{N} \sum_{i} \frac{1}{p_i}.$$

When index i is concerned, we will always reduce the set of indexes as in the previous formula, also for the product $\prod_{i=1}^{N} = \prod_i$. Given K_1, K_2 by the structure conditions (1.4), we will refer to the set of parameters $\{N, p_1, \dots, p_N, K_1, K_1\}$ as the set of (structural) *data*, and by saying that a constant γ depends only on the data we mean that it can be quantitatively determined *a priori* only in terms of the above quantities. As usual, the generic constant γ may change from line to line.

2.2 | Geometry

For fixed $r, \eta > 0$ and a point $(\bar{x}, \bar{t}) \in \mathbb{R}^{N+1}$, we define the standard cube

$$K_r(\bar{x}) = \left\{ x \in \mathbb{R}^N : \quad |x_i - \bar{x}_i| < r, \quad i = 1, \dots, N \right\},$$

the anisotropic cubes

$$\mathcal{K}_r(\bar{x}) := \left\{ x \in \mathbb{R}^N : \quad |x_i - \bar{x}_i| < r^{\frac{p}{p_i}}, \quad i = 1, \dots, N \right\},$$

and the cylinders

$$Q_{r,\eta}(\bar{x}, \bar{t}) := \mathcal{K}_r(\bar{x}) \times (\bar{t} - \eta, \bar{t}).$$

Finally, for fixed $k, r, \eta > 0$, we define the intrinsic anisotropic cubes

$$\mathcal{K}_r^k(\bar{x}) := \left\{ x \in \mathbb{R}^N : \quad |x_i - \bar{x}_i| < r^{\frac{p}{p_i}} k^{-\frac{p_N - p_i}{p_i}}, \quad i = 1, \dots, N \right\},$$

and cylinders

$$Q^k_{r,\eta}(\bar{x}, \bar{t}) := \mathcal{K}_r^k(\bar{x}) \times (\bar{t} - \eta, \bar{t}).$$

3 | FUNCTIONAL SETTING AND AUXILIARY RESULTS

3.1 | Functional anisotropic setting and parabolic embeddings

Equation (1.1) has to be understood in an appropriate variational meaning. For this aim, we define the anisotropic spaces of locally integrable functions as

$$W^{1,\mathbf{p}}(\Omega) = \left\{ u \in W^{1,1}(\Omega) : D_i u \in L^{p_i}(\Omega), \quad i = 1, \dots, N \right\},$$

$$L^{\mathbf{p}}(0, T; W^{1,\mathbf{p}}(\Omega)) = \left\{ u \in L^1(0, T; W^{1,1}(\Omega)) : D_i u \in L^{p_i}(0, T; L^{p_i}(\Omega)), \quad i = 1, \dots, N \right\},$$

and the respective spaces of functions with zero boundary data

$$W_0^{1,\mathbf{p}}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,\mathbf{p}}(\Omega),$$

$$L^{\mathbf{p}}\left(0, T; W_0^{1,\mathbf{p}}(\Omega)\right) = L^1\left(0, T; W_0^{1,1}(\Omega)\right) \cap L^{\mathbf{p}}(0, T; W^{1,\mathbf{p}}(\Omega)),$$

for $\mathbf{p} = (p_1, \dots, p_N)$. The following lemma was originally studied in [48], see [23] for a modern proof.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, $u \in W_0^{1,1}(\Omega)$ and let us assume*

$$\sum_i \int_\Omega |D_i |u|^{\alpha_i}|^{p_i} dx < \infty, \quad \text{for} \quad \alpha_i > 0 \quad \text{and} \quad p_i > 0 \text{ such that} \quad p < N.$$

Then there exists $c > 0$, depending on $\{N, p_1, \dots, p_N, \alpha_1, \dots, \alpha_N\}$, such that

$$\left(\int_\Omega |u|^{p^*_\alpha} dx \right)^{\frac{N-p}{N}} \leqslant c \prod_i \left(\int_\Omega |D_i |u|^{\alpha_i}|^{p_i} dx \right)^{\frac{p}{N p_i}}, \quad p^*_\alpha = \frac{N \alpha p}{N - p}, \quad \alpha = \frac{1}{N} \sum_i \alpha_i. \tag{3.1}$$

By using Hölder's inequality and (3.1), the following lemma is valid too.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, $u \in L^1(0, T; W_0^{1,1}(\Omega))$, $p < N$ and let also*

$$\sup_{0<t<T} \int_\Omega |u|^\beta \, dx < \infty, \quad \beta > 0,$$

and

$$\sum_i \int_0^T \int_\Omega |D_i |u|^{\alpha_i}|^{p_i} dx \, dt < \infty, \quad \alpha_i > 0, \quad p_i > 1, \quad i = 1, \dots, N.$$

Then there exists $c > 0$, depending on $N, p_1, \dots, p_N, \alpha_1, \dots, \alpha_N$, such that

$$\int_0^T \int_\Omega |u|^{p\left(\alpha+\frac{\beta}{N}\right)} dx\, dt \leqslant c\left(\sup_{0<t<T} \int_\Omega |u|^\beta\, dx\right)^{\frac{p}{N}} \prod_i \int_0^T \left(\int_\Omega |D_i|u|^{\alpha_i}|^{p_i} dx\right)^{\frac{p}{Np_i}} dt$$

$$\leqslant c\left(\sup_{0<t<T} \int_\Omega |u|^\beta\, dx\right)^{\frac{p}{N}} \left[\sum_i \int_0^T \int_\Omega |D_i|u|^{\alpha_i}|^{p_i} dx\, dt\right]. \tag{3.2}$$

A closer inspection of our proofs (Lemma 4.1 here below, and Lemma 3.4 of [10]) reveals, as in Remark 3.9 of [44], that the condition $p < N$ can be removed.

3.2 | Definition of weak solution

Here below we define the notion of local weak solution to Equation (1.3). For the motivation lying behind this definition and an alternative definition that takes into account an approximated time derivative, we refer to Section 8.2. A measurable function $u : \Omega_T \to [0, \infty)$ such that

$$\begin{cases} u \in C_{loc}\left(0, T; L_{loc}^{\frac{p_N}{p_N-1}}(\Omega)\right), \quad u^{\frac{1}{p_N-1}} \in L_{loc}^{\mathbf{p}}\left(0, T; W_{loc}^{1,\mathbf{p}}(\Omega)\right), \\ D_i(u^{\frac{1}{p_i-1}}) \in L_{loc}^{p_i}\left(0, T; L_{loc}^{p_i}(\Omega)\right), \qquad \text{for all} \quad i = 1, \dots, N, \end{cases} \tag{3.3}$$

is a nonnegative, local, weak sub (super)-solution to (1.3), if for every compact set $E \subset \Omega$ and every subinterval $[t_1, t_2] \subset (0, T]$

$$\int_E u\,\varphi\, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_E \left[u\,\varphi_t + \sum_i A_i(x, t, u, Du)\, D_i\varphi\right] dxdt \leqslant (\geqslant) 0, \tag{3.4}$$

for all nonnegative testing functions

$$\varphi \in W_{loc}^{1,p_N}(0, T; L^{p_N}(E)) \cap L_{loc}^{\mathbf{p}}(0, T; W_0^{1,\mathbf{p}}(E)).$$

We say that u is a nonnegative, local weak solution to Equation (1.3) if u is both a nonnegative, local weak sub- and supersolution.

3.3 | Local energy estimates

Lemma 3.3. *Let u be a nonnegative, local weak sub(super)-solution to (1.3)–(1.4). Then there exists a constant $\gamma > 0$, depending only on the data, such that, for every cylinder $Q_{r,\eta}(y, \tau) \subset \Omega_T$, every level $k > 0$, and every piecewise smooth cutoff function ζ vanishing on $\partial K_r(y)$ and such that $0 \leqslant \zeta \leqslant 1$, there holds*

$$\sup_{\tau-\eta\leqslant t\leqslant\tau} \int_{K_r(y)\times\{t\}} g_\pm(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}})\, \zeta^{p_N}\, dx +$$

$$+\gamma^{-1} \sum_i \iint_{Q_{r,\eta}(y,\tau)} u^{\frac{p_N-p_i}{p_N-1}} |D_i(u^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_\pm|^{p_i} \zeta^{p_N}\, dx\, dt$$

$$\leqslant \int_{K_r(y)\times\{\tau-\eta\}} g_\pm(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}})\, \zeta^{p_N}\, dx + \gamma \iint_{Q_{r,\eta}(y,\tau)} g_\pm(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}})\, |\zeta_t|\, dxdt +$$

$$+\gamma \sum_i \iint_{Q_{r,\eta}(y,\tau)} u^{\frac{p_N-p_i}{p_N-1}} (u^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_\pm^{p_i} |D_i\zeta|^{p_i}\, dx\, dt\,. \tag{3.5}$$

We postpone the nowadays standard proof of (3.5) to Section 8.3, in order to leave space to what is really new. At this stage, we only remark that the assumption $u \in L_{loc}^{\frac{p_N}{p_N-1}}(\Omega_T)$ of (3.3) implies that each integral above is convergent. Indeed, as usual, this assumption is crucial for the extrapolation of (3.5) from Equation (1.3)–(1.4).

3.4 | Local upper bound

Local weak subsolutions to (1.3)–(1.4) are bounded above quantitatively. This property can be extracted from the study performed in [10] for De Giorgi classes of functions satisfying doubly nonlinear parabolic equations that exhibit nonstandard growth conditions.

Lemma 3.4. *Let u be a nonnegative, local, weak subsolution to (1.3)–(1.4), then for any cylinder $Q_{8\rho,(8\rho)^p}(x_0,t_0) \subset \Omega_T$, there holds*

$$\sup_{Q_{\rho,\rho^p}(x_0,t_0)} u \leqslant \gamma \left(\frac{1}{\rho^{N+p}} \iint_{Q_{2\rho,(2\rho)^p}(x_0,t_0)} u^{\frac{p_N}{p_N-1}}\, dxdt \right)^{\frac{p_N-1}{p_N}}, \tag{3.6}$$

with a constant $\gamma > 0$ depending only on the data.

4 | A DE-GIORGI-TYPE LEMMA

The following lemma is an anisotropic version of the classic De-Giorgi-type lemma (see [18, 19, 21, 22]). Although a more general version of it could be extracted from the proof of [11, Lemma 2.4], we decided to report the full proof here, in order to show that for the special case (1.3), there is no need of intrinsic geometry (see [22] or [50] for more details) since here the parabolic and the elliptic energy terms suitably balance the De-Giorgi-type iteration.

Lemma 4.1. *Let $a \in (0,1)$ and $k > 0$ be fixed real numbers, and let u be a nonnegative, local weak supersolution to (1.3)–(1.4). Then, there exists $\nu_- \in (0,1)$, depending only on the data, a, r, and η as in (4.3), such that if*

$$|Q_{r,\eta}(y,\tau) \cap \{u \leqslant k\}| \leqslant \nu_- |Q_{r,\eta}(y,\tau)|, \tag{4.1}$$

then

$$u(x,t) \geqslant a\,k, \quad (x,t) \in Q_{\frac{r}{2},\frac{\eta}{2}}(y,\tau). \tag{4.2}$$

Proof. For $j = 0, 1, 2, \ldots$, we define

$$k_j^{\frac{1}{p_N-1}} = [a\,k]^{\frac{1}{p_N-1}} + (1 - a^{\frac{1}{p_N-1}}) k^{\frac{1}{p_N-1}} 2^{-j},$$

$$r_j := \frac{r}{2}(1+2^{-j}), \quad \eta_j := \frac{\eta}{2}(1+2^{-j}),$$

$$K_j := \mathcal{K}_{r_j}(y), \quad Q_j := K_j \times I_j, \quad I_j := (\tau - \eta_j, \tau).$$

Now, we consider piecewise smooth cutoff functions $\zeta_j := \zeta_j^{(1)}(x)\zeta_j^{(2)}(t)$, where $\zeta_j^{(1)}(x) \in C_0^1(K_j)$ is such that

$$\zeta_j^{(1)}(x) = 1 \quad \text{for} \quad x \in K_{j+1}, \qquad 0 \leqslant \zeta_j^{(1)}(x) \leqslant 1\,,$$

$$\text{and} \qquad \left|D_i \zeta_j^{(1)}(x)\right|^{p_i} \leqslant \frac{\gamma 2^j}{r^p}, i = 1, \dots, N,$$

while $\zeta_j^{(2)}(t) \in C^1(\mathbb{R})$ is such that

$$\zeta_j^{(2)}(t) = 0 \quad \text{for} \quad t \leqslant \tau - \eta_j; \qquad \zeta_j^{(2)}(t) = 1, \quad \text{for} \quad t \geqslant \tau - \eta_{j+1},$$

$$0 \leqslant \zeta_j^{(2)}(t) \leqslant 1, \quad \text{verifying} \quad \left|\zeta_{j,t}^{(2)}\right| \leqslant \frac{\gamma 2^j}{\eta}.$$

Using Lemmas 8.5 and 3.3, we obtain

$$\begin{aligned}
&\sup_{t\in I_j} \int_{K_j \times \{t\}} g_-\left(u^{\frac{1}{p_N-1}}, k_j^{\frac{1}{p_N-1}}\right)\zeta_j^{p_N}\, dx + \gamma^{-1}\sum_i \iint_{Q_j} u^{\frac{p_N-p_i}{p_N-1}} |D_i\left(u^{\frac{1}{p_N-1}} - k_j^{\frac{1}{p_N-1}}\right)_-|^{p_i}\, \zeta_j^{p_N} dx\, dt \\
&\leqslant \frac{\gamma 2^{j\gamma}}{\eta} \iint_{Q_j} g_-\left(u^{\frac{1}{p_N-1}}, k_j^{\frac{1}{p_N-1}}\right) dxdt + \frac{\gamma 2^{j\gamma}}{r^p}\sum_i \iint_{Q_j} u^{\frac{p_N-p_i}{p_N-1}}\left(u^{\frac{1}{p_N-1}} - k_j^{\frac{1}{p_N-1}}\right)_-^{p_i} dx\, dt \\
&\leqslant \gamma 2^{j\gamma}\, k^{\frac{p_N}{p_N-1}}\left(\frac{1}{\eta} + \frac{1}{r^p}\right)|Q_j \cap \{u \leqslant k_j\}|.
\end{aligned}$$

To estimate the left-hand side of this inequality, we define

$$v := \max(u, a\,k),$$

and note that each one of the terms on the left-hand side can be estimated in terms of v as follows:

$$\begin{aligned}
\int_{K_j \times \{t\}} g_-\left(u^{\frac{1}{p_N-1}}, k_j^{\frac{1}{p_N-1}}\right)\zeta_j^{p_N}\, dx &\geqslant \int_{K_j \times \{t\}} g_-\left(v^{\frac{1}{p_N-1}}, k_j^{\frac{1}{p_N-1}}\right)\zeta_j^{p_N}\, dx \geqslant \\
&\geqslant \frac{1}{2}(ak)^{\frac{p_N-2}{p_N-1}} \int_{K_j \times \{t\}} \left(v^{\frac{1}{p_N-1}} - k_j^{\frac{1}{p_N-1}}\right)_-^2 \zeta_j^{p_N}\, dx,
\end{aligned}$$

and also, for $\alpha_i = p_N/p_i$,

$$\begin{aligned}
&\iint_{Q_j} u^{\frac{p_N-p_i}{p_N-1}} |D_i\left(u^{\frac{1}{p_N-1}} - k_j^{\frac{1}{p_N-1}}\right)_-|^{p_i}\, \zeta_j^{p_N}\, dxdt \\
&\geqslant \iint_{Q_j \cap \{u \geqslant ak\}} u^{\frac{p_N-p_i}{p_N-1}} |D_i\left(u^{\frac{1}{p_N-1}} - k_j^{\frac{1}{p_N-1}}\right)_-|^{p_i}\, \zeta_j^{p_N}\, dxdt
\end{aligned}$$

$$= \iint_{Q_j \cap \{u \geqslant ak\}} v^{\frac{p_N - p_i}{p_N - 1}} |D_i \left(u^{\frac{1}{p_N-1}} - k_j^{\frac{1}{p_N-1}} \right)_- |^{p_i} \zeta_j^{p_N} \, dxdt$$

$$\geqslant \gamma^{-1} a^{\frac{p_N - p_i}{p_N - 1}} \iint_{Q_j} |D_i \left(v^{\frac{1}{p_N-1}} - k_j^{\frac{1}{p_N-1}} \right)_-^{\alpha_i} |^{p_i} \zeta_j^{p_N} \, dxdt.$$

Hence, collecting the previous estimates, we arrive at

$$(ak)^{\frac{p_N-2}{p_N-1}} \sup_{t \in I_j} \int_{K_j \times \{t\}} \left(v^{\frac{1}{p_N-1}} - k_j^{\frac{1}{p_N-1}} \right)_-^2 \zeta_j^{p_N} \, dx$$

$$+ \gamma^{-1} a^{\frac{p_N - p_1}{p_N - 1}} \sum_i \iint_{Q_j} |D_i \left(v^{\frac{1}{p_N-1}} - k_j^{\frac{1}{p_N-1}} \right)_-^{\alpha_i} |^{p_i} \zeta_j^{p_N} \, dxdt$$

$$\leqslant \gamma 2^{j\gamma} k^{\frac{p_N}{p_N-1}} \left(\frac{1}{\eta} + \frac{1}{r^p} \right) |Q_j \cap \{u \leqslant k_j\}|.$$

By means of Hölder's inequality together with Lemma 3.2, for $\beta = 2$ and $\alpha = \frac{1}{N} \sum_{i=1}^{N} \alpha_i = \frac{p_N}{p}$, we obtain

$$\left(k_j^{\frac{1}{p_N-1}} - k_{j+1}^{\frac{1}{p_N-1}} \right)^{p_N} |Q_{j+1} \cap \{u \leqslant k_{j+1}\}| = \left(k_j^{\frac{1}{p_N-1}} - k_{j+1}^{\frac{1}{p_N-1}} \right)^{\alpha p} |Q_{j+1} \cap \{v \leqslant k_{j+1}\}|$$

$$\leqslant \iint_{Q_j} \left[\left(v^{\frac{1}{p_N-1}} - k_j^{\frac{1}{p_N-1}} \right)_- \zeta_j^{p_N} \right]^{\alpha p} dxdt$$

$$\leqslant \left(\iint_{Q_j} \left[\left(v^{\frac{1}{p_N-1}} - k_j^{\frac{1}{p_N-1}} \right)_- \zeta_j^{p_N} \right]^{p \frac{\alpha N + 2}{N}} dxdt \right)^{\frac{N}{N+2/\alpha}} |Q_j \cap \{v \leqslant k_j\}|^{\frac{2/\alpha}{N+2/\alpha}}$$

$$\leqslant \gamma \left(\sup_{t \in I_j} \int_{K_j \times \{t\}} \left(v^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}} \right)_-^2 \zeta_j^{p_N} \, dx \right)^{\frac{p}{N+2/\alpha}} \times$$

$$\times \left(\sum_i \iint_{Q_j} |D_i \left[\left(v^{\frac{1}{p_N-1}} - k_j^{\frac{1}{p_N-1}} \right)_-^{\alpha_i} \zeta_j^{p_N} \right] |^{p_i} \, dxdt \right)^{\frac{N}{N+2/\alpha}} |Q_j \cap \{u \leqslant k_j\}|^{\frac{2/\alpha}{N+2/\alpha}}$$

$$\leqslant \gamma 2^{j\gamma} a^{-\gamma} k^{\frac{p_N}{p_N-1}} \left(\frac{1}{\eta} + \frac{1}{r^p} \right)^{\frac{N+p}{N+2/\alpha}} |Q_j \cap \{u \leqslant k_j\}|^{1 + \frac{p}{N+2/\alpha}}.$$

Set $y_j := \dfrac{|Q_j \cap \{u \leqslant k_j\}|}{|Q_j|}$, then from the previous, we obtain

$$y_{j+1} \leqslant \gamma 2^{j\gamma} a^{-\gamma} \left(\frac{r^p}{\eta} \right)^{\frac{N}{N+2/\alpha}} \left(1 + \frac{\eta}{r^p} \right)^{\frac{N+p}{N+2/\alpha}} y_j^{1 + \frac{p}{N+2/\alpha}},$$

which yields $\lim_{j\to\infty} y_j = 0$, provided that $y_0 \leqslant \nu_-$, where ν_- is chosen to satisfy

$$\nu_- := \frac{a^\gamma}{\gamma}\left(\frac{\eta}{r^p}\right)^{\frac{N}{p}}\left(1+\frac{\eta}{r^p}\right)^{-\frac{N+p}{p}}, \tag{4.3}$$

therefore proving proves (4.2). □

5 | EXPANSION OF POSITIVITY

Expansion of positivity is the main and crucial result to obtain Harnack's inequality, and reads as follows.

Proposition 5.1. *Let u be a nonnegative, local weak supersolution to (1.3)–(1.4) and let us assume that for some $r, k > 0$ and $\alpha_0 \in (0,1)$, the following measure information is at stake:*

$$|\mathcal{K}_r(y) \cap \{u(\cdot, s) \geqslant k\}| \geqslant \alpha_0 |\mathcal{K}_r(y)|. \tag{5.1}$$

Then, there exist numbers η_0, $\delta_0 \in (0,1)$ depending only on the data and α_0 such that

$$u(x,t) \geqslant \eta_0\, k, \quad x \in \mathcal{K}_{2r}(y), \tag{5.2}$$

for all times

$$s + \frac{\delta_0}{2} r^p \leqslant t \leqslant s + \delta_0 r^p, \tag{5.3}$$

provided that

$$\mathcal{K}_{8r}(y) \times (s, s + r^p) \subset \Omega_T.$$

By repeated application of Proposition 5.1, we obtain the following.

Corollary 5.1. *Under conditions stated in Proposition 5.1, for any $A > 0$, there exists number $\bar{\eta} \in (0,1)$ depending only on the data, α_0 and A such that*

$$u(x,t) \geqslant \bar{\eta}\, k, \quad x \in \mathcal{K}_{2r}(y),$$

for all t

$$s + \frac{A}{2} r^p \leqslant t \leqslant s + A r^p,$$

provided that (5.1) holds and $\mathcal{K}_{8r}(y) \times (s, s + A(8r)^p) \subset \Omega_T$.

The proof of Proposition 5.1 is performed in two major steps, given by Lemmas 5.1 and 5.2. We start by introducing the change of variables

$$t \to \frac{t-s}{r^p} = \tau, \quad x_i \to \frac{x_i - y_i}{r^{\frac{p}{p_i}}} = z_i, \quad i = 1, \dots, N,$$

which maps the cylinder $\mathcal{K}_{8r} \times (s, s+r^p)$ into $Q := K_8(0) \times (0,1)$. Function u keeps on being a local weak supersolution to (1.3)-(1.4), in Q, and inequality (5.1) translates into

$$|K_1(0) \cap \{u(\cdot,0) \geqslant k\}| \geqslant \alpha_0 |K_1(0)|. \tag{5.4}$$

5.1 | Propagation of positivity in time

Lemma 5.1. *Let u be a nonnegative, local weak supersolution to (1.3)–(1.4) in Q and assume that inequality (5.4) holds. Then, there exist ε_1, $\delta_1 \in (0,1)$, depending only on the data and on α_0, such that*

$$|K_1(0) \cap \{u(\cdot,\tau) \geqslant \varepsilon_1 k\}| \geqslant \frac{\alpha_0}{4} |K_1(0)|, \tag{5.5}$$

for all

$$0 \leqslant \tau \leqslant \delta_1.$$

Proof. The proof relies on the energy estimates of Lemma 3.3, written over the cylinder $K_1(0) \times (0, \delta_1]$, where the cutoff function $\zeta \in C_0^1(K_1(0))$ can be chosen as a time-independent function satisfying for a fixed $\sigma \in (0,1)$ the usual properties

$$\zeta(x) = 1 \text{ for } x \in K_{1-\sigma}(0), \qquad 0 \leqslant \zeta(x) \leqslant 1, \qquad |D_i \zeta(x)|^{p_i} \leqslant \frac{\gamma}{\sigma^{p_N}} \qquad \forall i = 1, \dots, N\,.$$

Now, by observing that, for any $0 \leqslant u < k$, we can reduce from above the following term:

$$g_-(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}}) = (p_N - 1) \int_{u^{\frac{1}{p_N-1}}}^{k^{\frac{1}{p_N-1}}} s^{p_N-2}(k^{\frac{1}{p_N-1}} - s)ds =$$

$$= \frac{1}{p_N} k^{\frac{p_N}{p_N-1}} - u\left(k^{\frac{1}{p_N-1}} - \frac{p_N-1}{p_N} u^{\frac{1}{p_N-1}}\right) \leqslant \frac{k^{\frac{p_N}{p_N-1}}}{p_N},$$

and when restricting on the set $\{u \leqslant \varepsilon_1 k\}$ for a small $0 < \varepsilon_1 < 1/p_N$, we can reduce it from below

$$g_-(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}}) \geqslant \frac{1}{p_N} k^{\frac{p_N}{p_N-1}} - \varepsilon_1 k^{\frac{p_N}{p_N-1}} = k^{\frac{p_N}{p_N-1}} (1/p_N - \varepsilon_1)\,,$$

we arrive at the following estimate, valid for all times $0 \leqslant \tau \leqslant \delta_1$:

$$k^{\frac{p_N}{p_N-1}} (1/p_N - \varepsilon_1) |K_{1-\sigma}(0) \cap \{u(\cdot,\tau) \leqslant \varepsilon_1 k\}| \leqslant \sup_{0 \leqslant \tau \leqslant \delta_1} \int_{K_1(0)} g_-(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}}) \zeta^{p_N}\, dz$$

$$\leqslant \int_{K_1(0)\times\{0\}} g_-(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}})\, dz + \frac{\gamma}{\sigma^{p_N}} \sum_i \int_0^{\delta_1} \int_{K_1(0)} u^{\frac{p_N-p_i}{p_N-1}} (u^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_-^{p_i}\, dz d\tau$$

$$\leqslant \left[\frac{1-\alpha_0}{p_N} + \frac{\gamma \delta_1}{\sigma^{p_N}}\right] k^{\frac{p_N}{p_N-1}} |K_1(0)|\,,$$

which by the measure estimate

$$|K_{1-\sigma}(0)\cap\{u(\cdot,\tau)\leqslant\varepsilon_1 k\}|\geqslant|K_1(0)\cap\{u(\cdot,t)\leqslant\varepsilon_1 k\}|-N\sigma|K_1(0)|,$$

implies an estimate from above of the sublevel sets of u as

$$|K_1(0)\cap\{u(\cdot,\tau)\leqslant\varepsilon_1 k\}|\leqslant\left[N\sigma+\frac{p_N}{1-\varepsilon_1 p_N}\left(\frac{1-\alpha_0}{p_N}+\frac{\gamma\,\delta_1}{\sigma^{p_N}}\right)\right]|K_1(0)|.$$

Now, by choosing

$$\sigma=\frac{\alpha_0}{8N},\qquad \varepsilon_1=\frac{\alpha_0}{(2-\alpha_0)p_N},\qquad \delta_1=\frac{\alpha_0(1-\alpha_0)\sigma^{p_N}}{16\gamma\,p_N},$$

we arrive at the required (5.5), which proves the lemma. □

5.2 | A measure shrinking lemma

Lemma 5.2. *Assume that the conditions of Lemma 5.1 are fulfilled and let $\delta_1,\varepsilon_1,\alpha_0$ given in lemma 5.1. Then, for any $\nu\in(0,1)$, there exists j_*, depending only on the data, ε_1, α_0, and ν, such that*

$$\left|K_4(0)\times\left(\frac{\delta_1}{4},\delta_1\right)\cap\{u\leqslant\frac{k}{2^{j_*}}\}\right|\leqslant\frac{3}{4}\nu\,\delta_1|K_4(0)|. \tag{5.6}$$

Proof. Let $j_*\in\mathbb{N}$ to be determined and, for $j\in\mathbb{N}$ such that $1\leqslant j\leqslant j_*$, let us set $k_j:=\varepsilon_1^j k$ and use Lemma 3.3 for the pair of cylinders

$$Q_1:=K_4(0)\times\left(\frac{\delta_1}{4},\,\delta_1\right)\qquad\text{and}\qquad Q_2:=K_6(0)\times(0,\,\delta_1),$$

with a piecewise smooth cutoff function $\zeta(x,t)$ vanishing on the parabolic boundary of Q_2 and verifying

$$|\zeta_{x_i}|^{p_i}\leqslant\gamma,\quad\forall i=1,\dots,N,\qquad\text{and}\qquad|\zeta_t|\leqslant\frac{\gamma}{\delta_1},$$

to obtain

$$\begin{aligned}\sum_i\iint_{Q_1\cap\{u\leqslant k_j\}}u^{\frac{p_N}{p_N-1}-p_i}|D_iu|^{p_i}\,dz\,d\tau&\leqslant\gamma\sum_i\iint_{Q_2}u^{\frac{p_N-p_i}{p_N-1}}|D_i(u^{\frac{1}{p_N-1}}-k_j^{\frac{1}{p_N-1}})_-|^{p_i}\zeta^{p_N}\,dz\,d\tau\\&\leqslant\gamma\iint_{Q_2}g_-(u^{\frac{1}{p_N-1}},k_j^{\frac{1}{p_N-1}})\,|\zeta_t|\,dzd\tau+\gamma\sum_i\iint_{Q_2}u^{\frac{p_N-p_i}{p_N-1}}(u^{\frac{1}{p_N-1}}-k_j^{\frac{1}{p_N-1}})_-^{p_i}|D_i\zeta|^{p_i}\,dz\,d\tau\\&\leqslant\frac{\gamma}{\delta_1}\,k_j^{\frac{p_N}{p_N-1}}|Q_1|.\end{aligned}$$

Thanks to the discrete isoperimetric inequality Lemma 8.3 and the assumption (5.4), we arrive at

$$k_{j+1}|K_4(0)\cap\{u(\cdot,\tau)\leqslant k_{j+1}\}|\leqslant\frac{\gamma}{\alpha_0}\sum_i\int_{K_4(0)\cap\{k_{j+1}\leqslant u(\cdot,\tau)\leqslant k_j\}}|D_iu|\,dz$$

for every time $\tau \in \left(\frac{\delta_1}{4}, \delta_1\right)$. Integrating this inequality over $\left(\frac{\delta_1}{4}, \delta_1\right)$ by usual manipulations, we get

$$
\begin{aligned}
\varepsilon_1 k_j\,|Q_1 \cap \{u \leqslant k_{j+1}\}| &\leqslant \frac{\gamma}{\alpha_0} \sum_i \iint_{Q_1 \cap \{k_{j+1} \leqslant u \leqslant k_j\}} |D_i u|\, dz d\tau \\
&\leqslant \frac{\gamma}{\alpha_0} \sum_i \left(\iint_{Q_1 \cap \{u \leqslant k_j\}} u^{\frac{p_N}{p_N-1} - p_i} |D_i u|^{p_i}\, dz d\tau \right)^{\frac{1}{p_i}} \left(\iint_{Q_1 \cap \{k_{j+1} \leqslant u \leqslant k_j\}} u^{\frac{p_i - \frac{p_N}{p_N-1}}{p_i - 1}}\, dx dt \right)^{1-\frac{1}{p_i}} \\
&\leqslant \frac{\gamma}{\alpha_0} \sum_i \left(k_j^{\frac{p_N}{p_N-1}} |Q_1|/\delta_1 \right)^{\frac{1}{p_i}} \left(k_j^{\frac{p_i - \frac{p_N}{p_N-1}}{p_i - 1}} |Q_1 \cap \{k_{j+1} \leqslant u \leqslant k_j\}| \right)^{1-\frac{1}{p_i}} \\
&= \frac{\gamma}{\alpha_0} k_j \sum_i |K_4|^{\frac{1}{p_i}}\, |Q_1 \cap \{k_{j+1} \leqslant u \leqslant k_j\}|^{1-\frac{1}{p_i}} \\
&\leqslant \gamma \alpha_0^{-1}\, k_j |Q_1| \left(\frac{|Q_1 \cap \{k_{j+1} \leqslant u \leqslant k_j\}|}{|Q_1|} \right)^{1-\frac{1}{p_1}}.
\end{aligned}
$$

Thereby we end up with an estimate of the sublevel sets of u such as

$$
\left(\frac{|Q_1 \cap \{u \leqslant k_{j_*}\}|}{|Q_1|} \right)^{\frac{p_1}{p_1-1}} \leqslant \gamma \alpha_0^{-2\frac{p_1}{p_1-1}}\, \frac{|Q_1 \cap \{k_{j+1} \leqslant u \leqslant k_j\}|}{|Q_1|}.
$$

Summing up the last inequalities over j, for $1 \leqslant j \leqslant j_*$, and using the properties of Lebesgue measure, we deduce the shrinking property

$$
|Q_1 \cap \{u \leqslant k_{j_*}\}| \leqslant \frac{\gamma \alpha_0^{-2}}{(j_* - 1)^{1-\frac{1}{p_1}}} |Q_1|.
$$

The proof is completed once we choose j_* large enough such that

$$
\frac{\gamma\, \alpha_0^{-2}}{(j_* - 1)^{1-\frac{1}{p_1}}} \leqslant \nu, \qquad \text{and} \qquad j_* = \left\lceil 1 + \left(\frac{\gamma}{\nu\, \alpha_0^2} \right)^{\frac{p_1}{p_1-1}} \right\rceil,
$$

where in the last definition, we used the upper integer part. □

De-Giorgi-type Lemma 4.1, for $a = \frac{1}{2}$, and Lemma 5.2 yield

$$
u(z,\tau) \geqslant k \varepsilon_1^{j_*}/2, \quad (z,\tau) \in K_2(0) \times \left(\frac{\delta_1}{2}, \delta_1 \right).
$$

Returning to the original variables, we arrive at the required (5.2)-(5.3) with $\delta_0 = \delta_1$ and $\eta_0 = \varepsilon_1^{j_*}/2$, completing the proof of Proposition 5.1.

6 | HARNACK'S INEQUALITY

Let us fix $(x_0, t_0) \in \Omega_T$ for which $u(x_0, t_0) > 0$ and construct the cylinder

$$\mathcal{K}_{8\rho}(x_0) \times (t_0 - \bar{C}(8\rho)^p, t_0 + \bar{C}(8\rho)^p) \subset \Omega_T$$

with $\bar{C} > 0$ to be chosen. The change of variables

$$t \to \frac{t - t_0}{\rho^p} = \tilde{t}, \quad x_i \to \frac{x_i - x_{0,i}}{\rho^{\frac{p}{p_i}}} = z_i, \quad i = 1, \dots, N$$

maps this cylinder into $Q := K_8(0) \times (-\bar{C}8^p, \bar{C}8^p)$, and turns $v(z, \tilde{t}) = \frac{1}{u(x_0,t_0)} u(x, y)$ in a bounded, nonnegative, weak solution to (1.3)-(1.4) in Q, verifying $v(0, 0) = 1$.

Consider the cylinder $Q_{1,1}(0, 0)$ and fix $\beta > 0$, to be determined later. By Lemma 8.4, there exist $\gamma_0 = \gamma_0(\beta) > 1$, $(y, \tau) \in Q_{1,1}(0, 0)$ and $r > 0$ such that

$$Q_{r,r^p}(y, \tau) \subset Q_{1,1}(0, 0), \quad r^\beta \sup_{Q_{r,r^p}(y,\tau)} v \leqslant \gamma_0, \quad r^\beta v(y, \tau) \geqslant \frac{1}{\gamma_0}. \tag{6.1}$$

Claim 1. There exist $\epsilon, \alpha \in (0, 1)$ depending only on the data and β such that

$$\left|Q_{r,r^p}(y, \tau) \cap \left\{v \geqslant \epsilon r^{-\beta}\right\}\right| \geqslant \alpha |Q_{r,r^p}(y, \tau)|. \tag{6.2}$$

Proof. Since v is a bounded, nonnegative, weak solution to (1.3)-(1.4), from Lemma 3.4 together with (6.1) and the fact that $|Q_{r,r^p}(y, \tau)| = r^{N+p}$, we get

$$\begin{aligned}
\frac{1}{\gamma_0} r^{-\beta} \leqslant v(y, \tau) &\leqslant \gamma \left(\frac{1}{r^{N+p}} \iint_{Q_{r,r^p}(y,\tau)} v^{\frac{p_N}{p_N-1}} \, dx\, dt \right)^{\frac{p_N-1}{p_N}} \leqslant \gamma \epsilon r^{-\beta} + \\
&+ \gamma \left(\frac{1}{r^{N+p}} \iint_{Q_{r,r^p}(y,\tau)\cap\{v\geqslant\epsilon r^{-\beta}\}} v^{\frac{p_N}{p_N-1}} \, dx\, dt \right)^{\frac{p_N-1}{p_N}} \\
&\leqslant \gamma \epsilon r^{-\beta} + \gamma\gamma_0 r^{-\beta} \left(\frac{|Q_{r,r^p}(y, \tau) \cap \{v \geqslant \epsilon r^{-\beta}\}|}{|Q_{r,r^p}(y, \tau)|} \right)^{\frac{p_N-1}{p_N}}.
\end{aligned}$$

We arrive at the required (6.2), with $\alpha = \left(\frac{1}{2\gamma\gamma_0^2}\right)^{\frac{p_N}{p_N-1}}$, once we choose $\epsilon = \frac{1}{2\gamma\gamma_0}$. □

Inequality (6.2) yields

$$\left|K_r(y) \cap \left\{v(\cdot, s) \geqslant \epsilon r^{-\beta}\right\}\right| \geqslant \alpha |K_r(y)|, \quad \text{for some} \quad s \in (\tau - r^p, \tau), \tag{6.3}$$

and then, by applying Proposition 5.1 (with $k = \epsilon r^{-\beta}$ and $\alpha_0 = \alpha$), we get

$$v(z, t_0) \geqslant \eta_0\, \epsilon r^{-\beta}, \quad x \in K_{2r}(y), \quad t_0 = s + \delta_0\, r^p, \tag{6.4}$$

for some $\eta_0, \delta_0 \in (0, 1)$ depending only on the data and on β.

We can now repeat this procedure (one and again) starting from (6.4). In fact, the expansion of positivity now applied for level $\eta_0\, \epsilon r^{-\beta}$, radius $2r$ and $\alpha_0 = 1$ yields

$$v(z,t_1) \geqslant \bar{\eta}\, \eta_0 \epsilon r^{-\beta}, \quad x \in K_{2^2 r}(y), \quad t_1 := t_0 + \bar{\delta}(2r)^p, \tag{6.5}$$

where $\bar{\eta}, \bar{\delta} \in (0,1)$ depend only on the data and are independent of r and β. By an iterative scheme, one arrives at

$$v(z,t_n) \geqslant \bar{\eta}^n\, \eta_0 \epsilon r^{-\beta}, \quad x \in K_{2^{n+1} r}(y), \quad t_n := t_{n-1} + \bar{\delta}(2^n r)^p. \tag{6.6}$$

Choosing n so large that $2^{n+1}\, r \geqslant 2$, the cube $K_{2^{n+1}r}(y)$ covers the cube $K_1(0)$, and then,

$$v(z,t_n) \geqslant \bar{\eta}^n\, \eta_0 \epsilon r^{-\beta}, \quad x \in K_1(0), \tag{6.7}$$

where time level t_n is given by

$$t_n = t_0 + \bar{\delta} \sum_{j=1}^{n} (2^j r)^p = s + \delta_0 r^p + \bar{\delta} \sum_{j=1}^{n} (2^j r)^p,$$

which can be estimated as

$$r^p \bar{\delta} 2^{np} - 1 \leqslant t_n \leqslant 1 + \bar{\delta}(2^{n+1} r)^p.$$

Choosing n such that $2 \leqslant \bar{\delta}(2^n r)^p \leqslant 2^{p+1}$, from (6.7), we obtain

$$v(z,\bar{t}) \geqslant \bar{\eta}^n\, \eta_0 \epsilon r^{-\beta} \geqslant \eta_0 \epsilon \left(\frac{\bar{\delta}}{2^{p+1}}\right)^{\frac{\beta}{p}} \left(2^{\beta}\, \bar{\eta}\right)^n = \eta_0\, \epsilon \left(\frac{\bar{\delta}}{2^{p+1}}\right)^{\frac{-\log_2(\bar{\eta})}{p}}, \quad x \in K_1(0), \quad 1 \leqslant \bar{t} \leqslant 2^{p+2}, \tag{6.8}$$

once we take $2^{\beta}\, \bar{\eta} = 1$. Returning to the original variables shows inequality (1.5) forward in time.

6.1 | Backward estimate

Let $(x_0,t_0) \in \Omega_T$ be such that $\mathcal{K}_{8\rho}(x_0) \times (t_0 - \bar{C}(8\rho)^p, t_0 + \bar{C}(8\rho)^p) \subset \Omega_T$, and let y be an arbitrary point in $\mathcal{K}_\rho(x_0)$. Applying the forward Harnack inequality in (1.5), which has already been established, with (x_0,t_0) replaced by $(y, t_0 - \bar{C}\rho^p)$, we obtain

$$u(y, t_0 - \bar{C}\rho^p) \leqslant C \inf_{\mathcal{K}_\rho(y)} u(\cdot, t_0) \leqslant C\, u(x_0,t_0). \tag{6.9}$$

By definition of supremum, for any $\epsilon \in (0,1)$, there exists $y_\epsilon \in \mathcal{K}_\rho(x_0)$ such that

$$\sup_{\mathcal{K}_\rho(x_0)} u(\cdot, t_0 - \bar{C}\rho^p) \leqslant u(y_\epsilon, t_0 - \bar{C}\rho^p) + \epsilon.$$

Since y is arbitrary in $\mathcal{K}_\rho(x_0)$, estimate (6.9) applies, in particular, to y_ϵ, which yields

$$\sup_{\mathcal{K}_\rho(x_0)} u(\cdot, t_0 - \bar{C}\rho^p) \leqslant C\, u(x_0,t_0) + \epsilon.$$

Letting $\epsilon \to 0$ shows the backward Harnack inequality in (1.5), completing the proof of Theorem 1.1.

7 | HÖLDER CONTINUITY

In this section, we prove Theorem 1.2. Let us fix a point $(x_0, t_0) \in \Omega_T$ and let $R > 0$ be such that $Q_{8R,(8R)^p}(x_0, t_0) \subset \Omega_T$. If for any $0 < \rho < R$,

$$\operatorname*{osc}_{Q_{\rho,\rho^p}(x_0,t_0)} u \leqslant \rho,$$

then there is nothing to prove. So, assume that there exists $0 < \rho_0 < R$ such that

$$\operatorname*{osc}_{Q_{\rho_0,\rho_0^p}(x_0,t_0)} u \geqslant \rho_0$$

and set

$$Q_{\rho_0} := Q_{\rho_0,\rho_0^p}(x_0, t_0), \qquad \mu_0^+ := \sup_{Q_{\rho_0}} u, \qquad \mu_0^- := \inf_{Q_{\rho_0}} u, \qquad \omega_0 := \mu_0^+ - \mu_0^-.$$

We will construct the decreasing sequences of numbers

$$\rho_j := \epsilon^j \rho_0, \qquad \omega_j := \sigma^j \omega_0 \qquad \text{for } \epsilon, \sigma \in (0, 1)$$

and cylinders $Q_{\rho_j} := Q_{\rho_j,\rho_j^p}(x_0, t_0)$ such that

$$\mu_j^+ := \sup_{Q_{\rho_j}} u, \qquad \mu_j^- := \inf_{Q_{\rho_j}} u, \qquad \mu_j^+ - \mu_j^- \leqslant \omega_j, \qquad j = 0, 1, 2, \dots . \tag{7.1}$$

Assume first that

$$\mu_0^+ \leqslant (1 + \eta)\omega_0, \tag{7.2}$$

where $\eta \in (0, 1)$ is to be determined depending only on the data. Observe that one of the following cases must be true: either

$$\left|Q_{\frac{3}{4}\rho_0} \cap \left\{u \geqslant \mu_0^+ - \frac{1}{2}\omega_0\right\}\right| = 0, \tag{7.3}$$

or

$$\left|Q_{\frac{3}{4}\rho_0} \cap \left\{u \geqslant \mu_0^+ - \frac{1}{2}\omega_0\right\}\right| > 0. \tag{7.4}$$

In the first case, we immediately get

$$u(x, t) \leqslant \mu_0^+ - \frac{1}{2}\omega_0, \qquad \forall (x, t) \in Q_{\frac{3}{4}\rho_0},$$

obtaining therefore the reduction of the oscillation

$$\operatorname*{osc}_{Q_{\frac{1}{2}\rho_0}} u \leqslant \frac{1}{2}\,\omega_0. \tag{7.5}$$

Assume now that (7.4) is in force.

Claim 2. There exist $\xi, \alpha \in (0,1)$ depending only on the data such that

$$|Q_{\rho_0} \cap \{u \geqslant \xi\,\omega_0\}| \geqslant \alpha |Q_{\rho_0}|. \tag{7.6}$$

Proof. From (7.4) and Lemma 3.4, applied to the pair of cylinders $Q_{\frac{3}{4}\rho_0}$ and Q_{ρ_0}, and recalling (7.2), we obtain

$$\begin{aligned}
\frac{1}{4}\omega_0 \leqslant \mu_0^+ - \frac{3}{4}\omega_0 \leqslant \sup_{Q_{\frac{3}{4}\rho_0}} u \leqslant \gamma \left(\frac{1}{\rho_0^{N+p}} \iint_{Q_{\rho_0}} u^{\frac{p_N}{p_N-1}}\, dxdt \right)^{\frac{p_N-1}{p_N}} \\
\leqslant \gamma\,\xi\,\omega_0 + \gamma \left(\frac{1}{\rho_0^{N+p}} \iint_{Q_{\rho_0} \cap \{u \geqslant \xi\,\omega_0\}} u^{\frac{p_N}{p_N-1}}\, dxdt \right)^{\frac{p_N-1}{p_N}} \\
\leqslant \gamma\xi\omega_0 + \gamma\mu_0^+ \left(\frac{|Q_{\rho_0} \cap \{u \geqslant \xi\omega_0\}|}{|Q_{\rho_0}|} \right)^{\frac{p_N-1}{p_N}} \leqslant \gamma\xi\omega_0 + 2\gamma\omega_0 \left(\frac{|Q_{\rho_0} \cap \{u \geqslant \xi\omega_0\}|}{|Q_{\rho_0}|} \right)^{\frac{p_N-1}{p_N}}.
\end{aligned}$$

We thereby derive (7.6) once we fix

$$\xi = \frac{1}{8\gamma}, \qquad \text{and} \qquad \alpha = \left(\frac{1}{16\gamma}\right)^{\frac{p_N}{p_N-1}}.$$

□

Claim 3. There exists a time level $s \in (t_0 - \rho_0^p, t_0 - \frac{\alpha}{2}\rho_0^p)$ such that

$$|\mathcal{K}_{\rho_0}(x_0) \cap \{u(\cdot, s) \leqslant \xi\omega_0\}| \leqslant \frac{1-\alpha}{1-\alpha/2} |\mathcal{K}_{\rho_0}(x_0)|. \tag{7.7}$$

Proof. The proof is quite simple and is conducted by means of a contradiction argument. Consider that, for all $t \in (t_0 - \rho_0^p, t_0 - \frac{\alpha}{2}\rho_0^p)$, inequality (7.7) is violated. Since (7.6) holds true,

$$\begin{aligned}
(1-\alpha)|Q_{\rho_0}| < |\mathcal{K}_{\rho_0}(x_0) \times (t_0 - \rho_0^p, t_0 - \frac{\alpha}{2}\rho_0^p) \cap \{u \leqslant \xi\,\omega_0\}| \leqslant \\
\leqslant |Q_{\rho_0} \cap \{u \leqslant \xi\omega_0\}| \leqslant (1-\alpha)|Q_{\rho_0}|,
\end{aligned}$$

reaching to a contradiction. □

Now we use Corollary 5.1, for $A = \frac{\alpha}{2}$ and $\bar\eta \in (0,1)$ a constant depending only on the data, to obtain the following inferior bound:

$$u(x,t) \geqslant \bar\eta\,\xi\omega_0, \quad (x,t) \in \mathcal{K}_{\rho_0}(x_0) \times (t_0 - \frac{\alpha}{4}\rho_0^p, t_0),$$

and then, recalling (7.2), we arrive at the reduction of the oscillation for this second case

$$\operatorname*{osc}_{K_{\rho_0}(x_0) \times (t_0 - \frac{\alpha}{4}\rho_0^p, t_0)} u \leqslant \mu_0^+ - \bar\eta\xi\omega_0 \leqslant (1 - \bar\eta\xi + \eta)\omega_0 = (1 - \frac{1}{2}\bar\eta\xi)\omega_0, \tag{7.8}$$

provided that

$$\eta = \frac{1}{2}\bar{\eta}\xi.$$

Combining both reduction of the oscillations (7.5) and (7.8), we arrive at

$$\operatorname*{osc}_{Q_{\epsilon\rho_0}} u \leqslant \sigma\omega_0 := \omega_1, \quad \epsilon = \left(\frac{\alpha}{4}\right)^{\frac{1}{p}}, \quad \sigma = 1 - \frac{1}{2}\bar{\eta}\xi. \tag{7.9}$$

This proves (7.1) for $j = 1$. We can repeat this procedure and prove (7.1), assuming on each step that

$$\mu_j^+ \leqslant (1 + \frac{1}{2}\bar{\eta}\xi)\,\omega_j\,, \tag{7.10}$$

where ϵ and $\sigma \in (0,1)$ are as in (7.9).

Assume now that $j_0 \geqslant 1$ is the first number such that (7.10) is violated, that is,

$$\mu_{j_0}^+ \geqslant (1 + \frac{1}{2}\bar{\eta}\xi)\,\omega_{j_0}, \quad \text{and hence,} \quad \mu_{j_0}^- \geqslant \left(\frac{\bar{\eta}\xi}{2+\bar{\eta}\xi}\right)\mu_{j_0}^+. \tag{7.11}$$

In this case, by the fact that (7.10) holds for $j = j_0 - 1$, we obtain

$$\left(1 + \frac{1}{2}\bar{\eta}\xi\right)\omega_{j_0} \leqslant \mu_{j_0}^+ \leqslant \mu_{j_0-1}^+ \leqslant \left(1 + \frac{1}{2}\bar{\eta}\xi\right)\omega_{j_0-1} = \left(\frac{2+\bar{\eta}\xi}{2\sigma}\right)\omega_{j_0},$$

and

$$\frac{1}{2}\bar{\eta}\xi\ \omega_{j_0} \leqslant \left(\frac{\bar{\eta}\xi}{2+\bar{\eta}\xi}\right)\mu_{j_0}^+ \leqslant \mu_{j_0}^- \leqslant \mu_{j_0}^+ \leqslant \left(\frac{1+\bar{\eta}\xi}{2\sigma}\right)\omega_{j_0}. \tag{7.12}$$

The function $v := u/\mu_{j_0}^-$ is a solution of the anisotropic p-Laplace evolution equation

$$v_t - \sum_i D_i a_i(x,t,v,Dv) = 0, \quad (x,t) \in Q_{\rho_{j_0}}, \tag{7.13}$$

where $a_i(x,t,v,Dv) := \frac{1}{\mu_{j_0}^-} A_i(x,t,\mu_{j_0}^- v, \mu_{j_0}^- Dv)$ and the following inequalities hold:

$$\begin{cases} a_i(x,t,v,Dv)D_i v \geqslant \bar{K}_1(\sigma,\bar{\eta},\xi)\sum_i |D_i v|^{p_i}, \\ |a_i(x,t,v,Dv)| \leqslant \bar{K}_2(\sigma,\bar{\eta},\xi)\,|D_i v|^{p_i-1}, \quad i = 1,\dots,N, \end{cases} \tag{7.14}$$

with positive constants $\bar{K}_1(\sigma,\bar{\eta},\xi)$, $\bar{K}_2(\sigma,\bar{\eta},\xi)$ depending only on σ, $\bar{\eta}$, ξ and on the constants K_1, K_2 appearing in (1.4).

To proceed with the study of the local Hölder continuity of u, we need to study separately the degenerate and the singular cases. In both studies, we will assume that

$$p_N - p_1 \leqslant \epsilon_*\,,$$

for a sufficiently small $\epsilon_* \in (0,1)$ as in Lemma 8.2.

7.1 Degenerate case — $p_N > 2$

Consider the real positive numbers $C_* > 1$, $b_2 > 0$, and $\epsilon_1 \in (0,1)$, to be defined, and introduce the following numbers:

$$\tilde{\mu}^+_{j_0} := \frac{\mu^+_{j_0}}{\mu^-_{j_0}}, \quad \tilde{\mu}^-_{j_0} := 1, \quad \tilde{\omega}_{j_0} := \frac{\omega_{j_0}}{\mu^-_{j_0}}, \tag{7.15}$$

and cylinders

$$\tilde{Q}^{(\tilde{\omega}_{j_0})}_{\epsilon_1\rho_{j_0}} := \mathcal{K}^{\tilde{\omega}_{j_0}/C_*}_{\epsilon_1\rho_{j_0}}(x_0) \times (t_0 - b_2(\epsilon_1\rho_{j_0})^p \tilde{\omega}^{2-p_N}_{j_0}, t_0).$$

We choose ϵ_1 so small that $\tilde{Q}^{(\tilde{\omega}_{j_0})}_{\epsilon_1\rho_{j_0}} \subset Q_{\rho_{j_0}}$, that is, by (7.12)

$$\epsilon_1\left(C_*\frac{1+\frac{1}{2}\bar{\eta}\xi}{\sigma}\right)^{\frac{p_N-p_1}{p}} \leqslant 1, \quad \text{and} \quad \epsilon_1^p\, b_2\left(\frac{1+\frac{1}{2}\bar{\eta}\xi}{\sigma}\right)^{p_N-2} \leqslant 1.$$

The following alternative cases are possible: either

$$\left|\mathcal{K}^{\tilde{\omega}_{j_0}/4}_{\epsilon_1\rho_{j_0}}(x_0) \cap \left\{v\left(\cdot, t_0 - b_2(\epsilon_1\rho_{j_0})^p\tilde{\omega}^{2-p_N}_{j_0}\right) \geqslant \tilde{\mu}^+_{j_0} - \frac{1}{4}\tilde{\omega}_{j_0}\right\}\right| \leqslant \frac{1}{2}\left|\mathcal{K}^{\tilde{\omega}_{j_0}/4}_{\epsilon_1\rho_{j_0}}(x_0)\right|,$$

or

$$\left|\mathcal{K}^{\tilde{\omega}_{j_0}/4}_{\epsilon_1\rho_{j_0}}(x_0) \cap \left\{v\left(\cdot, t_0 - b_2(\epsilon_1\rho_{j_0})^p\tilde{\omega}^{2-p_N}_{j_0}\right) \leqslant \tilde{\mu}^-_{j_0} + \frac{1}{4}\tilde{\omega}_{j_0}\right\}\right| \leqslant \frac{1}{2}\left|\mathcal{K}^{\tilde{\omega}_{j_0}/4}_{\epsilon_1\rho_{j_0}}(x_0)\right|.$$

By Lemma 8.2, there exist $C_* > 1$ and $0 < b_1 < b_2$, depending only on the data, σ and $\bar{\eta}$, such that

$$\text{either} \qquad v(x,t) \leqslant \tilde{\mu}^+_{j_0} - \frac{1}{C_*}\,\tilde{\omega}_{j_0}, \qquad \text{or} \qquad v(x,t) \geqslant \tilde{\mu}^-_{j_0} + \frac{1}{C_*}\,\tilde{\omega}_{j_0}, \tag{7.16}$$

for almost every point $(x,t) \in \mathcal{K}^{\tilde{\omega}_{j_0}/C_*}_{\epsilon_1\rho_{j_0}}(x_0) \times (t_0 - (b_2 - b_1)(\epsilon_1\rho_{j_0})^p\tilde{\omega}^{2-p_N}_{j_0}, t_0)$.

Now we set $\bar{\rho}_{j_0} := \epsilon_1\rho_{j_0}$, $\sigma_1 := 1 - \frac{1}{C_*}$, $\tilde{\omega}_{j_0+1} := \sigma_1\tilde{\omega}_{j_0}$ and choose $\epsilon_2 \in (0,1)$ such that

$$\epsilon_2^p \leqslant \sigma_1^{p_N-p_1} \quad \text{and} \quad \epsilon_2^p \leqslant \left(1 - \frac{b_1}{b_2}\right)\sigma_1^{p_N-2}$$

which, together with (7.16), guarantee that

$$\operatorname*{osc}_{\tilde{Q}^{(\tilde{\omega}_{j_0+1})}_{\bar{\rho}_{j_0+1}}} v \leqslant \tilde{\omega}_{j_0+1}, \quad \bar{\rho}_{j_0+1} := \epsilon_2\,\bar{\rho}_{j_0}; \tag{7.17}$$

and this is the starting point to an iterative scheme. Indeed, take

$$\bar{\rho}_{j+j_0} := \epsilon_2^j\bar{\rho}_{j_0} \quad \text{and} \quad \tilde{\omega}_{j+j_0} := \sigma_1^j\,\tilde{\omega}_{j_0}, \quad j \geqslant 0.$$

Then, starting from (7.17) and applying several times Lemma 8.2, we obtain

$$\operatorname*{osc}_{\tilde{Q}^{(\tilde{\omega}_{j+j_0})}_{\bar{\rho}_{j+j_0}}} v \leqslant \tilde{\omega}_{j+j_0}, \quad \text{for all} \quad j \geqslant 0. \tag{7.18}$$

Observe that under condition (7.12) and the additional assumption

$$\epsilon_2^p \leqslant \left(\frac{\bar{\eta}\xi}{2}\right)^{p_N - p_1} \quad \text{and} \quad \epsilon_2^p \leqslant \left(\frac{\bar{\eta}\xi}{2}\right)^{p_N - 2},$$

$$Q_{\bar{\rho}_{j+1+j_0}} \subset \tilde{Q}^{(\tilde{\omega}_{j+j_0})}_{\bar{\rho}_{j+j_0}}, \quad \text{for all} \quad j \geqslant 0,$$

and hence, redefining $\bar{\rho}_{j+j_0}$ and ω_{j+j_0} as $\bar{\rho}^{(1)}_{j+j_0} := \epsilon_2 \bar{\rho}_{j+j_0} = \epsilon_2^{j+1} \bar{\rho}_{j_0}$, $\omega_{j+j_0} := \sigma_1^j \omega_{j_0}$, inequality (7.18) yields

$$\operatorname*{osc}_{Q_{\bar{\rho}^{(1)}_{j+j_0}}} u = \operatorname*{osc}_{Q_{\bar{\rho}_{j+1+j_0}}} u \leqslant \omega_{j+j_0}, \quad \text{for all} \quad j \geqslant 0, \tag{7.19}$$

which completes the construction of the sequences as in (7.1) in the case if (7.11) holds. Now the Hölder continuity follows by standard iterative real analysis methods, see, for example, [22, Chap.3].

7.2 | Singular case — $p_N \leqslant 2$

Define numbers $\tilde{\mu}^+_{j_0}$, $\tilde{\mu}^-_{j_0}$, and $\tilde{\omega}_{j_0}$ as in (7.15), and for fixed numbers $C_* > 1$, $b_2 > 0$, $\epsilon_1 \in (0, 1)$ introduce the cylinders

$$\tilde{Q}^{(\tilde{\omega}_{j_0})}_{\epsilon_1 \rho_{j_0}} := \tilde{\mathcal{K}}^{(C_*)}_{\epsilon_1 \rho_{j_0}} \times (t_0 - b_2(\epsilon_1 \rho_{j_0})^p, t_0),$$

where

$$\tilde{\mathcal{K}}^{(C_*)}_{\epsilon_1 \rho_{j_0}} := \{x \,:\, |x_i - x_{i,0}| \leqslant C_*^{\frac{p_N - p_i}{p_i}} \tilde{\omega}_{j_0}^{-\frac{2-p_i}{p_i}} (\epsilon_1 \rho_{j_0})^{\frac{p}{p_i}}, \quad i = 1, \dots, N\}.$$

Note that redefining ρ_{j_0} by $r^p_{j_0} = \tilde{\omega}_{j_0}^{p_N - 2} \rho^p_{j_0}$, then $\tilde{\mathcal{K}}^{(C_*)}_{\epsilon_1 \rho_{j_0}}$ coincides with $\mathcal{K}^{\tilde{\omega}_{j_0}/C_*}_{\epsilon_1 r_{j_0}}$ and the cylinder $\tilde{Q}^{(\tilde{\omega}_{j_0})}_{\epsilon_1 \rho_{j_0}}$ coincides with the cylinder $\tilde{Q}^{(\tilde{\omega}_{j_0})}_{\epsilon_1 r_{j_0}}$ defined in the previous section.

Choose ϵ_1 small enough such that $\tilde{Q}^{(\tilde{\omega}_{j_0})}_{\epsilon_1 \rho_{j_0}} \subset Q_{\rho_{j_0}}$, that is, by (7.12)

$$\epsilon_1 C_*^{\frac{p_N - p_1}{p}} \left(\frac{2 + \bar{\eta}\xi}{2\sigma}\right)^{\frac{2-p_1}{p}} \leqslant 1 \quad \text{and} \quad \epsilon_1^p b_2 \leqslant 1.$$

We distinguish two different cases: either

$$|\tilde{\mathcal{K}}^{(4)}_{\epsilon_1 \rho_{j_0}} \cap \left\{v\left(\cdot, t_0 - b_2(\epsilon_1 \rho_{j_0})^p\right) \geqslant \tilde{\mu}^+_{j_0} - \frac{1}{4}\omega_{j_0}\right\}| \leqslant \frac{1}{2} |\tilde{\mathcal{K}}^{(4)}_{\epsilon_1 \rho_{j_0}}|,$$

or

$$\left|\tilde{\mathcal{K}}^{(4)}_{\epsilon_1\rho_{j_0}} \cap \left\{v\left(\cdot, t_0 - b_2(\epsilon_1\rho_{j_0})^p\right) \leqslant \tilde{\mu}^-_{j_0} + \frac{1}{4}\tilde{\omega}_{j_0}\right\}\right| \leqslant \frac{1}{2}|\tilde{\mathcal{K}}^{(4)}_{\epsilon_1\rho_{j_0}}|.$$

Lemma 8.2 yields the existence of $C_* > 1$ and $0 < b_1 < b_2$, depending only on the data, σ and $\bar{\eta}$, such that

$$\text{either} \qquad v(x,t) \leqslant \tilde{\mu}^+_{j_0} - \frac{1}{C_*}\tilde{\omega}_{j_0}, \qquad \text{or} \qquad v(x,t) \geqslant \tilde{\mu}^-_{j_0} + \frac{1}{C_*}\tilde{\omega}_{j_0}, \tag{7.20}$$

for almost every $(x,t) \in \tilde{\mathcal{K}}^{(C_*)}_{\epsilon_1\rho_{j_0}} \times (t_0 - (b_2 - b_1)(\epsilon_1\rho_{j_0})^p, t_0)$.

Set $\bar{\rho}_{j_0} := \epsilon_1\rho_{j_0}$, $\sigma_1 := 1 - \frac{1}{C_*}$, $\tilde{\omega}_{j_0+1} := \sigma_1\tilde{\omega}_{j_0}$ and choose $\epsilon_2 \in (0,1)$ such that

$$\epsilon_2^p \leqslant \sigma_1^{2-p_1}, \quad \text{and} \quad \epsilon_2^p \leqslant \left(1 - \frac{b_1}{b_2}\right).$$

Then, by (7.20), one obtains

$$\operatorname*{osc}_{\tilde{Q}^{(\tilde{\omega}_{j_0+1})}_{\bar{\rho}_{j_0+1}}} v \leqslant \tilde{\omega}_{j_0+1}, \quad \bar{\rho}_{j_0+1} := \epsilon_2\,\bar{\rho}_{j_0}. \tag{7.21}$$

We are just one step away from presenting the final part of the iterative scheme leading to Hölder's continuity. Let us first choose

$$\bar{\rho}_{j+j_0} := \epsilon_2^j\bar{\rho}_{j_0} \quad \text{and} \quad \tilde{\omega}_{j+j_0} := \sigma_1^j\,\tilde{\omega}_{j_0}, \quad j \geqslant 0,$$

then we repeat the previous procedure and

$$\operatorname*{osc}_{\tilde{Q}^{(\tilde{\omega}_{j+j_0})}_{\bar{\rho}_{j+j_0}}} v \leqslant \tilde{\omega}_{j+j_0}, \quad \text{for all} \quad j \geqslant 0, \tag{7.22}$$

and finally, by considering the additional assumption on ϵ_2

$$\epsilon_2^p \leqslant \left(\frac{\bar{\eta}\xi}{2}\right)^{2-p_1} \quad \text{and} \quad \epsilon_2^p \leqslant b_2,$$

we obtain that

$$Q_{\bar{\rho}_{j+1+j_0}} \subset \tilde{Q}^{(\tilde{\omega}_{j+j_0})}_{\bar{\rho}_{j+j_0}}, \quad \text{for all} \quad j \geqslant 0.$$

Redefine $\bar{\rho}_{j+j_0}$ and $\omega_{j+j_)}$ as $\bar{\rho}^{(1)}_{j+j_0} := \epsilon_2\bar{\rho}_{j+j_0} = \epsilon_2^{j+1}\bar{\rho}_{j_0}$ and $\omega_{j+j_0} := \sigma_1^j\omega_{j_0}$, then (7.22) yields

$$\operatorname*{osc}_{Q_{\bar{\rho}^{(1)}_{j+j_0}}} u = \operatorname*{osc}_{Q_{\bar{\rho}_{j+1+j_0}}} u \leqslant \omega_{j+j_0}, \quad \text{for any} \quad j \geqslant 0, \tag{7.23}$$

which completes the construction of the sequences as in (7.1) in the case if (7.11) holds. Once again and as in the degenerate case, the Hölder continuity follows by standard iterative real analysis methods. This completes the proof of Theorem 1.2.

APPENDIX

A.1 | Motivation for the functional setting

The functional assumption (3.3) is natural for the integrals of (3.4) to be convergent. Indeed, by by Hölder inequality, for any $0 < t < T$, we can estimate the first parabolic term as

$$\int_E u(x,t)\,\varphi(x,t)\,dx \leqslant \gamma \left(\sup_{[0,T]} \|u(\cdot,t)\|_{L^{\frac{p_N}{p_N-1}}(E)} \right) \left(\sup_{[0,T]} \|\varphi(\cdot,t)\|_{L^{p_N}(E)} \right) < \infty$$

since $u \in C_{loc}(0,T; L^{\frac{p_N}{p_N-1}}(E))$ and $\varphi \in C_{loc}(0,T; L^{p_N}(E)) \subset W^{1,p_N}_{loc}(0,T; L^{p_N}(E))$. Similarly, we bound the second parabolic term, thanks to the previous assumption since

$$u \in C_{loc}(0,T; L_{loc}^{\frac{p_N}{p_N-1}}(\Omega)) \subset L_{loc}^{\frac{p_N}{p_N-1}}(\Omega_T), \tag{A.1}$$

with

$$\int_{t_1}^{t_2} \int_E u \partial_t \varphi \, dxdt \leqslant \gamma \left(\int_{t_1}^{t_2} \int_E u^{\frac{p_N}{p_N-1}} \right)^{\frac{p_N-1}{p_N}} \left(\int_{t_1}^{t_2} \int_E |\partial_t \varphi|^{p_N} \right)^{\frac{1}{p_N}} < \infty .$$

Finally, the elliptic term is controlled by means of the structure conditions (1.4). For all $i \in \{1, \dots, N\}$,

$$\int_{t_1}^{t_2} \int_E A_i(x,t,u,Du)\, D_i \varphi \; dxdt \leqslant \gamma \|D_i(u^{\frac{1}{p_i-1}})\|_{L^{p_i-1}(E\times(t_1,t_2))} \, \|D_i\varphi\|^{p_i}_{L^{p_i}(E\times(t_1,t_2))} ,$$

showing therefore the necessity of the assumption $D_i u^{\frac{1}{p_i-1}} \in L^{p_i}_{loc}(0,T; L^{p_i}_{loc}(E))$. Finally, the assumption (A.1) allows similarly for the finiteness of the energy inequalities (3.5), and will be actually used crucially to prove them here below in Section A.3.

A.1.1 | Existence theory

Boundary value problems are well posed (see [44]) when considering Equations (1.3)–(1.4) within the special form of

$$A_i(x,t,u,Du) = a_{i,j}(x,t,u)|D_i(u^{\frac{1}{p_i-1}})|^{p_i-2} D_i(u^{\frac{1}{p_i-1}}) ,$$

a boundary datum $g \in L^{\mathbf{p}}(0,T; W^{1,\mathbf{p}}(\Omega)) \cap L^\infty(\Omega_T)$ such that $\partial_t g \in L^2(\Omega_T)$, together with an initial value $0 \leqslant u_0 \in L^\infty(\Omega)$. We refer to the study [44] with $m_i := 1/(p_i - 1)$. Two are the conditions required on the spread of p_is: the first one is

$$1 < p_i < 2 \qquad \forall i = 1, \dots, N ,$$

which is nevertheless claimed to be unnecessary in [44]; and the condition

$$\frac{1}{p_i - 1} \leqslant \frac{p_i}{p_i - 1} \left(\frac{1}{p_N - 1} \right) \qquad \Longleftrightarrow \qquad p_N < p_1 + 1 .$$

This last condition is implied by the assumption $p_N - p_1 < \epsilon$ of Theorem 1.2 and, in particular, it provides, for the solutions, the existence of a spatial gradient of $D(u^{\frac{1}{p_N-1}})$ together with the chain rule, see Section 4 of [44].

A.1.2 | The structure conditions revisited

In order to end up with positive exponents in the energy estimates, we rewrite the structure conditions (1.4) as follows:

$$\begin{cases} A_i(x,t,u,Du)\, D_i u^{\frac{1}{p_N-1}} \geqslant \tilde{K}_1\, u^{\frac{p_N-p_i}{p_N-1}}\, |D_i u^{\frac{1}{p_N-1}}|^{p_i}, \\ |A_i(x,t,u,Du)| \leqslant \tilde{K}_2\, u^{\frac{p_N-p_i}{p_N-1}}\, |D_i u^{\frac{1}{p_N-1}}|^{p_i-1}, \qquad \forall\, i=1,\dots,N, \end{cases} \tag{A.2}$$

where $\tilde{K}_1, \tilde{K}_2$ are positive constants, and $Du = (D_1 u, \dots, D_N u)$ is assumed (by the request (3.3)) to satisfy weakly the chain rule

$$D_i u = (p_N-1) u^{\frac{p_N-2}{p_N-1}}\, D_i\left(u^{\frac{1}{p_N-1}}\right), \qquad \forall i=1,\dots,N. \tag{A.3}$$

The fact that (A.2) implies (1.4) under the assumption of the chain rule can be seen by simple algebraic computations as

$$\left(\frac{u^{2-p_N}}{\gamma}\right) A_i\, D_i u = A_i\, D_i(u^{\frac{1}{p_N-1}}) \geqslant \tilde{K}_1 u^{\frac{p_N-p_i}{p_N-1}}\, |D_i(u^{\frac{1}{p_N-1}})|^{p_i} = \frac{\tilde{K}_1}{\gamma} u^{\frac{p_N-p_i}{p_N-1}+(\frac{2-p_N}{p_N-1})p_i}\, |D_i u|^{p_i}.$$

Manipulating the terms in the set $[u>0]\cap\Omega_T$, we obtain the first condition in (1.4); and analogously for the second one.

A.2 | Alternative weak formulation

Since we deal with measurable and bounded coefficients, the time derivative of a weak solution to (3.4) exists only in the sense of distribution (see for an example the diffraction problem in [39], Sec. 13 of Ch. III, pages 224–233). In order to overcome the lack of regularity in the time variable, we define the following mollification in time: let $v \in L^1(E\times(t_1,t_2))$ for a compact set $E \subset \Omega$, times $0 < t_1 < t_2 < T$, and let us set

$$[\![v]\!]_h(x,t) := \frac{1}{h}\int_{t_1}^{t} e^{\frac{\tau-t}{h}}\, v(x,\tau)\, d\tau, \qquad [\![v]\!]_{\bar h}(x,t) := \frac{1}{h}\int_{t}^{t_2} e^{\frac{t-\tau}{h}}\, v(x,\tau)\, d\tau.$$

We refer the reader to [31] and Appendix B of [3] for the proofs of the following properties.

Lemma A.1. *For any $s \geqslant 1$, we have*

(i) *If $v \in L^s(\Omega_T)$, then $[\![v]\!]_h \in L^s(\Omega_T)$. Moreover, $||[\![v]\!]_h||_{L^s(\Omega_T)} \leqslant ||v||_{L^s(\Omega_T)}$, and $[\![v]\!]_h \to v$ in $L^s(\Omega_T)$ and almost everywhere on Ω_T as $h \to 0$.*

(ii) $[\![v]\!]_h \in C(0,T;L^s(\Omega))$.

(iii) *Almost everywhere on Ω_T, there hold*

$$\frac{\partial}{\partial t}[\![v]\!]_h = \frac{1}{h}(v - [\![v]\!]_h).$$

(iV) *If $Dv \in L^s(\Omega_T)$, then $D[\![v]\!]_h = [\![Dv]\!]_h \to Dv$ in $L^s(\Omega_T)$ and almost everywhere on Ω_T as $h \to 0$.*

(V) *If $v \in C(0,T;L^s(\Omega))$, then $[\![v]\!]_h(\cdot,t) \to v(\cdot,t)$ in $L^s(\Omega)$ and almost everywhere on Ω for every $t \in (0,T)$ as $h \to 0$.*

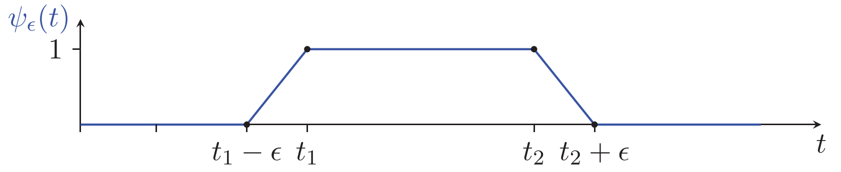

FIGURE 1 Illustration of the trapezoidal function ψ_ϵ.

From the differential inequalities (3.4), we deduce the mollified version as in Lemma 4.2 of [44],

$$\int_{t_1}^{t_2}\int_E \left(\frac{\partial}{\partial t}[\![u]\!]_h\,\varphi+\sum_i [\![A_i(x,\tau,u,Du)]\!]_h D_i\varphi\right)dxd\tau \leqslant (\geqslant) \int_E u(x,t_1)[\![\varphi]\!]_{\bar{h}}(x,t_1)\,dx\,, \tag{A.4}$$

for any nonnegative $\varphi\in L^{\mathbf{p}}(0,T;W_0^{1,\mathbf{p}}(E))$.

A.3 | Energy estimates proof of Lemma 3.6

We only consider the case of a local weak subsolution to (3.4), and the case of supersolutions can be considered completely analogously. For fixed $\tau-\eta<t_1<t_2<\tau$ and sufficiently small $\epsilon\in(0,1)$, we define (see Figure 1)

$$\psi_\epsilon(t):=\begin{cases}0, & \tau-\eta\leqslant t\leqslant t_1-\epsilon,\\ 1+\frac{t-t_1}{\epsilon}, & t_1-\epsilon\leqslant t\leqslant t_1,\\ 1, & t_1\leqslant t\leqslant t_2,\\ 1-\frac{t-t_2}{\epsilon}, & t_2\leqslant t\leqslant t_2+\epsilon,\\ 0, & t_2+\epsilon\leqslant t\leqslant \tau,\end{cases}$$

and choose in (A.4) the test function $\varphi=(u^{\frac{1}{p_N-1}}-k^{\frac{1}{p_N-1}})_+\zeta^{p_N}(x,t)\psi_\epsilon(t)$. So, we obtain

$$\begin{aligned}0\geqslant I_1+I_2-I_3 &= \int_{\tau-\eta}^{\tau}\int_E \partial_t[\![u]\!]_h(u^{\frac{1}{p_N-1}}-k^{\frac{1}{p_N-1}})_+\zeta^{p_N}\psi_\epsilon dxdt+\\ &+\sum_i\int_{\tau-\eta}^{\tau}\int_E [\![A_i]\!]_h D_i\left[(u^{\frac{1}{p_N-1}}-k^{\frac{1}{p_N-1}})_+\zeta^{p_N}\,\psi_\epsilon\right]dxdt-\\ &-\int_E u(x,\tau-\eta)\left(\int_{\tau-\eta}^{\tau}\frac{e^{\frac{\tau-\eta-s}{h}}}{h}\varphi(x,s)ds\right)dx\,.\end{aligned} \tag{A.5}$$

Our intention is to pass to the limit $h\downarrow 0$. The last integral vanishes, thanks to the vanishing property $\varphi(x,t)=0$ for $t\in[\tau-\eta\,,t_1-\epsilon]$, the time-continuity *á la Bochner*, Fubini–Tonelli's theorem and elementary integration, since

$$\int_E u(x,\tau-\eta)\left(\int_{\tau-\eta}^{\tau}\frac{e^{\frac{\tau-\eta-s}{h}}}{h}\varphi(x,s)ds\right)dx = \frac{1}{h}\int_{t_1-\epsilon}^{\tau} e^{\frac{\tau-\eta-s}{h}}\left(\int_E u(x,\tau-\eta)\varphi(x,s)\,dx\right)ds$$

$$\leqslant \left(\sup_{t_1-\epsilon<\tau<t_2}\|u(x,t_1)\varphi(x,\tau)\|_{L^1(E)}\right)\left(e^{\frac{\tau-\eta-t_1+\epsilon}{h}}\right)\to 0\,,$$

when h vanishes. For I_1, we compute

$$\int_{\tau-\eta}^{\tau}\int_E \frac{\partial}{\partial t}[\![u]\!]_h\,\varphi\,dxdt = \int_{\tau-\eta}^{\tau}\int_E \frac{\partial}{\partial t}[\![u]\!]_h\left([\![u]\!]_h^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}}\right)_+\zeta^{p_N}\,\psi_\epsilon dxdt +$$

$$+\int_{\tau-\eta}^{\tau}\int_E \frac{\partial}{\partial t}[\![u]\!]_h\left[(u^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_+ - \left([\![u]\!]_h^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}}\right)_+\right]\zeta^{p_N}\,\psi_\epsilon dxdt = I_{1,1}+I_{1,2}. \tag{A.6}$$

To estimate $I_{1,1}$, we use the properties of ψ_ϵ to get

$$\begin{aligned}
I_{1,1} &= \int_{\tau-\eta}^{\tau}\int_E \frac{\partial}{\partial t}\left[\int_k^{[\![u]\!]_h}(z^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_+\,dz\;\zeta^{p_N}\,\psi_\epsilon\right]dxdt - \\
&\quad - p_N\int_{\tau-\eta}^{\tau}\int_E\int_k^{[\![u]\!]_h}(z^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_+\,dz\;\zeta^{p_N-1}\zeta_t\,\psi_\epsilon dxdt - \\
&\quad - \int_{\tau-\eta}^{\tau}\int_E\int_k^{[\![u]\!]_h}(z^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_+\,dz\;\zeta^{p_N}\,\psi_\epsilon' dxdt \geqslant \\
&\geqslant -p_N\int_{t_1-\epsilon}^{t_2+\epsilon}\int_E\int_k^{[\![u]\!]_h}(z^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_+\,dz\;\zeta^{p_N-1}|\zeta_t|\,dxdt - \\
&\quad - \int_{t_1-\epsilon}^{t_2+\epsilon}\int_E\int_k^{[\![u]\!]_h}(z^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_+\,dz\;\zeta^{p_N}\,\psi_\epsilon' dxdt.
\end{aligned}$$

On the other hand, using property (iii) from Lemma A.1 and the fact that the function $s\to\max\{s^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}}\}$ is monotone increasing, we see that $I_{1,2}$ can be discarded since

$$I_{1,2} = \frac{1}{h}\int_{\tau-\eta}^{\tau}\int_E [u-[\![u]\!]_h]\left[(u^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_+ - \left([\![u]\!]_h^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}}\right)_+\right]\zeta^{p_N}\,\psi_\epsilon dxdt \geqslant 0.$$

We can let $h\to 0$ on I_1 and invoke Lebesgue's theorem of dominated convergence relying on the assumption

$$u^{\frac{1}{p_N-1}} \in L^1_{loc}(\Omega_T)\subset L^{\mathbf{p}}_{loc}(0,T;W^{1,\mathbf{p}}_{loc}(\Omega))$$

and property (i) of Lemma A.1. By using the fact that $g_+(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}}) = \int_k^u (z^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_+ dz$ is continuous w.r.t u, the limit can be computed

$$\lim_{h\to 0} \int_{\tau-\eta}^{\tau} \int_E \frac{\partial}{\partial t} [\![u]\!]_h \, \varphi \, dxdt \geqslant -p_N \int_{t_1-\epsilon}^{t_2+\epsilon} \int_E g_+(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}}) \, \zeta^{p_N-1} |\zeta_t| \, dxdt -$$
$$- \int_{t_1-\epsilon}^{t_2+\epsilon} \int_E g_+(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}}) \, \zeta^{p_N} \, \psi_\epsilon' dxdt.$$

Now we can pass to the limit $\epsilon \to 0$ in this inequality and obtain

$$\lim_{\epsilon\to 0} \lim_{h\to 0} \int_{\tau-\eta}^{\tau} \int_E \frac{\partial}{\partial t} [\![u]\!]_h \, \varphi \, dxdt \geqslant \int_E g_+(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}}) \, \zeta^{p_N} \, dx \Big|_{t_1}^{t_2} -$$
$$- p_N \int_{t_1}^{t_2} \int_E g_+(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}}) \, \zeta^{p_N-1} |\zeta_t| \, dxdt.$$

In order to pass to the limit for $h \downarrow 0$ the diffusion term I_2, we split it by the product of weak derivatives as

$$\begin{aligned} I_2 &= \int_{\tau-\eta}^{\tau} \int_E [\![A_i]\!]_h \left(D_i (u^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_+ \right) \zeta^{p_N} \, \psi_\epsilon dxdt + \\ &+ p_N \int_{\tau-\eta}^{\tau} \int_E [\![A_i]\!]_h \, (u^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_+ \, \zeta^{p_N-1} D_i \zeta \, \psi_\epsilon dxdt \\ &= I_{2,1} + I_{2,2}, \end{aligned}$$

and apply Lebesgue's dominated convergence theorem to pass to the limit, since by point (i) of Lemma A.1, structure conditions (3.4) and (3.3), we have

$$|A_i(x,t,u,Du)| \leqslant \gamma |D_i u^{\frac{1}{p_i-1}}|^{p_i-1} \quad \Rightarrow \quad [\![A_i]\!]_h \in L_{loc}^{\frac{p_i}{p_i-1}}(\Omega_T).$$

Hence, the approximated inequality (A.5) as soon as both h and ϵ vanish tends to

$$\begin{aligned} 0 \geqslant & \int_E g_+(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}}) \, \zeta^{p_N} \, dx \Big|_{t_1}^{t_2} - p_N \int_{t_1}^{t_2} \int_E g_+(u^{\frac{1}{p_N-1}}, k^{\frac{1}{p_N-1}}) \, \zeta^{p_N-1} |\zeta_t| \, dxdt + \\ &+ \sum_i \int_{\tau-\eta}^{\tau} \int_E A_i(x,t,u,Du) \, D_i (u^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_+ \, \zeta^{p_N} \, dxdt + \\ &- p_N \sum_i \int_{\tau-\eta}^{\tau} \int_E |A_i(x,t,u,Du)| \, (u^{\frac{1}{p_N-1}} - k^{\frac{1}{p_N-1}})_+ \, \zeta^{p_N-1} |D_i \zeta| \, dxdt \,. \end{aligned} \tag{A.7}$$

Now, the standard use of the structure conditions (3.4) and the Young inequality yield the desired estimate (3.5).

A.4 | Expansion of positivity for anisotropic p-Laplace evolution equation

For the proof of Hölder's continuity of solutions to (1.3)–(1.4), we will make use of the following result that can be found in [11]. For that purpose, consider the equation

$$u_t - \sum_i D_i a_i(x,t,u,Du) = 0, \quad (x,t) \in \Omega_T, \tag{A.8}$$

and let the following inequalities hold:

$$\begin{cases} \sum_{i=1}^{N} a_i(x,t,u,Du) D_i u \geqslant \bar{K}_1 \sum_i |D_i u|^{p_i}, \\ |a_i(x,t,u,Du)| \leqslant \bar{K}_2 \left(\sum_j |D_j u|^{p_j} \right)^{1-\frac{1}{p_i}}, \quad i = 1, \dots, N, \end{cases} \tag{A.9}$$

for fixed positive constants $\bar{K}_1$ and $\bar{K}_2$. For a point $(y,s) \in \Omega_T$, and constants $C_* > 1$ and $b, r > 0$, let μ^+, μ^-, ω be positive numbers satisfying

$$\mu^+ \geqslant \sup_{Q_{r,\theta}^{\omega/C_*}(y,s+\theta)} u, \qquad \mu^- \leqslant \inf_{Q_{r,\theta}^{\omega/C_*}(y,s+\theta)} u, \qquad \omega \geqslant \mu^+ - \mu^-,$$

and let θ be an intrinsic time parameter defined as

$$\theta := b\, r^p\, \omega^{2-p_N}.$$

Finally, let us set

$$v_+ := \mu^+ - u, \quad v_- := u - \mu^-.$$

The expansion of positivity is a property of locally bounded local weak solutions to (A.8)–(A.9), as shown in Theorems 3.1 and 3.4 of [11], at the price of restricting the range $p_N - p_1$ of a qualitative parameter ϵ_* that is determined only from the data.

Lemma A.2. *Let $u \in C_{loc}(0,T; L^2_{loc}(\Omega)) \cap L^{\mathbf{p}}_{loc}(0,T; W^{1,\mathbf{p}}_{loc}(\Omega)) \cap L^\infty_{loc}(\Omega_T)$ be a local weak solution to (A.8)–(A.9) in Ω_T, and also assume that*

$$\left| \mathcal{K}_r^{\xi\omega}(y) \cap \{ v_\pm(\cdot, s) \geqslant \xi\omega \} \right| \geqslant \alpha_0 |\mathcal{K}_r^{\xi\omega}(y)|, \tag{A.10}$$

with some $\xi, \alpha_0 \in (0,1)$. Then, there exist numbers $C_* > 1$, $0 < \bar{b} < b$, $\epsilon_* \in (0,1)$ depending only on the data, ξ and α_0 such that

$$v_\pm(x,t) \geqslant \frac{\omega}{C_*}, \quad (x,t) \in \mathcal{K}_r^{\omega/C_*}(y) \times (s + \bar{b} r^p \omega^{2-p_N}, s + b r^p \omega^{2-p_N}), \tag{A.11}$$

provided that $Q_{8r,8\theta}^{\omega/C_*}(y, s+\theta) \subset \Omega_T$ and

$$p_N - p_1 \leqslant \epsilon_*. \tag{A.12}$$

A.5 | Auxiliary lemmas

The following lemma is the well-known De Giorgi–Poincaré lemma (see [21, Lemma 2.2, Chap.2]), presented for standard cubes.

Lemma A.3. *Let $u \in W^{1,1}(K_1(y))$ and let k and l be real numbers such that $k < l$. Then, there exists a constant $\gamma > 0$, depending only on N, such that*

$$(l-k)|K_1(y)\cap\{u \leqslant k\}|\;|K_1(y)\cap\{u \geqslant l\}| \leqslant \gamma \int_{K_1(y)\cap\{k\leqslant u\leqslant l\}} |\nabla u| dx.$$

For the next lemma, we refer to [15], Lemma 4.1.

Lemma A.4. *For any $\beta > 0$, there exists a constant $\gamma_0 = \gamma_0(\beta) > 1$ such that, for any bounded $u : Q_{1,1}(x_0,t_0) \to \mathbb{R}$ with $u(x_0,t_0) \geqslant 1$, there exist $(y,\tau) \in Q_{1,1}(x_0,t_0)$ and $r > 0$ such that*

$$Q_{r,r^p}(y,\tau) \subset Q_{1,1}(x_0,t_0), \qquad r^\beta \sup_{Q_{r,r^p}(y,\tau)} u \leqslant \gamma_0 \qquad \textit{and} \qquad r^\beta u(y,\tau) \geqslant \frac{1}{\gamma_0}.$$

In order to manipulate the nonlinearity in time, we will also need the following result that can be found [1, Lemma 3.2]

Lemma A.5. *There exists a constant $\gamma > 0$, depending only on m, such that*

$$\frac{1}{\gamma}\,(k^m + u^m)^{\frac{1}{m}-1}(u^m - k^m)^2_\pm \leqslant g_\pm(u^m,k^m) \leqslant \gamma\,(k^m + u^m)^{\frac{1}{m}-1}(u^m - k^m)^2_\pm,$$

where

$$g_\pm(u^m,k^m) := \pm\frac{1}{m}\int_{k^m}^{u^m} s^{\frac{1}{m}-1}(s-k^m)_\pm\, ds, \qquad k,m > 0\,.$$

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