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On Vaulting: Heron's Manuals and Their Role in Roman Dome Design

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Abstract

The subject of domes in the Roman world is complex and can be studied from different perspectives. In this paper we focus on the relationship between Heron of Alexandria's manuals and the vaulting systems of the Hadrianic age. Our aim is to compare a selection of formulae from the critical edition by Johan Ludvig Heiberg with a series of buildings recently documented using photogrammetric and laser scanner technologies. The collection of writings *Heronis Alexandrini opera quae supersunt omnia* (mainly books IV and V) presents an interesting set of formulae for calculating vaults and domes: volumes and areas of niches, spherical segments, lunettes, as well as empirical strategies for calculating complex shapes. This approach, which integrates practical knowledge with Vitruvian graphic schemes, allows us to clarify the work of the ancient architect and consequently to investigate the architectural problem within the more general framework of archaeology with new conceptual tools.

Keywords Design analysis · Design theory · Historical treatise · Heron of Alexandria · Cupolas

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Heron's Legacy

Under the principate of Nero, in the second half of the first century BC, lived the mathematician and inventor Heron who operated in Alexandria, Egypt, as a technical subject teacher at the Alexandria Museum (Boyer 1991). His role in the coeval architectural debate represents a relatively new research topic, due to the serious absence of an up-to-date critical edition of the corpus of his works.

To date, the translation and commentary of the Alexandrian mathematician's texts by Heiberg (1900, 1914a, b) from the early twentieth century—as it was conceived by the Danish philologist—has been inadequate for effective dissemination to a wide and multidisciplinary audience.

Among the scholarly public, his studies in the fields of hydraulics and complex machine design are best known, but in the field of architecture he is a lesser-known author.

But in the *Heronis Alexandrini opera quae supersunt omnia*, many formulas deal with the calculation of surfaces and volumes directly related to architectural issues. These formulas explicitly relate to the computation of architectural elements, in particular domes and vaulted spaces. From the first century onwards, structural elements in Roman architectural were improved for better performance, reaching an unprecedented technical and compositional peak under Hadrianic principate (117–138 AD).

These considerations are the starting point for the present text, which relates digitally surveyed buildings to the formulas found in the Heronian corpus. The aim is to introduce important lexical elements concerning domes, and finally to fit irrational constructions (executed with ruler and compass) with the formulas that have arrived to us thanks to the patient work of copyists and philologists.

It is safe to say, that the architects of the three great emperors (Nero, Domitian, Hadrian) led the Roman urban and architectural renewal that took place in the first and second centuries. They had access to technical manuals or handbooks (Vitruvius) containing Heron's formulas for the establishing the quantity and, probably, the stability of buildings. In a span of less than a century, Roman architecture acquired an increasing originality and identity of its own, thanks to the use of economic, technological, and human resources that only the emperor could provide (Ward-Perkins 1974; Opper 2008). In the imperial court, in addition to technical figures such as architects and engineers, there were also other individuals who contributed to the development of these 'new' architectures such as mathematicians and geometers, like Heron.

The figures of Nero, Domitian, and Hadrian are inextricably tied to Greek art and culture and, as patrons, to the major architectural achievements of the time. In this context, Domes played an increasingly prominent role in the main archaeological sites of Latium and Campania.[1]

[1] The bibliography on the subject is extensive; a summary is present in Roca et al. (2023). For recent research on Hadrianic vaulted systems see Ottati (2017) and Eramo (2023).

We would like to mention a few examples. Severus and Celer were the authors of the Neronian Domus on the Oppian Hill with its very thin composite vault that originates from an octagonal impost in the form of a pavilion. Then it converts into a sphere approximated by skillful fitting with triangular surfaces (Fig. 1).[2]

On the nearby Palatine Hill, Rabirius made the twin octagonal halls (Fig. 2), covered with a cloister vault on the sides of which were alternately curved and rectangular niches for the Flavians (Ward-Perkins 1974: 70–73).

With Hadrian we have the widest experimentation on domes inserted in the most varied contexts. Thermal baths, scenographic triclinia, temples, and nymphaea (Viscogliosi 2020: 11–36; Cipriani et al. 2017: 431–434) are the places dedicated to the experimentation of innovative vaulted systems.

For Hadrianic Pantheon, the attribution to Apollodorus of Damascus still remains uncertain, although we know that the Nabatean architect had an active role for a good part of Hadrian's principate together with Decrianus (Dominic 2008).

The inspirational role played by the cultured and refined personalities of the patrons in each of the aforementioned examples is evident. It is equally clear that their architecture of the centrally planned halls reinterpreted ephemeral structures such as tents and pavilions (Calandra 2013). Naturalistic metaphors, being quotations from nature, are also present in the term 'gourds' adopted by Apollodorus,[3] in a derogatory way, to refer to the new domes designed by Hadrian himself.

This theatricalization of the architecture to express the pompous power of the *princeps* also required from architects an ever-increasing mastery of the *opus caementicium* characteristics, taking to the edge structural limits, as is the case in the Small Baths of Hadrian's Villa.

Related to the applicability of Heron's formulas, yet less considered in construction, is the cost of the building materials—often very expensive—used to build both the structure of the domes and their cladding (in general mosaic and glass paste for the intrados, lead, or copper for the extrados). The importance of accurately quantifying the volume of construction materials was relevant for the economic aspect of the project (*distributio*) and for the logistics of the construction site (supplies, workdays of skilled workers, etc.).

Finally, another matter should be considered. The domes and the subordinate surrounding halls were expensive because they encompassed many diversified functions related to entertainment and comfort: water features, dramatic use of natural light, ingenious mechanisms capable of generating movement in unexpected ways (Fig. 3).

[2] Roca et al. (2023) and Cipriani et al. (2017) presented the bibliography on the relationship between the architecture of Nero and Heron as well as useful graphic diagrams for the interpretation of the famous dome on the Oppian Hill.

[3] On the well-known episode of the diatribe between the young Hadrian and Apollodorus (Cassius Dio 69, 4, 1–2) and for the derogatory formal assimilation of domes with 'pumpkins' see Brown (1964).

Recent Studies

Part of the inexhaustible interest in the subject of domes in the ancient world can be attributed to the wave of surveys carried out with advanced devices since the late 90s (Gaiani et al. 2000), and to more recent restorations that have uncovered valuable clues to help understand both the original image and the structural philosophy of some of the most complex ancient cupolas (Adembri et al. 2021; Svenshon 2013) (Fig. 4).

Today, however, it is necessary to resume the reflection on domes and to revise the *status quaestionis* by adding different novelties to the body of technical knowledge on ancient design that, in recent decades, have been brought to the attention of the scientific community.

Measurements or number of modules systematically include the number 7 as diameters of centric compositions, as underlined by Fuchs' studies (2022, 2023). This result is consistent with Heron's writings, whose importance has already been highlighted by Conti and Martines (2010), who count among the initiators of this research method in architecture based on the examination of ancient technical literary sources. To these, we can add studies on geometry and construction of Byzantine domed architecture by Svenshon (2009). Here, too, the formulas traditionally held to be those of Heron provide important insight for planimetric dimensioning as well as for the calculation of building elements, with particular emphasis on numerical progressions based on $\sqrt{2}$, what we famously call '*ad quadratum*' construction.

Although archaeological finds representing architecture with a technical communication purpose (models, engravings, drawings) are very few (Azara (1997), common features can be found in them. In particular, we can recognize:

1. rigorous recourse to modularity;
2. intentionality in researching equivalence of areas in layout elements (Bianchini and Fantini 2015: 32–39);
3. patterns for drawing regular polygons.

The architect had at his disposal non-trivial geometric constructions as it emerges from the rare—but for us substantial—drawing analyzed by Savvides (2021) in his study of the Octagon of the Palace of Galerius in Thessaloniki. The author provides archaeological evidence for the use of patterns based on circles and squares, relating the building's plan to the design engraved on the underside of a marble slab (61 × 134 cm) found at the site.

The written sources that make it possible to analyze architectural design consistently with the line of studies just mentioned are limited. The work by Vitruvius prevails over them. Hints of architectural design do trace to the writings of the Gromatici Veteres[4] but without sufficient depth. The *De Architectura* must therefore be supplemented with a range of knowledge intentionally 'avoided' by the Latin author, because it cannot reconcile with the ennobling purpose of an

[4] The study on the Colosseum *ichnographia* by Docci (1999: 23–31) still is a clear example on how to relate surveying to technical manualism.

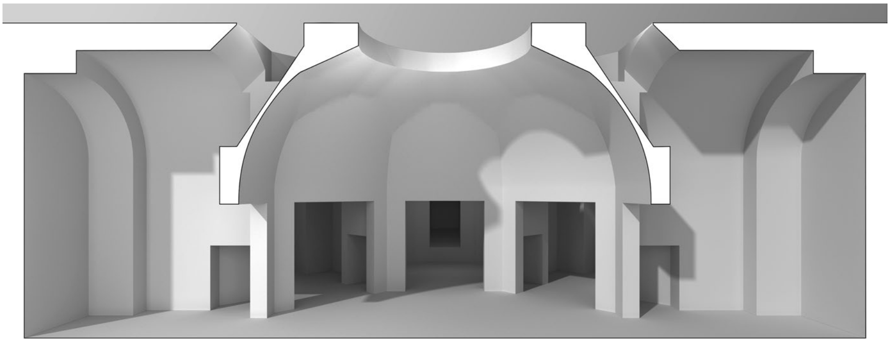

Fig. 1 The octagonal hall by Severus and Celer at Domus Aurea in Rome

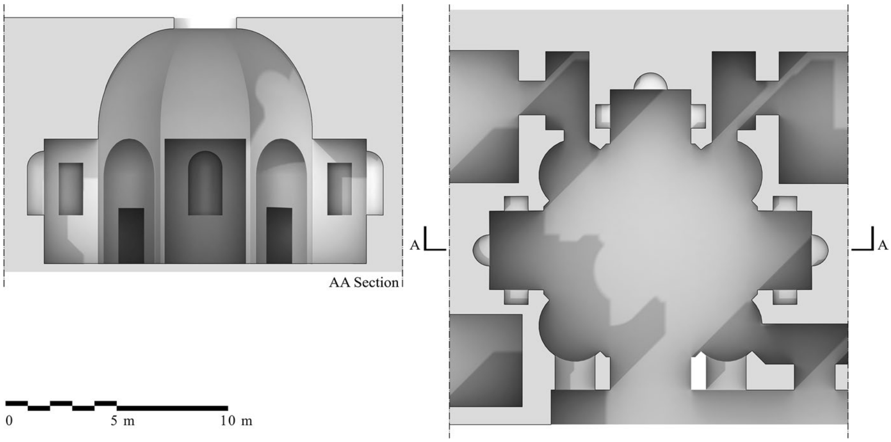

Fig. 2 The octagonal hall by Rabirius at Domus Augustana in Rome

eminently technical, and therefore inferior, knowledge to the *artes liberals* (Gross 1997:12–13, 64–65). It is likely that for this very reason, Vitruvius, while explaining the ichnographical construction of the Latin Theater, omits the formulas for *cavea* dimensioning that Heron mentioned instead. What we certainly know from the Latin author, however, is that the relationship between orchestra and theatrical building was regulated through the well-known geometric scheme based on the circle and inscribed triangles (or squares) (Salvatore 2007; Lara Ortega 1992). This construction, according to Vitruvius, defines dimensional limits, positions, and distributions of the various zones of the theater building, but it is also closely related to the modular structure that conceptually upholds the building: indeed, the diameter, divided by 6, provides the width of the *parodoi* (Vitruvius 2009: 139–141).

In the extensive literature on the geometric schemes used to design centrally planned buildings, it is particularly useful to mention Écochard (1977), who studied

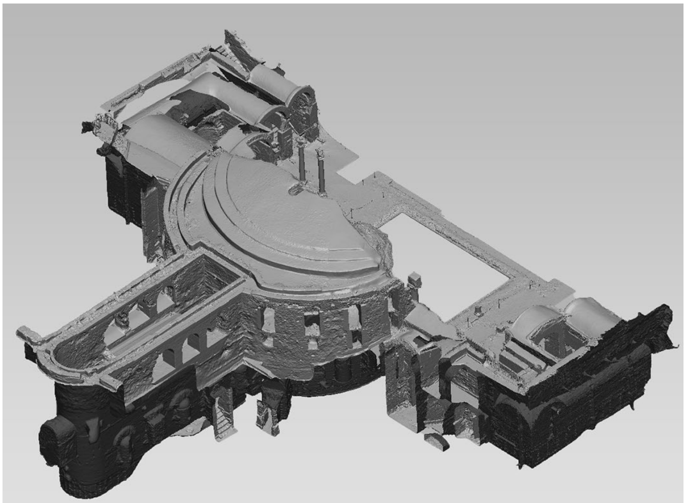

Fig. 3 Hadrian's Villa Serapeum in Tivoli, Rome showing a triclinium with a system for water supply and storage (on the extrados of the cupola) aimed for scenographic purposes

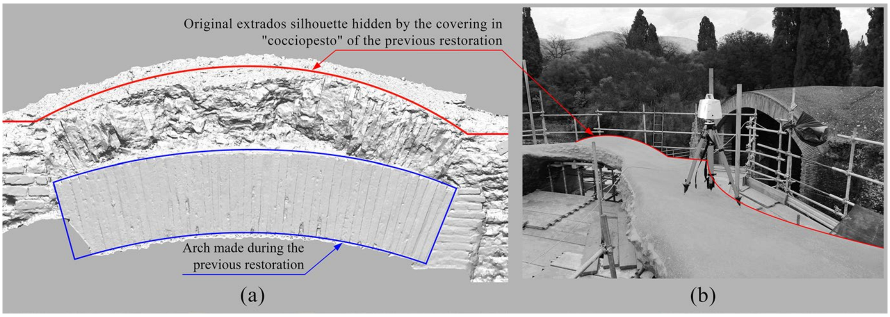

Fig. 4 Small Baths in Tivoli, Rome, **a** external silhouette of one of the four lunettes of the octagonal hall of (photogrammetric survey and modeling Gianna Bertacchi); **b** 'as built' surveying campaign in which the restored mixtilinear shape of the extrados is evident

the arrangement of octagonal floor plans of buildings such as the basilica of San Vitale in Ravenna and the Mosque of Omar in Jerusalem (The Dome of the Rock). The author finds compositional similarities despite their different origins and locations. Subsequently, many authors have analyzed the geometry of octagonal buildings of antiquity from patterns based on circles and squares.[5]

[5] The vestibule of the Golden Court presents several similarities with the early Christian chapel of Sant'Aquilino in the Basilica of San Lorenzo, Milan. Regarding geometric analysis of the layout Soler Sanz (2008). On the chapel see the recent in-depth study following the restorations carried out by the Soprintendenza: Ranaldi 2021.

Unlike Vitruvius, Heron's objective is the dissemination by means of 'dry,' applied handbooks with a minimal theoretical introduction, aimed for calculations and quantitative verifications (*distributio*). Certainly, Severus and Celerus were in possession of such texts, which they must have employed in their own creations, as did Apollodorus of Damascus, a few decades later in the context of the great Trajanic construction sites.

The Nabataean architect, though opposed by Hadrian, remained in his service at least in the early years of the principate, also collaborating with Decrianus, an engineer or architect who was responsible for the transportation of the colossal statue of Nero.[6] But the news that we would like to emphasize here appear at later times, coming from Eutocius of Ascalon (sixth century CE), who, while Isidore of Miletus and Antemius of Tralle were building/builded Hagia Sophia in Constantinople, wrote a commentary on Archimedes' text 'On the Sphere and Cylinder.' As is well known, Eutocius, addressing the second book of the Syracusan mathematician's work, refers to a text on vaults written by Heron (Cipriani et al. 2017).

In philological circles, however, the possibility that the Greek author really dedicated a real manual for *magistri et machinatores* to the subject is not unanimously shared (Papadopoulos 2007). Thanks to the work of the Danish philologist Johan Ludvig Heiberg (1854–1928), we have an account of a significant part of the body of scientific knowledge applicable to the construction of domes and vaulted spaces, referable to the Alexandrian period headed by Heron.

Of great interest for this purpose are Books IV and V of the German edition, edited by Heiberg (1914a, b) who, through patient work on the codices identifies the origin of each calculus, justifies its inclusion and compares his collection with other editions (Hultsch 1864). Here we consider Heiberg's edition as consisting of texts by Heron although, from a philological point of view, an aporia still exists.[7]

Definitiones and *Geometrica*—both contained in Book IV—are oriented to the study of plane figures. Book V contains *Stereometrica* (*Stereometrica I* and *Stereometrica II)* and *De Mensuris*. For our topic, Book V results of particular interest. It is a series of formulas and procedures for calculating areas of plane figures, surfaces, and volumes of three-dimensional figures (spheres, quarter spheres, spherical caps, shells), conical or pyramidal shapes. Among these, what Heron calls *trikentron*,[8] is a geometric figure defined by 'three centers,' that is, by three arcs of a circle assimilated as a spindle or fingernail, typical of vaulted spaces (Fig. 5).

[6] The quarrel that caused the death of Apollodorus seems to be placed at the time of the construction of the Temple of Venus and Rome, in the eastern part of the Forum (Rathbone 2008: 74–75).

[7] From here on Heiberg's texts and translations are selected as belonging to the Greek mathematician. However, uncertainties remain due to the lack of formal consistency among the various writings traditionally attributed to Heron (see *Praefatio* to Books IV and V).

[8] Can be defined as a double curvature surface enclosed by three curves, apparently three circles with centres belonging to different vertical planes.

Heron's Formulas on Vaults and Domes

Heiberg's (1914a, b) collation of Heron's books IV and V presents several versions of the same calculation; here we will present only one of some representative examples. When viewed with a contemporary gaze, the calculations are not optimized: They adopt an atypical formalization, but they make explicit the purpose for which these handbooks were written. It is evident how all the calculations are based on measurable or externally obtainable parameters. For example, the area of a triangle is obtained from the lengths of its sides (Heron's formula), just as the area of a circle is derived from the length of the circumference. The approximate value of π used in the formulas is $\frac{22}{7}$.

As mentioned in Roca et al. (2023), for the calculation of volumes and surfaces of building elements Heron uses three different approaches: subtraction (i.e., intrados minus extrados), calculation of an 'average' surfaces, halfway between the intrados and extrados to be multiplied by a thickness to obtain the volume, application of regularizable elements to be applied upon complex morphologies, then unwrapped, and measured.

In what follows we present some calculations from Book V. Our aim is to show how these calculations were accomplished. Only in the examples *Stereometrica I*, 59 and 60, to illustrate the flavor of Heron's writings, we include the entire translation of the items. The numbers refer to the sequence we find in Heiberg (1914a, b). Let's examine some transcribed examples from *Stereometrica I*[9]:

1. Calculation of the volume of a sphere (V_s), related to the volume of the cylinder (V_c) in which it is inscribed (Fig. 6a). Reference is made to the fact that this relationship appears in Archimedes (*On the sphere and the cylinder*, I, corollary 34). The calculations are done in the following form:

- Diameter of the sphere (d) = 10 *pedes*.
- The cylinder has a base equal to the great circle of the sphere.
- The height (h) is equal to the diameter.[10]

$$V_c = 10 \times 10 \times 11 \times \frac{1}{14} \times 10 \quad \text{(H)}$$

$$V_s = V_c \times \left(\frac{1}{2} + \frac{1}{6}\right) \quad \text{(H)}$$

Where[11] $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, then $V_s = \frac{2}{3} \times V_C$

[9] Heron's formalization is followed by the letter (H), the contemporary one by (C).

[10] $V_c = \pi \times \left(\frac{d}{2}\right)^2 \times h = \frac{22}{7} \times \left(\frac{10}{2}\right)^2 \times 10 = \frac{11}{14} \times 10^3 \quad \text{(C)}$

[11] $V_c \times \frac{2}{3} = d^3 \times \frac{11}{14} \times \frac{2}{3} = \left(\frac{d}{2}\right)^3 \times 4 \times \pi \times \frac{1}{3} = V_s \quad \text{(C)}$

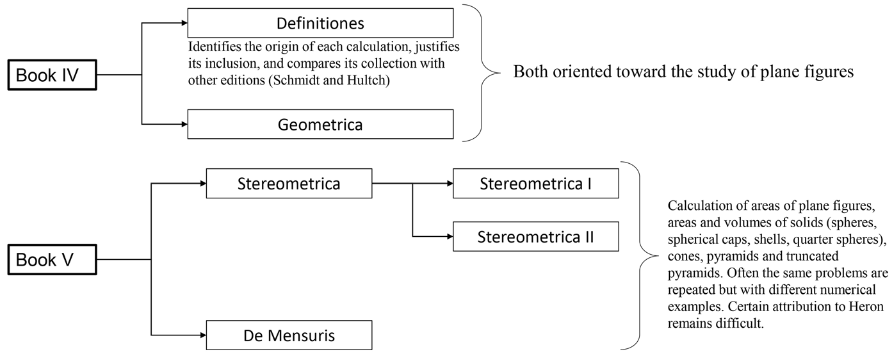

Fig. 5 Heiberg's arrangement of Heron's formulas in book IV and V of *Heronis Alexandrini, Opera quae supersunt omnia*

$$\mathrm{V_s} = \mathrm{V_c} \times \frac{2}{3} = 10 \times 10 \times 11 \times \frac{1}{14} \times 10 \times \frac{2}{3} = 523 + \frac{17}{21} \quad \text{(H)}$$

Accordingly, it is proved that: $11 \times \mathrm{d}^3 = 21 \times \mathrm{V_s}$

2. Calculation of the volume and area of a sphere, knowing the great circle (Fig. 6b).
Circumference (great circle) of the sphere = 22 *pedes*.
To find its volume[12]:
$22 = \pi \times \mathrm{d};\, \mathrm{d} = \frac{7}{22} \times 22; \mathrm{d} = 7$ (diameter)

$$V_s = 7 \times 7 \times 7 \times 11 \times \frac{1}{21} = 179 + \frac{2}{3} \quad \text{(H)}$$

Area of the sphere[13]:

$$A_s = 2 \times 7 \times 11 = 154 \quad \text{(H)}$$

19. Calculation of the volume (Vc) of a cylinder with:

- Diameter (d) = 7 *pedes*.
- Height (h) = 50 *pedes*.
- Circumference $\mathrm{L_{circ}}$ = 22 *pedes*.

From the circumference we find the surface (of the base circle) ($\mathrm{A_b}$), as of a circle (Book IV, Geometrica, 17):[14]

12 $V_s = \frac{4}{3} \times \pi \times r^3 = \frac{11}{21} \mathrm{x} d^3 = 179{,}67 \quad \text{(C)}$

13 $\mathrm{A_s} = 4 \times \pi \times \left(\frac{\mathrm{d}}{2}\right)^2 = 4 \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} = 22 \times 7 = 154 \quad \text{(C)}$

14 $\mathrm{A_b} = \pi \times \left(\frac{\mathrm{d}}{2}\right)^2 = \pi \times \mathrm{d} \times \frac{\mathrm{d}}{4} = \mathrm{L_{circ.}} \times \frac{\mathrm{d}}{4} \quad \text{(C)}$

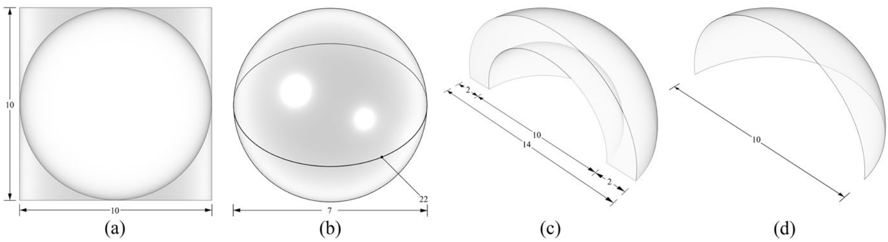

Fig. 6 Stereometrica I: **a** exercise 1; **b** Stereometrica I, exercise 2; **c** exercise 59 and 60; **d** exercise 61

$$A_b = L_{circ.} \times \frac{d}{4} = 22 \times \frac{7}{4} = 38 + \frac{1}{2} \quad (H)$$

then

$$V_c = A_b x h = (38 + \frac{1}{2}) \times 50 = 1925 pedes \quad (H)$$

is the volume of the cylinder.

20. Calculation of the volume (V_c) and area (A_c) of a cylinder with:

- Diameter (d) = 6 *pedes*
- Height (h) = 12 *pedes*.

The calculation of volume is done as before:
Base circle area (Ab)[15]:

$$A_b = L_{circ.} \times \frac{d}{4} = \pi \times d \times \frac{d}{4} = \frac{22}{7} \times 6 \times \frac{6}{4} = 28 + \frac{1}{4} + \frac{1}{28} \quad (H)$$

Cylinder volume (V_c)[16]:

$$V_c = A_b \times h = \left(28 + \frac{1}{4} + \frac{1}{28}\right) \times 12 = 339 + \frac{4}{7} \quad (H)$$

Cylinder surface (curved surface) (Sc),[17]:

$$S_c = L_{circ.} \times h = \left(3 + \frac{1}{7}\right) \times d \times h = \frac{22}{7} \times d \times h = \frac{22}{7} \times 6 \times 12 \quad (H)$$

56. Volume of a semi-sphere (V_{ss}):

15 $A_b = \pi \times \left(\frac{d}{2}\right)^2 = 28,2743 \quad (C)$

16 $V_c = \pi \times \left(\frac{d}{4}\right)^2 = 339,4286 \quad (C)$

17 $S_c = 2 \times \pi \times \frac{d}{2} \times h = \pi \times d \times h = \pi \times 6 \times 12 = 226,1947 \quad (C)$

(The volume of) a semi-sphere is computed as (that of) a sphere, but the result is divided by 42.

- Diameter (d) = 7 *pedes*
- Circumference (L_{circ}) = 22 *pedes*

To find the volume proceed as follows[18]:

$$V_{ss} = 7 \times 7 \times 7 \times \frac{11}{42} \quad \text{(H)}$$

58. Surface area of a semi-sphere (S_{ss})[19]:

- Diameter (d) = 10 *pedes*
- Circumference $L_{circ} = \frac{1}{4} + \frac{1}{7} + \frac{1}{28}$*pedes*

$$S_{ss} = 10 \times \left(31 + \frac{1}{4} + \frac{1}{7} + \frac{1}{28}\right) \times \frac{1}{2} = 157 + \frac{1}{8} + \frac{1}{56} \quad \text{(H)}$$

59. and **60**. Constructive volume (CV) of a niche (1/4 sphere):

Shorter diameter = 10 *pedes*
Wall thickness = 2 *pedes*
Larger diameter = 14 *pedes*

A niche or a quarter of a sphere is computed according to the method of a half-sphere, but dividing the result by 84:

Let us consider a niche with diameter of 14 pedes, including a wall thickness of 2 pedes. Then (the larger niche volume is V_{LN} and the smaller niche volume is V_{SN})[20],

$$V_{LN} = 14 \times 14 \times 14 \times \frac{11}{84} = 359 + \frac{1}{3} \quad \text{(H)}$$

This is the volume of the niche.

To find the volume of the hollow and the volume of the construction compute, according to the same method used to compute (the volume of) the larger niche, without the thickness of the wall[21]:

18 $V_{ss} = \frac{1}{2} \times \frac{4}{3}\pi \times \left(\frac{d}{2}\right)^3 = \frac{1}{2} \times \frac{4}{3} \times \frac{22}{7} \times \frac{7}{8}^3 = 7^3 \times \frac{11}{42}$ (C)

19 $S_{ss} = 2 \times \pi \times \left(\frac{d}{2}\right)^2 = \pi \times \frac{d}{2} \times d = L_{sc} \times d$ (C)

20 $V_{LN} = \frac{1}{3} \times \pi \times \left(\frac{d}{2}\right)^3 = \frac{1}{3} \times \frac{22}{7} \times \left(\frac{14}{2}\right)^3$ (C)

21 $V_{SN} = \frac{1}{3} \times \pi \times \left(\frac{d}{2}\right)^3 = \frac{1}{3} \times \frac{22}{7} \times \left(\frac{10}{2}\right)^3$ (C)

$$V_{SN} = 10 \times 10 \times 10 \times 11/84 = 130 + \frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{28} \quad \text{(H)}$$

This is the hollow of the niche.[22]

$$CV = V_{LN} - V_{SN} = 228 + \frac{1}{4} + \frac{1}{8} \quad \text{(H)}$$

This is the quantity of feet of the construction.

61. Surface of a niche (S_n) (Fig. 6d):

– Diameter (d) = 10 *pedes*:

To measure the surface of the same niche (smaller niche in item 60)[23]:

$$S_n = 10 \times 10 \times \frac{11}{14} = 78 + \frac{1}{2} + \frac{1}{14}\text{(H)}$$

71. Volume of spherical segment (V_{ss}) (Fig. 7a), smaller than a half-sphere:

– Diameter (d) = 12 *pedes*
– Height (h) = 4 *pedes*.

To find the volume I do the following[24]:

$$V_{ss} = \left[\left(\frac{1}{2} \times 12\right) \times \left(\frac{1}{2} \times 12\right) \times 3 + 4^2\right] \times 4 \times \frac{11}{21} = 259 + \frac{2}{3} + \frac{1}{7} \quad \text{(H)}$$

This example is particularly interesting in the context of Hadrian's architecture because of the obvious proportionality between Hero's numerical example and the section of the Pantheon. The spherical segment rising from the circular drum of the building (the extrados) results in a 1:11 ratio with the Alexandrian mathematician's example.

96. Surface area of a *trikentron* (ST) (Fig. 7b):

– Base (b) = 8 *pedes*
– Height (h) = 9 *pedes*

This figure is calculated as the area of a triangle ($S_{triangle}$) to which a 'supplement' is added to approximate the three-dimensionality of the surface.

22 $CV = \frac{1}{3} \times \pi \times \left(7^3 - 5^3\right) = 228,2891 \quad \text{(C)}$

23 $S_n = \frac{1}{2} \times S_{ss} = \frac{1}{2} \times L_{sc} \times d = \frac{1}{2} \times \pi \times d \times \frac{d}{2} = \frac{1}{2} \times \frac{11}{7} \times dxd \quad$ (see formula 58)

24 $V_{ss} = \left[3 \times \left(\frac{d}{2}\right)^2 + h^2\right] \times h \times \frac{\pi}{6} \quad \text{(C)}$

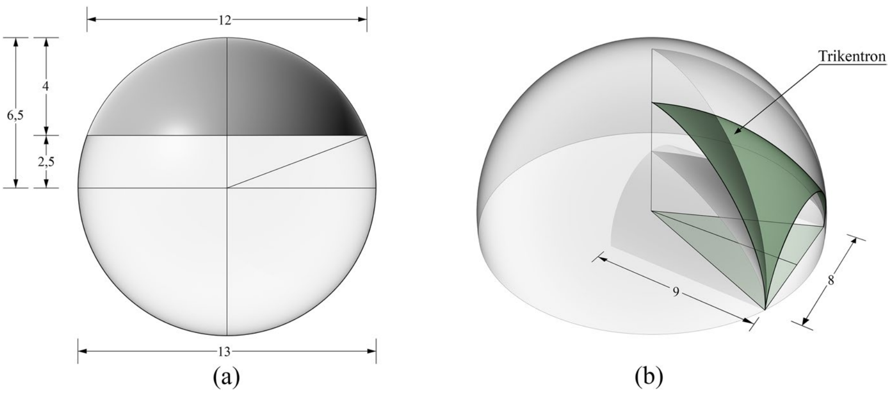

Fig. 7 Stereometrica I: **a** exercise 71; **b** exercise 96

$$S_{\text{triangle}} = b \times h \times \frac{1}{2} = 8 \times 9 \times \frac{1}{2} = 36 \quad \text{(H, C)}$$

$$S_T = 36 + 36 \times \frac{1}{3} = 48 \quad \text{(H)}$$

Through 3D models of Heron's exercise, it is possible to understand how the measurements of the base and height of the triangle of the vaulted surface approximate those of an isosceles triangle forming the octagon inscribed in a circle (b = 7.54 *pedes*, h = 9 *pedes*, area = 34 *pedes constrati*). It is also important to note that the surface of the *trikentron* closely resembles the average between the areas of the base triangle projected onto a sphere and that of a cylindrical groin.

The last example to be commented on is not numerical, but empirical, as it illustrates a method for calculating surfaces that cannot be fitted to primitive geometric shapes such as the sphere and cylinder (Roca et al. 2023). *De Mensuris* 46 discussed the application of regularizable patches such as cloths, animal skins, etc., for developing complex shapes which are then unwrapped and measured. In this case, one might speculate that Heron is alluding to the use of scale models of particularly complex areas to quantify aspects important to the design process.

The Relationship Between Irrational Constructions and *Symmetria*

The previous formulas formed part of the design process as practiced by the ancient architect, but as mentioned above, the problem was also to reconcile the diameters, areas, and volumes of curved shapes, with patterns (*ad quadratum*, Vitruvian scheme for the theatre).

From a theoretical point of view, such patterns existed before the codified technical drawing[25] that aimed at construction, as theoretical geometric schemes. In this sense, the pattern known as 'rotated square' (two squares rotated 45° relative to each other) is one of the oldest patterns used as a graphic algorithm to define the plan.

Such a scheme can be divided into smaller squares generating a modular grid of proportions that can be described by integers or rational numbers, easy to reconcile with the ancient numerical representation.

'Modularity' (Gros 1997: 27–29), functional for the computation of areas and volumes, facilitated the verification of the building requirements, as well as the fulfillment of that ideal of simple proportionality between geometric shapes that Vitruvius and Archimedes referred to as *symmetria* (Cipriani et al 2020: 1047).

In the first book of *De Architectura*, the term is used to describe a quality of the building that can be defined as metric consistency, or proportionality between the parts of the building, which occurs through the designer's calculation of a *koilòn métron* as Heron calls it in the *Definitiones*.

Case Studies

We propose to check to what an extent Heron's formulae were present in three cupolas and, above all, how all these assumptions (rotated square, modularity) were interrelated. We use as case studies singular buildings of Hadrian's Villa that were recently surveyed with digital technologies: the vestibule of Golden Court (Piazza d'Oro); the octagonal hall of Small Baths (Piccole Terme); and the Serapeum-Canopus complex[26]:

- The entire Golden Court complex was designed on a square grid of 5.5 *pedes* modules (11/2), evident in the rhythm of the half-columns leaning against the perimeter walls of the peristyle (Cipriani et al. 2020: 1053).
- The Small Baths complex was built upon a square grid of 5 *pedes* module, as demonstrated in previous studies (Adembri et al. 2021: 336). The 5×5 grid, in addition to generating dimensions of base 10, introduces the 7 in the layouts (the diagonal of the square of side 5 is approximately equal to 7: $5\text{x}\sqrt{2} = 7{,}07$).
- In the case of the Serapeum-Canopus, starting from the interaxle spacing of the north colonnade of the pool (about 14 *pedes*), we can assume a square grid of module 7 *pedes*.

[25] The 'plan' is a graphic scheme preceding the construction that, in some cultural contexts, may present an immaterial form (typologies or models handed down traditionally and evolved through extensive temporal refinement). On the broad topic of the relationship between ancient architecture and forms of representation see Azara (1997).

[26] Which forms part of an ongoing collaboration with the Dipartimento di Ingegneria Civile e Ingegneria Informatica of the Università degli Studi di Roma Tor Vergata.

The layout of the octagonal vestibule at the Golden Court starts from a circle whose diameter is 33 *pedes* and is equal to the side of a square of 6 modules consistent with the general grid (Fig. 8a). A larger circle circumscribed to the first system of rotated squares has a diameter of 46.67 *pedes* ($33 \times \sqrt{2}$) and defines the external boundaries of the system of radial niches characterizing the vestibule (Fig. 8b). The vertices of the inscribed octagon, projected according to the orthogonal directions of the grid, define the vertical planes of the outer faces of the rectangular niches (red color), as well as the planes tangent to the semi-cylinders of the semicircular niches (green color) (Fig. 8c). By extending the sides of the inner squares toward the outer squares we obtain the directions for the alignment of the planes that define the inner walls of the rectangular niches (thickness equal to 2 *pedes*, Fig. 8d). The irrational construction results in a circle with diameter 35.7 pedes (closely approximating a multiple of 7).

Several steps of this construction are almost identical to the one demonstrated by Soler Sanz (2008: 94–95) or the early Christian chapel of San Aquilino in Milan. A singular aspect of the dome is that, compared to the rigorous geometric layout of the plan, it has an intrados which on average has a diameter of 33 *pedes*, like the circle inscribed in the initial square of the *ichnographia*. The reasons for this choice may be many, probably also of a constructive nature due to the adjacent portico, which physically intersects the vestibule. However, considering the thickness of the cupola (2 *pedes*) we can observe how the intermediate hemisphere between the intrados and extrados has a diameter of 35 *pedes* and is therefore divisible by 7 (Fig. 9).

The vault opens on each side of the octagon by means of concave gores that could be measured with the *trikentron* formula presented (Stereometrica 96). The rest of the pavilion can be divided into uncomplicated construction elements: flat and cylindrical walls, spherical sectors (caps and shells with their thicknesses). In other words, the building can be semantically partitioned being consistent with Heron's formulas (Fig. 10).

The octagonal hall of the Small Baths has been examined several times, even in recent years (Verdiani et al. 2010; Marzuoli and Mollo 2020), but only through direct observation during the recent restoration has it been possible to present a clear view of its mixtilinear planimetric layout. In our hypothesis the geometric construction starts from a circle (C_e), consistent with the modular grid (Fig. 11a, b), that takes as its diameter the diagonal of the square of 8 modules on each side of the mesh (40 pedes on each side; 56.57 pedes in diameter $= 40 \times \sqrt{2}$). The two squares inscribed in C_e, rotated 45°, draw the octagon (in red) that defines the outer walls of the hall (Fig. 11c). The A′B′C′D′ is a square with side equal to 7 modules and presents a diagonal that can be approximated to 10 modules (A'B'C'D'). Once applied the *ad quadratum* construction to A′B′C′D′, we reach the octagon defining the interior walls (Fig. 11d). This results in a wall thickness of 2 1/3 *pedes*. The inscribed circumference of this octagon, with a diameter of 35 *pedes*, defines the intrados of the vault covering the central space. In this case, we can note coherence with both the Golden Court vestibule cupola and with the π approximation to $\frac{22}{7}$. Although it is difficult to establish the final geometric steps of the hall's *ichnographia*, due to several constraints present at the site of construction site (Adembri et al. 2021: 335),

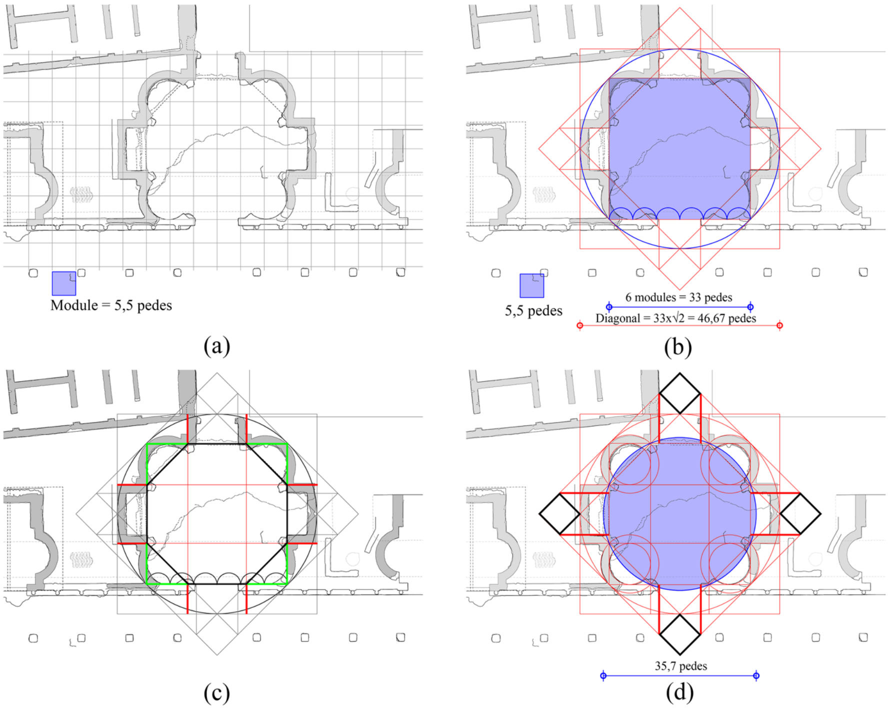

Fig. 8 Steps of the '*ad quadratum*' geometric construction used to define the vestibule plan layout

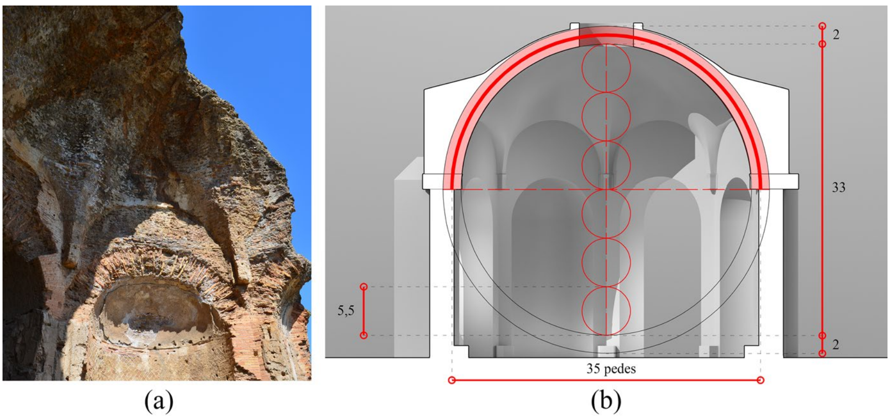

Fig. 9 **a** Trikentron of the Golden Court vestibule. The state of preservation of the upper part of the intrados is compromised by decay and prior restorations. **b** Hypothetical reconstruction of the elevation: the intermediate semicircular profile of the dome as proof (measured in *pedes*)

it is nevertheless possible to underline similarities with the procedure employed by Savvides (2021) for the definition of the convex sides. Once again, *ad quadratum* construction appear to be the solution: if we draw a circle inscribing the outer

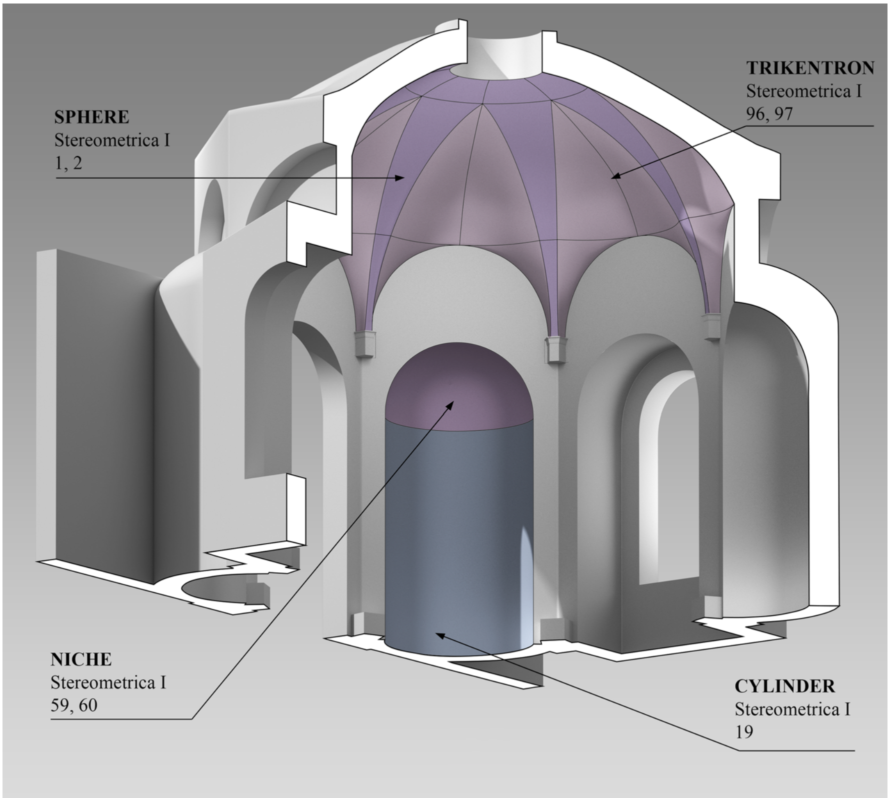

Fig. 10 Segmentation of the vestibule for calculating volumes in *Stereometrica* and *De Mensuris* by Heron

octagon (in red in Fig. 11e) and a new octagon inscribing this circle we get a set of new squares whose vertices belong to the circumference C_e and C_{e2} respectively (Fig. 11e). The vertices of the square belonging to both C_{e2} circumference and relative barycentric axes supply the centers of the 4 curved sides of the octagonal hall. Radii can be derived by extending the sides of the two octagons respectively inscribed and circumscribed by the red circle in (Fig. 11e). Although the mixtilinear nature, the state of preservation, and certain construction irregularities do not allow a clear view of the geometric shape of the *trikentrons* that are generated from the lunettes, interpretations have been proposed in previous studies (Cipriani et al. 2017). As with the *trikentrons* at the Golden Court vestibule, we point out how the connections between lunettes and spherical surfaces are defined by at least 5 or 6 edge curves (see Figs. 10 and 12) and not 3. Heron wrote in the period predating Hadrian, and it is legitimate to assume that the architects may have introduced stylistic and constructive innovations.

The semi-dome of the Serapeo-Canopus complex, unlike the other examples, presents a semicircular impost line along with other unique features. Here we can

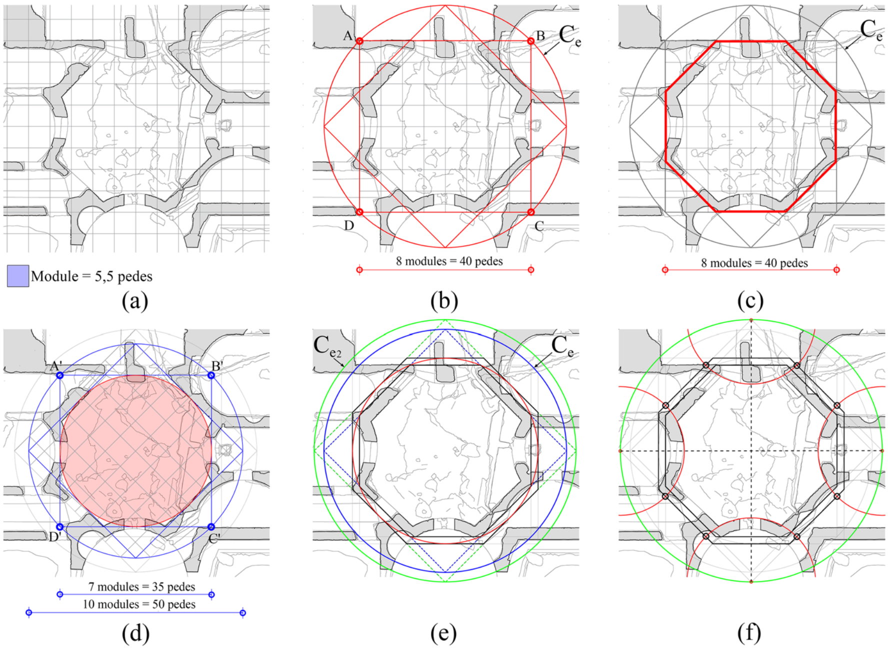

Fig. 11 Steps of the '*ad quadratum*' geometric construction used to define the octagonal hall plan layout

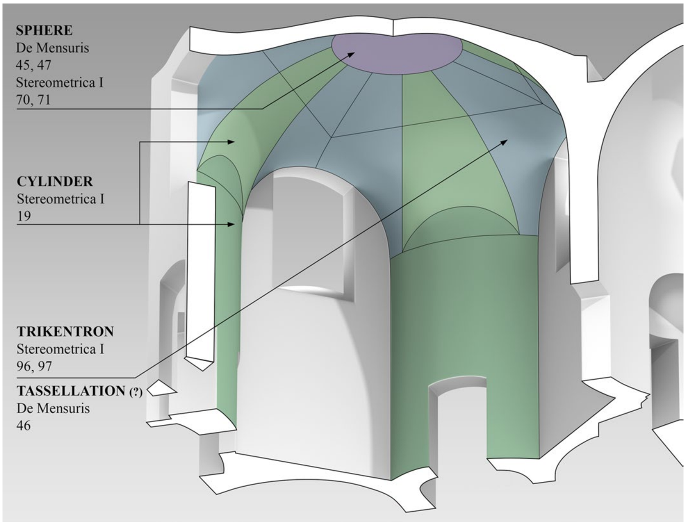

Fig. 12 Segmentation of the octagonal hall based on formulas for calculating volumes in *Stereometrica* and *De Mensuris* by Heron

see that the *trikentrons* are defined by 3 curves: two of them belong to vertical planes, while the other one is a non-planar curve, namely those of the lunettes and

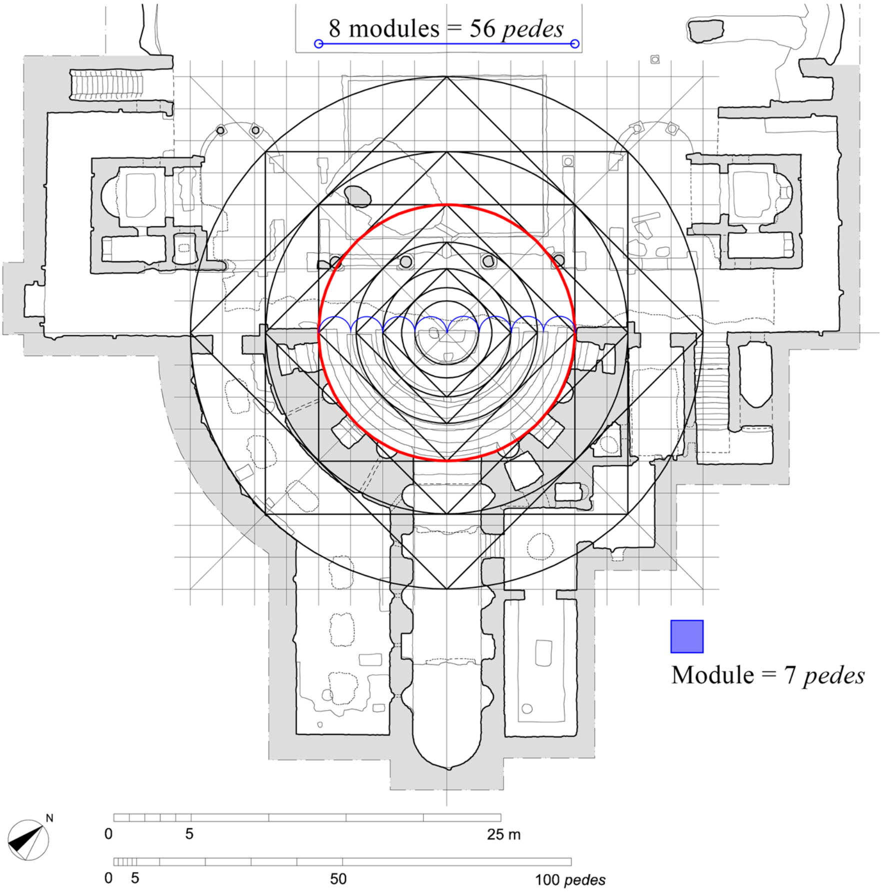

Fig. 13 Plan layout of the Serapeum-Canopus complex

that of the arch leading to the axial corridor. The inner circle of this great niche is consistent with both the approximation of $\pi = 22/7$ and the geometric progression of ratio $\sqrt{2}$ (*ad quadratum*) present in others octagonal layouts. The diameter of the internal perimeter of the tricliniar space measure 56 *pedes* (8 modules of 7 *pedes*) while inscribing successive squares and circles alternately, several structural elements appear to be aligned to this scheme (Fig. 13).

Conclusion

The analysis of the ancient cupolas covered in this article has undergone many modifications over time, as has happened to the texts of Heron, translated and regrouped by Heiberg in the attempt to provide thematic consistency to countless formulas that lacked their original context.

Restorations, especially those carried out in the middle of the last century, have produced alterations on their complex shapes, the expression of eminent personalities of ancient architecture, as well as the geometry and mathematics applied to construction.

What is perceived today concerning ancient architecture through the 'magnifying glass' of new surveying technologies, remains blurred. Or even worse, oscillates between an acceptable state of conservation and a total erasure of evidence useful for reconstructing all the necessary details of a full 'conceptual frame' of domical design.

Nevertheless, it is possible to establish certain 'cornerstones' that systematically emerge through present-day surveys: common aspects of domed buildings of the Roman imperial age. Among them we focused on a flexible and scalable use of geometric patterns based on the *ad quadratum* progression, the use of diameters that are multiples or submultiples of the number 7, and finally the use of modular grids introduced to reconcile irrational constructions with formulas for computing volumes and areas.

The vision that appeared to us, is of an ancient architect that had elementary mathematical means at his disposal, with highly scalable, flexible, and accurate knowledge. The set of solutions we detect in the design of the domes are thus simple, based on a limited number of regular shapes, but elegantly mastered. From the recursive use of modular quantities and patterns emerge how creativity had pragmatic roots, and not solely based on abstract schemes detached from causal relationship that always remains present in architectural design.

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