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The Cauchy–Szegő Projection for Domains in $\mathbb{C}^n$ with Minimal Smoothness: Weighted Theory

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Abstract

Let $D \subset \mathbb{C}^n$ be a bounded, strongly pseudoconvex domain whose boundary bD satisfies the minimal regularity condition of class C^2. A 2017 result of Lanzani & Stein [17] states that the Cauchy–Szegő projection $\mathcal{S}_\omega$ defined with respect to a bounded, positive continuous multiple ω of induced Lebesgue measure, maps $L^p(bD, \omega)$ to $L^p(bD, \omega)$ continuously for any $1 < p < \infty$. Here we show that $\mathcal{S}_\omega$ satisfies explicit quantitative bounds in $L^p(bD, \Omega_p)$, for any $1 < p < \infty$ and for any Ω_p in the maximal class of A_p-measures, that is for $\Omega_p = \psi_p \sigma$ where ψ_p is a Muckenhoupt A_p-weight and σ is the induced Lebesgue measure (with ω's as above being a sub-class). Earlier results rely upon an asymptotic expansion and subsequent pointwise estimates of the Cauchy–Szegő kernel, but these are unavailable in our setting of minimal regularity of bD; at the same time, more recent techniques that allow to handle domains with minimal regularity [17] are not applicable to A_p-measures. It turns out that the method of quantitative extrapolation is an appropriate replacement for the missing tools. To finish, we identify a class of holomorphic Hardy spaces defined with respect to A_p-measures for which a meaningful notion of Cauchy-Szegő projection can be defined when $p = 2$.

In memory of Joseph J. Kohn

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1 Introduction

Given a domain $D \Subset \mathbb{C}^n$ with rectifiable boundary and a reference measure μ on bD (the boundary of D), the *Cauchy-Szegő projection* S_μ is the (unique) orthogonal projection of the Hilbert space $L^2(bD, \mu)$ onto the holomorphic Hardy space $H^2(bD, \mu)$. The $L^p(bD, \mu)$*- regularity problem for* S_μ, that is, the study of the boundedness of S_μ on $L^p(bD, \mu)$ with $p \neq 2$, is a deep problem in harmonic analysis whose solution is highly dependent on the analytic and geometric properties of D and of the reference measure μ; see e.g., [12, 14, 17, 20, 25, 31].

In this paper we present a new approach to the L^p-regularity problem for the Cauchy–Szegő projection $\mathcal{S}_\omega$ defined with respect to $\omega = \Lambda\,\sigma$ (any) bounded, positive continuous multiple of the induced Lebesgue measure σ associated to a strongly pseudoconvex domain $D \Subset \mathbb{C}^n$, with $n \geq 2$, that satisfies the minimal regularity condition of class C^2. As an application, we obtain the quantitative bound

$$\|\mathcal{S}_\omega\, g\|_{L^p(bD,\Omega_p)} \lesssim [\Omega_p]_{A_p}^{3\cdot\max\left\{1,\, \frac{1}{p-1}\right\}} \|g\|_{L^p(bD,\Omega_p)}, \quad 1 < p < \infty, \tag{1.1}$$

for any $\omega \in \{\Lambda\sigma\}_\Lambda$ as above and for any Ω_p in the class of Muckenhoupt measures $A_p(bD)$, where $[\Omega_p]_{A_p}$ stands for the A_p-character of Ω_p while the implicit constant depends solely on p, ω and D; see Theorem 1.1 for the precise statement. We point out that the power 3 can be sharpened as $2 + \delta$ for any $\delta > 0$. However, it can not be reduced to 2 due to the structure of minimal smoothness of the domain. By contrast, the norm bounds obtained in the reference result [17]:

$$\|\mathcal{S}_\omega\, g\|_{L^p(bD,\omega)} \leq C(D, \omega, p)\, \|g\|_{L^p(bD,\omega)}, \quad 1 < p < \infty \quad \omega \in \{\Lambda\sigma\}_\Lambda \text{ as above}, \tag{1.2}$$

and its recent improvement for $\omega := \sigma$, [31, Theorem 1.1]:

$$\|\mathcal{S}_\sigma\, g\|_{L^p(bD,\Omega_p)} \leq \tilde{C}(D, [\Omega_p]_{A_p})\|g\|_{L^p(bD,\Omega_p)}, \quad 1 < p < \infty, \quad \Omega_p \in A_p(bD) \tag{1.3}$$

are unspecified functions of the stated variables.

As is well known, the A_p-measures are the maximal (doubling) measure-theoretic framework for a great variety of singular integral operators. Any positive, continuous multiple of σ is, of course, a member of $A_p(bD)$ for *any* $1 < p < \infty$, but the class $\{A_p(bD)\}_p$ far encompasses the family $\{\Lambda\,\sigma\}_\Lambda$. The classical methods for $\mathcal{S}_\omega$ rely on the asymptotic expansion of the Cauchy–Szegő kernel, see e.g., the foundational works

[3, 22] and [25], however this expansion is not available if D is non-smooth (below the class C^3). What's more, the Cauchy–Szegő projection $\mathcal{S}_\omega$ may not be Calderón–Zygmund, see [20], thus none of (1.1)–(1.3) can follow by a direct application of the Calderón–Zygmund theory. The proof of (1.2) given in [17] starts with the comparison of $\mathcal{S}_\omega$ with the members of an ad-hoc family $\{\mathcal{C}_\epsilon\}_\epsilon$ of non-orthogonal projections (the so-called Cauchy–Leray integrals):

$$\mathcal{C}_\epsilon = \mathcal{S}_\omega \circ \left[I - (\mathcal{C}_\epsilon^\dagger - \mathcal{C}_\epsilon)\right] \quad \text{in} \quad L^2(bD, \omega), \quad 0 < \epsilon < \epsilon(D), \tag{1.4}$$

where the upper-script $\dagger$ denotes the adjoint in $L^2(bD, \omega)$. The operators $\{\mathcal{C}_\epsilon\}_\epsilon$ are bounded on $L^p(bD, \omega)$, $1 < p < \infty$, by an application of the $T(1)$ theorem. Furthermore, an elementary, Hilbert space-theoretic observation yields the factorization

$$\mathcal{S}_\omega = \mathcal{C}_\epsilon \circ \left[I - (\mathcal{C}_\epsilon^\dagger - \mathcal{C}_\epsilon)\right]^{-1} \quad \text{in} \quad L^2(bD, \omega), \quad 0 < \epsilon < \epsilon(D), \tag{1.5}$$

along with its refinement

$$\mathcal{S}_\omega = \left(\mathcal{C}_\epsilon + \mathcal{S}_\omega \circ \left((\mathcal{R}_\epsilon^s)^\dagger - \mathcal{R}_\epsilon^s\right)\right) \circ \left[I - \left((\mathcal{C}_\epsilon^s)^\dagger - \mathcal{C}_\epsilon^s\right)\right]^{-1} \text{in } L^2(bD, \omega),\ 0 < \epsilon < \epsilon(D). \tag{1.6}$$

For the latter, the Cauchy–Leray integral $\mathcal{C}_\epsilon$ is decomposed as the sum $\mathcal{C}_\epsilon = \mathcal{C}_\epsilon^s + \mathcal{R}_\epsilon^s$ where the principal term $\mathcal{C}_\epsilon^s$ (bounded, by another application of the $T(1)$ theorem) enjoys the cancellation

$$\|(\mathcal{C}_\epsilon^s)^\dagger - \mathcal{C}_\epsilon^s\|_{L^p(bD,\,\omega) \to L^p(bD,\,\omega)} \le \epsilon^{1/2}\, C(p, D, \omega),\ 1 < p < \infty\,,\ 0 < \epsilon < \epsilon(D). \tag{1.7}$$

By contrast, the remainder $\mathcal{R}_\epsilon^s$ has $L^p \to L^p$ norm that may blow up as $\epsilon \to 0$ but is weakly smoothing in the sense that

$$\mathcal{R}_\epsilon^s \text{ and } (\mathcal{R}_\epsilon^s)^\dagger \text{ are bounded} : L^1(bD, \omega) \to L^\infty(bD, \omega) \quad \text{for any} \quad 0 < \epsilon < \epsilon(D). \tag{1.8}$$

Combining (1.7) and (1.8) with the bounded inclusions

$$L^2(bD, \omega) \hookrightarrow L^p(bD, \omega) \hookrightarrow L^1(bD, \omega), \quad 1 < p \le 2, \tag{1.9}$$

one concludes that there is $\epsilon = \epsilon(p)$ such that the right hand side of (1.6), and therefore $\mathcal{S}_\omega$, extends to a bounded operator on $L^p(bD, \omega)$ for any $\omega \in \{\Lambda\,\sigma\}_\Lambda$ whenever $1 < p < 2$; the proof for the full p-range then follows by duality.

Implementing this argument to *all* A_p-measures presents conceptual difficulties: for instance, there are no analogs of (1.8) nor of (1.9) that are valid with Ω_p in place of ω because A_p-measures may change with p. In [31, Lemma 3.3] it is shown that (1.7) is still valid for $p = 2$ if one replaces the measure ω in the $L^2(bD)$-space with an A_2-measure; the conclusion (1.3) then follows by Rubio de Francia's original extrapolation theorem [27].

Here we obtain a new cancellation in $L^p(bD, \Omega_p)$ where the dependence of the norm-bound on the A_p-character of the measure is completely explicit, namely

$$\|(\mathcal{C}^s_\epsilon)^\dagger - \mathcal{C}^s_\epsilon\|_{L^p(bD,\,\Omega_p)\to L^p(bD,\,\Omega_p)} \lesssim \epsilon^{1/2}[\Omega_p]_{A_p}^{\max\{1,\frac{1}{p-1}\}}, \quad 1 < p < \infty \tag{1.10}$$

where the implied constant depends on p, D and ω but is independent of Ω_p and of ϵ; see Proposition 3.2 for the precise statement. Combining (1.10) with the reverse Hölder inequality for $A_2(bD)$ we obtain

$$\|\mathcal{S}_\omega\|_{L^2(bD,\Omega_2)\to L^2(bD,\Omega_2)} \lesssim [\Omega_2]_{A_2} \quad \text{for any } \Omega_2 \in A_2(bD) \text{ and any } \omega \in \{\Lambda\,\sigma\}_\Lambda. \tag{1.11}$$

The conclusion (1.1) then follows by quantitative extrapolation [6, Theorem 9.5.3], which serves as an appropriate replacement for (1.8) and (1.9). Incidentally, we are not aware of other applications of extrapolation to several complex variables; yet quantitative extrapolation seems to provide an especially well-suited approach to the analysis of orthogonal projections onto spaces of holomorphic functions[1] because orthogonal projections are naturally bounded, with minimal norm, on the Hilbert space where they are defined; thus there is always a baseline choice of Ω_2 for which (1.11) holds trivially (that is, with $[\Omega_2]_{A_2} = 1$). We anticipate that this approach will give new insight into other settings where this kind of L^p-regularity problems are unsolved, such as the strongly $\mathbb{C}$-linearly convex domains of class $C^{1,1}$ ([16]) and the $C^{1,\alpha}$ model domains of [18].

Can the Cauchy–Szegő projection be defined for measures other than $\omega \in \{\Lambda\,\sigma\}_\Lambda$*?* Orthogonal projections are highly dependent upon the choice of reference measure for the Hilbert space where they are defined: it is therefore natural to seek the maximal measure-theoretic framework for which the notion of Cauchy–Szegő projection is meaningful. The A_p-measures are an obvious choice in view of their historical relevance to the theory of singular integral operators. It turns out that in this context the Cauchy–Szegő projection is meaningful only if it is defined with respect to A_2-measures that is, for $p = 2$: one may define $\mathcal{S}_{\Omega_2}$ which is bounded: $L^2(bD, \Omega_2) \to L^2(bD, \Omega_2)$ for any $\Omega_2 \in A_2(bD)$ with the minimal operator norm: $\|\mathcal{S}_{\Omega_2}\| = 1$; on the other hand there appears to be no well-posed notion of "$\mathcal{S}_{\Omega_p}$" if $p \neq 2$. The $L^p(bD, \Omega_p)$-regularity problem for $\mathcal{S}_{\Omega_2}$, while meaningful, is, at present, unanswered for $p \neq 2$.

1.1 Statement of the main results

We let σ denote induced Lebesgue measure for bD and we henceforth refer to the family

$$\{\Lambda\sigma\}_\Lambda \equiv \{\omega := \Lambda\,\sigma,\ \Lambda \in C(bD),\ 0 < c(D, \Lambda) \leq \Lambda(w) \leq C(D, \Lambda) < \infty \text{ for any } w \in bD\}$$

[1] The Cauchy–Szegő and Bergman projections being two prime examples.

as the *Leray Levi-like measures*. This is because the Leray Levi measure λ, which plays a distinguished role in the analysis [17] of the Cauchy–Leray integrals $\{\mathcal{C}_\epsilon\}_\epsilon$ and of their truncations $\{\mathcal{C}^s_\epsilon\}_\epsilon$, is a member of this family on account of the identity

$$d\lambda(w) = \Lambda(w)d\sigma(w), \quad w \in bD, \tag{1.12}$$

where $\Lambda \in C(\overline{D})$ satisfies the required bounds $0 < \epsilon(D) \leq \Lambda(w) \leq C(D) < \infty$ for any $w \in bD$ as a consequence of the strong pseudoconvexity and C^2-regularity and boundedness of D. Hence we may equivalently express any Leray Levi-like measure ω as

$$\omega = \varphi\lambda \tag{1.13}$$

for some $\varphi \in C(\overline{bD})$ such that $0 < m(D) \leq \varphi(w) \leq M(D) < \infty$ for any $w \in bD$. We refer to Section 2 for the precise definitions and to [26, Lemma VII.3.9] for the proof of (1.12) and a discussion of the geometric significance of λ.

For any Leray Levi-like measure ω, the holomorphic Hardy space $H^2(bD, \omega)$ is defined exactly as in the classical setting of $H^2(bD, \sigma)$, simply by replacing σ with ω, see e.g., [19, (1.1) and (1.2)]. In particular $H^2(bD, \omega)$ is a closed subspace of $L^2(bD, \omega)$ and we let $\mathcal{S}_\omega$ denote the (unique) *orthogonal* projection of $L^2(bD, \omega)$ onto $H^2(bD, \omega)$.

Our goal is to understand the behavior of $\mathcal{S}_\omega$ on $L^p(bD, \Omega_p)$ where $\{\Omega_p\}$ is any A_p*-measure* that is,

$$\Omega_p = \phi_p\,\sigma \tag{1.14}$$

where the density ϕ_p is a Muckenhoupt A_p-weight. The precise definition is given in Section 2; here we just note that the Leray Levi-like measures are strictly contained in the class $\{\Omega_p\}_p$ in the sense that each $\omega \in \{\Lambda\sigma\}_\Lambda$ is an A_p-measure for *every* $1 < p < \infty$. We may now state our main result.

Theorem 1.1 *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded, strongly pseudoconvex domain of class C^2; let ω be any Leray Levi-like measure for bD and let $\mathcal{S}_\omega$ be the Cauchy–Szegő projection associated to ω. We have that*

$$\|\mathcal{S}_\omega\|_{L^2(bD,\Omega_2)\to L^2(bD,\Omega_2)} \lesssim [\Omega_2]^3_{A_2} \quad \textit{for any } \ \Omega_2 \in A_2(bD), \tag{1.15}$$

where the implied constant depends solely on D and ω.

As before, the power 3 can be sharpened as $2 + \delta$ for any $\delta > 0$. However, it can not be reduced to 2 due to the structure of minimal smoothness of the domain. The L^p-estimate (1.1) follows from (1.15) by standard techniques, see [6, Theorem 9.5.3].

As mentioned earlier, extending the notion of Cauchy–Szegő projection to the realm of A_p-measures requires, first of all, a meaningful notion of holomorphic Hardy space for such measures, but this does not immediately arise from the classical theory [29]; here we adopt the approach of [19, (1.1)] and give the following

Definition 1.2 Suppose $1 \leq p < \infty$ and let Ω_p be an A_p-measure. We define $H^p(bD, \Omega_p)$ to be the space of functions F that are holomorphic in D with

$\mathcal{N}(F) \in L^p(bD, \Omega_p)$, and set

$$\|F\|_{H^p(bD,\Omega_p)} := \|\mathcal{N}(F)\|_{L^p(bD,\Omega_p)}. \tag{1.16}$$

Here $\mathcal{N}(F)$ denotes the non-tangential maximal function of F, that is

$$\mathcal{N}(F)(\xi) := \sup_{z\in\Gamma_\alpha(\xi)} |F(z)|, \quad \xi \in bD,$$

where $\Gamma_\alpha(\xi) = \{z \in D : |(z-\xi)\cdot\bar{\nu}_\xi| < (1+\alpha)\delta_\xi(z), |z-\xi|^2 < \alpha\delta_\xi(z)\}$, with $\bar{\nu}_\xi$ = the (complex conjugate of) the outer unit normal vector to $\xi \in bD$, and $\delta_\xi(z) =$ the minimum between the (Euclidean) distance of z to bD and the distance of z to the tangent space at ξ.

For the class of domains under consideration it is known that if Ω_p is Leray Levi-like, such definition agrees with the classical formulation [19, (1.2)]; see also [29].

Proposition 1.3 *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded, strongly pseudoconvex domain of class C^2. Then, for any $1 \leq p < \infty$ and any A_p-measure Ω_p we have that $H^p(bD, \Omega_p)$ is a closed subspace of $L^p(bD, \Omega_p)$. More precisely, suppose that $\{F_n\}_n$ is a sequence of holomorphic functions in D such that $\|\mathcal{N}(F_n) - f\|_{L^p(bD,\Omega_p)} \to 0$ as $n \to \infty$. Then, there is an F holomorphic in D such that $\mathcal{N}(F)(w) = f(w)$ $\Omega_p - a.e.\ w \in bD$.*

The proof relies on the following observation, which is of independent interest: for any $1 \leq p < \infty$ and any A_p-measure Ω_p with density ψ_p, there is $p_0 = p_0(\Omega_p) \in (1, p)$ such that

$$H^p(bD, \Omega_p) \subset H^{p_0}(bD, \sigma) \text{ with } \|F\|_{H^{p_0}(bD,\sigma)} \leq C_{\Omega_p,D}\|F\|_{H^p(bD,\Omega_p)}, \tag{1.17}$$

where

$$C_{\Omega_p,D} = \left(\int_{bD} \psi_p(w)^{-\frac{p_0}{p-p_0}}\, d\sigma(w)\right)^{\frac{p-p_0}{pp_0}}.$$

On the other hand in the context of A_p-measures the notion of Hilbert space is meaningful only in relation to A_2-measures (that is for $p = 2$). Hence Cauchy–Szegő projections can be associated only to such measures: on account of Proposition 1.3, for any A_2-measure Ω_2 there exists a unique, orthogonal projection:

$$\mathcal{S}_{\Omega_2} : L^2(bD, \Omega_2) \to H^2(bD, \Omega_2)$$

which is naturally bounded on $L^2(bD, \Omega_2)$ with minimal norm $\|\mathcal{S}_{\Omega_2}\| = 1$.

1.2 Further results

The proof of Theorem 1.1 also requires quantitative results for the Cauchy Leray integrals $\{\mathcal{C}_\epsilon\}_\epsilon$ that extend the scope of the earlier works [17] and [2] from Leray Levi-like measures, to A_p-measures: these are stated in Theorem 3.1 and Proposition 3.2.

1.3 Organization of this paper.

In Section 2 we recall the necessary background and give the proof of Proposition 1.3. All the quantitative results pertaining to the Cauchy–Leray integral are collected in Section 3. Theorem 1.1 is proved in Section 4.

2 Preliminaries and proof of Proposition 1.3

In this section we introduce notations and recall certain results from [2, 17] that will be used throughout this paper. We will henceforth assume that $D \subset \mathbb{C}^n$ is a bounded, strongly pseudoconvex domain of class C^2; that is, there is $\rho \in C^2(\mathbb{C}^n, \mathbb{R})$ which is strictly plurisubharmonic and such that $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ and $bD = \{w \in \mathbb{C}^n : \rho(w) = 0\}$ with $\nabla\rho(w) \neq 0$ for all $w \in bD$. (We refer to such ρ as a *defining function for* D; see e.g., [26] for the basic properties of defining functions. Here we assume that one such ρ has been fixed once and for all.) We will throughout make use of the following abbreviated notations:

$$\|T\|_p \equiv \|T\|_{L^p(bD,d\mu)\to L^p(bD,d\mu)}, \quad \text{and} \quad \|T\|_{p,q} \equiv \|T\|_{L^p(bD,d\mu)\to L^q(bD,d\mu)}$$

where the operator T and the measure μ will be clear from context.

• *The Levi polynomial and its variants.* Define

$$\mathcal{L}_0(w,z) := \langle \partial\rho(w), w - z\rangle - \frac{1}{2}\sum_{j,k} \frac{\partial^2\rho(w)}{\partial w_j \partial w_k}(w_j - z_j)(w_k - z_k),$$

where $\partial\rho(w) = (\frac{\partial\rho}{\partial w_1}(w), \ldots, \frac{\partial\rho}{\partial w_n}(w))$ and we have used the notation $\langle \eta, \zeta\rangle = \sum_{j=1}^n \eta_j\zeta_j$ for $\eta = (\eta_1, \ldots, \eta_n), \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$. The strict plurisubharmonicity of ρ implies that

$$2\,\mathrm{Re}\,\mathcal{L}_0(w,z) \geq -\rho(z) + c|w - z|^2,$$

for some $c > 0$, whenever $w \in bD$ and $z \in \bar{D}$ is sufficiently close to w. We next define

$$g_0(w,z) := \chi\mathcal{L}_0 + (1 - \chi)|w - z|^2 \tag{2.1}$$

where $\chi = \chi(w, z)$ is a C^∞-smooth cutoff function with $\chi = 1$ when $|w - z| \leq \mathfrak{c}/2$ and $\chi = 0$ if $|w - z| \geq \mathfrak{c}$. Then for $\mathfrak{c}$ chosen sufficiently small (and then kept fixed throughout), we have that

$$\operatorname{Re} g_0(w, z) \geq c(-\rho(z) + |w - z|^2) \tag{2.2}$$

for z in $\bar{D}$ and w in bD, with c a positive constant; we will refer to $g_0(w, z)$ as *the modified Levi polynomial*. Note that $g_0(w, z)$ is polynomial in the variable z, whereas in the variable w it has no smoothness beyond mere continuity. To amend for this lack of regularity, for each $\epsilon > 0$ one considers a variant g_ϵ defined as follows. Let $\{\tau^\epsilon_{jk}(w)\}$ be an $n \times n$-matrix of C^1 functions such that

$$\sup_{w \in bD} \left| \frac{\partial^2 \rho(w)}{\partial w_j \partial w_k} - \tau^\epsilon_{jk}(w) \right| \leq \epsilon, \quad 1 \leq j, k \leq n.$$

Set

$$c_\epsilon := \sup_{w \in bD, 1 \leq j,k \leq n} \left| \nabla \tau^\epsilon_{jk}(w) \right|. \tag{2.3}$$

For the convenience of our statement and proof, we may choose those $\{\tau^\epsilon_{jk}(w)\}$ such that

$$c_\epsilon \lesssim \epsilon^{-1}. \tag{2.4}$$

where the implicit constant is independent of ϵ. We also set

$$\mathcal{L}_\epsilon(w, z) = \langle \partial \rho(w), w - z \rangle - \frac{1}{2} \sum_{j,k} \tau^\epsilon_{jk}(w)(w_j - z_j)(w_k - z_k),$$

and define

$$g_\epsilon(w, z) = \chi \mathcal{L}_\epsilon + (1 - \chi)|w - z|^2, \quad z, w \in \mathbb{C}^n.$$

Now g_ϵ is of class C^1 in the variable w, and

$$|g_0(w, z) - g_\epsilon(w, z)| \lesssim \epsilon |w - z|^2, \quad w \in bD, z \in \overline{D}.$$

We assume that ϵ is sufficiently small (relative to the constant c in (2.2)), and this gives that

$$|g_0(w, z)| \leq |g_\epsilon(w, z)| \leq \tilde{C}\, |g_0(w, z)|, \quad w, z \in bD \tag{2.5}$$

where the constants C and $\tilde{C}$ are independent of ϵ; see [17, Section 2.1].

• *The Leray–Levi measure for bD.* Let j^* denote the pullback under the inclusion

$$j : bD \hookrightarrow \mathbb{C}^n.$$

Then the linear functional

$$f \mapsto \frac{1}{(2\pi i)^n} \int\limits_{bD} f(w) j^*(\partial\rho \wedge (\bar\partial\partial\rho)^{n-1})(w) =: \int\limits_{bD} f(w) d\lambda(w) \tag{2.6}$$

where $f \in C(bD)$, defines a measure λ with positive density given by

$$d\lambda(w) = \frac{1}{(2\pi i)^n} j^*(\partial\rho \wedge (\bar\partial\partial\rho)^{n-1})(w).$$

We point out that the definition of λ depends upon the choice of defining function for D, which here has been fixed once and for all; hence we refer to λ as "the" *Leray–Levi measure*.

• *A space of homogeneous type.* Consider the function

$$\mathtt{d}(w, z) := |g_0(w, z)|^{\frac{1}{2}}, \quad w, z \in bD. \tag{2.7}$$

It is known [17, (2.14)] that

$$|w - z| \lesssim \mathtt{d}(w, z) \lesssim |w - z|^{1/2}, \quad w, z \in bD$$

and from this it follows that the space of Hölder-type functions [17, (3.5)]:

$$|f(w) - f(z)| \lesssim \mathtt{d}(w, z)^\alpha \quad \text{for some } 0 < \alpha \le 1 \ \text{ and for all } w, z \in bD \tag{2.8}$$

is dense in $L^p(bD, \omega)$, $1 < p < \infty$ for any Leray Levi-like measure (we note that since bD is bounded, the space of Hölder-type functions contains the space of smooth functions in bD with compact support).

It follows from (2.5) that

$$\tilde C \mathtt{d}(w, z)^2 \le |g_\epsilon(w, z)| \le C \mathtt{d}(w, z)^2, \quad w, z \in bD \tag{2.9}$$

for any ϵ sufficiently small. It is shown in [17, Proposition 3] that $\mathtt{d}(w, z)$ is a quasi-distance: there exist constants $A_0 > 0$ and $C_{\mathtt{d}} > 1$ such that for all $w, z, z' \in bD$,

$$\begin{cases} 1) \ \mathtt{d}(w, z) = 0 \quad \text{iff} \quad w = z; \\ 2) \ A_0^{-1}\mathtt{d}(z, w) \le \mathtt{d}(w, z) \le A_0 \mathtt{d}(z, w); \\ 3) \ \mathtt{d}(w, z) \le C_{\mathtt{d}}\big(\mathtt{d}(w, z') + \mathtt{d}(z', z)\big). \end{cases} \tag{2.10}$$

Letting $B_r(w)$ denote the boundary balls determined via the quasi-distance $\mathtt{d}$,

$$B_r(w) := \{z \in bD : \ \mathtt{d}(z, w) < r\}, \quad \text{where } w \in bD, \tag{2.11}$$

we have that

$$c_\omega^{-1} r^{2n} \le \omega\big(B_r(w)\big) \le c_\omega r^{2n}, \quad 0 < r \le 1, \tag{2.12}$$

for some $c_\omega > 1$, see [17, p. 139]. It follows that the triples $\{bD, \mathtt{d}, \omega\}$, for any Leray Levi-like measure ω, are spaces of homogeneous type, where the measures ω have the doubling property:

Lemma 2.1 *The Leray Levi-like measures ω on bD are doubling, i.e., there is a positive constant C_ω such that for all $w \in bD$ and $0 < r \leq 1$,*

$$0 < \omega(B_{2r}(w)) \leq C_\omega \omega(B_r(w)) < \infty.$$

Furthermore, there exist constants $\epsilon_\omega \in (0, 1)$ and $C_\omega > 0$ such that

$$\omega(B_r(w) \backslash B_r(z)) + \omega(B_r(z) \backslash B_r(w)) \leq C_\omega \left(\frac{\mathtt{d}(w, z)}{r} \right)^{\epsilon_\omega}$$

for all $w, z \in bD$ such that $\mathtt{d}(w, z) \leq r \leq 1$.

Proof The proof is an immediate consequence of (2.12). □

We note that the above lemma is fundamental in the proof of the L^p boundedness throughout this paper, as we use the framework of Calderón–Zygmund theory in the context of spaces of homogeneous type.

We also recall the sharp maximal function on the space of homogeneous type $\{bD, \mathtt{d}, \omega\}$. For a locally integrable function $f \in L^1_{\mathrm{loc}}(bD)$, the *sharp maximal function* $f^\#$ is defined by

$$f^\#(x) = \sup_{B \ni x} \frac{1}{\omega(B)} \int_B |f(y) - f_B| \, d\omega(y), \tag{2.13}$$

where the supremum is taken over all balls $B \subset bD$ containing x, and

$$f_B = \frac{1}{\omega(B)} \int_B f(y) \, d\omega(y)$$

is the average of f over B.

A fundamental result of Fefferman–Stein states that for $1 < p < \infty$,

$$\|f\|_{L^p(bD,\omega)} \lesssim \|f^\#\|_{L^p(bD,\omega)} \lesssim \|Mf\|_{L^p(bD,\omega)},$$

where Mf denotes the Hardy–Littlewood maximal function.

• *A family of Cauchy-like integrals.* In [17, Sections 3 and 4] an ad-hoc family $\{\mathbf{C}_\epsilon\}_\epsilon$ of Cauchy–Fantappiè integrals is introduced (each determined by the aforementioned denominators $g_\epsilon(w, z)$) whose corresponding boundary operators $\{\mathcal{C}_\epsilon\}_\epsilon$ play a crucial role in the analysis of $L^p(bD, \lambda)$-regularity of the Cauchy–Szegő projection. We henceforth refer to $\{\mathcal{C}_\epsilon\}_\epsilon$ as the *Cauchy-Leray integrals*; we record here a few relevant points for later reference.

[i.] Each $\mathcal{C}_\epsilon$ admits a primary decomposition in terms of an "essential part" $\mathcal{C}_\epsilon^\sharp$ and a "remainder" $\mathcal{R}_\epsilon$, which are used in the proof of the $L^2(bD, \omega)$-regularity of $\mathcal{C}_\epsilon$. However, at this stage the magnitude of the parameter ϵ plays no role (this is because of the "uniform" estimates (2.9)) and we temporarily drop reference to ϵ and simply write $\mathcal{C}$ in lieu of $\mathcal{C}_\epsilon$; $C(w, z)$ for $C_\epsilon(w, z)$, etc.. Thus

$$\mathcal{C} = \mathcal{C}^\sharp + \mathcal{R}, \tag{2.14}$$

with a corresponding decomposition for the integration kernels:

$$C(w, z) = C^\sharp(w, z) + R(w, z). \tag{2.15}$$

The "essential" kernel $C^\sharp(w, z)$ satisfies standard size and smoothness conditions that ensure the boundedness of $\mathcal{C}^\sharp$ in $L^2(bD, \omega)$ by a $T(1)$-theorem for the space of homogeneous type $\{bD, \mathtt{d}, \omega\}$. On the other hand, the "remainder" kernel $R(w, z)$ satisfies improved size and smoothness conditions granting that the corresponding operator $\mathcal{R}$ is bounded in $L^2(bD, \omega)$ by elementary considerations; see [17, Section 4].

[ii.] One then turns to the Cauchy–Szegő projection, for which $L^2(bD, \omega)$-regularity is trivial but $L^p(bD, \omega)$-regularity, for $p \neq 2$, is not. It is in this stage that the size of ϵ in the definition of the Cauchy-type boundary operators of item [i.] is relevant. It turns out that each $\mathcal{C}_\epsilon$ admits a further, "finer" decomposition into (another) "essential" part and (another) "reminder", which are obtained by truncating the integration kernel $C_\epsilon(w, z)$ by a smooth cutoff function $\chi_\epsilon^s(w, z)$ that equals 1 when $\mathtt{d}(w, z) < s = s(\epsilon)$. One has:

$$\mathcal{C}_\epsilon = \mathcal{C}_\epsilon^s + \mathcal{R}_\epsilon^s \tag{2.16}$$

where

$$\|(\mathcal{C}_\epsilon^s)^\dagger - \mathcal{C}_\epsilon^s\|_p \lesssim \epsilon^{1/2} M_p \tag{2.17}$$

for any $1 < p < \infty$, where $M_p = \dfrac{p}{p-1} + p$. Here and henceforth, the upper-script "$\dagger$" denotes adjoint in $L^2(bD, \omega)$ (hence $(\mathcal{C}_\epsilon^s)^\dagger$ is the adjoint of $\mathcal{C}_\epsilon^s$ in $L^2(bD, \omega)$); see [17, Proposition 18]. Furthermore $\mathcal{R}_\epsilon^s$ and $(\mathcal{R}_\epsilon^s)^\dagger$ are controlled by $\mathtt{d}(w, z)^{-2n+1}$ and therefore are easily seen to be bounded

$$\mathcal{R}_\epsilon^s, \ \ (\mathcal{R}_\epsilon^s)^\dagger : L^1(bD, \omega) \to L^\infty(bD, \omega), \tag{2.18}$$

see [17, (5.2) and comments thereafter].

• *Muckenhoupt weights on bD.* Let $p \in (1, \infty)$. A non-negative locally integrable function ψ is called an *$A_p(bD, \sigma)$-weight*, if

$$[\psi]_{A_p(bD,\sigma)} := \sup_B \langle \psi \rangle_B \langle \psi^{1-p'} \rangle_B^{p-1} < \infty,$$

where the supremum is taken over all balls B in bD, and $\langle \phi \rangle_B := \dfrac{1}{\sigma(B)} \displaystyle\int_B \phi(z)\, d\sigma(z)$.

Moreover, ψ is called an $A_1(bD,\sigma)$-*weight* if $[\psi]_{A_1(bD,\sigma)} := \inf\{C \geq 0 : \langle \psi \rangle_B \leq C\psi(x), \forall x \in B, \forall \text{ balls } B \in bD\} < \infty$.

Similarly, one can define the $A_p(bD,\lambda)$-*weight* for $1 \leq p < \infty$.

As before, the identity (1.12) grants that

$$A_p(bD,\sigma) = A_p(bD,\lambda) \quad \text{with} \quad [\psi]_{A_p(bD,\sigma)} \approx [\psi]_{A_p(bD,\lambda)},$$

thus we will henceforth simply write $A_p(bD)$ and $[\psi]_{A_p(bD)}$. At times it will be more convenient to work with $A_p(bD,\lambda)$, and in this case we will refer to its members as A_p-*like weights*.

Proof of Proposition 1.3 To streamline notations, we write Ω for Ω_p, and ψ for ψ_p, where ψ_p is the density function of Ω_p. It is clear that $H^p(bD,\Omega)$ is a subspace of $L^p(bD,\Omega)$, where the density function ψ of Ω is in A_p.

Our first claim is that for every $F \in H^p(bD,\Omega)$, the non-tangential (also known as admissible) limit $F^b(w)$ exists Ω-a.e. $w \in bD$. In fact, note that for $\psi \in A_p$, there exists $1 < p_1 < p$ such that ψ is in A_{p_1}. Set $p_0 = p/p_1$. Then it is clear that $1 < p_0 < p$. Let $P = p/p_0$ and P' be the conjugate of P, which is $P' = p/(p-p_0)$. So we get $\psi^{1-P'} = \psi^{-p_0/(p-p_0)} \in A_{P'} = A_{p/(p-p_0)}$.

Hence, for $F \in H^p(bD,\Omega)$,

$$\begin{aligned}
\|F\|_{H^{p_0}(bD,\lambda)} &= \left(\int_{bD} |\mathcal{N}(F)(w)|^{p_0} \psi(w)^{\frac{p_0}{p}} \psi(w)^{-\frac{p_0}{p}} d\lambda(w)\right)^{\frac{1}{p_0}}\\
&\leq \left(\int_{bD} |\mathcal{N}(F)(w)|^{p} \psi(w) d\lambda(w)\right)^{\frac{1}{p}} \left(\int_{bD} \psi(w)^{-\frac{p_0}{p-p_0}} d\lambda(w)\right)^{\frac{p-p_0}{pp_0}}\\
&= \|F\|_{H^p(bD,\Omega)} \left(\psi^{-\frac{p_0}{p-p_0}}(bD)\right)^{\frac{p-p_0}{pp_0}}.
\end{aligned}$$

Since $\psi^{-p_0/(p-p_0)} \in A_{p/(p-p_0)}$ and bD is compact, we have that $\psi^{-\frac{p_0}{p-p_0}}(bD)$ is finite.

Hence, we see that $H^p(bD,\Omega) \subset H^{p_0}(bD,\lambda)$. Thus F has admissible limit F^b for λ–a.e. $z \in bD$ ([29, Theorem 10]), and hence F has admissible limit F^b for Ω–a.e. $z \in bD$ since the measure Ω (with the density function ψ: $d\Omega(z) = \psi(z)d\lambda(z)$) is absolutely continuous. So the boundary function F^b exists.

Next, from the definition of the non-tangential maximal function $\mathcal{N}(F)$, we have that

$$|F^b(z)| \leq \mathcal{N}(F)(z) \quad \text{for } \Omega - \text{a.e. } z \in bD.$$

Thus, $F^b \in L^p(bD,\Omega)$. Also note that with similar methods as in [29], see also [9] for a more precise argument, one can show that

$$\mathcal{N}(F)(z) \lesssim M(F^b)(z) \quad \text{for } \Omega - \text{a.e. } z \in bD,$$

where $M(F^b)$ is the Hardy–Littlewood maximal function on the boundary bD. Since the maximal function is bounded on $L^p(bD, \Omega)$, we obtain that

$$\|F\|_{H^p(bD,\Omega)} = \|\mathcal{N}(F)\|_{L^p(bD,\Omega)} \lesssim \|M(F^b)\|_{L^p(bD,\Omega)} \le C[\psi]_{A_p}\|F^b\|_{L^p(bD,\Omega)}.$$

Suppose now that $\{F_n\}$ is a sequence in $H^p(bD, \Omega)$ and $f \in L^p(bD, \Omega)$ such that

$$\|\mathcal{N}(F_n) - f\|_{L^p(bD,\Omega)} \to 0.$$

Then, it is clear that

$$\|\mathcal{N}(F_n) - f\|_{L^{p_0}(bD,\lambda)} \le \|\mathcal{N}(F_n) - f\|_{L^p(bD,\Omega)}\left(\psi^{-\frac{p_0}{p-p_0}}(bD)\right)^{\frac{p-p_0}{pp_0}} \to 0.$$

Since $H^{p_0}(D, \lambda)$ is a proper subspace of $L^{p_0}(bD, \lambda)$, then in particular $\{F_n\}_n$ is in $H^{p_0}(D, \lambda)$ and f is in $L^{p_0}(bD, \lambda)$, we see that there is an F holomorphic in D such that

$$F^b(w) = f(w) \quad \lambda - a.e.\ w \in bD.$$

Again, this implies that

$$F^b(w) = f(w) \quad \Omega - a.e.\ w \in bD.$$

Moreover, invoking the weighted boundedness of the Hardy–Littlewood maximal function, we see that

$$\|\mathcal{N}(F)\|_{L^p(bD,\Omega)} \lesssim \|f\|_{L^p(bD,\Omega)}.$$

The proof of Proposition 1.3 is complete. □

3 Quantitative estimates for the Cauchy–Leray integral

As before, in the proofs of all statements in this section we adopt the shorthand Ω for Ω_p, and ψ for ψ_p.

Theorem 3.1 *Let $D \subset \mathbb{C}^n$, $n \ge 2$, be a bounded, strongly pseudoconvex domain of class C^2. Then the Cauchy-type integral $\mathcal{C}_\epsilon$ is bounded on $L^p(bD, \Omega_p)$ for any $0 < \epsilon < \epsilon(D)$, any $1 < p < \infty$ and any A_p-measure Ω_p, with*

$$\|\mathcal{C}_\epsilon\|_{L^p(bD,\Omega_p)\to L^p(bD,\Omega_p)} \lesssim c_\epsilon \cdot [\Omega_p]_{A_p}^{\max\{1,\frac{1}{p-1}\}}, \tag{3.1}$$

where the implied constant depends on p and D, but is independent of ϵ or Ω_p, and c_ϵ is the constant in (2.3).

It follows that for any A_2-measure Ω_2, the $L^2(bD, \Omega_2)$-adjoint $\mathcal{C}_\epsilon^\spadesuit$ is also bounded on $L^p(bD, \Omega_p)$ with same bound.

Proof To begin with, we first recall that the Cauchy integral operator $\mathcal{C}_\epsilon$ can be split into the essential part and remainder, that is, $\mathcal{C}_\epsilon = \mathcal{C}_\epsilon^\sharp + \mathcal{R}_\epsilon$. Denote by $C_\epsilon^\sharp(w,z)$ and $R_\epsilon(w,z)$ the kernels of $\mathcal{C}_\epsilon^\sharp$ and $\mathcal{R}_\epsilon$, respectively.

Recall from [17, (4.9), (4.18) and (4.19)] that $C_\epsilon^\sharp(w,z)$ is a standard Calderón–Zygmund kernel, i.e. there exists a positive constant $\mathcal{A}_1$ such that for every $w, z \in bD$ with $w \neq z$,

$$\begin{cases} a)\ |C_\epsilon^\sharp(w,z)| \le \mathcal{A}_1 \dfrac{1}{\mathtt{d}(w,z)^{2n}}; \\ b)\ |C_\epsilon^\sharp(w,z) - C_\epsilon^\sharp(w',z)| \le c_\epsilon \cdot \mathcal{A}_1 \dfrac{\mathtt{d}(w,w')}{\mathtt{d}(w,z)^{2n+1}}, \quad \text{if } \mathtt{d}(w,z) \ge c\,\mathtt{d}(w,w'); \\ c)\ |C_\epsilon^\sharp(w,z) - C_\epsilon^\sharp(w,z')| \le \mathcal{A}_1 \dfrac{\mathtt{d}(z,z')}{\mathtt{d}(w,z)^{2n+1}}, \quad \text{if } \mathtt{d}(w,z) \ge c\,\mathtt{d}(z,z') \end{cases} \tag{3.2}$$

for an appropriate constant $c > 0$, where $\mathtt{d}(z,w)$ is a quasi-distance suitably adapted to bD, and c_ϵ is the constant in (2.3). Hence, the $L^p(bD)$ boundedness ($1 < p < \infty$) of $\mathcal{C}_\epsilon^\sharp$ follows from a version of the $T(1)$ Theorem. Moreover, from [17, (4.9)] we also get that there exists a positive constant $\mathcal{A}_2$ such that for every $w, z \in bD$ with $w \neq z$,

$$|C_\epsilon^\sharp(w,z)| \ge \mathcal{A}_2 \frac{1}{\mathtt{d}(w,z)^{2n}}. \tag{3.3}$$

However, the kernel $R_\epsilon(w,z)$ of $\mathcal{R}_\epsilon$ satisfies a size condition and a smoothness condition for only one of the variables as follows: there exists a positive constant C_R (independent of ϵ) such that for every $w, z \in bD$ with $w \neq z$,

$$\begin{cases} d)\ |R_\epsilon(w,z)| \le C_R \dfrac{1}{\mathtt{d}(w,z)^{2n-1}}; \\ e)\ |R_\epsilon(w,z) - R_\epsilon(w,z')| \le C_R \dfrac{\mathtt{d}(z,z')}{\mathtt{d}(w,z)^{2n}}, \quad \text{if } \mathtt{d}(w,z) \ge c_R \mathtt{d}(z,z'). \end{cases} \tag{3.4}$$

Since the kernel of $\mathcal{C}_\epsilon^\sharp$ is a standard Calderón–Zygmund kernel on $bD \times bD$, according to [10] (see also [15]), we can obtain that $\mathcal{C}_\epsilon^\sharp$ is bounded on $L^p(bD,\Omega)$ with

$$\|\mathcal{C}_\epsilon^\sharp\|_{L^p(bD,\Omega)\to L^p(bD,\Omega)} \lesssim c_\epsilon \cdot \mathcal{A}_1 [\psi]_{A_p}^{\max\{1, \frac{1}{p-1}\}}, \tag{3.5}$$

where c_ϵ and $\mathcal{A}_1$ are as in (3.2). Thus, it suffices to show that $\mathcal{R}_\epsilon$ is bounded on $L^p(bD,\Omega)$ with the appropriate quantitative estimate.

To see this, we claim that for every $f \in L^p(bD,\Omega)$, $\tilde{z} \in bD$, there exists a $q \in (1,p)$ such that

$$\left|(\mathcal{R}_\epsilon(f))^\#(\tilde{z})\right| \lesssim \Big(M(|f|^q)(\tilde{z})\Big)^{1/q}, \tag{3.6}$$

where $F^{\#}$ is the sharp maximal function of F as recalled in (2.13) Section 2, and the implied constant depends on p, D and c_R.

We now show this claim. Since bD is bounded, there exists $\overline{C} > 0$ such that for any $B_r(z) \subset bD$ we have $r < \overline{C}$. For any $\tilde{z} \in bD$, let us fix a ball $B_r = B_r(z_0) \subset bD$ containing $\tilde{z}$, and let z be any point of B_r. Now take $j_0 = \lfloor \log_2 \frac{\overline{C}}{r} \rfloor + 1$. Since $\mathtt{d}$ is a quasi-distance, there exists $i_0 \in \mathbb{Z}^+$, independent of z, r, such that $\mathtt{d}(w, z) > c_R r$ whenever $w \in bD \setminus B_{2^{i_0} r}$, where c_R is in (3.4). We then write

$$\mathcal{R}_\epsilon(f)(z) = \mathcal{R}_\epsilon\big(f\chi_{bD\cap B_{2^{i_0}r}}\big)(z) + \mathcal{R}_\epsilon\big(f\chi_{bD\setminus B_{2^{i_0}r}}\big)(z) =: I(z) + II(z).$$

For the term I, by using Hölder's inequality and the fact that $\mathcal{R}_\epsilon$ is bounded on $L^q(bD, \lambda)$, $1 < q < \infty$, we have

$$\begin{aligned}
\frac{1}{\lambda(B_r)} \int_{B_r} |I(z) - I_{B_r}| d\lambda(z) &\leq 2 \left(\frac{1}{\lambda(B_r)} \int_{B_r} \left|\mathcal{R}_\epsilon\big(f\chi_{bD\cap B_{2^{i_0}r}}\big)(z)\right|^q d\lambda(z) \right)^{\frac{1}{q}} \\
&\lesssim \left(\frac{1}{\lambda(B_r)} \int_{bD\cap B_{2^{i_0}r}} |f(z)|^q d\lambda(z) \right)^{\frac{1}{q}} \\
&\lesssim \big(M(|f|^q)(\tilde{z})\big)^{\frac{1}{q}},
\end{aligned}$$

where

$$I_{B_r} := \frac{1}{\lambda(B_r)} \int_{B_r} I(z)\lambda(z)$$

To estimate II, observe that if $i_0 \geq j_0$, then $bD \setminus B_{2^{i_0}r} = \emptyset$ and $|II(z) - II(z_0)| = 0$. If $i_0 < j_0$, then we have

$$\begin{aligned}
|II(z) - II(z_0)| &= \left|\mathcal{R}_\epsilon\big(f\chi_{bD\setminus B_{2^{i_0}r}}\big)(z) - \mathcal{R}_\epsilon\big(f\chi_{bD\setminus B_{2^{i_0}r}}\big)(z_0)\right| \\
&\leq \int_{bD\setminus B_{2^{i_0}r}} \left|R_\epsilon(w, z) - R_\epsilon(w, z_0)\right| |f(w)| d\lambda(w) \\
&\leq \mathtt{d}(z, z_0) \int_{bD\setminus B_{2^{i_0}r}} \frac{1}{\mathtt{d}(w, z_0)^{2n}} |f(w)| d\lambda(w) \\
&\leq r \left(\int_{bD\setminus B_{2^{i_0}r}} \frac{1}{\mathtt{d}(w, z_0)^{2n}} d\lambda(w) \right)^{\frac{1}{q'}}
\end{aligned}$$

$$\left(\int_{bD\setminus B_{2^{i_0}r}} \frac{1}{\mathtt{d}(w,z_0)^{2n}}|f(w)|^q d\lambda(w)\right)^{\frac{1}{q}}.$$

Since bD is bounded, we can obtain

$$\begin{aligned}\int_{bD\setminus B_{2^{i_0}r}} \frac{1}{\mathtt{d}(w,z_0)^{2n}}|f(w)|^q d\lambda(w) &\leq \sum_{j=i_0}^{j_0} \int_{2^j r\leq d(w,z_0)\leq 2^{j+1}r} \frac{1}{\mathtt{d}(w,z_0)^{2n}}|f(w)|^q d\lambda(w)\\ &\leq \sum_{j=i_0}^{j_0} \frac{1}{(2^j r)^{2n}} \int_{d(w,z_0)\leq 2^{j+1}r} |f(w)|^q d\lambda(w)\\ &\lesssim \sum_{j=i_0}^{j_0} \frac{1}{\lambda(B_{2^{j+1}r})} \int_{B_{2^{j+1}r}} |f(w)|^q d\lambda(w)\\ &\lesssim j_0 M(|f|^q)(\tilde{z}).\end{aligned}$$

Similarly, we have

$$\int_{bD\setminus B_{2^{i_0}r}} \frac{1}{\mathtt{d}(w,z_0)^{2n}} d\lambda(w) \lesssim \sum_{j=i_0}^{j_0} \frac{1}{\lambda(B_{2^{j+1}r})} \int_{B_{2^{j+1}r}} d\lambda(w) \lesssim j_0.$$

Thus, we get that $|II(z)-II(z_0)| \lesssim r j_0 \left(M(|f|^q)(\tilde{z})\right)^{\frac{1}{q}}$. Therefore,

$$\begin{aligned}\frac{1}{\lambda(B_r)}\int_{B_r} \left|II(z)-II_{B_r}\right| d\lambda(z) &\leq \frac{2}{\lambda(B_r)}\int_{B_r} \left|II(z)-II(z_0)\right| d\lambda(z)\\ &\lesssim r j_0 \left(M(|f|^q)(\tilde{z})\right)^{\frac{1}{q}}\\ &\lesssim r\left(\log_2\left(\frac{\overline{C}}{r}\right)+1\right)\left(M(|f|^q)(\tilde{z})\right)^{\frac{1}{q}}\\ &\lesssim \left(M(|f|^q)(\tilde{z})\right)^{\frac{1}{q}},\end{aligned}$$

where the last inequality comes from the fact that $r\log_2\left(\frac{\overline{C}}{r}\right)$ is uniformly bounded. Combining the estimates on I and II, we see that the claim (3.6) holds.

We now prove that $\mathcal{R}_\epsilon$ is bounded on $L^p(bD,\Omega)$. In fact, for $f\in L^p(bD,\Omega)$

$$\begin{aligned}\|\mathcal{R}_\epsilon(f)\|^p_{L^p(bD,\Omega)} &\le C\Big(\Omega(bD)(\mathcal{R}_\epsilon(f)_{bD})^p+\|\mathcal{R}_\epsilon(f)^{\#}\|^p_{L^p(bD,\Omega)}\Big)\\ &\le C\Big(\Omega(bD)(\mathcal{R}_\epsilon(f)_{bD})^p+\|\left(M(|f|^q)\right)^{\frac{1}{q}}\|^p_{L^p(bD,\Omega)}\Big)\\ &\lesssim \Omega(bD)(\mathcal{R}_\epsilon(f)_{bD})^p+[\psi]_{A_p}^{p\cdot\max\{1,\frac{1}{p-1}\}}\|f\|^p_{L^p(bD,\Omega)},\end{aligned}\tag{3.7}$$

where the second inequality follows from (3.6) and the last inequality follows from the fact that the Hardy–Littlewood function is bounded on $L^p(bD,\Omega)$. We point out that

$$\Omega(bD)(\mathcal{R}_\epsilon(f)_{bD})^p\lesssim\|f\|^p_{L^p(bD,\Omega)}.\tag{3.8}$$

In fact, by Hölder's inequality and that $\mathcal{R}_\epsilon$ is bounded on $L^q(bD,\lambda)$, $1<q<\infty$, we have

$$\begin{aligned}\Omega(bD)(\mathcal{R}_\epsilon(f)_{bD})^p &\le \Omega(bD)\left(\frac{1}{\lambda(bD)}\int_{bD}\left|\mathcal{R}_\epsilon(f)(z)\right|^q d\lambda(z)\right)^{\frac{p}{q}}\\ &\lesssim \Omega(bD)\left(\frac{1}{\lambda(bD)}\int_{bD}\left|f(z)\right|^q d\lambda(z)\right)^{\frac{p}{q}}\\ &\lesssim \Omega(bD)\inf_{z\in bD}\left(M(|f|^q)(z)\right)^{\frac{p}{q}}\\ &\lesssim \int_{bD}\left(M(|f|^q)(z)\right)^{\frac{p}{q}}d\Omega(z)\\ &\lesssim [\Omega]_{A_p}^{p\cdot\max\{1,\frac{1}{p-1}\}}\|f\|^p_{L^p(bD,\Omega)}.\end{aligned}$$

Therefore, (3.8) holds, which, together with (3.7), implies that $\mathcal{R}_\epsilon$ is bounded on $L^p(bD,\Omega)$ with the correct quantitative bounds. This, together with (3.5), gives that (3.1) holds. The proof of Theorem 3.1 is complete. □

We now turn to the proof of the new cancellation (1.10).

Proposition 3.2 *For any fixed $0<\epsilon<\epsilon(D)$ as in [17], there exists $s=s(\epsilon)>0$ such that*

$$\|(\mathcal{C}^s_\epsilon)^\dagger-\mathcal{C}^s_\epsilon\|_{L^p(bD,\Omega_p)\to L^p(bD,\Omega_p)}\lesssim\epsilon^{1/2}[\Omega_p]_{A_p}^{\max\{1,\frac{1}{p-1}\}}\tag{3.9}$$

for any $1<p<\infty$ and for any A_p-measure, Ω_p where the implied constant depends on D and p but is independent of Ω_p and of ϵ. As before, here $(\mathcal{C}^s_\epsilon)^\dagger$ denotes the adjoint in $L^2(bD,\omega)$.

Remark 3.3 We point out that the term $\epsilon^{1/2}$ can be improved to ϵ^{δ} for any fixed small $\delta > 0$, according to [17, Remark D] via choosing β there arbitrarily close to 1.

Proof Proof of Proposition 3.2

Recall from [17, (5.7)], that $\mathcal{C}_\epsilon^s$ is given by

$$\mathcal{C}_\epsilon^s(f)(z) = \mathcal{C}_\epsilon\big(f(\cdot)\chi_s(\cdot, z)\big)(z), \quad z \in bD,$$

where $\chi_s(w, z)$ is the cutoff function given by $\chi_s(w, z) = \widetilde{\chi}_{s,w}(z)\widetilde{\chi}_{s,z}(w)$ with

$$\widetilde{\chi}_{s,w}(z) = \chi\left(\frac{\operatorname{Im}\langle\partial\rho(w), w - z\rangle}{cs^2} + i\frac{|w - z|^2}{cs^2}\right).$$

Here χ is a non-negative C^1-smooth function on $\mathbb{C}$ such that $\chi(u + iv) = 1$ if $|u + iv| \le 1/2$, $\chi(u + iv) = 0$ if $|u + iv| \ge 1$, and furthermore $|\nabla\chi(u + iv)| \lesssim 1$.

Then we also have the essential part and the remainder of $\mathcal{C}_\epsilon^s$, which are given by

$$\mathcal{C}_\epsilon^{\sharp,s}(f)(z) = \mathcal{C}_\epsilon^{\sharp}\big(f(\cdot)\chi_s(\cdot, z)\big)(z), \quad z \in bD,$$

and hence we have $\mathcal{C}_\epsilon^s = \mathcal{C}_\epsilon^{\sharp,s} + \mathcal{R}_\epsilon^{\sharp,s}$, where $\mathcal{R}_\epsilon^{\sharp,s}$ is the remainder. Further,

$$\big(\mathcal{C}_\epsilon^s\big)^* = \big(\mathcal{C}_\epsilon^{\sharp,s}\big)^* + \big(\mathcal{R}_\epsilon^{\sharp,s}\big)^*,$$

where $\big(\mathcal{C}_\epsilon^s\big)^*$ is the adjoint of $\mathcal{C}_\epsilon^s$ with respect to $L^2(bD, \lambda)$.

To prove (3.9), we note that $(\mathcal{C}_\epsilon^s)^\dagger = \varphi^{-1}(\mathcal{C}_\epsilon^s)^*\varphi$, where φ is the density function of ω satisfying (1.13). Thus, $(\mathcal{C}_\epsilon^s)^\dagger - \mathcal{C}_\epsilon^s = (\mathcal{C}_\epsilon^s)^* - \mathcal{C}_\epsilon^s - \varphi^{-1}[\varphi, (\mathcal{C}_\epsilon^s)^*]$, which gives

$$\begin{aligned}
&\|(\mathcal{C}_\epsilon^s)^\dagger - \mathcal{C}_\epsilon^s\|_{L^p(bD,\Omega_p)\to L^p(bD,\Omega_p)}\\
&\le \|(\mathcal{C}_\epsilon^s)^* - \mathcal{C}_\epsilon^s\|_{L^p(bD,\Omega_p)\to L^p(bD,\Omega_p)} + \|\varphi^{-1}[\varphi, (\mathcal{C}_\epsilon^s)^*]\|_{L^p(bD,\Omega_p)\to L^p(bD,\Omega_p)}.
\end{aligned}$$

Thus, we claim that

$$\|(\mathcal{C}_\epsilon^s)^* - \mathcal{C}_\epsilon^s\|_{L^p(bD,\Omega_p)\to L^p(bD,\Omega_p)} \le \epsilon^{1/2}[\psi]_{A_p}^{\max\{1,\frac{1}{p-1}\}} M(p, D, \omega) \tag{3.10}$$

and that

$$\|\varphi^{-1}[\varphi, (\mathcal{C}_\epsilon^s)^*]\|_{L^p(bD,\Omega_p)\to L^p(bD,\Omega_p)} \le \epsilon^{1/2}[\psi]_{A_p}^{\max\{1,\frac{1}{p-1}\}} M(p, D, \omega). \tag{3.11}$$

We first prove (3.10). To begin with, we write $\mathcal{C}_\epsilon^s - \big(\mathcal{C}_\epsilon^s\big)^* = \mathcal{A}_\epsilon^s + \mathcal{B}_\epsilon^s$, where

$$\mathcal{A}_\epsilon^s = \mathcal{C}_\epsilon^{\sharp,s} - \big(\mathcal{C}_\epsilon^{\sharp,s}\big)^* \quad \text{and} \quad \mathcal{B}_\epsilon^s = \mathcal{R}_\epsilon^{\sharp,s} - \big(\mathcal{R}_\epsilon^{\sharp,s}\big)^*.$$

Let $A^s_\epsilon(w,z)$ be the kernel of $\mathcal{A}^s_\epsilon$. Then, from [17, (5.10) and (5.9)], we see that

$$\begin{cases} 1)\ |A^s_\epsilon(w,z)| \lesssim \epsilon \dfrac{1}{\mathtt{d}(w,z)^{2n}} & \text{for any } s \le s(\epsilon); \\ 2)\ |A^s_\epsilon(w,z) - A^s_\epsilon(w',z)| \lesssim \epsilon^{1/2} \dfrac{\mathtt{d}(w,w')^{1/2}}{\mathtt{d}(w,z)^{2n+1/2}} & \text{for any } s \le s(\epsilon); \end{cases}$$

for $\mathtt{d}(w,z) \ge c\mathtt{d}(w,w')$ with an appropriate constant $c > 0$. Moreover, 2) is true for w and z interchanged, that is,

$$3)\ |A^s_\epsilon(w,z) - A^s_\epsilon(w,z')| \lesssim \epsilon^{1/2} \frac{\mathtt{d}(z,z')^{1/2}}{\mathtt{d}(w,z)^{2n+1/2}}, \quad \text{for any } s \le s(\epsilon).$$

From the kernel estimates, we see that $\epsilon^{-1/2}\mathcal{A}^s_\epsilon$, $s \le s(\epsilon)$, is a standard Calderón–Zygmund operator on $bD \times bD$, according to [10] (see also [15]), we can obtain that $\epsilon^{-1/2}\mathcal{A}^s_\epsilon$ is bounded on $L^p(bD, \Omega_p)$ with

$$\|\epsilon^{-1/2}\mathcal{A}^s_\epsilon\|_{L^p(bD,\Omega_p)\to L^p(bD,\Omega_p)} \le [\psi]_{A_p}^{\max\{1,\frac{1}{p-1}\}} M(p,D,\omega),$$

where the constant $M(p, D, \omega)$ depends only on p, D and ω. Hence, we obtain that

$$\|\mathcal{A}^s_\epsilon\|_{L^p(bD,\Omega_p)\to L^p(bD,\Omega_p)} \le \epsilon^{1/2}[\psi]_{A_p}^{\max\{1,\frac{1}{p-1}\}} M(p,D,\omega) \qquad \text{for any } s \le s(\epsilon). \tag{3.12}$$

Thus, it suffices to consider $\mathcal{B}^s_\epsilon$. Note that now the kernel $R^s_\epsilon(w,z)$ of $\mathcal{R}^{\sharp,s}_\epsilon$ is given by

$$R^s_\epsilon(w,z) = R_\epsilon(w,z)\chi_s(w,z),$$

where $R_\epsilon(w,z)$ is the kernel of $\mathcal{R}_\epsilon$ satisfying (3.4). Thus, it is easy to see that $R^s_\epsilon(w,z)$ satisfies the following size estimate

$$4)\ |R^s_\epsilon(w,z)| \le \widetilde{c}_\epsilon \frac{\chi_s(w,z)}{\mathtt{d}(w,z)^{2n-1}},$$

where the constant $\widetilde{c}_\epsilon$ is large, depending on ϵ. For the regularity estimate of $R^s_\epsilon(w,z)$ on z, we get that for $\mathtt{d}(w,z) \ge c_R\mathtt{d}(z,z')$,

$$\begin{aligned} &|R^s_\epsilon(w,z) - R^s_\epsilon(w,z')| \\ &\le |R_\epsilon(w,z) - R_\epsilon(w,z')| \cdot \chi_s(w,z) + |R_\epsilon(w,z')| \cdot |\chi_s(w,z) - \chi_s(w,z')| \\ &\le \widetilde{c}_\epsilon \frac{\mathtt{d}(z,z')}{\mathtt{d}(w,z)^{2n}} \cdot \chi_s(w,z) + \widetilde{c}_\epsilon \frac{1}{\mathtt{d}(w,z)^{2n-1}} \cdot |\chi_s(w,z) - \chi_s(w,z')|, \end{aligned}$$

where the last inequality follows from (3.4). Note that from the definition of $\chi_s(w,z)$, we get that $|\chi_s(w,z) - \chi_s(w,z')|$ vanishes unless $\mathtt{d}(w,z) \approx \mathtt{d}(w,z') \approx s$. Moreover, under this condition, we further have

$$|\chi_s(w,z)-\chi_s(w,z')|\lesssim \frac{\mathtt{d}(z,z')}{\mathtt{d}(w,z)}.$$

Combining these estimates, we obtain that

$$5)\ |R^s_\epsilon(w,z)-R^s_\epsilon(w,z')|\le \widetilde{c}_\epsilon \frac{\mathtt{d}(z,z')\ \chi_s(w,z)}{\mathtt{d}(w,z)^{2n}},\quad \text{if } \mathtt{d}(w,z)\ge c_R\mathtt{d}(z,z').$$

We now consider the operator $\mathcal{T}=s^{-1}\mathcal{R}^{\sharp,s}_\epsilon$. Then we claim that

$$\|\mathcal{T}(f)\|_{L^p(bD,\Omega_p)}\le \widetilde{c}_\epsilon[\psi]_{A_p}^{\max\{1,\frac{1}{p-1}\}}M(p,D,\omega)\|f\|_{L^p(bD,\Omega_p)}. \tag{3.13}$$

To prove (3.13), following the proof of Theorem 3.1, we first show that for every $f\in L^p(bD,\Omega_p)$, $\tilde{z}\in bD$, there exists a $q\in(1,p)$ such that

$$\left|(\mathcal{T}(f))^\#(\tilde{z})\right|\lesssim \widetilde{c}_\epsilon\Big(M(|f|^q)(\tilde{z})\Big)^{1/q}, \tag{3.14}$$

where $F^\#$ is the sharp maximal function of F as recalled in (2.13) in Section 2.

We now show (3.14). Since bD is bounded, there exists $\overline{C}>0$ such that for any $B_r(z)\subset bD$ we have $r<\overline{C}$. For any $\tilde{z}\in bD$, let us fix a ball $B_r=B_r(z_0)\subset bD$ containing $\tilde{z}$, and let z be any point of B_r. Since $\mathtt{d}$ is a quasi-distance, there exists $i_0\in\mathbb{Z}^+$, independent of z,r, such that $\mathtt{d}(w,z)>c_Rr$ whenever $w\in bD\setminus B_{2^{i_0}r}$, where c_R is in (3.4). We then write

$$\mathcal{T}(f)(z)=\mathcal{T}\big(f\chi_{bD\cap B_{2^{i_0}r}}\big)(z)+\big(f\chi_{bD\setminus B_{2^{i_0}r}}\big)(z)=:I(z)+II(z).$$

For the term I, by using Hölder's inequality we have

$$\begin{aligned}\frac{1}{\lambda(B_r)}\int_{B_r}|I(z)-I_{B_r}|d\lambda(z)&\le 2\left(\frac{1}{\lambda(B_r)}\int_{B_r}\left|\mathcal{T}\big(f\chi_{bD\cap B_{2^{i_0}r}}\big)(z)\right|^q d\lambda(z)\right)^{\frac{1}{q}}\\&\lesssim \widetilde{c}_\epsilon\left(\frac{1}{\lambda(B_r)}\int_{bD\cap B_{2^{i_0}r}}|f(z)|^q d\lambda(z)\right)^{\frac{1}{q}}\\&\lesssim \widetilde{c}_\epsilon\big(M(|f|^q)(\tilde{z})\big)^{\frac{1}{q}},\end{aligned}$$

where the second inequality follows from the boundedness in [17, (4.22)] since the kernel of $\mathcal{T}$ satisfies the conditions in [17, (4.22)].

To estimate II, by using condition 5), we have

$$\begin{aligned}|II(z) - II(z_0)| &= \left|\mathcal{T}\big(f\chi_{bD\setminus B_{2^{i_0}r}}\big)(z) - \mathcal{T}\big(f\chi_{bD\setminus B_{2^{i_0}r}}\big)(z_0)\right| \\ &\le \int_{bD\setminus B_{2^{i_0}r}} \frac{\widetilde{c}_\epsilon}{s} \frac{\mathtt{d}(z,z_0)\ \chi_s(w,z_0)}{\mathtt{d}(w,z_0)^{2n}} |f(w)| d\lambda(w) \\ &\le \int_{(bD\setminus B_{2^{i_0}r})\cap B_s} \frac{\widetilde{c}_\epsilon}{s} \frac{\mathtt{d}(z,z_0)}{\mathtt{d}(w,z_0)^{2n}} |f(w)| d\lambda(w),\end{aligned}$$

where $B_s := B_s(z_0)$ and the last inequality follows from the property of the function $\chi_s(w, z_0)$. Thus, we see that if $2^{i_0}r \ge s$, then the last term is zero. So we just need to consider $2^{i_0}r < s$. In this case, we have

$$\begin{aligned}|II(z) - II(z_0)| &\le \frac{\widetilde{c}_\epsilon}{s}\ \mathtt{d}(z,z_0) \int_{B_s\setminus B_{2^{i_0}r}} \frac{1}{\mathtt{d}(w,z_0)^{2n}} |f(w)| d\lambda(w) \\ &\le \frac{\widetilde{c}_\epsilon\ r}{s} \left(\int_{B_s\setminus B_{2^{i_0}r}} \frac{1}{\mathtt{d}(w,z_0)^{2n}} d\lambda(w) \right)^{\frac{1}{q'}} \\ &\quad \left(\int_{B_s\setminus B_{2^{i_0}r}} \frac{1}{\mathtt{d}(w,z_0)^{2n}} |f(w)|^q d\lambda(w) \right)^{\frac{1}{q}}.\end{aligned}$$

We take $j_0 = \lfloor \log_2 \frac{s}{r} \rfloor + 1$. Continuing the estimate:

$$\begin{aligned}\int_{B_s\setminus B_{2^{i_0}r}} \frac{1}{\mathtt{d}(w,z_0)^{2n}} |f(w)|^q d\lambda(w) &\le \sum_{j=i_0}^{j_0} \int_{2^j r \le d(w,z_0) \le 2^{j+1}r} \frac{1}{\mathtt{d}(w,z_0)^{2n}} |f(w)|^q d\lambda(w) \\ &\le \sum_{j=i_0}^{j_0} \frac{1}{(2^j r)^{2n}} \int_{d(w,z_0)\le 2^{j+1}r} |f(w)|^q d\lambda(w) \\ &\lesssim \sum_{j=i_0}^{j_0} \frac{1}{\lambda(B_{2^{j+1}r})} \int_{B_{2^{j+1}r}} |f(w)|^q d\lambda(w) \\ &\lesssim j_0 M(|f|^q)(\tilde{z}).\end{aligned}$$

Similarly, we have

$$\int\limits_{B_s \setminus B_{2^{i_0}r}} \frac{1}{\mathtt{d}(w, z_0)^{2n}} d\lambda(w) \lesssim \sum_{j=i_0}^{j_0} \frac{1}{\lambda(B_{2^{j+1}r})} \int\limits_{B_{2^{j+1}r}} d\lambda(w) \lesssim j_0.$$

Thus, we get that $|II(z) - II(z_0)| \lesssim \widetilde{c}_\epsilon \frac{r}{s} j_0 \left(M(|f|^q)(\tilde{z})\right)^{\frac{1}{q}}$. Therefore,

$$\begin{aligned}
\frac{1}{\lambda(B_r)} \int\limits_{B_r} \left|II(z) - II_{B_r}\right| d\lambda(z) &\leq \frac{2}{\lambda(B_r)} \int\limits_{B_r} \left|II(z) - II(z_0)\right| d\lambda(z) \\
&\lesssim \widetilde{c}_\epsilon \frac{r}{s} j_0 \left(M(|f|^q)(\tilde{z})\right)^{\frac{1}{q}} \\
&\lesssim \widetilde{c}_\epsilon \frac{r}{s} \left(\log_2\left(\frac{s}{r}\right) + 1\right) \left(M(|f|^q)(\tilde{z})\right)^{\frac{1}{q}} \\
&\lesssim \widetilde{c}_\epsilon \left(M(|f|^q)(\tilde{z})\right)^{\frac{1}{q}},
\end{aligned}$$

where the last inequality comes from the fact that $A\left(\log_2(\frac{1}{A}) + 1\right)$ is uniformly bounded for $A \in (0, 1]$.

Combining the estimates on I and II, we see that the claim (3.14) holds. We now prove that $\mathcal{T}$ is bounded on $L^p(bD, \Omega_p)$. In fact, for $f \in L^p(bD, \Omega_p)$

$$\begin{aligned}
\|\mathcal{T}(f)\|^p_{L^p(bD,\Omega_p)} &\lesssim \left(\Omega_p(bD)(\mathcal{T}(f)_{bD})^p + \|\mathcal{T}(f)^\#\|^p_{L^p(bD,\Omega_p)}\right) \\
&\lesssim \left(\Omega_p(bD)(\mathcal{T}(f)_{bD})^p + \widetilde{c}_\epsilon^p \| \left(M(|f|^q)\right)^{\frac{1}{q}} \|^p_{L^p(bD,\Omega_p)}\right) \\
&\leq \Omega_p(bD)(\mathcal{T}(f)_{bD})^p \\
&\quad + \widetilde{c}_\epsilon^p [\psi]_{A_p}^{p\cdot\max\{1, \frac{1}{p-1}\}} M(p, D, \omega) \|f\|^p_{L^p(bD,\Omega_p)},
\end{aligned} \tag{3.15}$$

where the second inequality follows from (3.14) and the last inequality follows from the fact that the Hardy–Littlewood function is bounded on $L^p(bD, \Omega)$ with the constant $M(p, D, \omega)$ depending only on p, D and ω. We point out that

$$\Omega_p(bD)(\mathcal{T}(f)_{bD})^p \leq \widetilde{c}_\epsilon [\Omega_p]_{A_p}^{p\cdot\max\{1, \frac{1}{p-1}\}} M(p, D, \omega) \|f\|^p_{L^p(bD,\Omega_p)}. \tag{3.16}$$

In fact, by Hölder's inequality and the fact that $\mathcal{T}$ is bounded on $L^q(bD, \lambda)$ for all $1 < q < \infty$ (from [17, (4.22)]), we have

$$\begin{aligned}
\Omega_p(bD)(\mathcal{T}(f)_{bD})^p &\leq \Omega_p(bD)\left(\frac{1}{\lambda(bD)}\int\limits_{bD}\left|\mathcal{T}(f)(z)\right|^q d\lambda(z)\right)^{\frac{p}{q}} \\
&\lesssim \widetilde{c}_\epsilon^p\,\Omega_p(bD)\left(\frac{1}{\lambda(bD)}\int\limits_{bD}\left|f(z)\right|^q d\lambda(z)\right)^{\frac{p}{q}} \\
&\lesssim \widetilde{c}_\epsilon^p\,\Omega_p(bD)\inf_{z\in bD}\left(M(|f|^q)(z)\right)^{\frac{p}{q}} \\
&\lesssim \widetilde{c}_\epsilon^p\int\limits_{bD}\left(M(|f|^q)(z)\right)^{\frac{p}{q}} d\Omega_p(z) \\
&\leq \widetilde{c}_\epsilon^p[\Omega_p]_{A_p}^{p\cdot\max\{1,\frac{1}{p-1}\}}M(p, D, \omega)\|f\|^p_{L^p(bD,\Omega_p)}.
\end{aligned}$$

Therefore, (3.16) holds, which, together with (3.15), implies that the claim (3.13) holds.

As a consequence, we obtain that for every $1 < p < \infty$,

$$\left\|\mathcal{R}_\epsilon^{\sharp,s}(f)\right\|_{L^p(bD,\Omega_p)} \leq s\cdot\widetilde{c}_\epsilon\cdot[\Omega_p]_{A_p}^{\max\{1,\frac{1}{p-1}\}}M(p, D, \omega)\|f\|_{L^p(bD,\Omega_p)}. \tag{3.17}$$

By using the fact that $\Omega_p^{-\frac{1}{p-1}}$ is in $A_{p'}$ when Ω_p is in A_p (p' is the conjugate index of p), and by using duality, we obtain that

$$\left\|\left(\mathcal{R}_\epsilon^{\sharp,s}\right)^*(f)\right\|_{L^p(bD,\Omega_p)} \leq s\cdot\widetilde{c}_\epsilon\cdot[\Omega_p]_{A_p}^{\max\{1,\frac{1}{p-1}\}}M(p, D, \omega)\|f\|_{L^p(bD,\Omega_p)}. \tag{3.18}$$

Hence, we have

$$\left\|\mathcal{B}_\epsilon^{s}(f)\right\|_{L^p(bD,\Omega_p)} \leq s\cdot\widetilde{c}_\epsilon\cdot[\Omega_p]_{A_p}^{\max\{1,\frac{1}{p-1}\}}M(p, D, \omega)\|f\|_{L^p(bD,\Omega_p)}. \tag{3.19}$$

Since ϵ is a fixed small positive constant, we see that

$$s\cdot\widetilde{c}_\epsilon \approx \epsilon^{1/2}$$

if s is sufficiently small. Thus, we obtain that

$$\left\|\mathcal{B}_\epsilon^{s}(f)\right\|_{L^p(bD,\Omega_p)} \lesssim \epsilon^{1/2}\cdot[\Omega_p]_{A_p}^{\max\{1,\frac{1}{p-1}\}}M(p, D, \omega)\|f\|_{L^p(bD,\Omega_p)}. \tag{3.20}$$

Combining the estimates of (3.12) and (3.20), we obtain that the claim (3.10) holds.

We now prove (3.11). To begin with, following [17, Section 6.2], we consider a partition of $\mathbb{C}^n$ into disjoint cubes of side-length γ, given by $\mathbb{C}^n = \cup_{k\in\mathbb{Z}^{2n}} Q_k^\gamma$. Then we revert to our domain D. For a fixed $\gamma > 0$, we write 1_k for the characteristic function of $Q_k^\gamma \cap bD$. Then we consider $1_k(z)(\mathcal{C}_\epsilon^s)^*(1_j f)(z)$. we note that if z is not in the support of f, then

$$1_k(z)(\mathcal{C}_\epsilon^s)^*(1_j f)(z) = 1_k(z)\int_{bD} \overline{C_\epsilon(z,w)}\chi_s(w,z)f(w)1_j(w)d\lambda(w). \tag{3.21}$$

Thus, by choosing s small enough and $\gamma = cs$, where c is a fixed positive constant, we see that $1_k(z)(\mathcal{C}_\epsilon^s)^*(1_j f)(z)$ vanishes if Q_j^γ and Q_k^γ do not touch (the cubes Q_j^γ and Q_k^γ touch if $\overline{Q_j^\gamma} \cap \overline{Q_k^\gamma}$). Next, following the proof in [17, Section 6.3], and combining our result Theorem 3.1 we see that

$$\begin{aligned}\|1_k(\mathcal{C}_\epsilon^s)^*(1_j f)\|_{L^p(bD,\Omega_p)} &\lesssim \epsilon\|\mathcal{C}_\epsilon^s\|_{L^p(bD,\Omega_p)\to L^p(bD,\Omega_p)}\|f\|_{L^p(bD,\Omega_p)}\\ &\le \epsilon[\Omega_p]_{A_p}^{\max\{1,\frac{1}{p-1}\}} M(p,D,\omega)\|f\|_{L^p(bD,\Omega_p)}.\end{aligned}$$

As a consequence, by using [17, Lemma 24], we obtain that our claim (3.11) holds.

The proof of Proposition 3.2 is complete. □

4 Proofs of Theorem 1.1

We start with (1.6) and thus for any $\epsilon > 0$ we write

$$\mathcal{S}_\omega g = \mathcal{C}_\epsilon(\mathcal{T}_\epsilon^s)^{-1}g \;+\; \mathcal{S}_\omega \circ \big((\mathcal{R}_\epsilon^s)^\dagger - \mathcal{R}_\epsilon^s\big)\circ(\mathcal{T}_\epsilon^s)^{-1}g \;=: \mathfrak{A}_\epsilon g + \mathfrak{B}_\epsilon g,$$

where

$$\mathcal{T}_\epsilon^s h := \Big(I - \big((\mathcal{C}_\epsilon^s)^\dagger - \mathcal{C}_\epsilon^s\big)\Big)h.$$

We fix Ω_2 arbitrarily. To streamline the notations, we write $M(D,\omega)$ for $M(2,D,\omega)$ as in (3.9), see Proposition 3.2. We now choose $\epsilon = \epsilon(\Omega_2)$ such that

$$\frac{1}{4} \le \epsilon^{1/2}[\Omega_2]_{A_2} M(D,\omega) \le \frac{1}{2}, \tag{4.1}$$

where ψ_2 is the density of Ω_2. Thus, Proposition 3.2 grants

$$\|\big((\mathcal{C}_\epsilon^s)^\dagger - \mathcal{C}_\epsilon^s\big)^j g\|_{L^2(bD,\Omega_2)} \le \frac{1}{2^j}\|g\|_{L^2(bD,\Omega_2)}, \quad j = 0,1,2,\dots.$$

Hence for $\epsilon = \epsilon(\Omega_2)$ as in (4.1) we have that

$$\|(\mathcal{T}_\epsilon^s)^{-1}g\|_{L^2(bD,\Omega_2)} \le 2\|g\|_{L^2(bD,\Omega_2)}, \tag{4.2}$$

and by Theorem 3.1 (for $p = 2$) we conclude that

$$\|\mathfrak{A}_\epsilon g\|_{L^2(bD,\Omega_2)} \leq C_1(D,\omega)\cdot c_\epsilon \cdot [\Omega_2]_{A_2}\, \|g\|_{L^2(bD,\Omega_2)}.$$

Combining the estimate of c_ϵ in (2.4) and the choice of ϵ in (4.1), we obtain that

$$\|\mathfrak{A}_\epsilon g\|_{L^2(bD,\Omega_2)} \lesssim [\Omega_2]^3_{A_2}\, \|g\|_{L^2(bD,\Omega_2)}. \tag{4.3}$$

We point out that from Remark 3.3, the term $\epsilon^{1/2}$ can be improved to ϵ^δ for any small $\delta > 0$. Hence, the term $[\Omega_2]^3_{A_2}$ in (4.3) could be improved to $[\Omega_2]^{2+\delta}_{A_2}$ for any small $\delta > 0$.

We next proceed to bound the norm of $\mathfrak{B}_\epsilon g$; to this end we recall that the reverse Hölder inequality for $\Omega_2 = \psi_2 d\lambda$ ([6, Theorem 9.2.2])

$$\left(\frac{1}{\lambda(bD)}\int\limits_{bD} \psi_2^{1+\gamma}(z)\, d\lambda(z)\right)^{\frac{1}{\gamma+1}} \leq C(\Omega_2)\, \frac{1}{\lambda(bD)}\int\limits_{bD} \psi_2(z) d\lambda(z) \tag{4.4}$$

is true for for some $\gamma = \gamma(\Omega_2) > 0$.

Recall that by keeping track of the constants (see [6, (9.2.6) and (9.2.7)]), we have that

$$\sup_{\Omega_2\in A_2} \gamma(\Omega_2) \leq 1$$

and that there exists a positive constant C_2 such that

$$\sup_{\Omega_2\in A_2} C(\Omega_2) \leq C_2^2 < \infty.$$

Hence, we obtain that

$$\left(\int\limits_{bD} \psi_2^{1+\gamma}(z)\, d\lambda(z)\right)^{\frac{1}{\gamma+1}} \leq C_2^2\, \frac{1}{\lambda(bD)^{\frac{\gamma}{\gamma+1}}}\int\limits_{bD} \psi_2(z) d\lambda(z). \tag{4.5}$$

Recall that, for Leray Levi-like measures we have $d\omega(z) = \varphi(z)\, d\lambda(z)$, where

$$0 < m_\omega \leq \varphi(z) \leq M_\omega < \infty \quad \text{for any } \; z \in bD.$$

Hence, using the shorthand

$$B_\epsilon g = \mathcal{S}_\omega H, \quad H := \big((\mathcal{R}^s_\epsilon)^\dagger - \mathcal{R}^s_\epsilon\big)h, \quad h := (\mathcal{T}^s_\epsilon)^{-1} g$$

we have

$$\|B_\epsilon g\|_{L^2(bD,\Omega_2)} \leq m_\omega^{-\frac{\gamma}{\gamma+1}} \left(\int_{bD} |\mathcal{S}_\omega H|^2(z)\, \varphi(z)^{\frac{\gamma}{\gamma+1}}\, \psi_2(z)\, d\lambda(z) \right)^{\frac{1}{2}}.$$

Hölder's inequality for $q := \dfrac{\gamma+1}{\gamma}$, where $\gamma = \gamma(\Omega_2)$ is as in (4.4), now gives that

$$\|B_\epsilon g\|_{L^2(bD,\Omega_2)} \leq m_\omega^{-\frac{\gamma}{\gamma+1}} \left(\int_{bD} |\mathcal{S}_\omega(H)|^{\frac{2(\gamma+1)}{\gamma}}(z)\, d\omega(z) \right)^{\frac{\gamma}{2(\gamma+1)}} \left(\int_{bD} \psi_2^{1+\gamma}(z)\, d\lambda(z) \right)^{\frac{1}{2(1+\gamma)}}.$$

Comparing the above with (4.5) we obtain

$$\|B_\epsilon g\|_{L^2(bD,\Omega_2)} \leq m_\omega^{-\frac{\gamma}{\gamma+1}} C_2 \|\mathcal{S}_\omega(H)\|_{L^p(bD,\omega)} \frac{1}{\lambda(bD)^{\frac{\gamma}{2(1+\gamma)}}} \left(\int_{bD} \psi_2(z)\, d\lambda(z) \right)^{\frac{1}{2}},$$

where $p := \dfrac{2(1+\gamma)}{\gamma}$. We also note that

$$\|\mathcal{S}_\omega(H)\|_{L^p(bD,\omega)} \leq C(\omega, D)\|H\|_{L^p(bD,\omega)}$$

by the $L^p(bD,\omega)$-regularity of $\mathcal{S}_\omega$ [17], since H is shorthand for $\big((\mathcal{R}^s_\epsilon)^\dagger - \mathcal{R}^s_\epsilon\big)h$, and each of $(\mathcal{R}^s_\epsilon)$ and $(\mathcal{R}^s_\epsilon)^\dagger$ takes $L^1(bD,\omega)$ to $L^\infty(bD,\omega)$ (again by [17]). Hence, we further obtain that

$$\begin{aligned} \|\mathcal{S}_\omega(H)\|_{L^p(bD,\omega)} &\leq C(\omega, D)\Omega_p(bD)^{\frac{1}{p}} \|h\|_{L^1(bD,\omega)} \\ &\leq C(\omega, D)\Omega_p(bD)^{\frac{1}{p}} M_\omega \|h\|_{L^2(bD,\Omega_2)} \left(\int_{bD} \psi_2^{-1}(z) d\lambda(z) \right)^{\frac{1}{2}}. \end{aligned}$$

Noting that $h := (\mathcal{T}^s_\epsilon)^{-1} g$ and choosing $\epsilon = \epsilon(\Omega_2)$ as in (4.1), we then have that

$$\|h\|_{L^2(bD,\Omega_2)} \leq 2\|g\|_{L^2(bD,\Omega_2)} \quad \text{by} \quad (4.2).$$

Combining all the pieces we obtain

$$\|B_\epsilon g\|_{L^2(bD,\Omega_2)} \leq 2\, C_2\, C(\omega, D) M_\omega \left(m_\omega^{-1} \left(\frac{\Omega_p(bD)}{\lambda(bD)} \right)^{\frac{1}{2}} \right)^{\frac{\gamma}{\gamma+1}} [\Omega_2]_{A_2}^{\frac{1}{2}} \|g\|_{L^2(bD,\Omega_2)}.$$

Hence the desired bound for $\|B_\epsilon g\|_{L^2(bD,\Omega_2)}$ holds true because

$$\left(m_\omega^{-1}\left(\frac{\Omega_p(bD)}{\lambda(bD)}\right)^{\frac{1}{2}}\right)^{\frac{\gamma}{\gamma+1}} \le C_3(\omega, D) < \infty \quad \text{for any } 0 \le \gamma < \infty.$$

The proof of Theorem 1.1 is complete once we recall that $[\Omega_2]_{A_2} \ge 1$. □

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