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ON THE MEASURE OF PRODUCTS FROM THE MIDDLE-THIRD CANTOR SET

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ABSTRACT. We prove upper and lower bounds for the Lebesgue measure of the set of products xy with x and y in the middle-third Cantor set. Our method is inspired by Athreya, Reznick and Tyson, but a different subdivision of the Cantor set provides a more rapidly converging approximation formula.

1. INTRODUCTION

The middle-third Cantor set is the well-known set $\mathcal{K} \subset [0,1]$ of points of the form

$$x = \sum_{k=1}^{\infty} \frac{\alpha_k}{3^k}, \quad \text{where} \quad \alpha_k \in \{0,2\} \quad \text{for} \quad k \geq 1.$$

Define $P : \mathbb{R}^2 \to \mathbb{R}$ by $P(x,y) := xy$ and consider $P(\mathcal{K} \times \mathcal{K})$, that is the set of products xy with $x, y \in \mathcal{K}$, which is a closed set because P is continuous and $\mathcal{K}$ is compact. Denote $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}$. The main result of this paper is Theorem 1.1 below.

Theorem 1.1. *We have*

$$\left| \mathcal{L}\big(P(\mathcal{K} \times \mathcal{K})\big) - \frac{91782451}{113374080} \right| \leq \frac{1}{10^6}.$$

In [2], Athreya, Reznick and Tyson prove the bounds $17/21 < \mathcal{L}\big(P(\mathcal{K}\times\mathcal{K})\big) < 5/6$. Then, running a computer program, the authors of [2] get

(1.1) $$\mathcal{L}\big(P(\mathcal{K} \times \mathcal{K})\big) = 0,80955358 \pm 10^{-8}.$$

After, in [4], Gu, Jiang, Xi, Zhao can run a different computer program which confirms the same first 5 digits in (1.1). Theorem 1.1 confirms such first 5 digits by a rigorous proof.

Other arithmetic operations with $x, y \in \mathcal{K}$ are considered in [2], which describes the structure of quotients y/x with $x \neq 0$ and proves that $[0,1]$ is covered by products x^2y, so that in particular any element of $[0,1]$ is the product of 3 factors in $\mathcal{K}$. In [5] it is proved that sums $x_1^2 + x_2^2 + x_3^2 + x_4^2$ with $x_i \in \mathcal{K}$ for $i = 1,2,3,4$ cover $[0,4]$, which was conjectured in [2]. In [6] is described a general condition on maps $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(\mathcal{K} \times \mathcal{K})$ has non-empty interior, where such condition is obviously satisfied by the arithmetic operation mentioned above. For the image under affine maps, and in particular for $S(x,y) := x + y$, a much larger class of Cantor sets and other fractals have been studied. Specific *gap conditions* guarantee that the image is an interval. A first use of gap conditions appears in [7] and a more recent application in [3]. A gap condition is used for the product of Cantor sets in [10],

and for Lipschitz perturbations of $S(\cdot,\cdot)$ in Theorem 1.12 in [1]. Related ideas inspired by [2] are used also in this paper. Other techniques appear in [8] and [9].

1.1. **Approximation formula.** For $E \subset \mathbb{R}$ and $c > 0$ set $c \cdot E := \{cx : x \in E\}$. It is helpful to consider only products xy with $x, y \in [2/3, 1]$. For this reason we consider the right half $\mathcal{R} := \mathcal{K} \cap [2/3, 1]$ of $\mathcal{K}$. From $\mathcal{K} = \{0\} \cup \bigcup_{k=0}^{\infty} 3^{-k} \cdot \mathcal{R}$ we get

$$P(\mathcal{K} \times \mathcal{K}) = \{0\} \cup \bigcup_{k=0}^{\infty} 3^{-k} \cdot P(\mathcal{R} \times \mathcal{R}).$$

The union above is disjoint, because $P(\mathcal{R} \times \mathcal{R}) \subset [4/9, 1]$ and $1/3 < 4/9$. Therefore

$$(1.2) \qquad \mathcal{L}\big(P(\mathcal{K} \times \mathcal{K})\big) = \sum_{k=0}^{\infty} 3^{-k} \mathcal{L}\big(P(\mathcal{R} \times \mathcal{R})\big) = \frac{3}{2} \mathcal{L}\big(P(\mathcal{R} \times \mathcal{R})\big).$$

A natural way to estimate the measure of $P(\mathcal{R} \times \mathcal{R})$, and thus of $P(\mathcal{K} \times \mathcal{K})$ by (1.2) above, is to consider the iterative construction of the Cantor set $\mathcal{R}$ and estimate how the measure of the product set of the n-th step of the iteration converge to a limit. In order to do so, we introduce the following terminology. A *subdivision* for $\mathcal{R}$ is a nested family of compact sets $\mathcal{R}_0 \supset \cdots \supset \mathcal{R}_n \supset \mathcal{R}_{n+1} \ldots$ such that $\mathcal{R} = \bigcap_{n=0}^{\infty} \mathcal{R}_n$, where any $\mathcal{R}_{n+1}$ is obtained removing from $\mathcal{R}_n$ some open intervals (finitely or countably many). The standard subdivision $\mathcal{R}_n$ for $\mathcal{R}$ is obtained setting $\mathcal{R}_0 := [2/3, 1]$ and iteratively

$$\mathcal{R}_n := \mathcal{R}_{n-1} \setminus \bigcup_{j=2 \cdot 3^{n-1}+1}^{3^n} \left(\frac{3(j-1)+1}{3^{n+1}}, \frac{3(j-1)+2}{3^{n+1}} \right) \quad \text{for} \quad n \geq 1.$$

Remark 8 in [2] gives

$$(1.3) \qquad 0 \leq \mathcal{L}\big(P(\mathcal{R}_n \times \mathcal{R}_n)\big) - \mathcal{L}\big(P(\mathcal{R} \times \mathcal{R})\big) \leq \frac{1}{63}\left(\frac{2}{9}\right)^n.$$

The bound $17/21 < \mathcal{L}\big(P(\mathcal{K} \times \mathcal{K})\big) < 5/6$ in [2] follows from $\mathcal{L}\big(P(\mathcal{R}_0 \times \mathcal{R}_0)\big) = 5/9$ and (1.2), applying (1.3) with $n = 0$. Then (1.1) follows applying (1.3) with $n = 11$, where of course the estimate of the measure of $P(\mathcal{R}_{11} \times \mathcal{R}_{11})$ can only be done by a computer. In [4], approximation formulae like (1.3) are used in different applications. Our main tool to prove Theorem 1.1 is an approximation formula analogous to (1.3), where we replace the standard subdivision $(\mathcal{R}_n)_{n \geq 0}$ by the *fast subdivision* $(\mathcal{D}_n)_{n \geq 0}$ for $\mathcal{R}$, which is obtained by removing not just the middle third of each interval in each step, but rather countably many intervals simultaneously (the definition of $\mathcal{D}_n$ is given in the next § 2.2). According to Proposition 1.2 below, the fast subdivision provides a sequence of approximations for the measure of $P(\mathcal{R} \times \mathcal{R})$ which converges more rapidly than (1.3).

Proposition 1.2. *Let $(\mathcal{D}_n)_{n \geq 0}$ be the subdivision in (2.3). For any $n \geq 0$ we have*

$$0 \leq \mathcal{L}\big(P(\mathcal{D}_n \times \mathcal{D}_n)\big) - \mathcal{L}\big(P(\mathcal{R} \times \mathcal{R})\big) \leq \frac{1}{63}\left(\frac{1}{36}\right)^n.$$

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1.2. Proof of main Theorem 1.1. Theorem 1.1 follows directly applying (1.2), Proposition 1.2 with $n = 3$, and the next Proposition 1.3, which is proved in § 3 below. For the error observe that

$$\frac{3}{2}\left(\frac{1}{64 \cdot 9^6} + \frac{1}{63 \cdot 36^3}\right) = \frac{11}{7 \cdot 16 \cdot 3^{11}} < \frac{1}{10^6}.$$

Proposition 1.3. *We have*

$$\left|\mathcal{L}\big(P(\mathcal{D}_3 \times \mathcal{D}_3)\big) - \frac{91782451}{170061120}\right| \leq \frac{1}{64 \cdot 9^6}.$$

Structure of this paper. § 2 is devoted to the proof of Proposition 1.2. In § 2.2 we introduce the fast subdivision $(\mathcal{D}_n)_{n\geq0}$. The n-th generation product set $P(\mathcal{D}_n \times \mathcal{D}_n)$ differs from $P(\mathcal{R} \times \mathcal{R})$ because further subdivisions create gaps in $P(\mathcal{D}_n \times \mathcal{D}_n)$, that is intervals $\mathcal{G}$ with $\mathcal{G} \subset P(\mathcal{D}_n \times \mathcal{D}_n) \setminus P(\mathcal{D}_{n+1} \times \mathcal{D}_{n+1})$. According to Lemma 2.2, the new gaps only appear in product sets of the form $P(I \times I)$ for a given interval I of $\mathcal{D}_n$. In § 2.3 we describe such new gaps for an interval I which undergoes a single step of the fast subdivision. § 2.3 contains the technical notation and definitions which are used in the rest of the paper. The proof of Proposition 1.2 is completed in § 2.4.

§ 3 is devoted to the proof of Proposition 1.3. The analysis from § 2.3 only provides a local information, but some gaps $\mathcal{G}$ in $P(I \times I)$ generated by the subdivision of I may be covered by other regions of $P(\mathcal{R}\times\mathcal{R})$, that is $\mathcal{G} \subset P\big((\mathcal{R}\times\mathcal{R})\setminus(I\times I)\big)$. Such gaps are called *covered gaps*, and they don't give negative contribution to the measure of $P(\mathcal{R} \times \mathcal{R})$. A precise estimate of the measure of $P(\mathcal{D}_3 \times \mathcal{D}_3)$ requires detecting as many covered gaps as possible. The main tool is an arithmetic characterization of covered gaps, which is provided by (3.2) and by (3.3). According to Lemma 3.1, we have an exact description of all covered gaps of $P(\mathcal{D}_2 \times \mathcal{D}_2)$, and the exact value of its measure follows from (3.7). On the other hand, some covered gaps of $P(\mathcal{D}_3 \times \mathcal{D}_3)$ arise because of configurations not treated by Lemma 3.1. Such configurations are considered in Lemma 3.2 and Lemma 3.3, but we don't have an exact characterization of all covered gaps that they produce. For this reason we can only give an approximation for the measure of $P(\mathcal{D}_3 \times \mathcal{D}_3)$. This is done in § 3.4, which completes the proof of Proposition 1.3. A more detailed outline of the proof of such estimate is given in § 3.1.

2. The fast subdivision: proof of Proposition 1.2

For a closed interval $I = [a, b]$ set $|I| := |b - a|$ and let $A_I : [0, 1] \to I$ be the unique affine orientation preserving bijection between $[0, 1]$ and I, that is

$$A_I(x) := a + (b - a)x = A_I(0) + |I|x.$$

For a set $\mathcal{S} \subset [0, 1]$ which is the union of countably many points and countably many intervals, define

$$\mathcal{E}(\mathcal{S}) := \{I : \ I \text{ is a connected component of } \mathcal{S} \text{ with non-empty interior}\}.$$

that is the family of intervals of $\mathcal{S}$.

2.1. **Product sets along and outside the diagonal.** The set $\mathcal{R}$ is constructed by repeatedly removing the middle third of certain intervals. Therefore it is important to understand what happens to the product set $P(I \times J)$ when we remove the middle third of some intervals $I, J \subset [2/3, 1]$. We consider two cases separately: when $I = J$, that is *along the diagonal*, and when $I \cap J = \emptyset$, that is *outside the diagonal*.

For any closed interval $I \subset [2/3, 1]$ write $I = [a, a + 3t]$ with $2/3 \leq a < a + 3t \leq 1$, then set
$$\ddot{I} := [a, a + t] \cup [a + 2t, a + 3t].$$
Since $(a + t)^2 > a(a + 2t)$ and $(a + t)(a + 3t) = (a + 2t)^2 - t^2 < (a + 2t)^2$, then

(2.1) $$P(I \times I) \setminus P(\ddot{I} \times \ddot{I}) = \big((a + 2t)^2 - t^2, (a + 2t)^2\big),$$

that is the interval $\big((a + 2t)^2 - t^2, (a + 2t)^2\big)$ is a gap in the product set $P(\ddot{I} \times \ddot{I})$ with size $|I|^2/9$. On the other hand, consider intervals $I, J \subset [2/3, 1]$ with $I \cap J = \emptyset$ and assume that they have the same length $|I| = |J|$. Lemma 11 in [2] gives

(2.2) $$P(\ddot{I} \times \ddot{J}) = P(I \times J),$$

that is the product set $P(\ddot{I} \times \ddot{J})$ has no gap. The elementary proof follows computing the extremal values of $P(\cdot, \cdot)$ over the four connected components of $\ddot{I} \times \ddot{J}$ and checking that the images overlap. The next Lemma 2.1 is a standard argument, which is left to the reader.

Lemma 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and $(C_n)_{n \geq 0}$ be a sequence of compact sets of $\mathbb{R}^2$ with $C_{n+1} \subset C_n$ for any $n \geq 0$. Then*
$$f(C) = \bigcap_{n \geq 0} f(C_n) \quad \text{where} \quad C := \bigcap_{n \geq 0} C_n.$$

Recall that $\mathcal{K} \subset [0, 1]$ denotes the middle-third Cantor set.

Lemma 2.2. *Consider intervals $I, J \subset [2/3, 1]$ with the same length $|I| = |J|$ and assume that $I \cap J = \emptyset$. Then we have*
$$P\big(A_I(\mathcal{K}) \times A_J(\mathcal{K})\big) = P(I \times J).$$

Proof. Set $\mathcal{I}_0 := I$ and $\mathcal{J}_0 := J$ and for $n \geq 1$ define inductively the compact sets
$$\mathcal{I}_n := \bigcup_{E \in \mathcal{E}(\mathcal{I}_{n-1})} \ddot{E} \quad \text{and} \quad \mathcal{J}_n := \bigcup_{F \in \mathcal{E}(\mathcal{J}_{n-1})} \ddot{F},$$
where $\mathcal{I}_n \subset \mathcal{I}_{n-1}$ and $\mathcal{J}_n \subset \mathcal{J}_{n-1}$. We have $A_I(\mathcal{K}) = \bigcap_{n=0}^{\infty} \mathcal{I}_n$ and $A_J(\mathcal{K}) = \bigcap_{n=0}^{\infty} \mathcal{J}_n$. For any $n \geq 0$, any two intervals (E, F) with $E \in \mathcal{E}(\mathcal{I}_n)$ and $F \in \mathcal{E}(\mathcal{J}_n)$ satisfy the same assumption as the pair (I, J) in the statement. Then (2.2) implies
$$P(\mathcal{I}_{n+1} \times \mathcal{J}_{n+1}) = \bigcup_{E \in \mathcal{E}(\mathcal{I}_n), F \in \mathcal{E}(\mathcal{J}_n)} P(\ddot{E} \times \ddot{F}) = \bigcup_{E \in \mathcal{E}(\mathcal{I}_n), F \in \mathcal{E}(\mathcal{J}_n)} P(E \times F) = P(\mathcal{I}_n \times \mathcal{J}_n).$$
This implies $P(\mathcal{I}_n \times \mathcal{J}_n) = P(I \times J)$ for any $n \geq 0$. Lemma 2.1 gives
$$P\big(A_I(\mathcal{K}) \times A_J(\mathcal{K})\big) = \bigcap_{n \geq 0} P(\mathcal{I}_n \times \mathcal{J}_n) = P(I \times J).$$
$\square$

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2.2. Definition of the fast subdivision. Consider the compact set $\mathcal{D}$ below

$$\mathcal{D} := [0,1] \setminus \left(\left(\bigcup_{k \geq 1} \left(\frac{1}{3^{k+1}}, \frac{2}{3^{k+1}} \right) \right) \cup \left(\frac{1}{3}, \frac{2}{3} \right) \cup \left(\bigcup_{k \geq 1} \left(\frac{3^{k+1}-2}{3^{k+1}}, \frac{3^{k+1}-1}{3^{k+1}} \right) \right) \right),$$

which is the union of countably many intervals, together with the points $\{0\}$ and $\{1\}$. Recall that for $\mathcal{S} \subset [0,1]$ which is the union of countably many points and countably many intervals, we denote $\mathcal{E}(\mathcal{S})$ the family of intervals of $\mathcal{S}$. Define the *fast subdivision* $(\mathcal{D}_n)_{n \geq 0}$ of $\mathcal{R}$ setting $\mathcal{D}_0 := [2/3, 1]$ and iteratively

$$(2.3) \qquad \mathcal{D}_n := \bigcup_{I \in \mathcal{E}(\mathcal{D}_{n-1})} A_I(\mathcal{D}) \quad \text{for} \quad n \geq 1,$$

where we observe that such notation gives $\mathcal{D}_1 := A_{[2/3,1]}(\mathcal{D})$. Elementary arguments which are left to the reader easily give the next Lemma 2.3.

Lemma 2.3. (2.3) *defines a subdivision for* $\mathcal{R}$.

Observe that for any $n \geq 1$ we have

$$\sum_{I \in \mathcal{E}(\mathcal{D}_n)} |I|^2 = \sum_{J \in \mathcal{E}(\mathcal{D}_{n-1})} \left(\sum_{I \in \mathcal{E}(J \cap \mathcal{D}_n)} |I|^2 \right) = \sum_{J \in \mathcal{E}(\mathcal{D}_{n-1})} \left(|J|^2 \sum_{\widetilde{I} \in \mathcal{E}(\mathcal{D})} |\widetilde{I}|^2 \right)$$

$$= \sum_{J \in \mathcal{E}(\mathcal{D}_{n-1})} \left(|J|^2 \cdot 2 \sum_{k=2}^{\infty} \left(\frac{1}{3^k} \right)^2 \right) = \frac{1}{36} \sum_{J \in \mathcal{E}(\mathcal{D}_{n-1})} |J|^2 = \left| \left[\frac{2}{3}, 1 \right] \right|^2 \cdot \left(\frac{1}{36} \right)^n,$$

where the second equality holds because for any $J \in \mathcal{E}(\mathcal{D}_{n-1})$ the intervals $I \in \mathcal{E}(J \cap \mathcal{D}_n)$ are the images of the intervals $\widetilde{I} \in \mathcal{E}(\mathcal{D})$ under A_J, which is affine with $dA_J(t)/dt = |J|$ for any $t \in [0,1]$. Hence we get

$$(2.4) \qquad \sum_{I \in \mathcal{E}(\mathcal{D}_n)} |I|^2 = \frac{1}{9} \left(\frac{1}{36} \right)^n,$$

which we will use in the next § 2.4.

2.3. Gaps in the product set for a single step.
In this paragraph we study in detail a single step of the fast subdivision in (2.3), that is we consider any interval $I \subset [2/3, 1]$ and we describe the infinitely many gaps in the product set $P\big(A_I(\mathcal{D}) \times A_I(\mathcal{D})\big)$ generated by the subdivision of I. If I is an interval in $\mathcal{D}_n$, this describes the gaps appearing in $P(I \times I)$ when I is subdivided. According to the next Proposition 2.6, these gaps form the family $\mathcal{G}_I$ defined in (2.5) below. All the notation used in the rest of the paper is introduced in this paragraph, before Remark 2.5. Figure 1 represents the gaps in the products set, and provides a guide to such notation.

Fix an interval $I \subset [2/3, 1]$. Consider the map $A_I : [0,1] \to I$ and for $0 \leq x < y \leq 1$ set

$$I(x, y) := \big[A_I(x), A_I(y) \big].$$

For $k \geq 0$ define the intervals

$$\mathcal{P}_{(I,k,-)} := P\big(I(2/3^{k+1}, 1/3^k) \times I(0, 1/3^{k+1}) \big)$$
$$\mathcal{P}_{(I,k,+)} := P\big(I(1 - 1/3^{k+1}, 1) \times I(1 - 1/3^k, 1 - 2/3^{k+1}) \big),$$

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where $\mathcal{P}_{(I,0,-)} = \mathcal{P}_{(I,0,+)} = P\big(I(2/3,1) \times I(0,1/3)\big)$. Intervals $\mathcal{P}_{(I,k,\pm)}$ correspond to product sets arising from the light grey regions in Figure 1.

Lemma 2.4. *For any $I \subset [2/3,1]$ and any $k \geq 0$ we have*

$$\mathcal{P}_{(I,k,-)}, \mathcal{P}_{(I,k,+)} \subset P\big(A_I(\mathcal{K}) \times A_I(\mathcal{K})\big).$$

Proof. We have $A_{I(0,1/3)}(\mathcal{K}) \subset A_I(\mathcal{K})$ and $A_{I(2/3,1)}(\mathcal{K}) \subset A_I(\mathcal{K})$. The statement follows for $\mathcal{P}_{(I,0,-)} = \mathcal{P}_{(I,0,+)}$ because Lemma 2.2 gives

$$P\big(I(2/3,1) \times I(0,1/3)\big) = P\big(A_{I(2/3,1)}(\mathcal{K}) \times A_{I(0,1/3)}(\mathcal{K})\big).$$

The same argument applies to $\mathcal{P}_{(I,k,-)}$ and $\mathcal{P}_{(I,k,+)}$ for $k \geq 1$. $\qquad\square$

For $k \geq 0$ set $I_{(k,-)} := I(0, 1/3^k)$ and $I_{(k,+)} := I(1 - 1/3^k, 1)$, where $I_{(0,-)} = I_{(0,+)} = I$. Recall (2.1) and for $k \geq 0$ define the open intervals

$$\mathcal{G}_{(I,k,-)} := P(I_{(k,-)} \times I_{(k,-)}) \setminus P(\ddot{I}_{(k,-)} \times \ddot{I}_{(k,-)})$$

$$\mathcal{G}_{(I,k,+)} := P(I_{(k,+)} \times I_{(k,-)}) \setminus P(\ddot{I}_{(k,+)} \times \ddot{I}_{(k,+)}).$$

Observe that $\mathcal{G}_{(I,0,-)} = \mathcal{G}_{(I,0,+)} = P(I \times I) \setminus P(\ddot{I} \times \ddot{I})$. Referring to Figure 1, the intervals $\mathcal{G}_{(I,k,\pm)}$ correspond to product sets arising from the white regions spanned by hyperbolas not intersecting $A_I(\mathcal{D}) \times A_I(\mathcal{D})$. Set

(2.5) $$\mathcal{G}_I := \Big(\bigcup_{k=1}^{\infty} \mathcal{G}_{(I,k,-)} \Big) \cup \mathcal{G}_{(I,0,-)} \cup \Big(\bigcup_{k=1}^{\infty} \mathcal{G}_{(I,k,+)} \Big).$$

Finally for $k \geq 1$ denote the elements of $\mathcal{E}\big(A_I(\mathcal{D})\big)$ by

$$D_{(I,k,-)} := I(2/3^{k+1}, 1/3^k) \quad \text{and} \quad D_{(I,k,+)} := I(1 - 1/3^k, 1 - 2/3^{k+1})$$

and define the intervals

$$\mathcal{Q}_{(I,k,-)} := P(D_{(I,k,-)} \times D_{(I,k,-)}) \quad \text{and} \quad \mathcal{Q}_{(I,k,+)} := P(D_{(I,k,+)} \times D_{(I,k,+)}).$$

The intervals $\mathcal{Q}_{(I,k,\pm)}$ correspond to product sets arising from the black squares along the diagonal in Figure 1.

In order to describe the structure of $P\big(A_I(\mathcal{D}) \times A_I(\mathcal{D})\big)$ we compute the endpoints of its subsets $\mathcal{P}_{(I,k,\pm)}$, $\mathcal{G}_{(I,k,\pm)}$ and $\mathcal{Q}_{(I,k,\pm)}$. These endpoints are given by products $A_I(x)A_I(y)$, where x, y are endpoints of intervals in $\mathcal{D}$, and their reciprocal position is described in the next Remark 2.5. The Remark applies 8 times the next (2.6), and at each time the values for x, y, z, t to be plugged in (2.6) are given by the above definitions of the subsets $\mathcal{P}_{(I,k,\pm)}$, $\mathcal{G}_{(I,k,\pm)}$ and $\mathcal{Q}_{(I,k,\pm)}$. Alternatively, the correct values of x, y, z, t can be easily deduced from Figure 1.

Remark 2.5. *Fix I. For simplicity write $A = A_I$. It is easy to check that for $x, y, z, t \in \mathbb{R}$ we have*

(2.6) $$A(x)A(y) > A(z)A(t) \quad \Leftrightarrow \quad zt - xy < \frac{A(0)}{|I|}(x + y - z - t).$$
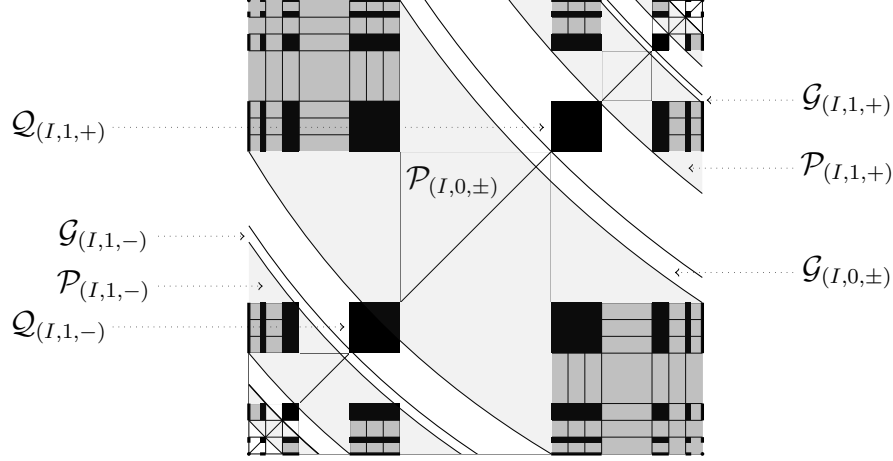
FIGURE 1. The big square represents $I \times I$, and the union of the black rectangles in the picture is $A_I(\mathcal{D}) \times A_I(\mathcal{D})$. By Lemma 2.2, each dark grey square has the same $P(\cdot, \cdot)$-image as its intersection with $A_I(\mathcal{D}) \times A_I(\mathcal{D})$, represented in black. Such images are the intervals $\mathcal{P}_{(I,k,\pm)}$, $k \geq 0$. White regions of hyperbolas not intersecting $A_I(\mathcal{D}) \times A_I(\mathcal{D})$ correspond to the gaps $\mathcal{G}_{(I,k,\pm)}$, $k \geq 0$. The intervals $\mathcal{Q}_{(I,k,\pm)}$, $k \geq 1$ are the images of the black squares $J \times J$ along the diagonal, where $J \in \mathcal{E}(A_I(\mathcal{D}))$. The superposition between such black squares and the light grey regions generates *covered gaps* (see § 3.1) at the next subdivision.

Using (2.6), and observing that $A(0)/|I| \geq 2$ for intervals $I \subset [2/3,1]$, it is easy to verify that for for any $k \geq 0$ we have

$$\sup \mathcal{G}_{(I,k+1,-)} = \inf \mathcal{Q}_{(I,k+1,-)} < \inf \mathcal{P}_{(I,k,-)} \Leftrightarrow A(2/3^{k+2})^2 < A(0)A(2/3^{k+1})$$

$$\inf \mathcal{P}_{(I,k,-)} < \sup \mathcal{Q}_{(I,k+1,-)} \Leftrightarrow A(0)A(2/3^{k+1}) < A(1/3^{k+1})^2$$

$$\sup \mathcal{Q}_{(I,k+1,-)} < \sup \mathcal{P}_{(I,k,-)} \Leftrightarrow A(1/3^{k+1})^2 < A(1/3^k)A(1/3^{k+1})$$

$$\sup \mathcal{P}_{(I,k,-)} = \inf \mathcal{G}_{(I,k,-)} < \sup \mathcal{G}_{(I,k,-)} \leq \sup \mathcal{G}_{(I,k,+)} = \inf \mathcal{Q}_{(I,k+1,+)}$$

and for any $k \geq 1$ we have

$$\inf \mathcal{Q}_{(I,k,+)} < \inf \mathcal{P}_{(I,k,+)} \Leftrightarrow A(1-1/3^k)^2 < A(1-1/3^{k+1})A(1-1/3^k)$$

$$\inf \mathcal{P}_{(I,k,+)} < \sup \mathcal{Q}_{(I,k,+)} \Leftrightarrow A(1-1/3^{k+1})A(1-1/3^k) < A(1-2/3^{k+1})^2$$

$$\sup \mathcal{Q}_{(I,k,+)} < \sup \mathcal{P}_{(I,k,+)} \Leftrightarrow A(1-2/3^{k+1})^2 < A(1)A(1-2/3^{k+1})$$

$$\sup \mathcal{P}_{(I,k,+)} = \inf \mathcal{G}_{(I,k,+)} < \sup \mathcal{G}_{(I,k,+)} = \inf \mathcal{Q}_{(I,k+1,+)}.$$

All the equalities above are immediate consequence of the definitions of $\mathcal{P}_{(I,k,\pm)}$, $\mathcal{G}_{(I,k,\pm)}$ and $\mathcal{Q}_{(I,k,\pm)}$. See Figure 1.

Remark 2.5 implies that $\mathcal{G}_I$ is a disjoint union. Then (2.1) implies

$$(2.7) \quad \mathcal{L}(\mathcal{G}_I) = \mathcal{L}(\mathcal{G}_{(I,0,-)}) + \sum_{k=1}^{\infty} \mathcal{L}(\mathcal{G}_{(I,k,-)}) + \mathcal{L}(\mathcal{G}_{(I,k,-)}) = \frac{|I^2|}{3^2} + 2\sum_{k=1}^{\infty} \frac{|I|^2}{3^{2k+2}} = \frac{5}{36}|I|^2.$$

Proposition 2.6. *Fix an interval $I \subset [2/3, 1]$. We have*

$$(2.8) \qquad P(I \times I) \setminus P\big(A_I(\mathcal{D}) \times A_I(\mathcal{D})\big) = \mathcal{G}_I.$$

Proof. Write $E^2 := E \times E$ for sets $E \subset [0, 1]$. It is also convenient to write $\mathcal{D}_I := A_I(\mathcal{D})$. The set $\mathcal{D}_I^2 \cap \{(x, y) \in \mathbb{R}^2 : y \leq x\}$ is contained in the union over $k \geq 0$ of the sets

$$D_{(I,k+1,-)}^2 \cup D_{(I,k+1,+)}^2 \cup$$
$$\Big(I(2/3^{k+1}, 1/3^k) \times I_{(k+1,-)}\Big) \cup \Big(I_{(k+1,+)} \times I(1 - 1/3^k, 1 - 2/3^{k+1})\Big).$$

This implies

$$(2.9) \qquad P(\mathcal{D}_I^2) \subset \Big(\bigcup_{k \geq 0} \mathcal{P}_{(I,k,-)} \cup \mathcal{P}_{(I,k,+)}\Big) \cup \Big(\bigcup_{k \geq 1} \mathcal{Q}_{(I,k,-)} \cup \mathcal{Q}_{(I,k,+)}\Big).$$

On the other hand $A_I(\mathcal{K}) \subset \mathcal{D}_I$ by Lemma 2.3. Hence Lemma 2.4 implies that $P(\mathcal{D}_I^2)$ contains $\mathcal{P}_{(I,k,-)}$ and $\mathcal{P}_{(I,k,+)}$ for $k \geq 0$. Obviously $P(\mathcal{D}_I^2)$ also contains $\mathcal{Q}_{(I,k,-)}$ and $\mathcal{Q}_{(I,k,+)}$ for any $k \geq 1$. Therefore the inclusion in (2.9) is indeed an equality between sets. The intervals in $\mathcal{G}_I$ fill the gaps in $P(\mathcal{D}_I^2)$ by Remark 2.5. This proves (2.8). $\qquad \square$

2.4. End of the proof of Proposition 1.2. Fix $n \geq 1$ and consider two intervals J_1, J_2 in $\mathcal{E}(\mathcal{D}_n)$ with $J_1 \neq J_2$. Without loss of generality assume $\sup J_2 < \inf J_1$. Let $m \leq n - 1$ be maximal such that there exists $I \in \mathcal{E}(\mathcal{D}_m)$ with $J_i \subset I$ for $i = 1, 2$. Maximality implies that J_1 and J_2 are included into different connected components of $A_I(\mathcal{D})$. If $J_1 \subset D_{(I,k,-)}$ for some $k \geq 1$, then Lemma 2.4 implies

$$P(J_1 \times J_2) \subset P\Big(I(2/3^{k+1}, 1/3^k) \times I(0, 1/3^{k+1})\Big) = \mathcal{P}_{(I,k,-)} \subset P(\mathcal{R} \times \mathcal{R}),$$

where we recall that $A_I(\mathcal{K}) \subset \mathcal{R}$ by Lemma 2.3. Otherwise there exists $l \geq 0$ with

$$J_1 \times J_2 \subset I(1 - 1/3^{l+1}, 1) \times I(1 - 1/3^l, 1 - 2/3^{l+1})$$

and we get again $P(J_1 \times J_2) \subset \mathcal{P}_{(I,l,+)} \subset P(\mathcal{R} \times \mathcal{R})$ by Lemma 2.4. Both inclusions cannot be derived directly from (2.2) because, a priori, J_1 and J_2 have different sizes. Thus

$$P\Big((\mathcal{D}_n \times \mathcal{D}_n) \setminus \bigcup_{I \in \mathcal{E}(\mathcal{D}_n)} I \times I\Big) \subset P(\mathcal{R} \times \mathcal{R}) \qquad \text{for any} \quad n \geq 0.$$

Since $P(\mathcal{R} \times \mathcal{R}) \subset P(\mathcal{D}_n \times \mathcal{D}_n)$ for any $n \geq 0$, then (2.3) and (2.8) give

$$P(\mathcal{D}_n \times \mathcal{D}_n) \setminus P(\mathcal{D}_{n+1} \times \mathcal{D}_{n+1}) \subset$$
$$\bigcup_{I \in \mathcal{E}(D_n)} P(I \times I) \setminus P(\mathcal{D}_{n+1} \times \mathcal{D}_{n+1}) \subset$$
$$\bigcup_{I \in \mathcal{E}(D_n)} P(I \times I) \setminus P\big(A_I(\mathcal{D}) \times A_I(\mathcal{D})\big) = \bigcup_{I \in \mathcal{E}(D_n)} \mathcal{G}_I.$$

Therefore (2.7) and (2.4) give

$$\mathcal{L}\big(P(\mathcal{D}_n \times \mathcal{D}_n)\big) - \mathcal{L}\big(P(\mathcal{D}_{n+1} \times \mathcal{D}_{n+1})\big) \leq \sum_{I \in \mathcal{E}(D_n)} \mathcal{L}(\mathcal{G}_I) = \frac{5}{36} \sum_{I \in \mathcal{E}(D_n)} |I|^2 = \frac{5}{9 \cdot 36^{n+1}}.$$

For any $n \geq 0$ and $m \geq n+1$ a telescopic argument gives

$$0 \leq \mathcal{L}\big(P(\mathcal{D}_n \times \mathcal{D}_n)\big) - \mathcal{L}\big(P(\mathcal{D}_m \times \mathcal{D}_m)\big) \leq \sum_{k=n+1}^{m} \frac{5/9}{36^k} \leq \sum_{k=n+1}^{\infty} \frac{5/9}{36^k} = \frac{1}{63 \cdot 36^n}.$$

We have $\mathcal{L}\big(P(\mathcal{D}_m \times \mathcal{D}_m)\big) \to \mathcal{L}\big(P(\mathcal{R} \times \mathcal{R})\big)$ as $m \to \infty$. Proposition 1.2 is proved.

3. Covered gaps: proof of Proposition 1.3

In this section we use the notation introduced in § 2.3.

3.1. **Definition of covered gaps and structure of the proof.** If E is an interval of some $\mathcal{D}_m$, its subdivision at time $m+1$ generates a family $\mathcal{G}_E$ of gaps in $P(E \times E)$, which is described by Proposition 2.6. A gap $\mathcal{G} \in \mathcal{E}(\mathcal{G}_E)$ is *covered* if $\mathcal{G} \subset P(\mathcal{R} \times \mathcal{R})$. Such gap $\mathcal{G}$ doesn't give negative contribution to the measure of $P(\mathcal{R} \times \mathcal{R})$. The next § 3.2 describes qualitatively the geometric configuration which produces covered gaps.

In § 3.3, we give an arithmetic characterization of covered gaps. More precisely we fix n and an interval I of $\mathcal{D}_n$, and we consider the two consecutive subdivisions of I at generations $n+1$ and $n+2$. Intervals E in $I \cap \mathcal{D}_{n+1}$ are labelled by an integer k, and for any such E a second integer l labels the gaps in $P\big((E \cap \mathcal{D}_{n+2}) \times (E \cap \mathcal{D}_{n+2})\big)$, which we can denote here by $\mathcal{G}_{(k,l)}$ (a more precise notation is used in § 3.3). Equation (3.2) (or its analogous (3.3)) gives a condition on k and l so that the corresponding gap $\mathcal{G}_{(k,l)}$ is covered. A relevant aspect of (3.2) (and of (3.3)) is that the same arithmetic condition determines the same inequality both for $\inf \mathcal{G}_{(k,l)}$ and for $\sup \mathcal{G}_{(k,l)}$, that is a gap $\mathcal{G}_{(k,l)}$ is either covered, or disjoint from $P\big((I \cap \mathcal{R}) \times (I \cap \mathcal{R})\big)$. This is resumed by Lemma 3.1.

Finally in § 3.4 we obtain the estimate for the measure of $P(\mathcal{D}_3 \times \mathcal{D}_3)$. A first observation is that Lemma 3.1 gives the complete description of all covered gaps for $P(\mathcal{D}_2 \times \mathcal{D}_2)$, so that by (3.7) we obtain the exact value of the measure of $P(\mathcal{D}_2 \times \mathcal{D}_2)$. For any interval E of $\mathcal{D}_1$, the 3-rd generation gaps $\mathcal{G}$ arising from E with $\mathcal{G} \subset P\big((E \cap \mathcal{R}) \times (E \cap \mathcal{R})\big)$ are again described completely by Lemma 3.1. The main technical problem is that there exist 3-rd generation gaps $\mathcal{G}$ arising from E with $\mathcal{G} \cap P\big((E \cap \mathcal{R}) \times (E \cap \mathcal{R})\big) = \emptyset$ but at the same time $\mathcal{G} \cap P\big((\mathcal{R} \times \mathcal{R}) \setminus (E \times E)\big) \neq \emptyset$. These $\mathcal{G}$ are considered in Lemma 3.2 and Lemma 3.3.

3.2. **How covered gaps appear.** In this paragraph we describe qualitatively the geometric configuration which produces covered gaps. This is also represented in Figure 1.

Fix $n \geq 0$ and $I \in \mathcal{E}(\mathcal{D}_n)$. For $k \geq 1$ consider $E := D_{(I,k,-)}$, which is an element of $\mathcal{E}\big(A_I(\mathcal{D})\big)$, and $\mathcal{Q}_{(I,k,-)} = P(E \times E)$. We have $\inf \mathcal{P}_{(I,k-1,-)} < \sup P(E \times E)$ by Remark 2.5. See also Figure 1. Recalling Lemma 2.4, for $l \gg 1$ we get

$$\mathcal{G}_{(E,l,+)} \subset \mathcal{P}_{(I,k-1,-)} \subset P(\mathcal{R} \times \mathcal{R}).$$

Hence the subdivision of E (at step $n+2$, after the subdivision of I, at step $n+1$) generates a tail of covered gaps $\mathcal{G}_{(E,l,+)}$. We also have $\inf P(E \times E) = \sup \mathcal{G}_{(i,k,-)} < \inf \mathcal{P}_{(I,k,-)}$, which follows again from Remark 2.5. Hence for $l \gg 1$ we have

$$\mathcal{G}_{(E,l,-)} \cap P\Big(\big(A_I(\mathcal{D}) \times A_I(\mathcal{D})\big) \setminus (E \times E)\Big) = \emptyset,$$

that is

$$\mathcal{G}_{(E,l,-)} \subset P\big((I \cap \mathcal{D}_{n+1}) \times (I \cap \mathcal{D}_{n+1})\big) \setminus P\big((I \cap \mathcal{D}_{n+2}) \times (I \cap \mathcal{D}_{n+2})\big).$$

The same holds for $F := D_{(I,k,+)}$ with $k \geq 1$, indeed $\sup \mathcal{G}_{(I,k-1,+)} = \inf \mathcal{Q}_{(I,k,+)}$ and $\inf \mathcal{P}_{(I,k,+)} < \sup \mathcal{Q}_{(I,k,+)}$ by Remark 2.5. Thus for $l \gg 1$ we have

$$\mathcal{G}_{(F,l,+)} \subset \mathcal{P}_{(I,k,+)} \subset P(\mathcal{R} \times \mathcal{R})$$

and

$$\mathcal{G}_{(F,l,-)} \cap P\Big(\big(A_I(\mathcal{D}) \times A_I(\mathcal{D})\big) \setminus (F \times F)\Big) = \emptyset,$$

that is

$$\mathcal{G}_{(F,l,+)} \subset P\big((I \cap \mathcal{D}_{n+1}) \times (I \cap \mathcal{D}_{n+1})\big) \setminus P\big((I \cap \mathcal{D}_{n+2}) \times (I \cap \mathcal{D}_{n+2})\big).$$

3.3. Arithmetic condition for covered gaps.
Fix $n \geq 0$ and $I \in \mathcal{E}(\mathcal{D}_n)$. The main result in this paragraph is Lemma 3.1, which considers the two consecutive subdivisions of I at generations $n+1$ and $n+2$. The Lemma determines arithmetically the gaps from generation $n+2$ which are contained in $P\big((I \cap \mathcal{R}) \times (I \cap \mathcal{R})\big)$, that is are covered. The Lemma also proves that all other gaps arising from I at generation $n+2$ are disjoint from $P\big((I \cap \mathcal{R}) \times (I \cap \mathcal{R})\big)$. These gaps may or may not be covered by some other part of $P(\mathcal{R} \times \mathcal{R} \setminus I \times I)$, but this is not determined by Lemma 3.1, which is a local result.

For $k \geq 1$ consider $E := D_{(I,k,-)}$, that is $E = I(2/3^{k+1}, 1/3^k)$. Consider the affine maps $A_I : [0,1] \to I$ and $A_E : [0,1] \to E$. For $x \in \mathbb{R}$ we have

$$A_I(x) = A_E\left(\frac{A_I(x) - A_E(0)}{|E|}\right) = A_E\left(\frac{3^{k+1}}{|I|}\left(x - \frac{2}{3^{k+1}}\right)|I|\right) = A_E(3^{k+1}x - 2).$$

We have $\inf \mathcal{P}_{(I,k-1,-)} = A_I(0)A_I(2/3^k) = A_E(-2)A_E(4)$. From the definition of $\mathcal{D}$ in § 2.2, it is clear that for $l \geq 0$ we have

$$(3.1) \qquad \inf \mathcal{G}_{(E,l,+)} = A_E(1)A_E\left(1 - \frac{2}{3^{l+1}}\right) \quad \text{and} \quad \sup \mathcal{G}_{(E,l,+)} = A_E\left(1 - \frac{1}{3^{l+1}}\right)^2.$$

Finally $A_E(0)/|E| = 3^{k+1}A_I(0)|I| + 2$. Hence (2.6) gives

$$A_E(-2)A_E(4) < A_E(1)A_E\left(1 - \frac{2}{3^{l+1}}\right) \Leftrightarrow -8 - 1 + \frac{2}{3^{l+1}} < -\frac{A_E(0)}{|E|}\frac{2}{3^{l+1}} \Leftrightarrow$$

$$-9 + \frac{2}{3^{l+1}} < -\left(3^{k+1}\frac{A_I(0)}{|I|} + 2\right)\frac{2}{3^{l+1}} \Leftrightarrow 3^{l+2} > 2 \cdot 3^k \frac{A_I(0)}{|I|} + 2.$$

$$A_E(-2)A_E(4) < A_E\left(1 - \frac{1}{3^{l+1}}\right)^2 \Leftrightarrow -8 - \left(\frac{3^{l+1} - 1}{3^{l+1}}\right)^2 < -\frac{A_E(0)}{|E|}\frac{2}{3^{l+1}} \Leftrightarrow$$

$$3^{l+3} + \frac{1}{3^{l+1}} > 2 \cdot 3^{k+1}\frac{A_I(0)}{|I|} + 6 \Leftrightarrow 3^{l+2} > 2 \cdot 3^k \frac{A_I(0)}{|I|} + 2,$$

where the last equivalence holds because $A_I(0)/|I|$ is integer (this can be seen by induction on n) and therefore we always have $3^{l+3} \neq 2 \cdot 3^{k+1}A_I(0)/|I| + 6$. We get

$$(3.2) \quad \inf \mathcal{P}_{(I,k-1,-)} < \inf \mathcal{G}_{(E,l,+)} \Leftrightarrow 3^{l+2} > 2 \cdot 3^k \frac{A_I(0)}{|I|} + 2 \Leftrightarrow \inf \mathcal{P}_{(I,k-1,-)} < \sup \mathcal{G}_{(E,l,+)}.$$

Similarly, for $k \geq 1$ consider $F := D_{(I,k,+)}$, that is $F = I(1 - 1/3^k, 1 - 2/3^{k+1})$. For $x \in \mathbb{R}$ the maps $A_I : [0,1] \to I$ and $A_F : [0,1] \to E$ satisfy

$$A_I(x) = A_F\left(\frac{3^{k+1}}{|I|}\left(x - 1 + \frac{1}{3^k}\right)|I|\right) = A_F\big(3^{k+1}(x-1) + 3\big).$$

We have $\inf \mathcal{P}_{(I,k,+)} = A_I(1 - 1/3^{k+1})A_I(1 - 1/3^k) = A_F(2)A_F(0)$. For $l \geq 0$ the expression of $\inf \mathcal{G}_{(F,l,+)}$ and $\sup \mathcal{G}_{(F,l,+)}$ in terms of A_F is as in (3.1). Thus (2.6) gives

$$A_F(2)A_F(0) < A_F(1)A_F\left(1 - \frac{2}{3^{l+1}}\right) \Leftrightarrow \frac{2}{3^{l+1}} - 1 < -\frac{A_F(0)}{|F|}\frac{2}{3^{l+1}} \Leftrightarrow$$

$$\frac{2}{3^{l+1}} - 1 < -\left(3^{k+1}\left(\frac{A_I(0)}{|I|} + 1\right) - 3\right)\frac{2}{3^{l+1}} \Leftrightarrow 3^{l+1} > 2 \cdot 3^{k+1}\left(\frac{A_I(0)}{|I|} + 1\right) - 4.$$

$$A_F(2)A_F(0) < A_F\left(1 - \frac{1}{3^{l+1}}\right)^2 \Leftrightarrow -\left(\frac{3^{l+1} - 1}{3^{l+1}}\right)^2 < -\frac{A_F(0)}{|F|}\frac{2}{3^{l+1}} \Leftrightarrow$$

$$3^{l+1} + \frac{1}{3^{l+1}} > 2 \cdot 3^{k+1}\left(\frac{A_I(0)}{|I|} + 1\right) - 4 \Leftrightarrow 3^{l+1} > 2 \cdot 3^{k+1}\left(\frac{A_I(0)}{|I|} + 1\right) - 4,$$

where again the last equivalence holds because $A_I(0)/|I|$ is integer and therefore we always have $3^{l+1} \neq 2 \cdot 3^{k+1}(A_I(0)/|I| + 1) - 4$. We get

$$(3.3) \quad \inf \mathcal{P}_{(I,k,+)} < \inf \mathcal{G}_{(F,l,+)} \Leftrightarrow 3^{l+1} > 2 \cdot 3^{k+1}\left(\frac{A_I(0)}{|I|} + 1\right) - 4 \Leftrightarrow \inf \mathcal{P}_{(I,k,+)} < \sup \mathcal{G}_{(F,l,+)}.$$

Lemma 3.1. *Fix $n \geq 0$, $I \in \mathcal{E}(\mathcal{D}_n)$ and $k \geq 1$. The following holds*

(1) *We have $L(k) := \min\{l \geq 0 : (3.2) \text{ holds }\} \geq k + 2n$. For $E := D_{(I,k,-)}$ and for any $l \geq L(k)$ we have $\mathcal{G}_{(E,l,+)} \subset P((I \cap \mathcal{R}) \times (I \cap \mathcal{R}))$. On the other hand*

$$(3.4) \qquad \left(\bigcup_{0 \leq l \leq L(k)-1} \mathcal{G}_{(E,l,+)} \cup \bigcup_{l=1}^{\infty} \mathcal{G}_{(E,l,-)}\right) \cap P((I \cap \mathcal{D}_{n+2}) \times (I \cap \mathcal{D}_{n+2})) = \emptyset.$$

(2) *We have $R(k) := \min\{l \geq 0 : (3.3) \text{ holds }\} \geq k + 2n$. For $F := D_{(I,k,+)}$ and for any $l \geq R(k)$ we have $\mathcal{G}_{(F,l,+)} \subset P((I \cap \mathcal{R}) \times (I \cap \mathcal{R}))$. On the other hand*

$$(3.5) \qquad \left(\bigcup_{0 \leq l \leq R(k)-1} \mathcal{G}_{(F,l,+)} \cup \bigcup_{l=1}^{\infty} \mathcal{G}_{(F,l,-)}\right) \cap P((I \cap \mathcal{D}_{n+2}) \times (I \cap \mathcal{D}_{n+2})) = \emptyset.$$

Proof. We have $A_I(0)/|I| \geq 2 \cdot 9^n$ for any $n \geq 0$ and any $I \in \mathcal{E}(\mathcal{D}_n)$. This can be easily proved by induction on n, observing that $A_I(0)/|I| = 2$ for $I = [2/3, 1] = \mathcal{D}_0$. Therefore (3.2) and (3.3) give $L(k) \geq k + 2n$ and $R(k) \geq k + 2n$. Moreover (3.2) implies

$$\mathcal{G}_{(E,L(k),+)} \subset \mathcal{P}_{(I,k-1,-)} \subset P((I \cap \mathcal{R}) \times (I \cap \mathcal{R}))$$

and

$$\mathcal{G}_{(E,L(k)-1,+)} \cap P\left((A_I(\mathcal{D}) \times A_I(\mathcal{D})) \setminus (E \times E)\right) = \emptyset.$$

The order between gaps established by Remark 2.5 gives

$$\mathcal{G}_{(E,l,+)} \subset P((I \cap \mathcal{R}) \times (I \cap \mathcal{R})) \quad \text{for} \quad l \geq L(k)$$

and

$$\left(\bigcup_{0 \leq l \leq L(k)-1} \mathcal{G}_{(E,l,+)} \cup \bigcup_{0 \leq l < \infty} \mathcal{G}_{(E,l,-)}\right) \cap P\left((A_I(\mathcal{D}) \times A_I(\mathcal{D})) \setminus (E \times E)\right) = \emptyset.$$

According to Proposition 2.6, all gaps $\mathcal{G}_{(E,l,\pm)}$ are removed from $P(E \times E)$ at generation $n + 2$, thus (3.4) follows. Part (1) is proved. Part (2) follows by a similar argument. $\quad\square$

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In the notation of Lemma 3.1, gaps $\mathcal{G}_{(E,l,+)}$ with $l \geq L(k)$ and gaps $\mathcal{G}_{(F,l,+)}$ with $l \geq R(k)$ don't give negative contribution to the measure of $P\big((I \cap \mathcal{D}_{n+2}) \times (I \cap \mathcal{D}_{n+2})\big)$. For a sharp measure estimate we cannot take into account the measure of these gaps. Therefore, for a generic interval J and and integer $N \geq 1$, we replace (2.7) by

$$(3.6) \quad \mathcal{L}\Big(\bigcup_{0 \leq l \leq N-1} \mathcal{G}_{(J,l,+)} \cup \bigcup_{1 \leq l < \infty} \mathcal{G}_{(J,l,-)} \Big) = \sum_{0 \leq l \leq N-1} \frac{|J|^2}{9^{l+1}} + \sum_{1 \leq l \leq \infty} \frac{|J|^2}{9^{l+1}} = \frac{|J|^2}{8}\Big(\frac{10}{9} - \frac{1}{9^N} \Big).$$

We apply (3.6) with $J := E$ and $N := L(k)$ if we are as in Part (1) of Lemma 3.1, while we set $J := F$ and $N := R(k)$ if we are as in Part (2) of Lemma 3.1.

3.4. **End of the proof of Proposition 1.3.** Set $I := [2/3, 1]$. We have $A_I(0)/|I| = 2$. For any $n \geq 1$ define

$$\mu_n := \mathcal{L}\big(P(I \times I) \setminus P(\mathcal{D}_n \times \mathcal{D}_n) \big).$$

According to (2.3) we have $\mathcal{D}_1 = A_{[2/3,1]}(\mathcal{D})$, thus (2.8) and (2.7) give

$$\mu_1 = \mathcal{L}(\mathcal{G}_{[2/3,1]}) = \frac{5}{36} \big| [2/3, 1] \big|^2 = \frac{5}{4 \cdot 81}.$$

For $k \geq 1$ consider $E_k := I(2/3^{k+1}, 1/3^k)$ and $F_k := I(1 - 1/3^k, 1 - 2/3^{k+1})$. We have $|E_k| = |F_k| = |I|/3^{k+1} = 1/3^{k+2}$. We apply Lemma 3.1 with $n = 0$ and $I = [2/3, 1]$, and since $A_I(0)/|I| = 2$, it is easy to see that we get $L(k) = k$ and $R(k) = k+2$. Thus Lemma 3.1 and (3.6) give

$$(3.7) \qquad \mu_2 - \mu_1 = \sum_{k=1}^{\infty} \frac{|E_k|^2}{8}\Big(\frac{10}{9} - \frac{1}{9^k} \Big) + \sum_{k=1}^{\infty} \frac{|F_k|^2}{8}\Big(\frac{10}{9} - \frac{1}{9^{k+2}} \Big) = \frac{859}{9^4 \cdot 5 \cdot 64}.$$

Fix $k \geq 1$ and $F = F_k$. For $m \geq 1$ let $D(m, \pm) := D_{(F,m,\pm)}$ be the intervals arising from the subdivision of F at the 2-nd step. At the 3-rd step any $D = D(m, \pm)$ generates gaps $\mathcal{G}_{(D,l,\pm)}$ with $l \geq 0$. These gaps appear in the product sets $\mathcal{Q}_{(F,m,\pm)} \subset P(\mathcal{D}_2 \times \mathcal{D}_2)$, where we recall that in our notation $\mathcal{Q}_{(F,m,\pm)} = P\big(D(m, \pm) \times D(m, \pm) \big)$. We have $A_F(0)/|F| = 3^{k+2} - 3$. Thus (3.2) and (3.3) give

$$\mathcal{G}_{(D(m,-),l,+)} \subset \mathcal{P}_{(F,m-1,-)} \Leftrightarrow 3^{l+2} > 2 \cdot 3^m(3^{k+2} - 3) + 2 \Leftrightarrow l \geq m + k + 1.$$

$$\mathcal{G}_{(D(m,+),l,+)} \subset \mathcal{P}_{(F,m,+)} \Leftrightarrow 3^{l+1} > 2 \cdot 3^{m+1}(3^{k+2} - 2) - 4 \Leftrightarrow l \geq m + k + 3.$$

The two conditions above determine those gaps $\mathcal{G}_{(D,l,\pm)}$ with

$$\mathcal{G}_{(D,l,\pm)} \cap P\big((F \cap \mathcal{R}) \times (F \cap \mathcal{R}) \big) = \emptyset.$$

But such gaps are subsets of the product sets $\mathcal{Q}_{(F,m,\pm)}$, which can have non-empty intersection with $P\big((\mathcal{R} \times \mathcal{R}) \setminus (F \times F) \big)$. Therefore in the family of non-covered 3-rd generation gaps arising from $F = F_k$ some values of m are excluded because of a configuration which is not treated by Lemma 3.1. The next Lemma 3.2 determines the relevant values of m.

Lemma 3.2. *The following holds.*

(1) *For any $m \geq k + 3$ we have*

$$\mathcal{Q}_{(F,m,+)} \subset \mathcal{P}_{(I,k,+)}.$$

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(2) *On the other hand*

$$\Big(\bigcup_{m=1}^{\infty} \mathcal{Q}_{(F,m,-)} \cup \bigcup_{m=1}^{k+2} \mathcal{Q}_{(F,m,+)} \Big) \cap P\big((\mathcal{R} \times \mathcal{R}) \setminus (F \times F)\big) = \emptyset.$$

Proof. Part (1) follows because $R(k) = k + 2$, so that for $m \geq k + 3$ we have

$$\inf \mathcal{G}_{(F,k+1,+)} < \inf \mathcal{P}_{(I,k,+)} < \inf \mathcal{G}_{(F,k+2,+)} < \inf \mathcal{Q}_{(F,k+3,+)} \leq \inf \mathcal{Q}_{(F,m,+)},$$

where the second to last inequality follows from Remark 2.5. In order to prove Part (2), we observe that $\mathcal{Q}_{(F,k+2,+)} \cap \mathcal{P}_{(I,k,+)} = \emptyset$, indeed using (2.6) as in § 3.3 we get

$$\sup \mathcal{Q}_{(F,k+2,+)} < \inf \mathcal{P}_{(I,k,+)} \Leftrightarrow A_F(2)A_F(0) > A_F\Big(1 - \frac{2}{3^{k+3}}\Big)^2 \Leftrightarrow$$

$$\Big(\frac{3^{k+3} - 2}{3^{k+3}} \Big)^2 < \frac{A_F(0)}{|F|} \frac{4}{3^{k+3}} \Leftrightarrow 8 + \frac{4}{3^{k+3}} < 3^{k+2},$$

which is true for any $k \geq 1$. Part (2) follows because for any $m \geq 1$ we have

$$\sup \mathcal{P}_{(I,k-1,+)} < \inf P(F \times F) < \inf \mathcal{Q}_{(F,m,-)} < \sup \mathcal{Q}_{(F,k+2,+)}.$$

$\square$

Let β_k be the total measure of non-covered gaps generated by intervals $D(m, \pm) \in \mathcal{E}(\mathcal{D}_F)$, where $F = F_k$ and $m \geq 1$. The discussion above and (3.6) imply

$$\beta_k = \sum_{m=1}^{\infty} \frac{|D(m,-)|^2}{8} \Big(\frac{10}{9} - \frac{1}{9^{m+k+1}} \Big) + \sum_{m=1}^{k+2} \frac{|D(m,+)|^2}{8} \Big(\frac{10}{9} - \frac{1}{9^{m+k+3}} \Big)$$

$$= \frac{1}{8} \sum_{m=1}^{\infty} \frac{1}{9^{k+m+3}} \Big(\frac{10}{9} - \frac{1}{9^{m+k+1}} \Big) + \frac{1}{8} \sum_{m=1}^{k+2} \frac{1}{9^{k+m+3}} \Big(\frac{10}{9} - \frac{1}{9^{m+k+3}} \Big)$$

$$= \frac{1}{64} \Big(\frac{20}{9^{k+4}} - \frac{91/5}{9^{2k+6}} + \frac{1/10}{9^{4k+10}} \Big).$$

Now fix $k \geq 1$ and $E = E_k$. For $m \geq 1$ let $D(m, \pm) := D_{(E,m,\pm)}$ be the intervals arising from the subdivision of E at the 2-nd step. At the 3-rd step any $D = D(m, \pm)$ generates gaps $\mathcal{G}_{(D,l,\pm)}$ with $l \geq 0$. We have $A_E(0)/|E| = 2 \cdot 3^{k+1} + 2$. Thus (3.2) and (3.3) give

$$\mathcal{G}_{(D(m,-),l,+))} \subset \mathcal{P}_{(F,m-1,-)} \Leftrightarrow 3^{l+2} > 2 \cdot 3^m (2 \cdot 3^{k+1} + 2) + 2 \Leftrightarrow l \geq m + k + 1.$$

$$\mathcal{G}_{(D(m,+),l,+))} \subset \mathcal{P}_{(F,m,+)} \Leftrightarrow 3^{l+1} > 2 \cdot 3^{m+1}(2 \cdot 3^{k+1} + 3) - 4 \Leftrightarrow l \geq m + k + 3.$$

The two conditions above determine those gaps $\mathcal{G}_{(D,l,\pm)}$ with

$$\mathcal{G}_{(D,l,\pm)} \cap P\big((E \cap \mathcal{R}) \times (E \cap \mathcal{R})\big) = \emptyset.$$

But such gaps are subsets of the product sets $\mathcal{Q}_{(E,m,\pm)}$, which can have non-empty intersection with $P\big((\mathcal{R} \times \mathcal{R}) \setminus (E \times E)\big)$. Therefore in the family of non-covered 3-rd generation gaps arising from $E = E_k$ some values of m are excluded, because of a configuration which is not treated by Lemma 3.1. The next Lemma 3.3 determines the relevant values of m.

Lemma 3.3. *The following holds.*

(1) *For any $m \geq k + 1$ we have*

$$\mathcal{Q}_{(E,m,+)} \subset \mathcal{P}_{(I,k-1,-)},$$

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(2) *For $m = k$ we have*

$$\mathcal{Q}_{(E,k,+)} \cap \mathcal{P}_{(I,k-1,-)} = \emptyset \ \text{if } k \geq 2$$
$$\mathcal{Q}_{(E,k,+)} \cap \mathcal{P}_{(I,k-1,-)} \neq \emptyset \ \text{if } k = 1.$$

(3) *Finally*

$$\left(\bigcup_{m=1}^{\infty} \mathcal{Q}_{(E,m,-)} \cup \bigcup_{m=1}^{k-1} \mathcal{Q}_{(E,m,+)} \right) \cap P\left(\big((I \cap \mathcal{R}) \times (I \cap \mathcal{R}) \big) \setminus E \times E \right) = \emptyset,$$

where of course $\bigcup_{m=1}^{k-1} \mathcal{Q}_{(E,m,+)} = \emptyset$ for $k = 1$.

Proof. Part (1) follows because $L(k) = k$, so that for $m \geq k + 1$ we have

$$\inf \mathcal{G}_{(E,k-1,+)} < \inf \mathcal{P}_{(I,k-1,-)} < \inf \mathcal{G}_{(E,k,+)} < \inf \mathcal{Q}_{(E,k+1,+)} \leq \inf \mathcal{Q}_{(E,m,+)},$$

where the second to last inequality follows from Remark 2.5. Part (3) follows because for any $m \geq 1$ we have

$$\sup \mathcal{P}_{(I,k,-)} < \inf P(E \times E) < \inf \mathcal{Q}_{(E,m,-)} < \sup \mathcal{Q}_{(E,k-1,+)} < \inf \mathcal{Q}_{(E,k,+)} < \inf \mathcal{P}_{(I,k-1,-)},$$

where the last inequality follows observing that applying (2.6) as in § 3.3 we get

$$\inf \mathcal{Q}_{(E,k,+)} < \inf \mathcal{P}_{(I,k-1,-)} \Leftrightarrow A_E(-2)A_E(4) > A_E\left(1 - \frac{1}{3^k} \right)^2 \Leftrightarrow$$

$$\left(\frac{3^k - 1}{3^k} \right)^2 + 8 < \frac{A_E(0)}{|E|} \frac{2}{3^k} \Leftrightarrow \frac{1}{3^k} < 3^{k+1} + 6,$$

which is true for any $k \geq 1$. In order to prove Part (2) we apply again (2.6) as above and we get

$$\sup \mathcal{Q}_{(E,k,+)} < \inf \mathcal{P}_{(I,k-1,-)} \Leftrightarrow A_E(-2)A_E(4) > A_E\left(1 - \frac{2}{3^{k+1}} \right)^2 \Leftrightarrow$$

$$\left(\frac{3^{k+1} - 2}{3^{k+1}} \right)^2 + 8 < \frac{A_E(0)}{|E|} \frac{4}{3^{k+1}} \Leftrightarrow 3^{k+1} + \frac{4}{3^{k+1}} < 12,$$

which is true for $k = 1$ and false for $k \geq 2$. $\qquad \square$

Recall that in our notation $E = E_k$ and $\mathcal{E}(\mathcal{D}_E) = \{ D(m, \pm) = D_{(E,m,\pm)} : m \geq 1 \}$. In particular $D(k, +) = D_{(E_k,k,+)} \subset E_k$. Let α_k be the total measure of non covered gaps generated by intervals $D(m, \pm) \in \mathcal{E}(\mathcal{D}_E) \setminus \{ D(k, +) \}$. The discussion above and (3.6) give

$$\alpha_k = \sum_{m=1}^{\infty} \frac{|D(m, -)|^2}{8} \left(\frac{10}{9} - \frac{1}{9^{m+k+1}} \right) + \sum_{m=1}^{k-1} \frac{|D(m, +)|^2}{8} \left(\frac{10}{9} - \frac{1}{9^{m+k+3}} \right)$$

$$= \beta_k - \sum_{m=k}^{k+2} \frac{|D(m, +)|^2}{8} \left(\frac{10}{9} - \frac{1}{9^{m+k+3}} \right) = \beta_k - \frac{1}{8} \sum_{j=3,4,5} \frac{1}{9^{2k+j}} \left(\frac{10}{9} - \frac{1}{9^{2k+j}} \right).$$

Let γ_k be the total measure of non covered gaps generated only by $D(k, +)$. For $k = 1$ we have $\mathcal{Q}_{(E,1,+)} \cap \mathcal{P}_{(I,0,-)} = \emptyset$. Then applying (3.6) with $N = k + m + 3$ for $m = k = 1$ we get

$$\gamma_1 = \frac{|D(1, +)|^2}{8} \left(\frac{10}{9} - \frac{1}{9^{2+3}} \right) = \frac{5}{4 \cdot 9^6} - \frac{1}{8 \cdot 9^{10}}.$$

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Conversely $\mathcal{Q}_{(E,k,+)} \cap \mathcal{P}_{(I,k-1,-)} \neq \emptyset$ for $k \geq 2$. Thus $\mathcal{P}_{(I,k-1,-)}$ cover some gaps arising from $D(k,+)$ and we can only give an upper bound on γ_k using (2.7). We obtain

$$0 \leq \mu_3 - \left(\mu_2 + \gamma_1 + \sum_{k \geq 1}(\beta_k + \alpha_k)\right) \leq \sum_{k \geq 2} \gamma_k \leq \sum_{k \geq 2} \mathcal{L}(\mathcal{G}_{D(k,+)})$$
$$= \sum_{k \geq 2} \frac{5}{36}\frac{1}{9^{2k+3}} = \frac{1}{64 \cdot 9^6}.$$

We have

$$\sum_{k \geq 1}\beta_k + \alpha_k = \frac{1}{8}\sum_{k=1}^{\infty}\frac{5}{9^{k+4}} - \frac{910}{9^{2k+6}} - \frac{91/20}{9^{2k+6}} + \frac{1/40}{9^{4k+10}} + \frac{6643}{9^{4k+10}} = \frac{157}{32 \cdot 9^6} - \epsilon$$

where

$$\epsilon := \frac{1}{8}\sum_{k \geq 1}\frac{91/20}{9^{2k+6}} - \frac{1/40}{9^{4k+10}} - \frac{6643}{9^{4k+10}} \in \left(0, \frac{1}{128 \cdot 9^6}\right).$$

Set $M := \mu_2 + \gamma_1 + 157/(32 \cdot 9^6)$, so that

$$\frac{5}{9} - M = \frac{5}{9} - \left(\frac{5}{4 \cdot 81} + \frac{859}{9^4 \cdot 5 \cdot 64} + \frac{5}{4 \cdot 9^6} - \frac{1}{8 \cdot 9^{10}} + \frac{157}{32 \cdot 9^6}\right) = \frac{91782451}{170061120} + \frac{1}{8 \cdot 9^{10}}.$$

We have $\mathcal{L}(P(\mathcal{D}_3 \times \mathcal{D}_3)) = 5/9 - \mu_3$. Therefore

$$\frac{-1}{64 \cdot 9^6} < \frac{-1}{64 \cdot 9^6} + \frac{1}{8 \cdot 9^{10}} + \epsilon \leq \mathcal{L}\big(P(\mathcal{D}_3 \times \mathcal{D}_3)\big) - \frac{91782451}{170061120} \leq \frac{1}{8 \cdot 9^{10}} + \epsilon < \frac{1}{64 \cdot 9^6}.$$

Proposition 1.3 is proved.

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