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Rogue traders

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Abstract
Investing on behalf of a firm, a trader can feign personal skill by committing fraud that with high probability remains undetected and generates small gains, but with low probability bankrupts the firm, offsetting ostensible gains. Honesty requires enough *skin in the game*: if two traders with isoelastic preferences operate in continuous time and one of them is honest, the other is honest as long as the respective fraction of capital is above an endogenous *fraud threshold* that depends on the trader's preferences and skill. If both traders can cheat, they reach a Nash equilibrium in which the fraud threshold of each of them is lower than if the other one were honest. More skill, higher risk aversion, longer horizons and higher volatility all lead to honesty on a wider range of capital allocations between the traders.

Keywords Rogue trading · Internal fraud · Operational risk · Stochastic differential games

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1 Introduction

The expression "rogue trader" entered popular culture in 1995 when Nicholas W. Leeson, a trader of an overseas office of Barings Bank in Singapore, made unauthorised bullish bets on the Japanese stock market, concealing his losses in an error account. At first, losses were recovered with a profit, but in the aftermath of the Kobe earthquake, they reached $1.4 billion (Brown and Steenbeek [7]), forcing the 233 years old bank into bankruptcy. Earlier episodes of rogue trading *ante litteram* include the losses of Robert Citron in 1994 for Orange County ($1.7 billion, Jorion [27]) and of Toshihide Iguchi in 1983–1995 for Daiwa Bank ($1.1 billion, Iguchi [22, Prologue']). The earliest case is possibly that involving the law firm of Grant & Ward in 1884, which embarrassed former president Ulysses S. Grant, one of the firm's partners (Krawiec [32]).

Since the demise of Barings Bank, rogue trading episodes have increased in frequency and magnitude. In 2008, Jerome Kerviel, a junior trader at Société Générale who had been exceeding position limits through fictitious trades to avoid detection, eventually lost $7.6 billion, the largest rogue trading loss in history. In his defense, he claimed that colleagues also engaged in unauthorised trading (The New York Times [41] and Reuters [39]). Most recently, in September 2021, Keith A. Wakefield, the former head of the fixed income trading desk at the broker–dealer IFS Securities, was charged by the U.S. Securities and Exchange Commission with unauthorised speculative trading and creating fictitious trading profits, leading to the closure of IFS Securities and substantial losses to both IFS Securities and one dozen counter-parties to the trades (U.S. Securities and Exchange and Commission [42]).

The rise in rogue trading and its threat to both financial institutions and financial stability is recognised by the Basel Committee as operational risk, defined as *"the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events"* (BCBS [3, Clause 10 of Sect. "Principles for the management of operational risk"]). The Capital Accord of Basel II – and Basel III, to be enacted in 2023 – includes provisions for protection from operational losses: while insurance can cover high-frequency, low-impact events, rogue trading falls squarely in the low-frequency, high-impact category of uninsurable risks, which incur capital charges. Such charges are in turn based on standardised approaches or statistical models, due in part to the absence of consensus on the origin of rogue trading, which is the focus of this paper.

Our starting point is that *"The continued existence of rogue trading [. . .] presents a mystery for many scholars and industry observers."* (Krawiec [32]). *"Operational risk is unlike market and credit risk; by assuming more of it, a financial firm cannot expect to generate higher returns."* (Crouhy et al. [11]). In other words, *prima facie* it is hard to reconcile rogue traders' actions with the optimising behaviour of sophisticated rational agents.

We propose a model in which rational, self-interested, risk-averse traders deliberately engage in fraudulent activity that has zero risk premium. While undetected, fraud allows a trader to feign superior returns, ostensibly without additional risk. In reality, higher returns are exactly offset by a higher probability of bankruptcy, thereby creating no value for the firm. Yet, under some circumstances, fraud may be optimal for a trader because while its benefits are personal, potential bankruptcy costs are shared with other traders. Furthermore, a trader who understands the circumstances leading to others' fraud can anticipate them and act accordingly, leading to a dynamic Nash equilibrium.

In equilibrium, each trader abstains from fraud as long as the respective share of wealth under management exceeds an endogenous *fraud threshold* that depends on both traders' preferences (risk aversions and average horizon) and investment characteristics (expected returns and volatilities). Thus a trader must have enough *skin in the game* to remain honest: when the share of managed assets drops below the fraud threshold, the marginal utility of fraudulent trades becomes positive, and a trader cheats as little and as quickly as possible to restore the wealth share to the honesty region. Importantly, such fraudulent activity does not generate extra volatility; so it cannot be detected by monitoring wealth before bankruptcy occurs.

These results bring several insights. First, our model suggests that rogue trading has an important social component: A sole trader investing all the firm's capital would not engage in fraud because such a trader would bear in full both the costs and the benefits of fraudulent activity (Proposition 2.1). Furthermore, the fraud threshold is higher if a trader knows that nobody else is cheating (Lemma 3.8 and Theorem 3.10).

Second, the model emphasises the risk that traders with relatively small amounts of capital can pose to a financial institution, due to their insufficient stakes in the firm. This concern is confirmed by the cases of the junior traders Jerome Kerviel and Nick Leeson. By reviewing Mr. Leeson's trading record and the investigation reports from Singaporean authorities, Brown and Steenbeek [7] suggest that he had excluded the error account (meant for traders to settle minor trading mismatches) from the market reports to headquarters and had built up unauthorised speculative positions since taking the post at Baring's office in Singapore in 1992.

Third, our comparative statics offer some clues for assessing and mitigating rogue trading risk. The incidence of fraud is higher in less skilled traders, which means that emphasis on performance evaluation has the indirect benefit of fraud reduction. Fraud also declines significantly as risk aversion increases, suggesting that, *ceteris paribus*, the most fearless traders are also the ones most tempted by fraud, and that the most dangerous combination is found in a trader with *high* risk tolerance and *low* share of managed assets. In addition, fraud declines when the horizon is long enough.

Fourth, our model hints at a subtle trade-off between investment performance and operational risk. Classical portfolio theory implies that diversification can only increase performance; hence the addition of a trader with expertise in a new asset class always improves the risk–return trade-off. Yet, our results caution that a higher number of traders, each with a lower share of assets under management, may also increase the appeal of fraud for each of them, potentially worsening the firm's risk profile. (The quantitative analysis of the trade-off between diversification and fraud requires very different technical tools, hence is deferred to future research.)

This paper offers the first structural model of rogue trading, in which fraud arises from agency issues between traders and their firms. A priori, it is traders' hidden action that enables fraudulent activity. A posteriori, the traders' optimal strategies imply that fraud is both continuous and of finite variation, which makes it hard to detect even for a hypothetical observer who could continuously monitor traders' wealth.

In the interest of both simplicity and relevance, the model assumes that each trader is compensated with a fraction of trading profits, i.e., contracts are linear. As a result, the fraudulent activity that arises in the model does not stem from nonlinear incentives that may encourage risk-taking (Carpenter [9]), but merely from the asymmetric opportunity of taking personal credit from fraudulent gains while sharing bankruptcy costs. In this sense, each trader's fraud represents an externality for other traders and the firm, whence the overall demand for fraud is socially suboptimal (i.e., nonzero).

At the technical level, this paper contributes to the theory of nonzero-sum stochastic differential games with singular controls. A distinctive feature of our model is that both players are free to perform simultaneous discontinuous actions, a possibility that is often excluded in the literature for technical convenience. We also provide a continuous-time formulation of Nash equilibrium with singular controls and construct an equilibrium explicitly through Skorokhod reflection.

The results in the paper also bear a curious analogy with portfolio choice with proportional transaction costs in that, similar to Davis and Norman [12], the solution to the present model leads to an inaction region, surrounded by two regions in which actions are performed as little as necessary to return to the inaction region. Although the mechanisms underlying the two models are very different, it is worth pointing out the common feature that leads to the common structure. In both cases (and in many other singular control problems), an action is performed only in a positive amount (fraud of either trader in this paper, buying or selling in portfolio choice). As a result, the inaction region arises when each action is counterproductive for its agent, while the action regions are visited at their boundaries because costs are linear in the action performed (bankruptcy probability in this paper, trading costs in portfolio choice).

In the present model, a trader's marginal value of fraud depends on that trader's share of the firm's wealth. When the wealth share is large enough (skin in the game is high), the marginal value is negative, hence the optimal amount of fraud is zero. Vice versa, the marginal value of fraud is positive when the share is low: since fraud yields a reward proportional to its amount, the optimal amount would be infinite. However, as fraud (before causing bankruptcy) increases wealth, it occurs in equilibrium only as the wealth share is at that level for which its marginal value is exactly zero, and only in the infinitesimal amounts necessary to keep the wealth share at such a level.

The literature on rogue trading is relatively sparse. Most existing works explore the legal (Krawiec [32, 33]), regulatory (Moodie [36]) and social-psychological (Wexler [43]) aspects of rogue trading, and offer a number of hypotheses for mechanisms that may foster malfeasance in trading. Armstrong and Brigo [2] find that common risk measures are ineffective in preventing excessive risk-taking by traders with tail-risk-seeking preferences. In a similar vein, Gwilym and Ebrahim [20] argue that position limits are inadequate in restraining rogue trading. Taking the perspective of a firm's management, Xu et al. [38] use stochastic control to minimise operational risk through preventative and corrective policies, while Kim and Xu [31] design inspection policies to manage operational risk losses. Xu et al. [37] review the recent literature on operational risk.

In contrast to single-agent singular stochastic control problems, which date back to the finite-fuel problem of Bather and Chernoff [4], research on singular stochastic differential games is relatively recent. Guo and Xu [19] generalise the finite-fuel problem to an n-player stochastic game and a mean-field game, in which each player minimises the distance of an object to the center of N objects, while minimising the total amount of control applied. Guo et al. [18] extend this analysis to a larger class of games with potentially moving reflecting boundaries in Nash equilibria. Kwon [34] analyses the game of contribution to the common good and discovers Nash equilibria of mixed type, i.e., the strategies in equilibrium consist of both absolutely continuous and singular components. De Angelis and Ferrari [13] establish a connection between a class of stochastic games with singular controls and a certain optimal stopping game, where the underlying state processes differ but the reflecting and exit boundaries coincide. Kwon and Zhang [35] and Ekström et al. [16] study optimal stopping games in which all or one of the players control an exit time that terminates the game. Note that the fraud in Ekström et al. [16] differs from that considered here in that their model entails an agent stealing from another one, who seeks to detect fraud and can terminate the game. In these papers, players are forbidden to execute discontinuous actions simultaneously, whereas our model does not impose such a restriction. In addition, the present work provides a structural formulation of Nash equilibrium in the presence of singular controls. Adopting BSDE techniques, Karatzas and Li [28] investigate existence and uniqueness of Nash equilibrium in games of control and stopping, while Hamadène and Mu [21] establish existence for games without exit but with unbounded drift. Dianetti and Ferrari [14] employ fixed-point methods for the monotone-follower games with submodular costs.

The rest of this paper is organised as follows. Section 2 describes our model of rogue trading and its rationale. Section 3 constructs a Nash equilibrium with two traders and states the main result. Section 4 discusses the interpretation of the results and their implications. Concluding remarks are in Sect. 5, and all proofs are in the Appendix.

2 A model of rogue trading

Krawiec [32] offers the following definition: "A rogue trader is a market professional who engages in unauthorised purchases or sales of securities, commodities or derivatives, often for a financial institution's proprietary trading account."

Most episodes of fraudulent trading share some distinctive features. First, they involve violations of a firm's internal rules or external regulations. Second, fraud often remains concealed and results in modest (relative to the firm's size) gains that are ascribed to the skill of the perpetrator. Third, fraud generates substantial risk without expected return for the firm, and is revealed only when catastrophic losses eventually materialise.

To reproduce these features, it is useful to think of a small fraud as a (forbidden) bet that a trader wagers on the whole firm's capital. With a small chance (say ε), the bet bankrupts the firm (a return of -100%), but most of the time (with probability $1-\varepsilon$), it results in a return of $1/(1-\varepsilon)-1 \approx \varepsilon$ for which the trader can take credit. Of course, the bet's overall return for the firm is zero as $(1-\varepsilon)\cdot(1/(1-\varepsilon)-1)-1\cdot\varepsilon = 0$. Such asymmetric outcomes (likely small gains against unlikely large losses) are in fact common in both illicit and licit trading strategies (for example, selling deep out-of-the-money options), and have attracted the label of "picking up nickels in front of a steamroller" (Duarte et al. [15]).

Thus the dilemma of an unscrupulous but profit-driven and risk-averse trader is to what degree to engage in fraud, as cheating too little may forego some easy profits, but cheating too much may result in likely bankruptcy. If one imagines the small fraud above as the outcome of a (heavily biased) coin-toss, the trader essentially ponders how many coins to toss. For example, tossing two coins would generate a likely payoff of $(1-\varepsilon)^{-2}$, but may also lead to bankruptcy with probability $2\varepsilon-\varepsilon^2$.

If the trader is the firm's sole owner, it is not hard to see that fraud does not pay: when one bears both gains and losses in full, wagering fair bets on one's capital merely replaces a payoff with another one, more uncertain but with the same mean – an inferior choice by risk aversion.

In this sense, fraud arises from social interactions, both through the incentives implied by traders' compensation contracts or by each trader's ability to take risks with other people's money (Kay [30, Chap. 2]), with the awareness that colleagues may also engage in fraud. The present model focuses on the latter motive by assuming that each trader receives a fixed fraction of individual profits and losses, which is a common arrangement for bonuses with clawback provisions. The model envisages multiple traders; each of them has the mandate to invest a share of the firm's capital in some risky asset with a positive risk premium and is paid with a fraction of the terminal payoff. Thus except for fraudulent behaviour, each trader's objective is aligned with the firm's. For the sake of tractability and clarity, the paper focuses on the case of two traders.

The moral hazard stems from the asymmetric effects of fraud on a trader's reward: as long as the fraudulent activity is successful, the trader can disguise its revenues as the fruit of personal skill in performing the investment mandate. In reality, such additional revenues merely compensate for the fraudulent bets that the trader wagers on the capital of the *whole* firm, rather than personal capital (e.g. exceeding risk limits by either collateralising the firm's asset or assuming excess liabilities). Of course, such bets are possible exactly because they are fraudulent, and are explicitly forbidden by the firm's regulations; they nonetheless exist, due to "inadequate or failed internal processes, people and systems" embodied in the definition of operational risk (BCBS [3, Clause 10 of Sect. "Principles for the management of operational risk"]).

The appeal of fraud – privatising gains while socialising losses – thus varies with a trader's share of the firm's capital: intuitively, the temptation of fraudulently enriching oneself is much stronger for a small trader, who has little to lose and much to gain from gambling with others' wealth, than for a large trader who has significant skin in the game. For this reason, in the present continuous-time model, each trader can cheat with varying intensity in response to changes in one's and others' wealth.

After this informal description, the precise definition of the model follows.

2.1 Investment and fraud in continuous time

We fix a stochastic basis $(\Omega,\mathcal{F},\mathbb{P})$ equipped with the natural filtration $\mathbb{F}=(\mathcal{F}_t)_{t\ge 0}$ of an N-dimensional ($N\ge 1$) Brownian motion $B=(B_t)_{t\ge 0}$, satisfying the usual hypotheses of right-continuity and completeness, and set $\mathcal{F}_\infty := \sigma(\bigcup_{t\geq 0}\mathcal{F}_t) \subseteq \mathcal{F}$. As all processes considered here are at least right-continuous and $\mathbb{P}$ is fixed, we write "a.s. for all $t \geq 0$" for the equivalent properties "for all $t \geq 0$, $\mathbb{P}$-a.s." and "$\mathbb{P}$-a.s., for each $t \geq 0$".

Assuming a zero safe rate to ease notation, in the absence of fraud, the capital Y^i of the ith trader ($1 \leq i \leq N$) evolves as

$$dY_t^{i,x} = \mu_i Y_t^{i,x} dt + \sigma_i Y_t^{i,x} dB_t^i, \qquad Y_0^{i,x} = x_i > 0,$$

reflecting the trader's average ability $\mu_i > 0$ to deliver excess returns with the volatility $\sigma_i > 0$ that the firm's risk management is willing to accept. For simplicity, assume that B^i and B^j are independent for $i \neq j$, which means that traders take uncorrelated risks (for example, one invests in stocks and the other in bonds).

To describe how each trader may engage in fraud by endangering the firm's capital, define the class of processes

$$\mathcal{A} := \{(A_t)_{t\geq 0} : \mathbb{F}\text{-adapted, right-continuous, nondecreasing, } A_{0-} = 0\}.$$

For $A \in \mathcal{A}$, A_t represents the cumulative amount of "bets" wagered by a trader on the firm's capital up to time t. To understand this representation, suppose that $A_t = \int_0^t \lambda_s ds$, which means that in the interval $[s, s+ds]$, the trader wagers a fair bet that has the probability $\lambda_s ds$ of bankrupting the firm. Because the fraud is illicitly wagered on the firm's capital (thereby exceeding the capital $Y^{i,x}$ that the trader has been assigned), if bankruptcy does not occur, that fraud yields a profit of $Y_s^S \lambda_s ds$, where $Y^{S,x} := \sum_{k=1}^N Y^{k,x}$ is the total capital of the firm.

Although this description is intuitive, it has two limitations. First, it encompasses only the case of fraud with a finite rate λ_s, excluding bursts of rogue trades at any instant. Second, the bankruptcy probability cannot incorporate the impact of fraud over an arbitrary time interval as the value of $\int_s^t \lambda_u du$ can exceed 1. For these reasons, a more careful but also more technical description is necessary.

To make precise the intuition that dA_s drives the bankruptcy rate, note first that any $A \in \mathcal{A}$ is right-continuous and of finite variation. Therefore, it has the representation $A_t = A_t^c + \sum_{0\leq s\leq t} \Delta A_s$ for any $t \geq 0$, where $\Delta A_s = A_s - A_{s-}$ and A^c is the continuous part of the process A with $A_0^c = 0$. For a set of N traders' fraud processes $(A^1, \dots, A^N) \in \mathcal{A}^N$, denote the *total fraud* process by $A^S = \sum_{k=1}^N A^k$. The *bankruptcy time* is then defined as

$$\tau_A = \inf\{t \geq 0 : A_t^S \geq \theta\}, \tag{2.1}$$

where θ is an $\mathcal{F}$-measurable exponential random variable with rate 1, independent of the filtration $\mathbb{F}$. (Recall the convention that $\inf\emptyset = \infty$.) Lemma A.3 below shows that the survival probability satisfies $\mathbb{P}[\tau_A > t|\mathcal{F}_t] = e^{-A_t^S}$ for all $t \geq 0$. At time τ_A, the wealth of all agents becomes zero.

Before bankruptcy occurs, the wealth of each trader follows the dynamics

$$dY_t^{i,x} = \mu_i Y_t^{i,x} dt + \sigma_i Y_t^{i,x} dB_t^i + Y_{t-}^S d\tilde{A}_t^i, \qquad Y_{0-}^{i,x} = x_i > 0, 1 \leq i \leq N, \tag{2.2}$$

where the integral with respect to $\tilde{A}^i$ in (2.2) is understood in the Lebesgue–Stieltjes sense, and $\tilde{A}^i_t := A^{i,c}_t + \sum_{0\le s\le t}(e^{\Delta A^i_s} - 1)$ reflects the fact that the *simple* return of a jump in fraud is not Δ itself but rather $e^{\Delta} - 1$. Such a distinction is immaterial with continuous fraud because $e^{\Delta} - 1 \approx \Delta$ for Δ close to zero. The final expression for wealth, which includes the effect of bankruptcy at τ_A, is

$$X^{i,x}_t = \mathbf{1}_{\{t<\tau_A\}} Y^{i,x}_t \qquad \text{with } X^{i,x}_{0-} = x_i.$$

Lemma A.2 in Appendix A.1 formally verifies that the pre-bankruptcy wealth in (2.2) is well defined by showing that $Y^x = (Y^{1,x}, \dots, Y^{N,x})$ is the unique strong solution to the N-dimensional linear stochastic differential equation (SDE) in (2.2). Upon bankruptcy on the event $\{t \ge \tau_A\}$, the wealth of all traders vanishes and remains null thereafter; hence the dynamics of the fraud processes beyond τ_A is irrelevant for the model. Effectively, fraud is described by the stopped process $(A^i_{t\wedge\tau_A})_{t\ge 0}$.

Note that the bankruptcy time τ_A is not an $\mathbb{F}$-stopping time. Thus to accommodate the wealth process $X^x = (X^{1,x}, \dots, X^{N,x})$, it is necessary to make the minimal enlargement of the filtration $\mathbb{F}$ that makes τ_A a stopping time. To this end, let $\mathbb{H}^A = (\mathcal{H}^A_t)_{t\ge 0}$ be the natural filtration of the indicator process $(\mathbf{1}_{\{t\ge\tau_A\}})_{t\ge 0}$ and define the enlarged filtration $\mathbb{G}^A = (\mathcal{G}^A_t)_{t\ge 0}$ as $\mathcal{G}^A_t = \bigcap_{s>t}(\mathcal{F}_s \vee \mathcal{H}^A_s)$, which is the smallest right-continuous filtration containing $\mathbb{F}$ such that τ_A is a stopping time. Such an extension is known as 'progressive filtration enlargement' (cf. Jeanblanc and Le Cam [25] and Jeulin [26, Chap. IV]). Moreover, the bankruptcy time τ_A is $\mathbb{G}^A$-predictable *if and only if* fraud does not occur after time 0 (Lemma A.6).

As wagering bets on one's own wealth means bearing their risks in full, thereby earning a zero risk premium, a trader who owns the whole firm ($N = 1$) has a wealth process that is a $\mathbb{G}^A$-martingale in the absence of investment skill.[1] (See Proposition A.4, which additionally justifies the choice of the return from jump fraud.)

The goal of each trader is to maximise expected utility over a random horizon τ, which is an $\mathcal{F}$-measurable exponential random variable with rate $\lambda > 0$, independent of both $\mathcal{F}_\infty$ and θ (and hence of the bankruptcy time τ_A). This random horizon models a trader with an open-ended contract, whose mandate is to maximise profits in the long term. The arrival rate λ captures the likelihood that business may end for exogenous reasons (that is, independently of traders' performance).

A trader's attitude to risk is represented by a utility function of power type

$$U^i(x_i) = \frac{x_i^{1-\gamma_i}}{1-\gamma_i} \qquad \text{with } 0 < \gamma_i < 1.$$

In particular, the relative risk aversion parameter γ_i is below one so that the utility is finite also upon bankruptcy ($x_i = 0$), and the problem is nontrivial. If γ_i were greater or equal to one, then zero wealth would be completely unacceptable ($U^i(0) = -\infty$) and fraud would disappear. In fact, as shown below (Remark 3.11), fraud does vanish as γ_i converges to one.

[1] In principle, one could consider the case of fraud with a negative risk premium. Our model focuses on the parsimonious case of zero risk premium, which maximises the propensity for a trader to cheat. If the risk premium were positive, the bet would be a legitimate investment opportunity, for which the label of "fraud" may not be justified.

As anticipated in the description, an important implication of this model is that a rational and strictly risk-averse trader abstains from fraud if no other trader is present. Its significance is to confirm that in this model, fraud stems from the ability to share losses but not gains, and hence disappears when such sharing disappears.

Proposition 2.1 *Let $N = 1$, $\kappa \geq 0$ and τ be an $\mathcal{F}$-measurable, a.s. finite random horizon independent of $\mathcal{F}_\infty$ and θ such that*

$$\mathbb{E}[e^{((1-\gamma_1)(\mu_1-\gamma_1\sigma_1^2/2)-\kappa)\tau}] < \infty.$$

If the sole trader maximises

$$\mathbb{E}[e^{-\kappa\tau}U^1(X_\tau^{1,x_1})]$$

over all fraud processes $A^1 \in \mathcal{A}$, then $A^{1,\star}$ is optimal if and only if $A_t^{1,\star} = 0$ a.s. for all $t \geq 0$ such that $\mathbb{P}[\tau \geq t] > 0$. In particular:

(i) *If τ is unbounded, then $A_t^{1,\star} = 0$ a.s. for all $t \geq 0$.*

(ii) *If $\tau \leq T^1$ a.s. for some $T^1 > 0$, then $A_{T^1-}^{1,\star} = 0$. If $\mathbb{P}[\tau = T^1] > 0$, then also $A_{T^1}^{1,\star} = 0$ a.s.*

Note that for this result, the assumption of an exponential horizon made in the rest of the paper can be dropped. Note also that Proposition 2.1 fails if $N \geq 2$ because the coupling term $Y_{t-}^{S,x}$ in (2.2) rescinds the martingale property (Proposition A.4) for each trader's wealth in the absence of drift ($\mu_i = 0$). For example, if all but the ith trader abstain from fraud, then $X^{i,x}$ can become a submartingale if the ith trader cheats; in this case, the wealth processes of other traders become supermartingales as they share the bankruptcy risk from the ith trader's actions. As shown in Sect. 3, engaging in fraud may be optimal, depending on traders' shares of capital, risk aversions, drifts and volatilities.

3 Main result

While the presentation in the previous section considered an arbitrary number N of traders, the main result in this section focuses on two traders to simplify both the exposition and the proofs. (A model with N traders implies that relative wealth shares follow an $(N-1)$-dimensional diffusion, which reduces to a scalar diffusion for two traders.) Thus henceforth $N = 2$, and for clarity, the indices $\{a, b\}$ replace $\{1, 2\}$ to identify traders. The wealth processes are denoted by either $X^x(A^a, A^b)$ or X^x (respectively, $Y^x(A^a, A^b)$ or Y^x), depending on the need to specify the fraud process (A^a, A^b) in context.

3.1 Definition of Nash equilibrium

For any i, j in $\{a, b\}$ with $i \neq j$ (henceforth abbreviated as 'for any $i \neq j \in \{a, b\}$'), the goal of trader i is to maximise expected utility over a random horizon τ as the other trader j chooses the respective fraud process A^j, i.e.,

$$J^i(x; A^i, A^j) := \mathbb{E}\big[e^{-\kappa\tau} U^i\big(X_\tau^{i,x}(A^i, A^j)\big)\big],$$

over $A^i \in \mathcal{A}$. Here $\kappa \geq 0$ is the discount rate and the random horizon τ is independent of $\mathbb{F}$ and θ and exponentially distributed with rate λ (meaning that $\frac{1}{\lambda}$ represents traders' *average horizon*). Let thus

$$V^i(x; A^j) := \sup_{A^i \in \mathcal{A}} J^i(x; A^i, A^j) \tag{3.1}$$

be the *value function* for the ith trader for trader j's fraud process A^j and initial wealth $x \in \mathbb{R}^2_{++}$. The next assumption concerning minimum risk aversion and maximum skill stands **throughout the paper** and ensures that the optimisation problem is well posed.

Assumption 3.1 Let $\lambda^\kappa = \kappa + \lambda$ and assume that $\lambda^\kappa > (1 - \gamma_a \wedge \gamma_b)(\mu_a \vee \mu_b)$.

The value function V^i satisfies the following basic properties.

Lemma 3.2 *For any* $i \neq j \in \{a, b\}$, $x \in \mathbb{R}^2_{++}$ *and* $(A^i, A^j) \in \mathcal{A}^2$, *we have*:

(i) $0 < V^i(x; A^j) \leq \dfrac{\lambda U^i(x_a + x_b)}{\lambda^\kappa - (1 - \gamma_i)(\mu_a \vee \mu_b)}$.

(ii) $J^i(x; A^i, A^j) = \lambda\, \mathbb{E}\left[\int_0^\infty e^{-\lambda^\kappa t - A_t^S} U^i\big(Y_t^{i,x}(A^i, A^j)\big) dt\right]$.

(iii) *For any* $c > 0$, $J^i(cx; A^i, A^j) = c^{1-\gamma_i} J^i(x; A^i, A^j)$.

Most importantly, Lemma 3.2 (i) ensures that under Assumption 3.1, the value function (3.1) is finite, rendering a well-posed optimisation problem. Furthermore, (ii) reveals that we only need to use the pre-bankruptcy wealth Y^x as the state processes of the optimisation problem, as the random horizon τ is exponentially distributed. Finally, (iii) reveals the scale-invariance of the value function, which allows reducing the resulting Hamilton–Jacobi–Bellman (HJB) equations to ordinary differential equations (see Appendix B).

At time t, the ith trader observes the history of personal wealth $(Y_s^{i,x})_{s\in[0,t)}$ and personal fraud $(A_s^i)_{s\in[0,t)}$, as well as the wealth history of the other trader j, i.e., $(Y_s^{j,x})_{s\in[0,t]}$, so that trader i can respond to trader j's instant wealth change $\Delta Y_t^{j,x}$. Formally, for $t \geq 0$, let $\mathcal{D}_+([0,t])$ denote the set of $\mathbb{R}_{++}$-valued càdlàg functions on $[0,t]$ with a left limit at $t = 0$. Let $\mathcal{D}^\uparrow([0,t])$ be the set of $\mathbb{R}_+$-valued nondecreasing, right-continuous functions on $[0,t]$ with zero left limit at $t = 0$. The sets $\mathcal{D}_+([0,t))$ and $\mathcal{D}^\uparrow([0,t))$ are defined analogously. For any process $(Z_t)_{t\geq 0}$ with left limit at 0, $Z_{[0,t)}$ (resp. $Z_{[0,t]}$) denotes the restrictions of the paths of Z to the interval $[0,t)$ (resp. $[0,t]$). Denote by $\mathcal{H}_t^+$, $\mathcal{H}_{t-}^+$ and $\mathcal{H}_{t-}^\uparrow$ the smallest σ-algebras generated by all $\mathbb{F}$-adapted processes with trajectories in $\mathcal{D}_+([0,t])$, $\mathcal{D}_+([0,t))$ and $\mathcal{D}^\uparrow([0,t))$, respectively.

To construct a Nash equilibrium of closed-loop form, we consider a special class of fraud strategies that constitute a trader's possible responses to the fraudulent activities of the other trader, but depend on the latter only through the *wealth* of both traders and one's *own strategy*.

Definition 3.3 Let $i \neq j \in \{a, b\}$. The set Λ^i is the collection of maps $\Psi = (\Psi_t)_{t \geq 0}$ which are for any $t \geq 0$ of the form

$$\Psi_t : \mathcal{D}_+\big([0,t)\big) \times \mathcal{D}_+\big([0,t]\big) \times \mathcal{D}^{\uparrow}\big([0,t)\big) \to \mathbb{R}_+ \quad \mathcal{H}^+_{t-} \otimes \mathcal{H}^+_t \otimes \mathcal{H}^{\uparrow}_{t-}\text{-measurable}$$

such that for any $x = (x_i, x_j) \in \mathbb{R}^2_{++}$ and any $A^j \in \mathcal{A}$, there exists a unique $A^i \in \mathcal{A}$ satisfying

$$A^i_t = \Psi_t(Y^{i,x}_{[0,t)}, Y^{j,x}_{[0,t]}, A^i_{[0,t)}) \qquad \text{a.s. for all } t \geq 0, \tag{3.2}$$

where $(Y^{i,x}, Y^{j,x})$ is the pre-bankruptcy wealth associated with (A^i, A^j).

Lemma A.2 (i) yields a unique strong solution $(Y^{i,x}, Y^{j,x})$ to the SDE (2.2) for a given pair of fraud processes (A^a, A^b) and initial wealth $x \in \mathbb{R}^2_{++}$.

We are now ready to define Nash equilibria in the context of this paper. See Carmona [8, Sect. III.5] for an overview of Nash equilibria in stochastic settings with absolute continuous controls.

Definition 3.4 A pair $(\Psi^{\star,a}, \Psi^{\star,b}) \in (\Lambda^a, \Lambda^b)$ is a *Nash equilibrium* if for any initial capital $x \in \mathbb{R}^2_{++}$, there exists a unique pair $(A^{a,\star}, A^{b,\star}) \in \mathcal{A}^2$ such that for any $i \neq j \in \{a, b\}$,

(i) $A^{i,\star}_t = \Psi^{\star,i}_t(Y^{i,x,\star}_{[0,t)}, Y^{j,x,\star}_{[0,t]}, A^{i,\star}_{[0,t)})$ a.s. for all $t \geq 0$, where $(Y^{a,x,\star}, Y^{b,x,\star})$ denotes the wealth associated with $(A^{a,\star}, A^{b,\star})$;

(ii) non-cooperative optimality holds, that is, for any $A^i \in \mathcal{A}$, the response A^j satisfying (3.2) with $\Psi^j = \Psi^{\star,j}$ makes A^i sub-optimal, i.e.,

$$J^i(x; A^i, A^j) \leq J^i(x; A^{i,\star}, A^{j,\star}).$$

The pair $(A^{a,\star}, A^{b,\star})$ is referred to as *equilibrium fraud processes*.

Remark 3.5 A Nash equilibrium $(\Psi^{\star,a}, \Psi^{\star,b}) \in (\Lambda^a, \Lambda^b)$ does not necessarily yield best-response maps: It is not necessarily true that for any $i \neq j \in \{a, b\}$ and $A^i \in \mathcal{A}$,

$$J^j(x; A^{j,'}, A^i) = \sup_{A^j \in \mathcal{A}} J^j(x; A^j, A^i),$$

with the response map $A^{j,'}_t = \Psi^{\star,j}_t(Y^{j,x}_{[0,t)}, Y^{i,x}_{[0,t]}, A^{j,'}_{[0,t)})$ for all $t \geq 0$ satisfying (3.2). In other words, the response $\Psi^{\star,j}$ of trader j need not be optimal for any fraud process of trader i, but merely sufficient to deter the other trader from deviating from $A^{i,\star}$. For the specific equilibrium fraud process $A^{i,\star}$ of trader i, (3.3) holds true in view of Definition 3.4, condition (ii).

3.2 Construction of Nash equilibrium

In the Nash equilibrium described below, each trader cheats as little as necessary to keep the personal share of wealth above a certain threshold. To rigorously define this behaviour, it is necessary to recall the notion of Skorokhod reflection. For any $x = (x_a, x_b) \in \mathbb{R}^2_{++}$ and any $i \in \{a, b\}$, define $r_i(x) = \frac{x_i}{x_a+x_b}$. Then $r_i(Y_t^x)$ is trader i's share of the firm's capital at time t. (See Lemma A.9 for the SDE identification of $r_i(Y^x)$.) Define by $W_t^{i,w_i}(A^i, A^j) = r_i(Y_t^x(A^i, A^j))$ for any $t \geq 0$ with the initial wealth share $W_{0-}^{i,w_i}(A^i, A^j) = r_i(x) = w_i$.

Definition 3.6 Let $i \neq j \in \{a, b\}$ and $m_i \in (0, 1)$. A function $\Psi^{i,m_i} \in \Lambda^i$ *solves the (one-sided) Skorokhod reflection problem* (henceforth $\mathrm{SP}^i_{m_i+}$) if for any $A^j \in \mathcal{A}$ and any $x \in \mathbb{R}^2_{++}$, the pair (A^i, Y^x) associated to Ψ^{i,m_i} is the unique pair satisfying

(i) $m_i \leq W_t^{i,w_i}(A^i, A^j) < 1$ a.s. for all $t \geq 0$;
(ii) $\int_{\mathbb{R}_+} \mathbf{1}_{\{W_t^{i,w_i}(A^i,A^j)>m_i\}} dA_t^i = 0$ a.s.

By (i), $W_t^{i,w_i}(A^i, A^j) \geq m_i$ a.s. for all $t \geq 0$, while (ii) means that as A^i increases, $W^{i,w_i}(A^i, A^j)$ can reach m_i but without spending any positive amount of time at this point. Because $W^{i,w_i} = 1 - W^{j,1-w_i}$ for any $i \neq j \in \{a, b\}$, W^{i,w_i} is reflected upward at m_i if and only if the other trader's fraction of wealth $W^{j,1-w_i}$ is reflected downward at $1 - m_i$. Moreover, the solution to $\mathrm{SP}^i_{m_i+}$ is unique in that it identifies a unique pair (A^i, Y^x).

For any $i \neq j \in \{a, b\}$, let $m_i \in (0, 1)$ and define $\Psi^{i,m_i} \in \Lambda^i$ as follows. For all $t \geq 0$ and $(y^i_{[0,t)}, y^j_{[0,t]}, a^i_{[0,t)}) \in \mathcal{D}_+([0, t)) \times \mathcal{D}_+([0, t]) \times \mathcal{D}^{\uparrow}([0, t))$, set

$$\begin{aligned}
&\Psi_t^{i,m_i}(y^i_{[0,t)}, y^j_{[0,t]}, a^i_{[0,t)}) \\
&:= \frac{1}{1-m_i}\Bigg(\sup_{s\in[0,t]}\Big(m_i - w_s^{i-} + (1-m_i)a_s^{i,c} + \sum_{0\leq u<s}(m_i - w_u^{i-})^+\Big)^+ \\
&\qquad - \sum_{0\leq s\leq t}(m_i - w_s^{i-})^+\Bigg) \\
&\quad + \sum_{0\leq s\leq t}\left(\ln\left(1 + \frac{w_{s-}^{i-}}{1-m_i}\left(\frac{m_i}{w_s^{i-}} - 1\right)\right)\right)^+,
\end{aligned} \tag{3.3}$$

where $w_t^{i-} := r_i(y^i_{t-}, y^j_t)$ for $t \geq 0$ and $a^{i,c}$ denotes the continuous part of a^i. The first and second term on the right-hand side of (3.3) govern the continuous and discontinuous components of the path $t \mapsto \Psi_t^{i,m_i}(y^i_{[0,t)}, y^j_{[0,t]}, a^i_{[0,t)})$, respectively. Proposition A.10 in Appendix A.5 proves that Ψ^{i,m_i} is the solution to $\mathrm{SP}^i_{m_i+}$. It also establishes conditions under which the separate Skorokhod reflections can be combined to form a two-sided Skorokhod reflection, which ultimately yields a Nash equilibrium.

At this point, it is necessary to introduce some notation.

Definition 3.7 For any $i \neq j \in \{a, b\}$, define the threshold

$$\hat{w}_i := \frac{-\alpha_i(1-\gamma_i)}{\gamma_i - \alpha_i}, \tag{3.4}$$

where

$$\alpha_i := \frac{1}{\sigma^2}\Big(k_i - \sqrt{k_i^2 + 2\sigma^2 p_i}\Big), \qquad \sigma^2 := \sigma_a^2 + \sigma_b^2,$$

$$p_i := \lambda^\kappa - (1-\gamma_i)\bigg(\mu_j - \frac{\gamma_i \sigma_j^2}{2}\bigg), \qquad k_i := \mu_j - \mu_i - \gamma_i \sigma_j^2 + \frac{\sigma^2}{2}.$$

Furthermore, set

$$q_i := \lambda^\kappa - (1-\gamma_i)\bigg(\mu_i - \frac{\gamma_i \sigma_i^2}{2}\bigg), \qquad a_i := 1 - \gamma_i - \alpha_i,$$

$$\beta_i := \frac{1}{\sigma^2}\Big(k_i + \sqrt{k_i^2 + 2\sigma^2 p_i}\Big), \qquad b_i := 1 - \gamma_i - \beta_i.$$

Let $\Delta := \{(w_a, w_b) \in (0,1)^2 : w_a + w_b < 1\}$ and for any $i \neq j \in \{a, b\}$, define the map $F^i : \Delta \to \mathbb{R}$ by

$$\begin{aligned} F^i(w_i, w_j) := {} & a_i\big(\alpha_i(1-\gamma_i-w_i) + \gamma_i w_i\big)(w_j - b_i)\bigg(\frac{w_i}{1-w_i}\bigg)^{b_i}\bigg(\frac{w_j}{1-w_j}\bigg)^{-\beta_i} \\ & - b_i\big(\beta_i(1-\gamma_i-w_i) + \gamma_i w_i\big)(w_j - a_i)\bigg(\frac{w_i}{1-w_i}\bigg)^{a_i}\bigg(\frac{w_j}{1-w_j}\bigg)^{-\alpha_i} \\ & + (a_i - b_i)\big(w_i(\alpha_i + \beta_i - 1) - \alpha_i\beta_i\big) w_j^{\gamma_i}(1-w_j)^{1-\gamma_i}. \end{aligned}$$

Note that F^i implicitly depends on the rate λ of the exponentially distributed random horizon (through α_i and β_i) and on the parameters of both traders', *except* trader j's risk aversion γ_j. The next result identifies the fraud thresholds used in Theorem 3.9 below to construct the Nash equilibrium.

Lemma 3.8 *There exists* $(\tilde{w}_a, \tilde{w}_b) \in \Delta$ *such that*

$$F^a(\tilde{w}_a, \tilde{w}_b) = F^b(\tilde{w}_b, \tilde{w}_a) = 0.$$

Moreover, any such pair $(\tilde{w}_a, \tilde{w}_b)$ *satisfies* $\tilde{w}_k < \hat{w}_k$ *for all* $k \in \{a, b\}$.

Theorem 3.9 *For* $(\tilde{w}_a, \tilde{w}_b)$ *as in Lemma 3.8, the pair* $(\Psi^{a,\tilde{w}_a}, \Psi^{b,\tilde{w}_b})$ *is a Nash equilibrium. In particular, for any* $i \neq j \in \{a, b\}$, *trader* i *cheats, if necessary, at time* 0 *so as to bring the wealth share instantly to* $\tilde{w}_i$. *Thereafter, the trader minimally cheats to keep that share above* $\tilde{w}_i$ *(the no-fraud region). The corresponding game values satisfy, for any* $i \neq j \in \{a, b\}$ *and* $x \in \mathbb{R}^2_{++}$,

$$V^i(x; A^{j,\star}) = \lambda(x_a + x_b)^{1-\gamma_i} \varphi^i\big(r_i(x)\big),$$

where $(A^{a,\star}, A^{b,\star})$ *is the equilibrium fraud process and*

$$\varphi^i(w) = \begin{cases} c_0^i(1-w)^{-\gamma_i}, & w \in (0, \tilde{w}_i), \\ c_1^i w^{\alpha_i}(1-w)^{a_i} + c_2^i w^{\beta_i}(1-w)^{b_i} + \frac{U^i(w)}{q_i}, & w \in [\tilde{w}_i, 1-\tilde{w}_j), \\ c_3^i w^{-\gamma_i}, & w \in [1-\tilde{w}_j, 1), \end{cases}$$

with the constants

$$c_0^i = \frac{-a_i b_i (1-\tilde{w}_i)^{\gamma_i} U^i(\tilde{w}_i)}{q_i((\alpha_i+\beta_i-1)\tilde{w}_i - \alpha_i\beta_i)} > 0,$$

$$c_1^i = \frac{-b_i \tilde{w}_i^{a_i}(1-\tilde{w}_i)^{-a_i}(\gamma_i\tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\beta_i)}{(1-\gamma_i)q_i(\beta_i-\alpha_i)((\alpha_i+\beta_i-1)\tilde{w}_i - \alpha_i\beta_i)} > 0,$$

$$c_2^i = \frac{a_i \tilde{w}_i^{b_i}(1-\tilde{w}_i)^{-b_i}(\gamma_i\tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\alpha_i)}{(1-\gamma_i)q_i(\beta_i-\alpha_i)((\alpha_i+\beta_i-1)\tilde{w}_i - \alpha_i\beta_i)} < 0,$$

$$c_3^i = (1-\tilde{w}_j)^{\gamma_i}\left(c_1^i(1-\tilde{w}_j)^{\alpha_i}\tilde{w}_j^{a_i} + c_2^i(1-\tilde{w}_j)^{\beta_i}\tilde{w}_j^{b_i} + \frac{U^i(1-\tilde{w}_j)}{q_i}\right) > 0.$$

For the purpose of comparative statics, it is also useful to consider the case when only one trader can commit fraud. Indeed, depending on circumstances, access to fraud may be uneven. For instance, Nick Leeson was able to conceal his unauthorised trades because he was allowed to settle his own trades (controlling both the front- and the back-office), a privilege that other traders of the firm did not share. In this regard, assuming that one of the two traders cannot cheat, the other trader maximises expected utility by cheating in a similar way, but with a different fraud threshold.

Theorem 3.10 *For any* $i \neq j \in \{a, b\}$, *if* $A^j \equiv 0$, *the optimal fraud process for trader* i *is* $A_t^{i,\star} = \Psi_t^{i,\hat{w}_i}(Y_{[0,t)}^{i,x}, Y_{[0,t)}^{j,x}, A_{[0,t)}^{i,\star})$ *for all* $t \geq 0$, *and the corresponding value function satisfies*

$$V^i(x;0) = \lambda(x_a+x_b)^{1-\gamma_i}\hat{\varphi}^i\big(r_i(x)\big)$$

for any $x \in \mathbb{R}_{++}^2$, *where*

$$\hat{\varphi}^i(w) = \begin{cases} s_0^i(1-w)^{-\gamma_i}, & w \in (0, \hat{w}_i), \\ s_1^i w^{\alpha_i}(1-w)^{a_i} + \frac{U^i(w)}{q_i}, & w \in [\hat{w}_i, 1), \end{cases}$$

with

$$s_0^i = \frac{a_i(1-\hat{w}_i)^{\gamma_i}}{q_i(\hat{w}_i-\alpha_i)}U^i(\hat{w}_i) > 0, \qquad s_1^i = \frac{1-\gamma_i-\hat{w}_i}{(1-\gamma_i)q_i(\hat{w}_i-\alpha_i)}\left(\frac{\hat{w}_i}{1-\hat{w}_i}\right)^{a_i} > 0.$$

Remark 3.11 Because $\hat{w}_i > \tilde{w}_i$ (Lemma 3.8), a rogue trader who knows that the other is honest has a higher cheating threshold than one who knows that the other can also cheat. The fraud region of trader i is indeed smaller in the Nash equilibrium, where both cheat as little as necessary (Theorem 3.9) to keep their proportion of wealth above $\tilde{w}_i$. Furthermore, $\lim_{\gamma_i \uparrow 1} \hat{w}_i = 0$ follows by (3.4), and then $\lim_{\gamma_i \uparrow 1} \tilde{w}_i = 0$ due to $\hat{w}_i > \tilde{w}_i$, which shows that fraud disappears with log-utility for both solo and dual rogue traders, as bankruptcy becomes unacceptable.

4 Discussion

This section brings to life the theoretical results in Sect. 3.2 by examining the properties of the Nash equilibrium for concrete parameter values.

4.1 Comparative statics

A trader's fraud threshold is relatively insensitive to the profitability of personal investments (Fig. 1, upper left), even as that profitability increases from 10% to 60%.

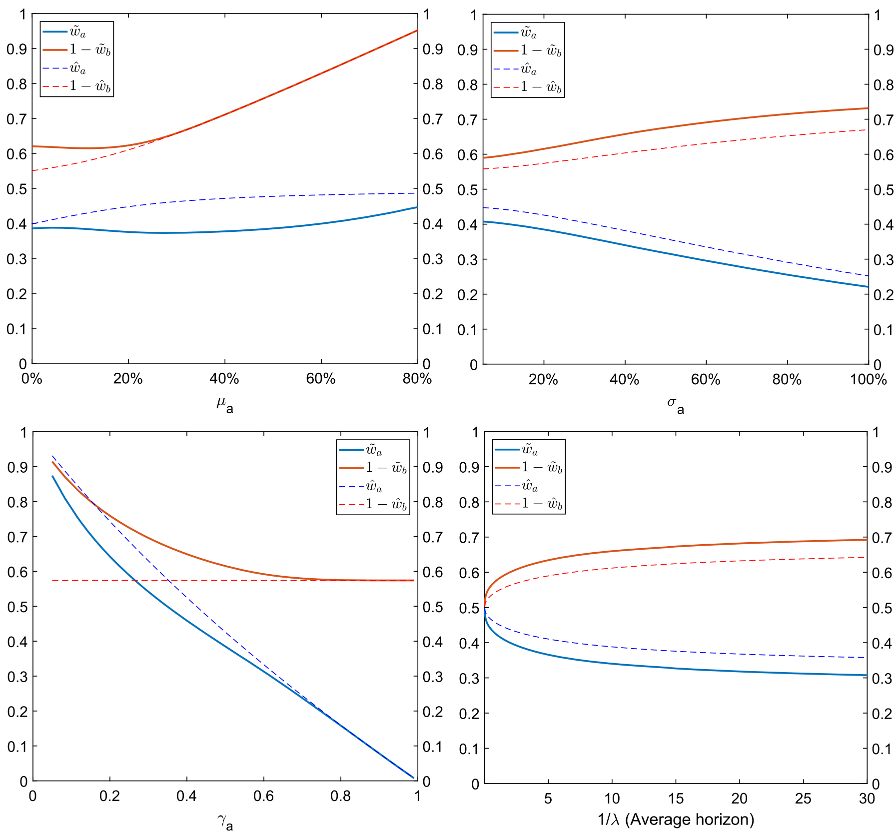


Fig. 1 Fraud thresholds for trader a (blue) and b (red), in view of trader a's share of wealth (vertical axis), in Nash equilibrium (solid line), and when the other trader is honest (dashed line), against trader a's expected return (upper left, $0\% \leq \mu_a \leq 60\%$), volatility (upper right, $0\% < \sigma_a \leq 100\%$), risk aversion (bottom left, $0 < \gamma_a < 1$) and average horizon (bottom right, $0 < 1/\lambda \leq 20$). Other parameters are $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$, $\gamma_a = \gamma_b = 0.5$, $\lambda = 1/3$, $\kappa = 10\%$

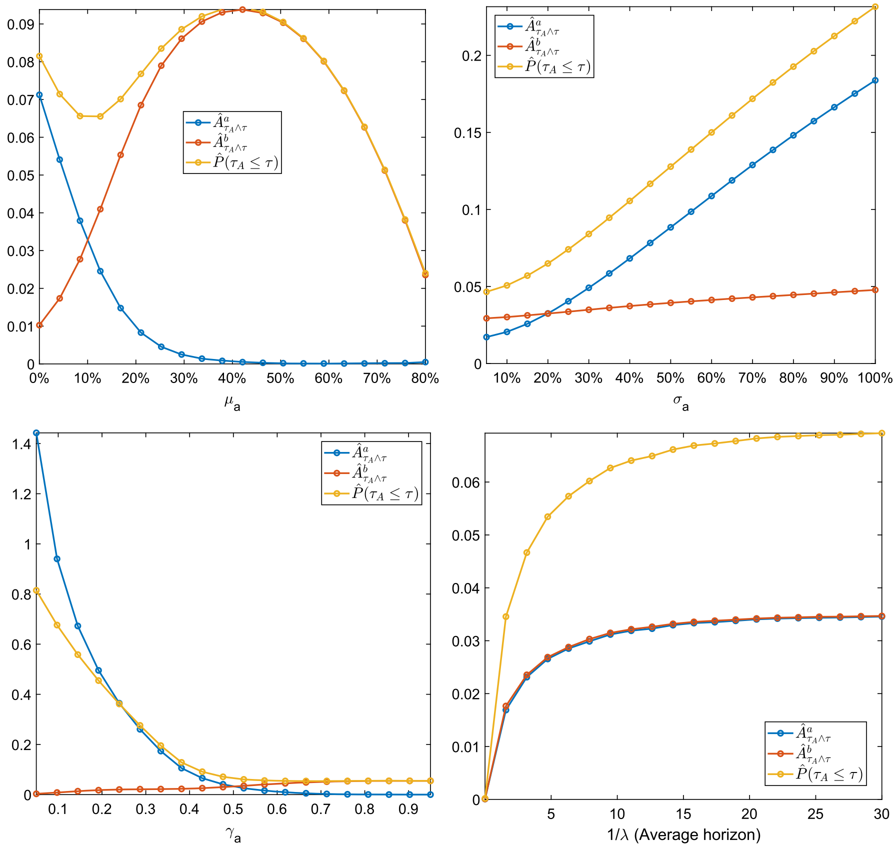


Fig. 2 Equilibrium average fraud, up to horizon or bankruptcy, of traders a (blue) and b (red), and bankruptcy probability (orange) against trader a's expected return (upper left, $0\% \le \mu_a \le 60\%$), volatility (upper right, $0\% < \sigma_a \le 100\%$), risk aversion (bottom left, $0.1 < \gamma_a < 0.9$) and average horizon (bottom right, $0 < 1/\lambda \le 20$). Results obtained from the simulation of 10^4 paths, each with step size $5 \cdot 10^{-4}$. Other parameters are $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$, $\gamma_a = \gamma_b = 0.5$, $w_a = w_b = 0.5$, $\lambda = 1/3$, $\kappa = 10\%$

The flatness of the threshold, however, does not imply the flatness of average fraud, which instead declines rapidly as profitability increases (Fig. 2, upper left). The explanation of this phenomenon lies in the dynamics of relative wealth shares: when one trader's profitability is high, that trader's wealth share tends to increase over time, thereby reaching the fraud threshold less often, hence generating lower fraud.

By contrast, the fraud threshold of the other trader (whose profitability remains constant) rapidly shifts upwards; hence this trader cheats when the respective wealth share falls below a lower threshold. Again, this does not imply a decline in the amount of personal fraud, because that trader's typical wealth share also tends to decline. In fact, Fig. 2 shows that the amount of fraud first increases up to $\mu_a \approx 40\%$, at which insolvency risk peaks, and then decreases: The initial rise is understood as a *short-term* appropriation, whereby the less skilled trader's higher fraud pilfers the other's profits. The subsequent decline is more akin to a *long-term* appropriation:

the less skilled trader recognises that the other's skill is so high that it is overall more profitable to limit the amount of fraud per unit of time so as to let the other's wealth grow faster, and that future fraud can be even more profitable. Put differently, the less skilled trader establishes a sort of parasite–host relationship with the more skilled trader, thereby avoiding excessive cheating, lest the host perish. Note also that the threshold of the more skilled trader is more sensitive to the honesty (or lack thereof) of the other trader, while the less skilled trader becomes indifferent to the other's honesty when the profitability is sufficiently high. Furthermore, the equality of traders' skills corresponds to a local minimum for bankruptcy risk, but the global minimum (approximately 2%) is achieved when one trader markedly outperforms the other one ($\mu_a = 80\%$ versus $\mu_b = 10\%$).

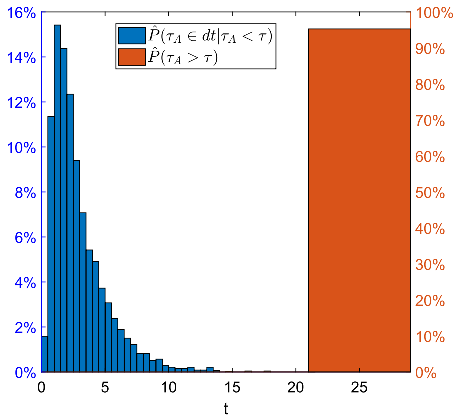
As the volatility of a trader's investments increases (upper right, Figs. 1 and 2), that trader's fraud threshold recedes aggressively, but total fraud and hence the probability of bankruptcy increase significantly. Increased volatility is qualitatively similar to lower skill, which makes the trader more reliant on fraud to generate profits. Vice versa, the other trader can still rely on a personal payoff with lower volatility, which would be significantly degraded by the additional asymmetry generated by more fraud.

Risk aversion (lower left, Figs. 1 and 2) has a major impact on the propensity to fraud. Holding the opponent's risk aversion constant at 0.5, as a trader's risk aversion increases from zero to one, the fraud threshold declines very rapidly from one (incessant fraud) to zero (no fraud). Note that as one fraud threshold declines, the other threshold also declines, not to zero, but to the threshold that assumes the other's honesty. Put differently, a fearless trader's propensity to fraud forces the other, more prudent trader to withdraw from fraud as the overall risk is already too high. The implication is that when the two traders have very different risk aversions but similar investment opportunities, it is the least risk-averse (in particular, if it is below 0.5) that has the most potential for fraud. Vice versa, when risk aversions are similar, the overall potential for fraud is evenly distributed between traders. Note that the insolvency probability is insensitive to the risk aversion when it is above 0.5 because the reduction of fraud from the more risk-averse trader is offset completely by the increase of fraud from the other trader with risk aversion 0.5.

Fraud completely disappears with unit risk aversion (i.e., logarithmic preferences). In this case, the dread of bankruptcy is so high that traders abstain from fraud regardless of its potential rewards. Note that this phenomenon stems from the fraud's inherent discontinuity, which always implies a probability, however small, that wealth may vanish. Put differently, for the logarithmic investor, the marginal utility of any amount of fraud is infinitely negative, regardless of expected profits.

The average horizon is also an important determinant of fraud (lower right, Figs. 1 and 2). Fraud thresholds recede as the horizon increases (λ decreases) and with it the expected reward for delaying fraud. In fact, the average amount of fraud increases sharply, up to a horizon of about five years, climbing steadily thereafter and eventually stabilising. The implication is that while a longer horizon helps in reducing fraud *per unit of time*, it does not reduce overall fraud, which in fact increases the most in the medium term – the typical turnover of traders in financial institutions.

Fig. 3 Distribution of the bankruptcy time τ_A in Nash equilibrium, conditionally on bankruptcy occurring before the terminal horizon τ against time ($0 \le t \le 20$) with a bin size of half a year (blue). The probability of survival (red) $\mathbb{P}[\tau_A > \tau]$ is $\approx 95\%$. Results are obtained from the simulation of 10^5 paths, each with step size $5 \cdot 10^{-4}$. Other parameters are $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$, $\gamma_a = \gamma_b = 0.5$, $w_a = w_b = 0.5$, $\lambda = 1/3$, $\kappa = 10\%$



4.2 Bankruptcy

Figures 1 and 2 demonstrate the dependence of the fraud thresholds on model parameters, the average amount of fraud of each trader and the bankruptcy probability. A direct application of the Doob–Meyer decomposition (Lemma A.5) reveals that in the Nash equilibrium, the bankruptcy probability satisfies

$$\mathbb{P}[\tau_a \le \tau] = \begin{cases} \mathbb{E}[A^{a,\star}_{\tau\wedge\tau_A} + A^{b,\star}_{\tau\wedge\tau_A}] - \ln\frac{1-w_a}{1-\tilde{w}_a} + \frac{\tilde{w}_a - w_a}{1-w_a}, & w_a < \tilde{w}_a, \\ \mathbb{E}[A^{a,\star}_{\tau\wedge\tau_A} + A^{b,\star}_{\tau\wedge\tau_A}], & \tilde{w}_a \le w_a < 1-\tilde{w}_b, \\ \mathbb{E}[A^{a,\star}_{\tau\wedge\tau_A} + A^{b,\star}_{\tau\wedge\tau_A}] - \ln\frac{1-w_b}{1-\tilde{w}_b} + \frac{\tilde{w}_b - w_b}{1-w_b}, & w_a \ge 1-\tilde{w}_b, \end{cases}$$

showing that the bankruptcy probability is the sum of the (stopped) fraud processes and an extra term if the initial share of wealth is in the fraud region. In Fig. 2, the initial share of wealth is 50%, which lies in the fraud-free region $[\tilde{w}_a, 1-\tilde{w}_b]$, except for falling into the personal fraud region $(0, \tilde{w}_a]$ when trader a's risk aversion is below 0.26.

Figure 3 depicts the distribution of the bankruptcy time τ_A conditionally on the event that bankruptcy occurs before the random horizon τ_A, assuming the traders are identical in risk aversion, skill and initial wealth. The distribution is skewed to the right: more than half of the insolvencies occur within the first 3 years (coinciding with the average time horizon $\mathbb{E}[\tau] = \frac{1}{\lambda} = 3$ years), reaching the peak of approximately 30% in the second year and quickly decreasing below 2% after the sixth year. The survival probability (red bar) is approximately 95%.

4.3 Welfare

Figure 4 compares trader a's expected utility in three scenarios: (i) both traders abstain from fraud, (ii) only trader a commits fraud, and (iii) both commit fraud in a Nash equilibrium. Once becoming the solo rogue trader, trader a's utility increases dramatically when the share of wealth is low; in contrast, that increment is insignificant when the share is high. The presence of the additional rogue trader b reduces the value function of the sole cheater across the span of the initial wealth share. This

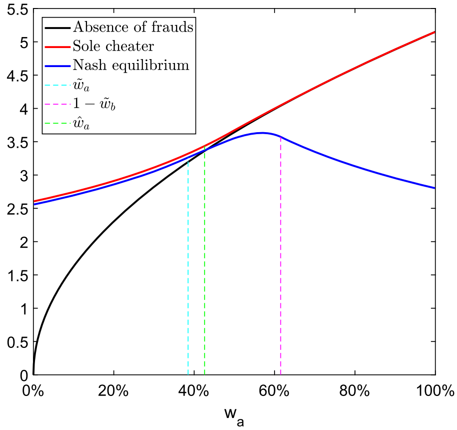


Fig. 4 Value functions of trader a in the absence of fraud (black), when trader a is the sole cheater (red) with fraud threshold $\hat{w}_a$ (dashed line in green) as in Theorem 3.10, and in the Nash equilibrium (blue) with fraud threshold $\tilde{w}_a$ (dashed line in cyan) and trader b's fraud threshold in view of trader a's wealth share $1-\tilde{w}_b$ (dashed line in purple) as in Theorem 3.9, against the initial share of the wealth of trader a ($0\% < w_a < 100\%$). Other parameters are $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$, $\gamma_a = \gamma_b = 0.5$, $\lambda = 1/3$, $\kappa = 10\%$, $x_a + x_b = \lambda^{-(1-\gamma_a)^{-1}}$

reduction is most significant when trader a has the most skin in the game (which coincides in this example with the fraud zone of trader b). Near the 50% wealth share, *both traders are better off abstaining completely from fraud* and even the prospect of solitary fraud yields little benefit. Thus in the case of two traders with similar ability, an equal allocation of managed wealth mitigates the potential for fraud.

4.4 Uncertain opponent's skill

In practice, a trader may not have perfect information about the other's investment skill and portfolio risk, but may be able to estimate them. Volatility can be determined rather precisely from frequent (say daily) observations of wealth history; indeed, in the model, volatility follows directly from the quadratic variation of the logarithmic wealth process, which is insensitive to fraud (which is a finite-variation process).

The situation is more delicate for the skill μ_j. As Theorem 3.9 proves that a rational trader cheats only when the respective wealth share drops below some boundary (and spends approximately zero time at that boundary), the cumulative return of the opponent satisfies

$$\frac{dY_t^j}{Y_t^j} = \mu_j dt + \sigma_j dB_t^j + dU_t, \qquad Y_0^j > 0,$$

where the continuous, nondecreasing process $(U_t)_{t\geq 0}$ (reflecting the contribution of fraud to returns) increases only on the set $\{(t,\omega) : r_j(Y_t^x(\omega)) = w_j\}$, where w_j is the fraud threshold. Thus the opponent's return includes the contributions of both skill and fraud, but the latter can be removed by excluding the returns that take place near the minimum of r_j. In practice, if the discrete-time observations are $(Y_{t_k}^j)_{0\leq k\leq n}$, the trader calculates the minimum $\underline{r} = \min_{1\leq k\leq n} r_j(Y^x_{t_{k-1}})$ and then estimates the opponent's skill μ_j from the returns as

$$\hat{\mu}_j = \frac{1}{m} \sum_{r_j(Y^x_{t_{k-1}})>\underline{r}+\varepsilon} \left(\frac{Y^j_{t_k}}{Y^j_{t_{k-1}}} - 1 \right), \qquad \text{where } m = \#\{1 \leq k < n : r_j(Y^x_{t_{k-1}}) > \underline{r} + \varepsilon\}$$

and the parameter ε is chosen so that the probability that $r_j(Y^x)$ reaches $\underline{r}$ between $r_j(Y^x_{t_{k-1}})$ and $r_j(Y^x_{t_k})$ is negligible; therefore the estimator of μ_j is approximately unbiased. Indeed, the probability that an Itô process with diffusion coefficient σ moves from $x > y$ to $z > y$ in time Δt without reaching y is approximately $e^{-2(x-y)(z-y)/(\sigma^2\Delta t)}$ (cf. Borodin and Salminen [6, 1.2.8 in Part II, Sect. 2.1]). Thus choosing $x - y, z - y \approx 2\sigma\sqrt{\Delta t}$, this probability is about $e^{-8} \approx 0.03\%$, which corresponds for daily observations to a frequency of less than one day in ten years ($0.03\% \cdot 252 \cdot 10 \approx 0.8$). Hence a reasonable choice for ε is two standard deviations of the daily change in wealth share.

The large-sample distribution of $\hat{\mu}_j$ is close to normal, but the trader recognises that the exact normal distribution is ill-suited to estimate the skill μ_j which is assumed to be positive and to satisfy Assumption 3.1. Instead, a viable alternative distribution that is close to normal while preserving positivity is the binomial distribution, so that trader i can more plausibly posit that

$$\mu_j \sim \mathrm{Bin}(n_j, p_j),$$

where the parameters n_j and p_j are identified by the first two moments $n_j p_j = \hat{\mu}_j$ and $n_j p_j(1 - p_j) = \hat{v}_j$, and $\hat{v}_j$ is the variance associated to the opponent's skill. (A frequentist trader who estimates the variance only from returns would choose $\hat{v}_j$ to be their sample variance, i.e., $\frac{1}{m-1} \sum_{r_j(Y_{t_{k-1}})>\underline{r}+\varepsilon} (Y^j_{t_k}/Y^j_{t_{k-1}} - 1 - \hat{\mu}^j)^2$. A Bayesian trader may use different estimators for $\hat{\mu}_j$ and $\hat{v}_j$, depending on the relative weight of the prior on the opponent's skill.) Then the trader can choose a personal cheating threshold that maximises the expected utility for an uncertain opponent's skill with the prescribed distribution.

Figure 5 highlights the impact of uncertainty on the opponent's skill on fraud. The left panel displays the probability mass function of the drift estimator, while the right panel displays the dependence of the average amount of fraud of each trader on the estimation error, holding the opponent's estimator of the trader's drift constant with mean 10% and error 5%. As a trader's estimation error of the opponent's skill increases from 1% to 10% (horizontal axis), fraud reduces significantly (approximately 10% with the chosen parameters), while the opponent's behaviour remains nearly constant.

This phenomenon arises because when the opponent's skill is uncertain, a hypothetical high skill implies a significant reduction in fraud, while a hypothetical low skill has little effect on fraud (cf. upper left panels in Figs. 1 and 2). This asymmetry implies that uncertainty on the opponent's skill is akin to its overestimation and partially mitigates fraud: traders who are unsure of each other's abilities behave as if their peers were more skilled than they actually are on average.

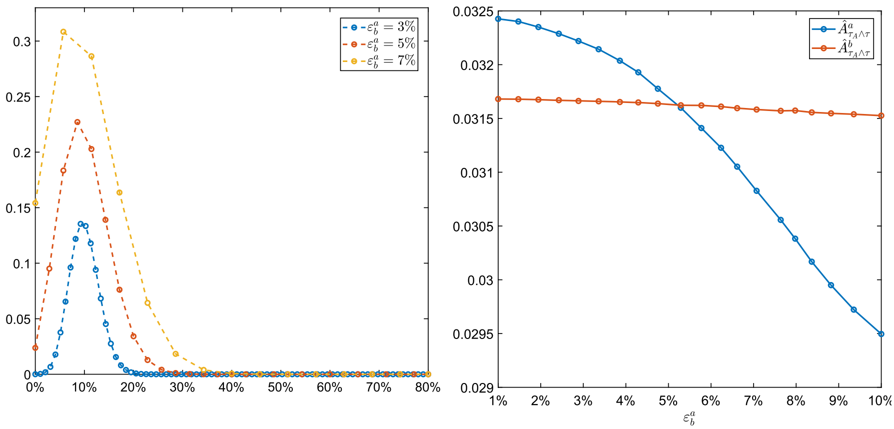


Fig. 5 (Left) Probability mass function of trader a's estimator $\hat{\mu}_b^a$ with mean 10% and standard deviation ε_b^a (3%, 5% and 7%, from top to bottom). (Right) Equilibrium average fraud (vertical axis) with estimated drifts, up to horizon or bankruptcy, of traders a (blue) and b (red) against trader a's estimation error ($1\% \leq \varepsilon_b^a \leq 10\%$). Results obtained from the simulation of 10^4 paths, each with step size $5 \cdot 10^{-4}$. Other parameters are $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$, $\gamma_a = \gamma_b = 0.5$, $w_a = w_b = 0.5$, $\lambda = 1/3$, $\kappa = 10\%$ and $\hat{\mu}_a^b$ has mean 10% and standard deviation $\varepsilon_a^b = 5\%$

4.5 The shareholders' problem

Suppose that if no bankruptcy occurs, each trader receives at the terminal horizon τ a fixed portion $p \in (0, 1)$ of wealth, with the remainder $1 - p$ distributed to shareholders. (Up to subtracting the initial wealth, this formulation is equivalent to (more realistically) rewarding traders with a fraction of *gains* rather than *wealth*. As an additive constant does not change the optimisation problem but complicates the notation, we do not discuss this variant.) For traders, the individual objective function is

$$\mathbb{E}\big[e^{-\kappa\tau} U^i\big(pX_\tau^{i,x}(A^i, A^j)\big)\big] = \frac{p^{1-\gamma_i}}{1-\gamma_i}\mathbb{E}\big[e^{-\kappa\tau} U^i\big(X_\tau^{i,x}(A^i, A^j)\big)\big].$$

The Nash equilibrium strategies of Theorem 3.9 remain unchanged under such constant scaling, while the game values are obtained by multiplying V^i (as in Theorem 3.9) with the constant $\frac{p^{1-\gamma_i}}{1-\gamma_i}$.

If shareholders are risk-neutral (as is customary in the corporate finance literature, in view of their ability to diversify investments across a multitude of assets), their objective is to maximise

$$J(x; A^a, A^b) := \mathbb{E}[e^{-\kappa\tau}(1-p)X_\tau^{S,x}(A^i, A^j)] \tag{4.1}$$

over all $(A^a, A^b) \in \mathcal{A}^2$. Denote by $J^S(x; A^a) := \mathbb{E}[e^{-\kappa\tau}(1-p)X_\tau^{a,x}(A^a)]$ the value of a sole trader's cheating strategy A^a.

Proposition 4.1 *Let $\lambda^\kappa > \mu_a \geq \mu_b$.*

(i) *If $\mu_a = \mu_b$, then a pair $(A^a, A^b) \in \mathcal{A}^2$ maximises the value in (4.1) if and only if it satisfies $\Delta A_t^a \Delta A_t^b = 0$ a.s. for all $t \geq 0$.*

(ii) *If $\mu_a > \mu_b$, then for any $(x_a, x_b) \in \mathbb{R}^2_{++}$ and $A^a \in \mathcal{A} \setminus \{0\}$, we have*

$$J\big((x_a, x_b); 0, 0\big) < J\big((x_a, x_b); A^a, 0\big), \tag{4.2}$$

with the inequality reversing if $\mu_a < \mu_b$. Moreover, the value function coincides with the value of a sole trader's cheating strategy, that is, for any $(x_a, x_b) \in \mathbb{R}^2_{++}$,

$$\sup_{(A^a, A^b) \in \mathcal{A}^2} J\big((x_a, x_b); A^a, A^b\big) = J^S(x_a + x_b; A^a) \qquad \text{for any } A^a \in \mathcal{A}.$$

However, the value function is unattainable in $\mathcal{A}^2$.

In Proposition 4.1, (i) states that unless there are simultaneous jumps of the fraud processes, shareholders are indifferent to any fraud. This is in particular the case for the Nash equilibrium in Theorem 3.9, where the only jumps may arise at inception, albeit not simultaneously.

By (ii), for shareholders, fraud of the more skilled trader is preferable to no fraud at all, which is in turn preferable to fraud by the less skilled trader. *Prima facie*, such a result is counterintuitive as fraud risk does not carry any premium. However, the fraud of the more highly skilled, accidentally rewarded as skill, helps in reducing the wealth share managed by the less skilled trader, thereby increasing the return on the firm's capital.

However, due to the final statement of (ii), the optimisation problem is ill posed. Nevertheless, the above intuition can be strengthened by including an additional control, which represents the initial share of assets under the management of trader a, to make the problem well posed. Let this control be $w_a \in [0, 1]$, where the firm recruits only trader a whenever $w_a = 1$ and only trader b whenever $w_a = 0$. Then by choosing $w_a = 1$, the supremum of $J((x_a, x_b); A^a, A^b)$ is indeed attained. Note that if the firm employs only a sole trader (say trader a), risk-neutral shareholders are indifferent to fraud as wagering bets with one's own capital has *zero* (rather than negative) risk premium, because wealth from only rogue trading (i.e., without legitimate investment) $X^a_t = \mathbf{1}_{\{t<\tau_A\}} e^{A^a_t}$, $t \geq 0$, is a $\mathbb{G}^A$-*martingale*. However, shareholders would be averse to fraud if it carried a negative risk premium because wealth would be a true supermartingale. To see this, modify the bankruptcy time (2.1) as

$$\tau^\epsilon_A = \inf\{t \geq 0 : (1+\epsilon) A^S_t \geq \theta\} \tag{4.3}$$

for some constant $\epsilon \geq 0$ representing the unit cost of fraud. Then fraud is indeed undesirable for risk-neutral shareholders because $(\mathbf{1}_{\{t<\tau_A\}} e^{A^a_t})$ is a true supermartingale for $\epsilon > 0$. Indeed, by viewing $(1+\epsilon)A^a$ as the fraud process, Proposition A.4 (i) implies that $(\mathbf{1}_{\{t<\tau^\epsilon_A\}} e^{(1+\epsilon)A^a_t})$ is a $\mathbb{G}^A$-martingale, whence

$$\mathbf{1}_{\{s<\tau^\epsilon_A\}} e^{A^a_s} = \mathbb{E}\big[\mathbf{1}_{\{t<\tau^\epsilon_A\}} e^{A^a_t + \epsilon(A^a_t - A^a_s)} \big| \mathcal{G}_s\big] \geq \mathbb{E}\big[\mathbf{1}_{\{t<\tau^\epsilon_A\}} e^{A^a_t} \big| \mathcal{G}_s\big]$$

for any $0 \leq s < t < \infty$, where the inequality is strict if and only if $\mathbb{P}[A^a_t > A^a_s] > 0$.

Denoting by $J^{S,\epsilon}$ the reward function corresponding to the modified bankruptcy time and with only trader a, it follows that for $\epsilon > 0$,

$$\sup_{A^a \in \mathcal{A}} J^{S,\epsilon}(x_a + x_b; A^a) = J^{S,\epsilon}(x_a + x_b; 0)$$

and for any $A^a \neq 0$ in $\mathcal{A}$,

$$J^{S,\epsilon}(x_a + x_b; A^a) < J^{S,\epsilon}(x_a + x_b; 0).$$

Since the survival processes satisfy $\mathbf{1}_{\{t<\tau_A^\epsilon\}} \leq \mathbf{1}_{\{t<\tau_A\}}$ a.s. for all $t \geq 0$ with equality if and only if $A^S \equiv 0$, then $J^\epsilon((x_a, x_b); A^a, A^b) \leq J((x_a, x_b); A^a, A^b)$ for any $(A^a, A^b) \in \mathcal{A}^2$ and $J^S(x_a + x_b; 0) = J^{S,\epsilon}(x_a + x_b; 0)$. By (ii), it follows that

$$\sup_{(A^a, A^b) \in \mathcal{A}^2, w_a \in [0,1]} J^\epsilon\big((x_a, x_b); A^a, A^b\big) = J^{S,\epsilon}(x_a + x_b; 0).$$

Therefore, using the model modification (4.3), the optimal policy for risk-neutral shareholders is to recruit only the more skilled trader a who, being alone, will then abstain from fraud (Proposition 2.1).

5 Conclusion

This paper develops a structural model of rational rogue trading. Self-interested, risk-averse traders can deliberately engage in fraudulent trading activity that can be concealed as superior performance while successful, but may lead to a firm's bankruptcy if unsuccessful. Traders abstain from fraud when they have sufficient skin in the game, suggesting that effective mitigation of rogue trading episodes should not focus on large traders alone.

Appendix A

A.1 Wealth of rogue traders

Recall the definition of the stochastic exponential (Jacod and Shiryaev [24, Eq. I.4.62]) of a general semimartingale.

Definition A.1 For any $\mathbb{R}$-valued semimartingale S with $S_{0-} \in \mathbb{R}$, the stochastic exponential of S is the process

$$\mathcal{E}(S)_t := \exp\left(S_t - S_{0-} - \frac{1}{2}[S^c]_t\right) \prod_{0 \leq s \leq t} e^{-\Delta S_s}(1 + \Delta S_s), \qquad t \geq 0,$$

where $\mathcal{E}(S)_{0-} = 1$ and S^c denotes the continuous local martingale part of S.

All stochastic exponentials in this article are a.s. strictly positive because the jump sizes are bounded away from -1 (cf. [24, Theorem I.4.61 (c)]. If the total variation of the jumps of S is finite, recall that

$$S_t = S_t^c + \sum_{0 \leq s \leq t} \Delta S_s$$

a.s. for all $t \geq 0$. Therefore the stochastic exponential then simplifies to

$$\mathcal{E}(S)_t = \exp\left(S_t^c - \frac{1}{2}[S^c]_t\right) \prod_{0 \leq s \leq t} (1 + \Delta S_s)$$

for $S_{0-} = 0$. The following result shows that the pre-bankruptcy wealth (2.2) is well defined and provides an expression in terms of a stochastic exponential.

Lemma A.2 *For any $k \in \{1, \ldots, N\}$, let $r_k(x) = \frac{x_k}{\sum_{i=1}^N x_i}$ for any $x \in \mathbb{R}_{++}^N$. Then*:

(i) *There exists a unique strong solution $Y^x = (Y^{1,x}, \ldots, Y^{N,x})$ of* (2.2), *and for all* $1 \leq i \leq N$, $\mathbb{P}[Y_t^{i,x} > 0$ *for all* $t \geq 0] = 1$.

(ii) *For all* $1 \leq i \leq N$ *and any* $t \geq 0$, *we have a.s. with* $\tilde{A}^S = \sum_{k=1}^N \tilde{A}^k$ *that*

$$Y_t^{i,x} = x_i \mathcal{E}\left(\mu_i \cdot + \sigma_i B_\cdot^i + \int_{[0,\cdot]} r_i(Y_{s-}^x)^{-1} d\tilde{A}_s^i\right)_t,$$

$$Y_t^{S,x} = \left(\sum_{k=1}^N x_k\right) \mathcal{E}\left(\sum_{k=1}^N \left(\int_0^\cdot \mu_k r_k(Y_s^x) ds + \int_0^\cdot \sigma_k r_k(Y_s^x) dB_s^k\right) + \tilde{A}_\cdot^S\right)_t. \tag{A.1}$$

Proof Denote by I_N the $N \times N$ identity matrix. The SDE (2.2) can be written in vector form as

$$dY_t^x = \operatorname{diag}(Y_t) dR_t + \operatorname{trace}\big(\operatorname{diag}(Y_{t-})\big) I_N d\tilde{A}_t, \qquad Y_{0-} = x \in \mathbb{R}_{++}^N, \tag{A.2}$$

where we define the process $R = (R^1, \ldots, R^N)$ by $R_t^i = \mu_i t + \sigma_i B_t^i$ for $1 \leq i \leq N$ and set $\tilde{A} = (\tilde{A}^1, \ldots, \tilde{A}^N)$. The linearity of the coefficients of (A.2) implies uniform Lipschitz-continuity, hence the existence and uniqueness of a strong solution (cf. Cohen and Elliott [10, Theorem 16.3.11]).

For any $1 \leq i \leq N$, let $Z_t^i = R_t^i + \tilde{A}_t^i$ and $H_t^i = x_i + \int_{[0,t]} \sum_{j \neq i}^N Y_{s-}^j d\tilde{A}_s^i$ for all $t \geq 0$, with $Z_{0-}^i = 0$ and $H_{0-}^i = x_i$. Rewriting (2.2) yields

$$Y_t^i = H_t^i + \int_{[0,t]} Y_{s-}^i dZ_t^i.$$

By Jacod [23, Theorem 6.8], it follows that

$$Y_t^i = \mathcal{E}(Z^i)_t \left(x_i + \int_{[0,t]} \mathcal{E}(Z^i)_{s-}^{-1} d\bar{H}_s^i\right), \tag{A.3}$$

where

$$\begin{aligned} \bar{H}_t^i &= H_t^i - \sum_{0 \leq s \leq t} \frac{\Delta H_s^i \Delta \tilde{A}_s^i}{1 + \Delta \tilde{A}_s^i} \\ &= H_t^{i,c} + \sum_{0 \leq s \leq t} \frac{\Delta H_s^i}{1 + \Delta \tilde{A}_s^i} = x_i + \int_{[0,t]} \sum_{j \neq i}^N Y_{s-}^j d\bar{A}_s^i \end{aligned} \tag{A.4}$$

and $\bar{A}^i_t = A^{i,c}_t + \sum_{0\le s\le t} \frac{\Delta \tilde{A}^i_s}{1+\Delta \tilde{A}^i_s}$ with $\bar{A}^i_{0-} = 0$. Substituting (A.4) into (A.3) yields

$$Y^i_t = \mathcal{E}(Z^i)_t \left(x_i + \int_{[0,t]} \mathcal{E}(Z^i)^{-1}_{s-} \sum_{j\ne i}^{N} Y^j_{s-} d\bar{A}^i_s \right). \tag{A.5}$$

Define the exit time $\tau_0 = \inf\{t \ge 0 : \min_{1\le k\le N} Y^k_t \le 0\}$. Suppose for a contradiction that $\mathbb{P}[0 \le \tau_0 < \infty] > 0$. Then for any $\omega \in \{\omega \in \Omega : 0 \le \tau_0(\omega) < \infty\}$, there exists $1 \le q \le N$ such that $Y^q_{\tau_0(\omega)}(\omega) \le 0$ and $Y^q_{\tau_0(\omega)-}(\omega) \ge 0$ because Y^q is a càdlàg process. Since $x_q > 0$, (A.5) implies that $\sum_{j\ne q}^N Y^j_s(\omega) < 0$ for some $s < \tau_0(\omega)$ which contradicts the definition of τ_0, thereby completing the proof of (i).

Thus rewrite (2.2) and the firm's total pre-bankruptcy wealth Y^S as

$$\frac{dY^i_t}{Y^i_{t-}} = dR^i_t + \frac{Y^S_{t-}}{Y^i_{t-}} d\tilde{A}^i_t, \qquad Y^i_{0-} = x_i,$$

$$\frac{dY^S_t}{Y^S_{t-}} = \sum_{k=1}^N \frac{Y^k_t}{Y^S_{t-}} dR^k_t + d\tilde{A}^S_t, \qquad Y^S_{0-} = \sum_{k=1}^N x_k.$$

An application of [23, Theorem 6.8] yields (ii). □

The following lemma characterises the conditional probability of bankruptcy in relation to total fraud.

Lemma A.3 *The following hold a.s. for all* $t \ge 0$:

$$\mathbb{P}[\tau_A > t | \mathcal{F}_t] = e^{-A^S_t}, \tag{A.6}$$

$$\mathbb{P}[\tau_A > t | \mathcal{F}_\infty] = \mathbb{P}[\tau_A > t | \mathcal{F}_t]. \tag{A.7}$$

Proof First, we show that

$$\{t < \tau_A\} = \{A^S_t < \theta\}. \tag{A.8}$$

On the one hand, $\{t < \tau_A\} \subseteq \{A^S_t < \theta\}$ by the definition of τ_A. On the other hand, let $\omega \in \Omega$ be such that $A^S_t(\omega) < \theta(\omega)$. If $\tau_A(\omega) < \infty$, then $\theta(\omega) \le A^S_{\tau_A(\omega)}(\omega)$. Hence $t < \tau_A(\omega)$ because $A^S_t(\omega) < A^S_{\tau_A(\omega)}(\omega)$ and A^S is nondecreasing. If we have instead $\tau_A(\omega) = \infty$, then trivially $t < \tau_A(\omega)$.

As θ is independent of $\mathcal{F}_\infty$, and exponentially distributed with unit mean, (A.8) implies that

$$\mathbb{P}[\tau_A > t | \mathcal{F}_\infty] = \mathbb{P}[A^S_t < \theta | \mathcal{F}_\infty] = e^{-A^S_t}.$$

Because A^S is $\mathbb{F}$-adapted and by the tower property of conditional expectations, (A.6) and (A.7) follow. □

We next show that the treatment of jumps in the definition of $\tilde{A}$ is the only one consistent with the martingale property for wealth in the absence of skill.

Proposition A.4 *Let* $N = 1$, $\mu_1 = 0$ *and* $A^g_t := A^{1,c}_t + \sum_{0\le s\le t} g(\Delta A^1_s)$, *where the function* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *is measurable with* $g(0) = 0$. *Let* $\bar{Y}$ *be the solution of the SDE*

$$d\bar{Y}^{1,x}_t = \bar{Y}^{1,x}_{t-}(\sigma_1 dB^1_t + dA^g_t), \qquad \bar{Y}^{1,x}_{0-} = x_1 > 0,$$

and

$$\bar{X}^{1,x}_t := \mathbf{1}_{\{t<\tau_A\}}\bar{Y}^{1,x}_t \qquad \textit{with } \bar{X}^{1,x}_{0-} = x_1.$$

Then:

(i) *If* $A^1_t = A^{1,c}_t$ *a.s. for all* $t \ge 0$ *or if* $g(a) = e^a - 1$, *then* $\bar{X}^{1,x_1}$ *is a* $\mathbb{G}^A$*-martingale.*

(ii) *If* $\bar{X}^{1,x_1}$ *is a* $\mathbb{G}^A$*-martingale for any* $A^1 \in \mathcal{A}$, *then* $g(a) = e^a - 1$ *for any* $a \ge 0$.

Proof (i) By Aksamit and Jeanblanc [1, Lemma 3.8], (A.7) implies the immersion property, i.e., any $\mathbb{F}$-martingale remains a martingale in the enlarged filtration $\mathbb{G}^A$. It then follows by Lévy's characterisation theorem that B^1 remains a Brownian motion in $\mathbb{G}^A$. Lemma A.2 (ii) and Cohen and Elliott [10, Corollary 15.1.9] yield that

$$X^{i,x}_t = x_i \mathbf{1}_{\{t<\tau_A\}}\mathcal{E}(\sigma_1 B^1 + \tilde{A}^1)_t = x_i \mathbf{1}_{\{t<\tau_A\}}\mathcal{E}(\tilde{A}^1)_t \mathcal{E}(\sigma_1 B^1)_t$$

a.s. for all $t \ge 0$. If A^1 has a.s. continuous paths or $g(a) = e^a - 1$, then $\mathcal{E}(A^g)_t = e^{A^1_t}$, and $(\mathbf{1}_{\{t<\tau_A\}}e^{A^1_t})$ is a $\mathbb{G}^A$-martingale by Bielecki and Rutkowski [5, Lemma 5.1.7]. Because the covariation between $(\mathbf{1}_{\{t<\tau_A\}}e^{A^1_t})$ and $\mathcal{E}(\sigma_1 B^1)$ is zero and $X^{1,x_1}_t = X^{1,x_1}_{t\wedge\tau_A}$ a.s. for all $t \ge 0$, it follows that $X^{i,x}$ is a $\mathbb{G}^A$-martingale.

(ii) Consider the family A^ξ of strategies indexed by $\xi \ge 0$, defined for $t \ge 0$ by $A^\xi_t = 1_{\{t\ge 1\}}\xi$. By construction, $A^\xi \in \mathcal{A}$ for all $\xi \ge 0$. Denote the corresponding wealth by $\bar{X}^{1,x_1,\xi}$. By assumption, this is a $\mathbb{G}^A$-martingale for any $\xi \ge 0$. One can factorise it as $\bar{X}^{1,x_1,\xi} = MU$, where for any $t \ge 0$,

$$M_t = x_1 \mathbf{1}_{\{t<\tau_A\}} e^{A^1_t}\mathcal{E}(\sigma_1 B^1)_t, \tag{A.9}$$

$$U_t = \prod_{0\le s\le t\wedge\tau_A} e^{-\Delta A^1_s}\big(1 + g(\Delta A^1_s)\big) = \mathbf{1}_{\{t<1\}} + e^{-\xi}\big(1 + g(\xi)\big)\mathbf{1}_{\{t\ge 1\}}.$$

Note that M is a $\mathbb{G}^A$-martingale and by Lemma A.5, the finite-variation process U is $\mathbb{G}^A$-predictable. Integration by parts (cf. [1, Proposition 1.16]) yields

$$\bar{X}^{1,x_1}_t = M_t U_t = x + \int_{[0,t]} U_s dM_s + \int_{[0,t]} M_{s-} dU_s.$$

The process $(\int_0^t U_s dM_s)_{t\ge 0}$ is a $\mathbb{G}^A$-local martingale. By Jacod and Shiryaev [24, Proposition I.3.5], the process $(\int_0^t M_{s-} dU_s)_{t\ge 0}$ inherits the $\mathbb{G}^A$-predictability and the finite-variation property from its integrator U. Because $\bar{X}^{1,x_1,\xi}$ is a $\mathbb{G}^A$-martingale, $(\int_0^t M_{s-}dU_s)_{t\geq 0}$ is a $\mathbb{G}^A$-local martingale. Then by [10, Lemma 10.3.9], the process $(\int_0^t M_{s-}dU_s)_{t\geq 0}$ is constant. Since $M_{1-} > 0$ with positive probability and

$$\int_0^t M_{s-}dU_s = \begin{cases} 0, & t < 1, \\ M_{1-}(e^{-\xi}(1+g(\xi)) - 1), & t \geq 1, \end{cases}$$

it follows that $e^{-\xi}(1+g(\xi)) = 1$ for all $\xi \geq 0$. Note that in (A.9), all quantities except for the indicator function are strictly positive and $\mathbb{P}[\tau_A \geq 1] > 0$ because in view of (A.8), $0 < \mathbb{P}[A_1^\xi < \theta] \leq \mathbb{P}[\bigcap_{\varepsilon\in(0,1)}\{A_{1-\varepsilon} < \theta\}] = \mathbb{P}[\bigcap_{\varepsilon\in(0,1)}\{\tau_A > 1-\varepsilon\}]$. $\square$

A.2 Proof of Proposition 2.1

By Lemma A.2, trader 1's wealth is of the form

$$X_t^{1,x_1} = \mathbf{1}_{\{t<\tau_A\}} x_1 e^{A_t^1 + (\mu_1 - \sigma_1^2/2)t + \sigma_1 B_t^1}, \qquad t \geq 0.$$

Hence by Lemma A.3,

$$\begin{aligned}\mathbb{E}[U^1(X_t^{1,x_1})|\mathcal{F}_t] &= \mathbb{E}[\mathbf{1}_{\{t<\tau_A\}} U^1(Y_t^{1,x_1})|\mathcal{F}_t] = e^{-A_t^1} U^1(Y_t^{1,x_1}) \\ &= \frac{x_1^{1-\gamma_1}}{1-\gamma_1} e^{-\gamma_1 A_t^1 + (1-\gamma_1)((\mu_1 - \sigma_i^2/2)t + \sigma_1 B_t^1)} \leq \frac{x_1^{1-\gamma_1}}{1-\gamma_1} e^{(1-\gamma_1)((\mu_1 - \sigma_i^2/2)t + \sigma_1 B_t^1)}.\end{aligned}$$

Therefore, by the tower property of conditional expectations,

$$\mathbb{E}[U^1(X_t^{1,x_1})] \leq \frac{x_1^{1-\gamma_1}}{1-\gamma_1} e^{(1-\gamma_1)(\mu_1 - \gamma_1\sigma_1^2/2)t} \tag{A.10}$$

and

$$\mathbb{E}[U^1(X_t^{1,x_1})] = \frac{x_1^{1-\gamma_1}}{1-\gamma_1} e^{(1-\gamma_1)(\mu_1 - \gamma_1\sigma_1^2/2)t} \quad \text{if and only if} \quad A_t^1 = 0 \text{ a.s.} \tag{A.11}$$

Let $\mathbb{P}_\tau$ be the law of τ, i.e., $\mathbb{P}_\tau[U] = \mathbb{P}[\tau \in U]$ for any Borel-measurable set $U \subseteq \mathbb{R}_+$. Then by the law of total probability and the independence of τ from B and θ,

$$\begin{aligned}\mathbb{E}[e^{-\kappa\tau} U^1(X_\tau^{1,x_1})] &= \int_0^\infty \mathbb{E}[e^{-\kappa\tau} U^1(X_\tau^{1,x_1})|\tau = t] d\mathbb{P}_\tau(dt) \\ &= \int_0^\infty e^{-\kappa t} \mathbb{E}[U^1(X_t^{1,x_1})] d\mathbb{P}_\tau(dt).\end{aligned} \tag{A.12}$$

Thus (A.10) implies that

$$\begin{aligned}\mathbb{E}[e^{-\kappa\tau} U^1(X_\tau^{1,x_1})] &\leq \frac{x_1^{1-\gamma_1}}{1-\gamma_1} \int_0^\infty e^{((1-\gamma_1)(\mu_1 - \gamma_1\sigma_1^2/2) - \kappa)t} \mathbb{P}_\tau[dt] \\ &= \frac{x_1^{1-\gamma_1}}{1-\gamma_1} \mathbb{E}[e^{((1-\gamma_1)(\mu_1 - \gamma_1\sigma_1^2/2) - \kappa)\tau}],\end{aligned} \tag{A.13}$$

and due to (A.11) and (A.12),

$$\mathbb{E}[e^{-\kappa\tau}U^1(X_\tau^{1,x_1})] = \frac{x_1^{1-\gamma_1}}{1-\gamma_1}\mathbb{E}[e^{((1-\gamma_1)(\mu_1-\gamma_1\sigma_1^2/2)-\kappa)\tau}]$$

if and only if $\mathbb{P}[A_t^1 = 0] = 1$ for all $t > 0$ for which $\mathbb{P}[\tau \geq t] > 0$. In fact, suppose that on the contrary, there exists some $t_0 \geq 0$ for which $\mathbb{P}[\tau \geq t_0] > 0$, but $\mathbb{P}[A_{t_0}^1 > 0] > 0$. Because A^1 is nondecreasing a.s., we have $\mathbb{P}[A_t^1 \geq A_{t_0}^1 > 0] = \mathbb{P}[A_{t_0}^1 > 0] > 0$ for $t \geq t_0$ and so (A.11) implies that

$$\mathbb{E}[U^1(X_t^{1,x_1})] < \frac{x_1^{1-\gamma_1}}{1-\gamma_1}e^{(1-\gamma_1)(\mu_1-\gamma_1\sigma_1^2/2)t}, \qquad t \geq t_0.$$

As $\mathbb{P}[\tau \geq t_0] > 0$, integration (cf. (A.12)) yields the strict inequality in (A.13). □

A.3 Doob–Meyer decomposition

Let $\bar{A}_t^S = A_t^{S,c} + \sum_{0\leq s\leq t}(1-e^{-\Delta A_s^S})$ for all $t \geq 0$ and note that $\bar{A}^S \in \mathcal{A}$. In the absence of fraud jumps, the total fraud process A^S is the $\mathbb{G}^A$-compensator of the bankruptcy time τ_A. However, in the presence of jumps, the compensator is in fact $\bar{A}^S$.

Lemma A.5 *The process M^A defined as*

$$M_t^A = \mathbf{1}_{\{t\geq\tau_A\}} - \bar{A}_{t\wedge\tau_A}^S, \qquad t \geq 0, \tag{A.14}$$

is a uniformly integrable $\mathbb{G}^A$-martingale. Furthermore, $(\bar{A}_{t\wedge\tau_A}^S)_{t\geq 0}$ is the unique $\mathbb{G}^A$-predictable, integrable and nondecreasing process B such that $(\mathbf{1}_{\{t\geq\tau_A\}} - B_t)_{t\geq 0}$ is a $\mathbb{G}^A$-martingale and $B_{0-} = 0$.

Proof The nondecreasing process $(\mathbf{1}_{\{t\geq\tau_A\}})_{t\geq 0}$ is a $\mathbb{G}^A$-submartingale. Define the process $(Z_t)_{t\geq 0}$ by $Z_t := \mathbb{P}[t < \tau_A|\mathcal{F}_t]$. Because $\mathbb{F} \subseteq \mathbb{G}^A$ and all $\mathbb{F}$-martingales are continuous by the martingale representation theorem (Karatzas and Shreve [29, Theorem 3.4.2]), the dual $\mathbb{F}$-predictable projection of $(\mathbf{1}_{\{t\geq\tau_A\}})_{t\geq 0}$ is $1 - Z$ by Aksamit and Jeanblanc [1, Proposition 3.9 (b)]. It follows by [1, Proposition 2.15] that the $\mathbb{G}^A$-compensator of τ_A is the process $(\int_{[0,t\wedge\tau_A]} Z_{s-}^{-1}d(1-Z_s))_{t\geq 0}$. So by Lemma A.3 and the Itô formula, we get a.s. for all $t \geq 0$ that

$$\int_{[0,t\wedge\tau_A]} Z_{s-}^{-1}d(1-Z_s) = \int_{[0,t\wedge\tau_A]} e^{A_{s-}^S}d(-e^{-A_{s-}^S}) = A_{t\wedge\tau_A}^{S,c} + \sum_{0\leq s\leq t\wedge\tau_A}(1-e^{-\Delta A_s^S}).$$

□

Lemma A.6 *The bankruptcy time τ_A is $\mathbb{G}^A$-predictable if and only if $A_{t\wedge\tau_A}^S = A_0^S$ a.s. for all $t \geq 0$.*

Proof Suppose τ_A is a $\mathbb{G}^A$-predictable stopping time. It follows that the process $(\mathbf{1}_{\{t\geq\tau_A\}})_{t\geq 0}$ is $\mathbb{G}^A$-predictable. Lemma A.5 implies that the $\mathbb{G}^A$-martingale M^A defined by (A.14) is $\mathbb{G}^A$-predictable. A predictable martingale of finite variation must be constant; hence $M_t^A = M_0^A$ a.s. for all $t \geq 0$. It follows that for any $t \geq 0$,

$$\mathbf{1}_{\{t\geq\tau_A\}} - \mathbf{1}_{\{0=\tau_A\}} = \bar{A}^S_{t\wedge\tau_A} - \bar{A}^S_0 \qquad \text{a.s.}$$

On the event $\{0 < \tau_A < \infty\}$, for any $t < \tau_A$, $\bar{A}^S_{t\wedge\tau_A} - \bar{A}^S_0 = 0$ and thus $\Delta\bar{A}^S_{\tau_A} = 1$ a.s., which contradicts $\Delta\bar{A}^S_t = 1 - e^{-A^S_t} < 1$. Hence $\mathbb{P}[0 < \tau_A < \infty] = 0$. However, on the events $\{\tau_A = 0\}$ and $\{\tau_A = \infty\}$, it is clear that $\bar{A}^S_{t\wedge\tau_A} = \bar{A}^S_0$ and thus $A^S_{t\wedge\tau_A} = A^S_0$ a.s. for all $t \geq 0$.

Conversely, let $A^S_{t\wedge\tau_A} = A^S_0$ a.s. for all $t \geq 0$. Then the definition (2.1) of bankruptcy implies that $\tau_A \in \{0, \infty\}$ a.s. As the events $\{\tau_A = 0\}$ and $\{\tau_A = \infty\}$ are in $\mathcal{G}^A_0$, the increasing sequence (τ_n) of $\mathbb{G}^A$-stopping times defined by

$$\tau_n = \mathbf{1}_{\{\tau_A=0\}} + n\mathbf{1}_{\{\tau_A=\infty\}}$$

announces τ_A. □

Remark A.7 The preceding result implies that the bankruptcy time is announced by a strictly increasing sequence of stopping times if and only if either all traders abstain from fraud or all fraudulent trades are performed at the initial time $t = 0$.

A.4 Value function (Proof of Lemma 3.2)

Lemma A.8 *For any $p \in (0, 1)$ and $t \geq 0$,*

$$\mathbb{E}[(X_t^{S,x})^p] \leq \left(\sum_{k=1}^N x_k\right)^p e^{pt\max_{1\leq k\leq N}\mu_k}.$$

Proof Using the expression of $Y^{S,x}$ in (A.1), since the covariation between the terms inside $\mathcal{E}(\cdot)$ in (A.1) is zero, it follows by [10, Corollary 15.1.9] that

$$Y_t^{S,x} = \left(\sum_{k=1}^N x_k\right)\mathcal{E}(\tilde{A}^S_\cdot)_t\,\mathcal{E}\left(\sum_{k=1}^N \mu_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}}ds\right)_t \mathcal{E}\left(\sum_{k=1}^N \sigma_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}}dB_s^k\right)_t$$

and, rearranging terms,

$$\begin{aligned}
&Y_t^{S,x}\mathcal{E}(\tilde{A}^S_\cdot)_t^{-1}\\
&= \left(\sum_{k=1}^N x_k\right)\mathcal{E}\left(\sum_{k=1}^N \mu_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}}ds\right)_t \mathcal{E}\left(\sum_{k=1}^N \sigma_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}}dB_s^k\right)_t\\
&\leq \left(\sum_{k=1}^N x_k\right)\mathcal{E}\left(\left(\max_{1\leq k\leq N}\mu_k\right)\sum_{k=1}^N \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}}ds\right)_t \mathcal{E}\left(\sum_{k=1}^N \sigma_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}}dB_s^k\right)_t\\
&= \left(\sum_{k=1}^N x_k\right)e^{(\max_{1\leq k\leq N}\mu_k)t}\,\mathcal{E}\left(\sum_{k=1}^N \sigma_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}}dB_s^k\right)_t. \qquad \text{(A.15)}
\end{aligned}$$

For any $1 \le k \le N$ and any $t \ge 0$,

$$\exp\left(\frac{\sigma_k^2}{2}\int_0^t \left(\frac{Y_s^{k,x}}{Y_s^{S,x}}\right)^2 ds\right) \le \exp\left(\frac{\sigma_k^2}{2}t\right) < \infty \qquad \text{a.s.}$$

Hence Novikov's condition is satisfied and

$$\mathcal{E}\left(\sum_{k=1}^N \sigma_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}} dB_s^k\right)_{t\ge 0}$$

is a true martingale. By taking expectations on both sides of (A.15),

$$\mathbb{E}[Y_t^{S,x}\mathcal{E}(\tilde{A}_\cdot^S)_t^{-1}] \le \left(\sum_{k=1}^N x_k\right) e^{(\max_{1\le k\le N}\mu_k)t}. \tag{A.16}$$

The stochastic exponential $\mathcal{E}(\tilde{A}^S)$ satisfies

$$\begin{aligned}\mathcal{E}(\tilde{A}^S)_t &= e^{\tilde{A}_t^S}\prod_{0\le s\le t}(1+\Delta\tilde{A}_s^S)e^{-\Delta\tilde{A}_s^S} = e^{A_t^{S,c}}\prod_{0\le s\le t}\left(1+\sum_{k=1}^N(e^{\Delta A_s^k}-1)\right)\\ &\le e^{A_t^{S,c}}\prod_{0\le s\le t} e^{\sum_{k=1}^N \Delta A_s^k} = e^{A_t^S},\end{aligned} \tag{A.17}$$

where the inequality (A.17) holds due to the inequality

$$e^{\sum_{k=1}^N y_k} - 1 \ge \sum_{k=1}^N (e^{y_k}-1)$$

for any $(y_1,\ldots,y_N) \in \mathbb{R}_{++}^N$. By Lemma A.3 (i), for any $t \ge 0$,

$$\mathbb{E}[X_t^{S,x}] = \mathbb{E}[\mathbf{1}_{\{t<\tau_A\}}Y_t^{S,x}] = \mathbb{E}\big[Y_t^{S,x}\mathbb{E}[\mathbf{1}_{\{t<\tau_A\}}|\mathcal{F}_t]\big] = \mathbb{E}[e^{-A_t^S}Y_t^{S,x}]. \tag{A.18}$$

Using (A.18) and the estimate (A.17), one obtains

$$\mathbb{E}[X_t^{S,x}] \le \mathbb{E}[Y_t^{S,x}\mathcal{E}(\tilde{A}_\cdot^S)_t^{-1}]. \tag{A.19}$$

Finally, for any $0 < p \le 1$, Jensen's inequality, (A.16) and (A.19) yield

$$\mathbb{E}[(X_t^{S,x})^p] \le (\mathbb{E}[X_t^{S,x}])^p \le \left(\sum_{k=1}^N x_k\right)^p e^{pt\max_{1\le k\le N}\mu_k}.$$

□

Proof of Lemma 3.2 The independence of τ and $\mathcal{F}_\infty \vee \sigma(\theta)$, the tower property of conditional expectations and Lemma A.3 yield for any $i \neq j \in \{a, b\}$ and $A^i, A^j \in \mathcal{A}$ that

$$\begin{aligned}
\mathbb{E}\big[e^{-\kappa\tau} U^i\big(X_\tau^{i,x}(A^i, A^j)\big)\big] &= \mathbb{E}\Big[\mathbb{E}\big[e^{-\kappa\tau} U^i\big(X_\tau^{i,x}(A^i, A^j)\big)\big|\mathcal{F}_\infty \vee \sigma(\theta)\big]\Big] \\
&= \mathbb{E}\Big[\int_0^\infty \lambda e^{-\lambda^\kappa t} U^i\big(X_t^{i,x}(A^i, A^j)\big)dt\Big] \\
&= \lambda\mathbb{E}\Big[\int_0^\infty e^{-\lambda^\kappa t}\mathbb{E}[\mathbf{1}_{\{t<\tau_A\}}|\mathcal{F}_t] U^i\big(Y_t^{i,x}(A^i, A^j)\big)dt\Big] \\
&= \lambda\mathbb{E}\Big[\int_0^\infty e^{-\lambda^\kappa t - A_t^S} U^i\big(Y_t^{i,x}(A^i, A^j)\big)dt\Big]
\end{aligned}$$

which establishes (ii).

By Lemma A.2 (ii), the processes $Y^{i,cx}$ and $cY^{i,x}$ are indistinguishable and thus the scale-invariance (iii) holds. Finally, Tonelli's theorem and Lemma A.8 yield

$$\begin{aligned}
J^i(x; A^i, A^j) &= \frac{\lambda}{1-\gamma_i}\int_0^\infty e^{-\lambda^\kappa t}\mathbb{E}\big[\big(X_t^{i,x}(A^i, A^j)\big)^{1-\gamma_i}\big]dt \\
&\le \frac{\lambda(x_a + x_b)^{1-\gamma_i}}{1-\gamma_i}\int_0^\infty e^{-\lambda^\kappa t + (1-\gamma_i)(\mu_a \vee \mu_b)t}dt \\
&= \frac{\lambda U^i(x_a + x_b)}{\lambda^\kappa - (1-\gamma_i)(\mu_a \vee \mu_b)}.
\end{aligned}$$

Taking the supremum over $A^i \in \mathcal{A}$, (i) follows. □

A.5 Skorokhod reflection

Assume **for the rest of this chapter** that there are $N = 2$ traders. First, we study the SDE that identifies the fractions of the wealth of each trader. For any $i \neq j \in \{a, b\}$, define the coefficient functions (of a single variable)

$$\begin{aligned}
\bar{b}_i(w) &:= w(1-w)\big(\sigma_j^2(1-w) - \sigma_i^2 w + \mu_i - \mu_j\big), \\
\bar{\sigma}_i(w) &:= \begin{cases} (\sigma_a w(1-w), -\sigma_b w(1-w)) & \text{if } i = a, \\ (-\sigma_a w(1-w), \sigma_b w(1-w)) & \text{if } i = b. \end{cases}
\end{aligned}$$

Introduce also the processes $(\tilde{Q}_t^i)_{t\ge 0}$ given by $\tilde{Q}_t^i = A_t^{i,c} + \sum_{0\le s\le t} q_i(\Delta\tilde{A}_s)$, where $q_i : \mathbb{R}_+^2 \to \mathbb{R}_+$ is defined as $q_i(a_1, a_2) = \frac{a_i}{1+a_1+a_2}$.

Lemma A.9 (i) *For any $i \neq j \in \{a,b\}$ and $x \in \mathbb{R}^2_{++}$, the trader's share of wealth $r_i(Y^x)$ is the unique strong solution of the SDE*

$$W_t^{i,w_i} = w_i + \int_0^t \bar{b}_i(W_s^{i,w_i})ds + \int_0^t \bar{\sigma}_i(W_s^{i,w_i})dB_s + \int_{[0,t]} (1 - W_{s-}^{i,w_i})d\tilde{Q}_s^i - \int_{[0,t]} W_{s-}^{i,w_i} d\tilde{Q}_s^j, \tag{A.20}$$

with $w_i = r_i(x)$.

(ii) *For all $t \geq 0$ and $w_i \in (0,1)$, $W_t^{i,w_i} \in (0,1)$ a.s.*

Proof For (i), an application of Itô's formula shows that the fractions $r_i(Y^x)$ satisfy the SDE (A.20), and uniqueness follows by the local Lipschitz-continuity of its coefficients (cf. [10, Theorem 16.3.11]). Furthermore, the strict positivity of the pre-bankruptcy wealth Y^x (Lemma A.2) proves (ii). □

The next result constructs the solution to the Skorokhod reflection problem $\mathrm{SP}^i_{m_i+}$.

Proposition A.10 *Let $i \neq j \in \{a,b\}$ and $m_i \in (0,1)$. The mapping $\Psi^{i,m_i} \in \Lambda^i$ with*

$$\Psi_t^{i,m_i} = \Psi_t^{i,c,m_i} + \Psi_t^{i,d,m_i} : \mathcal{D}_+\big([0,t)\big) \times \mathcal{D}_+([0,t]) \times \mathcal{D}^{\uparrow}\big([0,t)\big) \to \mathbb{R}_+$$

given by

$$\begin{aligned}
&\Psi_t^{i,c,m_i}(y^i_{[0,t)}, y^j_{[0,t]}, a^i_{[0,t)}) \\
&= \frac{1}{1-m_i}\Bigg(\sup_{s\in[0,t]}\Big(m_i - w_s^{i-} + (1-m_i)a_s^{i,c} + \sum_{0\leq u<s}(m_i - w_u^{i-})_+\Big)^+ \\
&\qquad - \sum_{0\leq s\leq t}(m_i - w_s^{i-})^+\Bigg)
\end{aligned}$$

and

$$\Psi_t^{i,d,m_i}(y^i_{[0,t)}, y^j_{[0,t]}, a^i_{[0,t)}) = \sum_{0\leq s\leq t}\left(\ln\left(1 + \frac{w_{s-}^{i-}}{1-m_i}\Big(\frac{m_i}{w_s^{i-}} - 1\Big)\right)\right)^+,$$

where $w_t^{i-} := r_i(y^i_{t-}, y^j_t)$ for all $t \geq 0$ and $a^{i,c}$ denotes the continuous part of a^i, solves $SP^i_{m_i+}$. In particular, for any $A^j \in \mathcal{A}$ and $x \in \mathbb{R}^2_{++}$, the response map Ψ^{i,m_i} defines the response A^i with associated wealth Y^x of the form

$$\begin{aligned}
A_t^{i,c,\prime} &= \Psi_t^{i,c,m_i}(Y^j_{[0,t)}, Y^j_{[0,t]}, A^{i,\prime}_{[0,t)}), \\
\Delta A_t^{i,\prime} &= \Delta\Psi_t^{i,d,m_i}(Y^j_{[0,t)}, Y^j_{[0,t]}, A^{i,\prime}_{[0,t)}) \\
&= \left(\ln\left(1 + \frac{r_i(Y^i_{t-}, Y^j_{t-})}{1-m_i}\Big(\frac{m_i}{r_i(Y^i_{t-}, Y^j_t)} - 1\Big)\right)\right)^+.
\end{aligned} \tag{A.21}$$

Furthermore, if $m_a + m_b < 1$, *there exists a unique tuple* $(Y^x, A^{i,'}, A^{j,'})$ *such that* $Y^{i,x}$ *is the unique strong solution of the SDE* (2.2) *with*

$$A_t^{i,'} = \Psi_t^{i,m_i}(Y^i_{[0,t]}, Y^j_{[0,t]}, A^{i,'}_{[0,t)}), \qquad A_t^{j,'} = \Psi_t^{i,m_j}(Y^j_{[0,t]}, Y^i_{[0,t]}, A^{j,'}_{[0,t)})$$

for all $t \geq 0$ (*known as two-sided Skorokhod reflection problem*). *In this case, the expression of* $\Delta A_t^{k,'}$ *simplifies to* $\mathbf{1}_{\{t=0\}}(\ln \frac{1-w_k}{1-m_k})^+$ *for all* $k \in \{a, b\}$.

Proof Fix $x \in \mathbb{R}^2_{++}$ and let $w_i = r_i(x)$. If $A^i \equiv 0$, then the process $W^{i,w_i}(0, A^j)$ with $W^{i,w_i}_{0-}(0, A^j) = w_i \in (0, 1)$ satisfies

$$W_t^{i,w_i}(0, A^j) = w_i + \int_0^t \bar{b}_i\big(W_s^{i,w_i}(0, A^j)\big)ds + \int_0^t \bar{\sigma}_i\big(W_s^{i,w_i}(0, A^j)\big)dB_s - \int_{[0,t]} W_{s-}^{i,w_i}(0, A^j)d\bar{A}_s^j. \tag{A.22}$$

Now, slightly generalise the SDE (A.22) by adding a process $P^i \in \mathcal{A}$ to get

$$W_t^{i,w_i}(P^i, A^j) = w_i + \int_0^t \bar{b}_i\big(W_s^{i,w_i}(P^i, A^j)\big)ds + \int_0^t \bar{\sigma}_i\big(W_s^{i,w_i}(P^i, A^j)\big)dB_s - \int_{[0,t]} W_{s-}^{i,w_i}(P^i, A^j)d\bar{A}_s^j + P_t^i. \tag{A.23}$$

Note that $W^{i,w_i}(P^i, A^j)$ is not necessarily bounded above by 1 (this depends on the process P^i). By De Angelis and Ferrari [13, Lemma 2.2], there exists a unique pair $(W^{i,w,}(P^{i,'}, A^j), P^{i,'})$ such that $W^{i,w,}(P^{i,'}, A^j)$ is the unique strong solution to the SDE (A.23) with $P^{i,'} \in \mathcal{A}$ given by

$$\begin{aligned} P_t^{i,'} &= \sup_{0\leq s\leq t}\bigg(m_i - w_i - \int_0^t \bar{b}_i\big(W_s^{i,w_i}(P^{i,'}, A^j)\big)ds - \int_0^t \bar{\sigma}_i\big(W_s^{i,w_i}(P^{i,'}, A^j)\big)dB_s \\ &\qquad + \int_{[0,t]} W_{s-}^{i,w_i}(P^{i,'}, A^j)d\bar{A}_s^j\bigg)^+ \\ &= \sup_{0\leq s\leq t}\big(m_i - W_s^{i,w_i}(P^{i,'}, A^j) + P_s^{i,'}\big)^+ \end{aligned} \tag{A.24}$$

a.s. for all $t \geq 0$ satisfying

(i) $m_i \leq W_t^{i,w_i}(P^{i,'}, A^j) < 1$ a.s. on the event $\{0 \leq t < \tau_1\}$ for all $t \geq 0$,

(ii) $\int_{[0,\tau_1)} \mathbf{1}_{\{W_t^{i,w_i}(P^{i,'},A^j)>m_i\}} dP_t^{i,'} = 0$ a.s.,

with the exit time $\tau_1 = \inf\{t \geq 0 : W_t^{i,w_i}(P^{i,'}, A^j) \geq 1\}$.

Let $P^{i,c,'}$ denote the continuous part of the process $P^{i,'}$. Then there exists a unique process $(A_t^{i,c,'})_{t\geq 0} \in \mathcal{A}$ with continuous paths such that

$$P_t^{i,c,'} = \int_0^t \big(1 - W_s^{i,w_i}(P^{i,'}, A^j)\big)dA_s^{i,c,'} \tag{A.25}$$

or, equivalently,

$$
\begin{aligned}
&A_t^{i,c,'}\\
&=\int_0^t \big(1-W_s^{i,w_i}(P^{i,'},A^j)\big)^{-1} dP_s^{i,c,'}\\
&=\int_0^t \big(\mathbf{1}_{\{W_s^{i,w_i}(P^{i,'},A^j)>m_i\}}+\mathbf{1}_{\{W_s^{i,w_i}(P^{i,'},A^j)=m_i\}}\big)\big(1-W_s^{i,w_i}(P^{i,'},A^j)\big)^{-1} dP_s^{i,c,'}\\
&=\frac{P_t^{i,c,'}}{1-m_i},
\end{aligned}
\tag{A.26}
$$

where the third equality follows by condition (ii) above. Notice that for any $t \geq 0$, the jumps satisfy $\Delta W_t^{i,w_i}(P^{i,'},A^j) = \Delta P_t^{i,'} - W_{t-}^{i,w_i}(P^{i,'},A^j)\Delta\bar{A}_t^j$ a.s. Condition (ii) implies that if $\Delta P_t^{i,'}(\omega) > 0$ for some $\omega \in \Omega$, then $W_t^{i,w_i}(P^{i,'},A^j)(\omega) = m_i$ for any $t \in [0, \tau_1(\omega))$, which in turn implies that

$$\Delta P_t^{i,'}(\omega) = m_i - e^{-\Delta A_t^j(\omega)} W_{t-}^{i,w_i,'}(P^{i,'},A^j)(\omega) > 0.$$

Otherwise, if $\Delta P_t^{i,'}(\omega) = 0$ for some $\omega \in \Omega$, then $W_t^{i,w_i}(P^{i,'},A^j)(\omega) \geq \tilde{w}_i$. Hence

$$\Delta P_t^{i,'} = \big(m_i - e^{-\Delta A_t^j} W_{t-}^{i,w_i,'}(P^{i,'},A^j)\big)^+ \qquad \text{a.s. for all } t \geq 0. \tag{A.27}$$

Let $p^{i,j}(a_i,a_j,w) = \frac{a_i(1+a_j-w)}{(1+a_j)(1+a_i+a_j)}$ for $(a_i,a_j,w) \in \mathbb{R}_+^2 \times (0,1)$. Because the mapping $a_i \mapsto p^{i,j}(a_i,a_j,w)$ is strictly increasing, there exists for any $t \geq 0$ a unique random variable $\Delta A_t^{i,'}$ such that

$$\Delta P_t^{i,'} = p^{i,j}\big(\Delta\tilde{A}_t^{i,'}, \Delta\tilde{A}_t^j, W_{t-}^{i,w_i}(P^{i,'},A^j)\big) \tag{A.28}$$

or, equivalently,

$$
\begin{aligned}
\Delta\tilde{A}_t^{i,'} &= \frac{\Delta P_t^{i,'}(\Delta\tilde{A}_t^j+1)}{1-\Delta P_t^{i,'}-(\Delta\tilde{A}_t^j+1)^{-1}W_{t-}^{i,w_i}(P^{i,'},A^j)}\\
&= \frac{1}{1-m_i}\big(m_i e^{\Delta A_t^j} - W_{t-}^{i,w_i}(P^{i,'},A^j)\big)^+,
\end{aligned}
\tag{A.29}
$$

where the second equality follows by (A.27).

Defining the process $(\tilde{A}_t^{i,'})_{t\geq 0}$ by $\tilde{A}_t^{i,'} = A_t^{i,c,'} + \sum_{0\leq s\leq t}\Delta\tilde{A}_s^{i,'}$ and substituting (A.25) and (A.28) into (A.23) shows that the process $W^{i,w_i}(P^{i,'},A^j)$ solves the SDE (A.20). Hence we can set $W_t^{i,w_i}(A^{i,'},A^j) = W_t^{i,w_i}(P^{i,'},A^j)$ for all $t \geq 0$. As the solution to the SDE (A.20) with initial data $w_i \in (0,1)$ never leaves the interval $(0,1)$ for any $(A^i,A^j) \in \mathcal{A}^2$ with probability 1 (Lemma A.9 (ii)), it follows that $\tau_1 = \infty$.

Lemma A.2 (i) implies that there exists a unique strong solution $Y^x(A^{i,'},A^j)$ to the SDE (2.2), and Lemma A.9 (i) implies that $r_i(Y^x(A^{i,'},A^j))$ and $W^{i,w_i}(A^{i,'},A^j)$

are indistinguishable. Let $W_t^{i-,w_i}(A^{i,'}, A^j) = r_i(Y_{t-}^{i,x}, Y_t^j)$ for all $t \geq 0$ and note that

$$W_t^{i-,w_i}(A^{i,'}, A^j) = \frac{Y_{t-}^{i,x}}{Y_{t-}^{i,x} + Y_t^j} = \frac{Y_{t-}^{i,x}}{Y_{t-}^{i,x} + Y_{t-}^j} e^{-\Delta A_t^j}$$
$$= W_{t-}^{i,w_i}(A^{i,'}, A^j) e^{-\Delta A_t^j} \tag{A.30}$$

a.s. for all $t \geq 0$. Also, it follows by (A.23) that

$$W_{t-}^{i,w_i}(A^{i,'}, A^j) - P_{t-}^{i,'} = w_i + \int_0^t \bar{b}_i\big(W_s^{i,w_i}(A^{i,'}, A^j)\big)ds$$
$$+ \int_0^t \bar{\sigma}_i\big(W_s^{i,w_i}(A^{i,'}, A^j)\big)dB_s$$
$$- \int_{[0,t)} W_{s-}^{i,w_i}(A^{i,'}, A^j)d\bar{A}_s^j, \tag{A.31}$$

and subtracting $W_{t-}^{i,w_i}(A^{i,'}, A^j)\Delta\bar{A}_t^j$ from both sides of (A.31) yields by (A.30) that

$$W_{t-}^{i-,w_i}(A^{i,'}, A^j) - P_{t-}^{i,'} = w_i + \int_0^t \bar{b}_i\big(W_s^{i,w_i}(A^{i,'}, A^j)\big)ds$$
$$+ \int_0^t \bar{\sigma}_i\big(W_s^{i,w_i}(A^{i,'}, A^j)\big)dB_s$$
$$- \int_{[0,t]} W_{s-}^{i,w_i}(A^{i,'}, A^j)d\bar{A}_s^j$$
$$= W_t^{i,w_i}(A^{i,'}, A^j) - P_t^{i,'} \tag{A.32}$$

a.s. for all $t \geq 0$. By substituting (A.32) into (A.24), it follows that

$$P_t^{i,'} = \sup_{s\in[0,t]} \big(m_i - W_s^{i-,w_i}(A^{i,'}, A^j) + P_{s-}^{i,'}\big)^+$$

a.s. for all $t \geq 0$, which in turn yields

$$P_t^{i,c,'} + \sum_{0\leq s\leq t} \Delta P_s^{i,'} = \sup_{s\in[0,t]} \Bigg(m_i - W_s^{i-,w_i}(A^{i,'}, A^j) + P_s^{i,c,'} + \sum_{0\leq u<s} \Delta P_u^{i,'}\Bigg)^+. \tag{A.33}$$

The equality (A.30) implies

$$\Delta P_t^{i,'} = \big(m_i - W_t^{i-,w_i,'}(P^{i,'}, A^j)\big)^+, \tag{A.34}$$

and together with (A.29), it follows a.s. for all $t \geq 0$ that

$$\Delta\tilde{A}_t^{i,'} = \frac{W_{t-}^{i-,w_i}(A^{i,'}, A^j)}{1 - m_i}\Bigg(\frac{m_i}{W_t^{i-,w_i}(A^{i,'}, A^j)} - 1\Bigg)^+.$$

To complete the proof, substituting (A.26) and (A.34) into (A.33) yields

$$A_t^{i,c,'} = \Psi_t^{i,c,m_i}\big(Y^j_{[0,t)}(A^{i,'}, A^j), Y^j_{[0,t]}(A^{i,'}, A^j), A^{i,'}_{[0,t)}\big)$$

a.s. for all $t \geq 0$, and due to $\Delta \tilde{A}_t^{i,'} = e^{\Delta A_t^{i,'}} - 1$, it follows that

$$A_t^{i,d,'} = \Psi_t^{i,d,m_i}\big(Y^j_{[0,t)}(A^{i,'}, A^j), Y^j_{[0,t]}(A^{i,'}, A^j), A^{i,'}_{[0,t)}\big).$$

For the second part of the claim, consider for any processes $P^i, P^j \in \mathcal{A}$ the SDE

$$W_t^{i,w_i}(P^i, P^j) = w_i + \int_0^t \bar{b}_i\big(W_s^{i,w_i}(P^i, P^j)\big)ds + \int_0^t \bar{\sigma}_i\big(W_s^{i,w_i}(P^i, P^j)\big)dB_s \\ + P_t^i - P_t^j. \tag{A.35}$$

By Tanaka [40, Theorem 4.1], there exists for any $w_i \in (m_i, 1-m_j)$ a unique triplet $(W^{i,w_i}(P^{i,'}, P^{j,'}), P^{i,'}, P^{j,'})$ with continuous paths such that $W^{i,w_i}(P^{i,'}, P^{j,'})$ is the unique strong solution to (A.35) with $(P^{i,'}, P^{j,'}) \in \mathcal{A}^2$, and a.s. for all $t \geq 0$, we have

(a) $m_i \leq W_t^{i,w_i}(P^{i,'}, P^{j,'}) < 1 - \tilde{w}_j$,

(b) $\int_{\mathbb{R}_+} \mathbf{1}_{\{W_t^{i,w_i}(P^{i,'}, P^{j,'}) > m_i\}} dP_t^{i,'} = 0$,

(c) $\int_{\mathbb{R}_+} \mathbf{1}_{\{W_t^{i,w_i}(P^{i,'}, P^{j,'}) < 1-m_j\}} dP_t^{j,'} = 0$.

Therefore, define a unique pair $(A^{i,c,'}, A^{j,c,'}) \in \mathcal{A}^2$ with continuous paths such that

$$P_t^{i,'} = \int_0^t \big(1 - W_s^{i,w_i}(P^{i,'}, P^{j,'})\big)dA_s^{i,c,'} = (1-m_i)A_t^{i,c,'}, \tag{A.36}$$

$$P_t^{j,'} = \int_0^t W_s^{i,w_i}(P^{i,'}, P^{j,'})dA_s^{j,c,'} = (1-m_j)A_t^{j,c,'}. \tag{A.37}$$

Substituting (A.36) and (A.37) into (A.35) reveals that the process $W^{i,w_i}(P^{i,'}, P^{j,'})$ satisfies the SDE (A.20) with $A^k = A^{k,c,'}$ for any $k \in \{a, b\}$. For any $w_i \in (0,1)$, define $A_t^{i,'} = A_t^{i,',c} + (\ln \frac{1-w_i}{1-m_i})^+$ and $A_t^{j,'} = A_t^{j,',c} + (\ln \frac{1-w_i}{m_j})^+$. Note that $(A_0^{i,'}, A_0^{j,'})$ are the unique functions of w_i such that

$$W_0^{i,w_i}(A^{i,'}, A^{j,'}) \in [m_i, 1-m_j],$$

$$A_0^{i,'} \mathbf{1}_{\{W_0^{i,w_i}(A^{i,'}, A^{j,'}) > m_i\}} = 0 \text{ and } A_0^{j,'} \mathbf{1}_{\{W_0^{i,w_i}(A^{i,'}, A^{j,'}) < 1-m_j\}} = 0.$$

The properties (a)–(c), Lemma A.2 (i) and Lemma A.9 (i) imply that

$$A_t^{i,'} = \Psi_t^{i,m_i}\big(Y^i_{[0,t)}(A^{i,'}, A^{j,'}), Y^j_{[0,t]}(A^{i,'}, A^{j,'}), A^{i,'}_{[0,t)}\big)$$

for any $i \neq j \in \{a, b\}$. As $\Delta A_t^{k,'} = 0$ for any $k \in \{a, b\}$ and a.s. for all $t \geq 0$, the expression of (A.21) simplifies to $\mathbf{1}_{\{t=0\}}(\ln \frac{1-w_k}{1-m_k})^+$. □

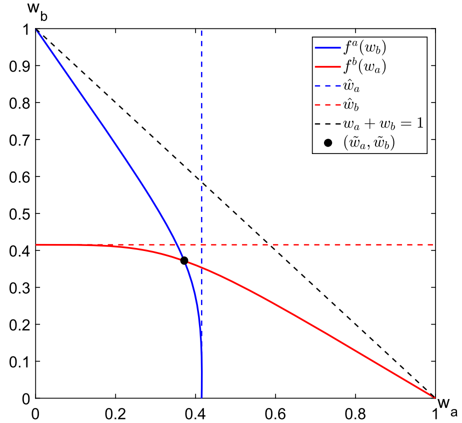


Fig. A.1 The implicit curves $f^a(w_b)$ and $f^b(w_a)$ in Lemma A.11 that satisfy $F^a(f^a(w_b), w_b) = 0$ for any $0 < w_b < 1$ (vertical axis) and $F^b(w_a, f^b(w_a)) = 0$ for any $0 < w_a < 1$ (horizontal axis) such that $w_a + w_b < 1$ with parameters $\gamma_a = \gamma_b = 0.5$, $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$ and $\lambda^\kappa = 1/3$

A.6 Cheating thresholds (Lemma 3.8)

For any $i \neq j \in \{a, b\}$, let $\hat{\Delta}^i = \{(w_i, w_j) \in (0,1)^2 : w_i < \min(\hat{w}_i, 1 - w_j)\} \subseteq \Delta$. The following result proves the existence of fraud thresholds.

Lemma A.11 *The following hold for any $i \neq j \in \{a, b\}$:*
(i) $\alpha_i < 0$, $a_i > 1 - \gamma_i$, $\beta_i > 1 - \gamma_i$, $b_i < 0$ *and* $\hat{w}_i \in (0, 1 - \gamma_i)$.
(ii) *There exists a function f^i whose graph satisfies*

$$\big\{\big(f^i(w_j), w_j\big) : w_j \in (0,1)\big\} \subseteq \hat{\Delta}^i$$

and

$$\{(w_i, w_j) \in \Delta : F^i(w_i, w_j) = 0\} = \big\{\big(f^i(w_j), w_j\big) : w_j \in (0,1)\big\}. \tag{A.38}$$

(iii) *f^i is differentiable.*
(iv) $\lim_{w_j \uparrow 1} f^i(w_j) = 0$ *and* $\lim_{w_j \downarrow 0} f^i(w_j) = \hat{w}_i$.
(v) *There exists $(\tilde{w}_a, \tilde{w}_b) \in \Delta$ such that*

$$F^a(\tilde{w}_a, \tilde{w}_b) = F^b(\tilde{w}_b, \tilde{w}_a) = 0. \tag{A.39}$$

Moreover, any such pair $(\tilde{w}_a, \tilde{w}_b)$ satisfies $\tilde{w}_k < \hat{w}_k$ for $k \in \{a, b\}$.

Figure A.1 displays the functions f^a, f^b and the solution $(\tilde{w}_a, \tilde{w}_b)$ to (A.39), where one can observe that f^a and f^b satisfy (ii)–(iv). In this case, the pair $(\tilde{w}_a, \tilde{w}_b)$ satisfying (v) is unique.

Proof of Lemma A.11 Starting with (i), Assumption 3.1 implies that

$$p_i > (1 - \gamma_i)\left((\mu_i - \mu_j)^+ + \frac{\gamma_i \sigma_j^2}{2}\right),$$

$$q_i > (1 - \gamma_i)\left((\mu_j - \mu_i)^+ + \frac{\gamma_i \sigma_i^2}{2}\right), \tag{A.40}$$

and the inequality (A.40) yields

$$\alpha_i < \frac{1}{\sigma^2}\Big(k_i - \sqrt{k_i^2 + \sigma^2(1-\gamma_i)\big(2(\mu_i-\mu_j)^+ + \gamma_i\sigma_j^2\big)}\Big) < 0, \qquad a_i > 1-\gamma_i,$$

$$\beta_i > \frac{1}{\sigma^2}\Big(k_i + \sqrt{k_i^2 + \sigma^2(1-\gamma_i)\big(2(\mu_i-\mu_j)^+ + \gamma_i\sigma_j^2\big)}\Big) =: \hat{\beta}_i(\gamma_i),$$

$$\hat{w}_i < 1-\gamma_i.$$

Note that k_i also depends on γ_i. Because $(\hat{\beta}_i(\gamma_i) - (1-\gamma_i))' > 0$ for $0 < \gamma_i < 1$, it follows that $\hat{\beta}_i(\gamma_i) - (1-\gamma_i) \geq \hat{\beta}_i(0) - 1 \geq 0$, whence $\beta_i > 1-\gamma_i$ and $b_i < 0$.

(ii) First, we show the inclusion '$\supseteq$' in (A.38). For any $w_j \in (0,1)$,

$$\lim_{w_i \downarrow 0} F^i(w_i, w_j) = \lim_{w_i \downarrow 0} \alpha_i a_i (1-\gamma_i)\left(\frac{1-w_j}{w_j}\right)^{\beta_i}\left(\frac{1-w_i}{w_i}\right)^{-b_i}(w_j - b_i)$$
$$= -\infty \tag{A.41}$$

and

$$\lim_{w_i \uparrow 1-w_j} F^i(w_i, w_j) = -a_i b_i (a_i - b_i)\gamma_i\left(\frac{1-w_j}{w_j}\right)^{1-\gamma_i} > 0.$$

Next, to show that

$$F^i(\hat{w}_i, w_j) > 0 \qquad \text{for any } w_j < 1-\hat{w}_i, \tag{A.42}$$

decompose F^i as

$$F^i(\hat{w}_i, w_j) = \frac{\alpha_i b_i (a_i - b_i)(\frac{w_j}{1-w_j})^{-\alpha_i}}{(1-\alpha_i)(\gamma_i - \alpha_i)}\ell^i(w_j),$$

where

$$\ell^i(w_j) = (1-w_j)(1-\alpha_i)^2\left(\frac{w_j}{1-w_j}\right)^{\gamma_i+\alpha_i} - (1-\gamma_i)^2(a_i - w_j)\left(\frac{\gamma_i(1-\alpha_i)}{-\alpha_i(1-\gamma_i)}\right)^{\gamma_i+\alpha_i}.$$

Note that $\frac{\alpha_i b_i (a_i - b_i)(\frac{w_j}{1-w_j})^{-\alpha_i}}{(1-\alpha_i)(\gamma_i - \alpha_i)} > 0$ so that $\operatorname{sgn}(F^i(\hat{w}_i, w_j)) = \operatorname{sgn}(\ell^i(w_j))$. The first and second derivatives of ℓ^i are

$$\ell^i_{w_j}(w_j) = (\alpha_i - 1)^2 w_j^{-1}(\alpha_i + \gamma_i - w_j)\left(\frac{w_j}{1-w_j}\right)^{\gamma_i+\alpha_i}$$
$$+ (1-\gamma_i)^2\left(\frac{\gamma_i(1-\alpha_i)}{-\alpha_i(1-\gamma_i)}\right)^{\gamma_i+\alpha_i},$$

$$\ell^i_{w_j w_j}(w_j) = -\frac{a_i(\gamma_i + \alpha_i)(\alpha_i - 1)^2}{w_j^2(1-w_j)}\left(\frac{w_j}{1-w_j}\right)^{\gamma_i+\alpha_i}$$
$$\geq 0 \quad \text{if and only if } \alpha_i + \gamma_i \leq 0. \tag{A.43}$$

We now distinguish two cases. If $\alpha_i + \gamma_i \le 0$, it follows by (A.43) that

$$\ell^i_{w_j}(w_j) \le \ell^i_{w_j}(1-\hat{w}_i) = -a_i\gamma_i^{-1}(\gamma_i-\alpha_i)^2\left(\frac{\gamma_i(1-\alpha_i)}{-\alpha_i(1-\gamma_i)}\right)^{\gamma_i+\alpha_i} < 0,$$

and therefore

$$\ell^i(w_j) > \ell^i(1-\hat{w}_i) = a_i(1-\gamma_i)(\gamma_i-\alpha_i)\left(\frac{\gamma_i(1-\alpha_i)}{-\alpha_i(1-\gamma_i)}\right)^{\gamma_i+\alpha_i} > 0.$$

If $\alpha_j + \gamma_j > 0$, then

$$\lim_{w_j \downarrow 0} \ell^i(w_j) = 0.$$

The concavity of ℓ^i (implied by (A.43)) in combination with $\ell^i(1-\hat{w}_i) > 0$ yields

$$\ell^i(w_j) > 0 \qquad \text{for } w_j < 1-\hat{w}_i,$$

whence (A.42) follows.

Due to (A.41) and (A.42), the intermediate value theorem implies that for any $w_j \in (0,1)$, there exists $u_i \in \min(\hat{w}_i, 1-w_j)$ such that $F^i(u_i, w_j) = 0$. Thus there exists a function f^i with its graph in $\hat{\Delta}^i$ such that the inclusion '$\supseteq$' in (A.38) holds.

To prove the inclusion '$\subseteq$' in (A.38), we first show that for any fixed $w_j \in (0,1)$, if $u_i \in (0, \min(\hat{w}_i, 1-w_j))$ satisfies $F^i(u_i, w_j) = 0$, then $F^i_{w_i}(u_i, w_j) > 0$. The equation $F^i(u_i, w_j) = 0$ expands to

$$\begin{aligned}
&a_i\big(\alpha_i(1-\gamma_i-u_i)+\gamma_i u_i\big)(w_j-b_i)\left(\frac{1-u_i}{u_i}\right)^{-b_i}\left(\frac{1-w_j}{w_j}\right)^{\beta_i} \\
&= -(a_i-b_i)\big(u_i(\alpha_i+\beta_i-1)-\alpha_i\beta_i\big)w_j^{\gamma_i}(1-w_j)^{1-\gamma_i} \\
&\quad + b_i\big(\beta_i(1-\gamma_i-u_i)+\gamma_i u_i\big)(w_j-a_i)\left(\frac{u_i}{1-u_i}\right)^{a_i}\left(\frac{w_j}{1-w_j}\right)^{-\alpha_i},
\end{aligned} \tag{A.44}$$

and differentiating F^i with respect to w_i yields for any $(w_i, w_j) \in \Delta$ that

$$\begin{aligned}
&-(1-w_i)w_i F^i_{w_i}(w_i, w_j) \\
&= a_i(w_j-b_i)\big(\gamma_i w_i(w_i-b_i-1)+b_i\gamma_i\alpha_i+(1-w_i)(w_i-b_i)\alpha_i\big) \\
&\quad \times \left(\frac{1-w_i}{w_i}\right)^{-b_i}\left(\frac{1-w_j}{w_j}\right)^{\beta_i} \\
&\quad - b_i(a_i-w_j)\big(\gamma_i w_i(1+a_i-w_i)-a_i\gamma_i\beta_i-(1-w_i)(w_i-a_i)\beta_i\big) \\
&\quad\quad \times \left(\frac{w_i}{1-w_i}\right)^{a_i}\left(\frac{w_j}{1-w_j}\right)^{-\alpha_i} \\
&\quad - (a_i-b_i)(\alpha_i+\beta_i-1)(1-w_i)w_i(1-w_j)^{1-\gamma_i}w_j^{\gamma_i}.
\end{aligned} \tag{A.45}$$

Substituting (A.44) into (A.45) yields

$$\operatorname{sgn}\big(\partial_{w_i} F^i(u_i, w_j)\big) = \operatorname{sgn}\big(\rho^i(u_i, w_j)\ell^i(u_i)\big),$$

where

$$\rho^i(w_i, w_j) = -(1-\gamma_i - w_i)(a_i - w_j)\left(\frac{w_i}{1-w_i}\right)^{a_i}\left(\frac{w_j}{1-w_j}\right)^{a_i} + (w_i - \alpha_i)w_j$$

and

$$\ell^i(w_i) = -(1-\gamma_i)\alpha_i\beta_i + \big(\alpha_i\beta_i + \gamma_i(1-\alpha_i-\beta_i)\big)w_i$$

for $w_i \in (0, \min(\hat{w}_i, 1-w_j))$. If $\alpha_i\beta_i + \gamma_i(1-\alpha_i-\beta_i) \geq 0$, then $\ell^i(u_i) > 0$; if instead $\alpha_i\beta_i + \gamma_i(1-\alpha_i-\beta_i) < 0$, then by the inequality $u_i < \hat{w}_i$,

$$\ell^i(u_i) > \frac{\alpha_i(\alpha_i - 1)(1-\gamma_i)\gamma_i}{\gamma_i - \alpha_i} > 0.$$

Hence we get

$$\operatorname{sgn}\big(\partial_{w_i} F^i(u_i, w_j)\big) = \operatorname{sgn}\big(\rho^i(u_i, w_j)\big).$$

It is clear that $\rho^i(u_i, w_j) > 0$ if $w_j \geq a_i$. Thus assume $w_j < a_i$. Since $w_j + u_i < 1$, $w_j \geq a_i$ is satisfied if and only if $w_j < \min\{a_i, 1-u_i\}$. Note that

$$\lim_{w_j \downarrow 0} \rho^i(u_i, w_j) = 0 \tag{A.46}$$

and

$$\rho^i\big(u_i, \min(a_i, 1-u_i)\big) = \begin{cases} a_i(w_i - \alpha_i) > 0 & \text{if } a_i < 1-u_i, \\ \gamma_i a_i > 0 & \text{if } a_i \geq 1-u_i, \end{cases} \tag{A.47}$$

and $\rho^i_{w_j w_j}(u_i, w_j)$ has the same sign as

$$\begin{aligned} &-(a_i - 1)a_i(1-\gamma_i - u_i)(a_i + w_j)\left(\frac{u_i}{1-u_i}\right)^{a_i}\left(\frac{w_j}{1-w_j}\right)^{a_i} \\ &\leq 0 \quad \text{if and only if } a_i \geq 1. \end{aligned}$$

If $a_i \geq 1$, then the concavity of ρ^i, (A.46) and (A.47) yield

$$\rho^i(u_i, w_j) \geq \frac{\rho^i(u_i, \min\{a_i, 1-u_i\})}{\min\{a_i, 1-u_i\}} w_j > 0.$$

If $a_i < 1$, then $(\frac{u_i}{1-w_j})^{a_i} > \frac{u_i}{1-w_j}$ and $(\frac{w_j}{1-u_i})^{a_i} > \frac{w_j}{1-u_i}$ give

$$\begin{aligned} \rho^i(u_i, w_j) &> -(1-\gamma_i - u_i)(a_i - w_j)\frac{u_i}{1-u_i}\frac{w_j}{1-w_j} + (u_i - \alpha_i)w_j \\ &= \frac{w_j\hat{\rho}(u_i, w_j)}{(1-u_i)(1-w_j)}, \end{aligned}$$

where for any $w_i \in (0, \min(\hat{w}_i, 1 - w_j))$,

$$\hat{\rho}^i(w_i, w_j) := -w_i(1 - \gamma_i - w_i)(a_i - w_j) + (1 - w_i)(1 - w_j)(w_i - \alpha_i).$$

Taking the partial derivatives with respect to w_i yields

$$\hat{\rho}^i_{w_i}(u_i, w_j) = (1 - a_i)(2 - \gamma_i - 2u_i - w_j) > 0.$$

(The inequality follows from $u_i + w_j < 1$ and $u_i < \hat{w}_i < 1 - \gamma_i$.) Thus

$$\partial_{w_i} F^i(u_i, w_j) > 0 \tag{A.48}$$

for any $u_i \in (0, \min(\hat{w}_i, 1 - w_j))$ such that $F^i(u_i, w_j) = 0$. Define f^i via

$$f^i(w_j) = \inf\{w_i \in \big(0, \min(\hat{w}_i, 1 - w_j)\big) : F^i(w_i, w_j) = 0\},$$

which is the minimal zero of $w_i \mapsto F^i(w_i, w_j)$. Suppose by contradiction that there exists $w_i > f^i(w_j)$ such that $F^i(w_i, w_j) = 0$ and let

$$v_i = \inf\{w_i \in \big(f^i(w_j), \min(\hat{w}_i, 1 - w_j)\big) : F^i(w_i, w_j) = 0\}, \tag{A.49}$$

which is the first zero of $w_i \mapsto F^i(w_i, w_j)$ after $f^i(w_j)$. Then by (A.48), we obtain $F^i_{w_i}(f^i(w_j), w_j) > 0$ and $F^i_{w_i}(v_i, w_j) > 0$. The smoothness of F^i implies that there exists $\epsilon \in (0, \frac{v_i - f^i(w_j)}{2})$ such that $F^i(f^i(w_j) + \epsilon, w_j) > 0$ and $F^i(v_i - \epsilon, w_j) < 0$. However, the intermediate value theorem implies that there must exist some point $z_i \in (f^i(w_j) + \epsilon, v_i - \epsilon)$ with $F^i(z_i, w_j) = 0$, which is impossible in view of the definition (A.49).

To establish the claim, it remains to show that $f^i(w_j)$ is the unique solution in the larger domain $(0, 1 - w_j) \supseteq (0, \min(\hat{w}_i, 1 - w_j))$ such that $F^i(f^i(w_j), w_j) = 0$. This fact follows by showing that there does not exist $(w_i, w_j) \in \Delta \backslash \hat{\Delta}^i$ such that $F^i(w_i, w_j) = 0$. Note that $\Delta \backslash \hat{\Delta}^i = \{(w_i, w_j) \in \Delta : w_i \geq \hat{w}_i\}$. By virtue of (A.42), it suffices to consider $w_i \in (\hat{w}_i, 1)$.

Differentiating F^i with respect to w_j reveals that $F^i_{w_j}(w_i, w_j)$ has the same sign as

$$\begin{aligned}
&a_i\big(b_i\beta_i + (1 - w_j - \beta_i)w_j\big)\big((1 - \gamma_i - w_i)\alpha_i + \gamma_i w_i\big)\left(\frac{1 - w_j}{w_j}\right)^{\beta_i}\left(\frac{1 - w_i}{w_i}\right)^{-b_i} \\
&- b_i\big(a_i\alpha_i + w_j(1 - w_j - \alpha_i)\big)\big(\gamma_i w_i + (1 - \gamma_i - w_i)\beta_i\big)\left(\frac{w_j}{1 - w_j}\right)^{-\alpha_i}\left(\frac{w_i}{1 - w_i}\right)^{a_i} \\
&- (a_i - b_i)\big(\gamma + (1 - 2\gamma_i)w_j\big)\big((\alpha_i + \beta_i - 1)w_i - \alpha_i\beta_i\big)(1 - w_j)^{1 - \gamma_i} w_j^{\gamma_i}.
\end{aligned} \tag{A.50}$$

Take $(u_i, u_j) \in \Delta$ with $F^i(u_i, u_j) = 0$. Putting $F^i(w_i, u_j) = 0$ into (A.50) yields

$$\operatorname{sgn}\big(F^i_{w_j}(u_i, u_j)\big) = \operatorname{sgn}\big(G^i(u_i, u_j)\big), \tag{A.51}$$

where for any $(w_i, w_j) \in \Delta$,

$$\begin{aligned} G^i(w_i, w_j) &\\ := (\gamma_i + \alpha_i)\big(\beta_i(1-\gamma_i - w_i) + \gamma_i w_i\big) &\\ - (\gamma_i + \beta_i)\big(\alpha_i(1-\gamma_i - w_i) + \gamma_i w_i\big)\left(\frac{1-w_i}{w_i}\right)^{\beta_i - \alpha_i}\left(\frac{1-w_j}{w_j}\right)^{\beta_i - \alpha_i}. \quad \text{(A.52)} \end{aligned}$$

Note that $\alpha_i(1-\gamma_i - w_i) + \gamma_i w_i > 0$ for any $w_i \in (\hat{w}_i, 1)$, and it follows that

$$\operatorname{sgn}\big(G^i_{w_j}(w_i, w_j)\big) = \operatorname{sgn}\big(\alpha_i(1-\gamma_i - w_i) + \gamma_i w_i\big) > 0. \quad \text{(A.53)}$$

Fix $w_i \in (\hat{w}_i, 1)$ and suppose by contradiction that there exists $u_j \in (0, 1 - w_i)$ such that $F^i(w_i, u_j) = 0$. Let v_j be the smallest u_j, i.e.,

$$v_j = \inf\{u_j \in (0, 1-w_i) : F^i(w_i, u_j) = 0\}. \quad \text{(A.54)}$$

Then by (A.51) and (A.53), we obtain $F^i_{w_j}(w_i, v_j) > 0$, implying that there exists an $\epsilon \in (0, v_j)$ such that $F^i(w_i, v_j - \epsilon) < 0$. Because for any $w_i \in (\hat{w}_i, 1)$, we have

$$\lim_{u_j \downarrow 0} F^i(w_i, u_j) = \infty,$$

there hence exists an intermediate point $z_j \in (0, v_j - \epsilon)$ such that $F^i(w_i, z_j) = 0$, which contradicts the definition (A.54) and shows the non-existence of a solution in $\Delta \backslash \hat{\Delta}^i$, hence the inclusion in (A.38).

(iii) For any $(w_i^0, w_j^0) \in \Delta$ with $w_i^0 < \min(\hat{w}_i, 1 - w_j^0)$ and $F^i(w_i^0, w_j^0) = 0$, the implicit function theorem (whose assumption is satisfied due to (A.48)) implies that in a neighbourhood of (w_i^0, w_j^0), there exists a unique smooth (in fact, analytic) function $f_0^i(w_j)$ satisfying $f_0^i(w_j^0) = w_i^0$ and such that

$$f_0^i(w_j) \in \big(0, \min(\hat{w}_i, 1 - w_j)\big) \quad \text{and} \quad F^i\big(f_0^i(w_j), w_j\big) = 0.$$

Suppose by contradiction that f^i is not analytic. Then there exists $w_j^0 \in (0, 1)$ where the local function $f_0^i(w_j^0) \neq f^i(w_j^0)$. But this implication contradicts uniqueness, and thus $w_j \mapsto f^i(w_j)$ is indeed analytic on the open domain $(0, 1)$.

(iv) Since $0 < f^i(w_j) < 1 - w_j$ for all $w_j \in (0, 1)$, we get $\lim_{w_j \uparrow 1} f^i(w_j) = 0$. Moreover, for any $w_i \in (0, \hat{w}_i)$,

$$\lim_{w_j \downarrow 0} F^i(w_i, w_j) = -\infty,$$

$$\lim_{w_j \uparrow 1 - w_i} F^i(w_i, w_j) = -\gamma_i a_i b_i (a_i - b_i)\left(\frac{w_i}{1-w_i}\right)^{1-\gamma_i} > 0.$$

Thus by the intermediate value theorem, for any $w_i \in (0, \hat{w}_i)$, there exists $u_j < 1 - w_i$ with $F^i(w_i, u_j) = 0$, and so there exists a function $g^i : (0, \hat{w}_i) \to (0, 1)$ such that

$$\big\{\big(w_i, g^i(w_i)\big) : w_i \in (0, \hat{w}_i)\big\} \subseteq \{(w_i, w_j) \in \hat{\Delta}^i : F^i(w_i, w_j) = 0\}.$$

Property (ii) yields

$$\big\{\big(w_i, g^i(w_i)\big) : w_i \in (0, \hat{w}_i)\big\} \subseteq \big\{\big(f^i(w_j), w_j\big) : w_j \in (0,1)\big\},$$

which implies $\sup_{w_j \in (0,1)} f^i(w_j) \geq \hat{w}_i$ or, equivalently,

$$\max_{w_j \in (0,1)} f^i(w_j) \geq \hat{w}_i \quad \text{or} \quad \lim_{w_j \downarrow 0} f^i(w_j) \geq \hat{w}_i.$$

On the other hand, $f^i(w_j) < \hat{w}_i$ for all $w_j \in (0,1)$ implies $\max_{w_j \in (0,1)} f^i(w_j) < \hat{w}_i$, and by the continuity of f^i, $\lim_{w_j \downarrow 0} f^i(w_j) \leq \hat{w}_i$. Thus $\lim_{w_j \downarrow 0} f^i(w_j) = \hat{w}_i$.

(v) First we establish the existence of a point $(\tilde{w}_a, \tilde{w}_b) \in \Delta$ satisfying the equation $F^a(\tilde{w}_a, \tilde{w}_b) = F^a(\tilde{w}_b, \tilde{w}_a) = 0$. For any $i \neq j \in \{a, b\}$, by using (iv), we extend f^i continuously to the boundary 0 by setting $f^i(0) = \lim_{w_j \downarrow 0} f^i(w_j) = \hat{w}_i$, and to 1 by setting $f^i(1) = \lim_{w_j \uparrow 1} f^i(w_j) = 0$. Define the set $\mathcal{D} = (0,1)^2$ and the function $H : \bar{\mathcal{D}} \to \bar{\mathcal{D}}$ by $H(w_a, w_b) := (f^a(w_b), f^b(w_a))$. Property (ii) implies for any $w_j \in [0,1]$ that $f^i(w_j) \in [0, \hat{w}_i] \subseteq [0,1]$. Therefore H is well defined. Since $\bar{\mathcal{D}}$ is compact and H is continuous due to (iii), Brouwer's fixed point theorem implies the existence of a point $(\tilde{w}_a, \tilde{w}_b) \in \bar{\mathcal{D}}$ with $(f^a(\tilde{w}_b), f^b(\tilde{w}_a)) = (\tilde{w}_a, \tilde{w}_b)$. Next, we show that $(\tilde{w}_a, \tilde{w}_b) \notin \partial\mathcal{D}$. Note that $\partial\mathcal{D} = \mathcal{D}_1^{a,b} \cup \mathcal{D}_1^{b,a} \cup \mathcal{D}_2^{a,b} \cup \mathcal{D}_2^{b,a}$, where

$$\mathcal{D}_1^{a,b} = \{0\} \times [0,1], \qquad \mathcal{D}_1^{b,a} = [0,1] \times \{0\},$$
$$\mathcal{D}_2^{a,b} = \{1\} \times [0,1], \qquad \mathcal{D}_2^{b,a} = [0,1] \times \{1\}.$$

For $i \neq j \in \{a, b\}$, we have $f^i(f^j(0)) = f^i(\hat{w}_j) \neq 0$ on $\mathcal{D}_1^{i,j}$ as $f^i(w_j) \in (0, \hat{w}_i)$ for any $w_j \in (0,1)$; and $f^i(f^j(1)) = f^i(0) \neq 1$ on $\mathcal{D}_2^{i,j}$ because $f^i(0) = \hat{w}_i$. Hence $(\tilde{w}_a, \tilde{w}_b) \notin \partial\mathcal{D}$ and so $(\tilde{w}_a, \tilde{w}_b) \in \mathcal{D}$. Finally, (ii) implies that for any $i \neq j \in \{a, b\}$, we have $\tilde{w}_i = f^i(\tilde{w}_j) < 1 - \tilde{w}_j$ and thus $(\tilde{w}_a, \tilde{w}_b) \in \Delta$.

To conclude the proof, notice that if a pair $(\tilde{w}_a, \tilde{w}_b) \in \Delta$ satisfies the equality $F^a(\tilde{w}_a, \tilde{w}_b) = F^a(\tilde{w}_b, \tilde{w}_a) = 0$, then $(\tilde{w}_a, \tilde{w}_b) = (f^a(\tilde{w}_b), f^b(\tilde{w}_a))$ is in $\hat{\Delta}^a \cap \hat{\Delta}^b$ by (ii), meaning that $\tilde{w}_k < \hat{w}_k$ for any $k \in \{a, b\}$. □

In the following result, we first obtain some properties of the function G^i (given by (A.52)); then we find that the function f^i is strictly decreasing. (For an illustration, see Fig. A.2 below.)

Lemma A.12 *The following statements hold for any $i \neq j \in \{a, b\}$:*

(i) *If $\alpha_i < -\gamma_i$, then*

(a) *there exists a function $g^i : (0, \frac{1+\gamma_i}{2}) \to (0, \hat{w}_i)$ such that*

$$\{(w_i, w_j) \in \hat{\Delta}^i : G^i(w_i, w_j) = 0\} = \left\{\big(g^i(w_j), w_j\big) : w_j \in \left(0, \frac{1+\gamma_i}{2}\right)\right\};$$

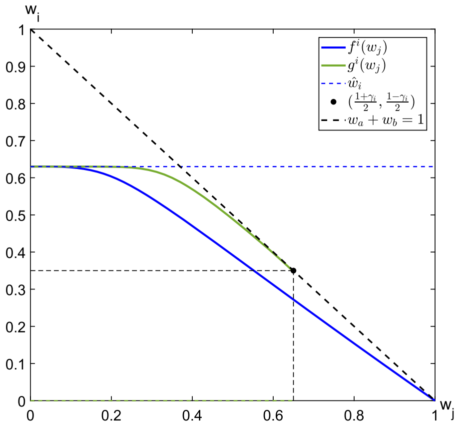


Fig. A.2 The function $f^i : (0, 1) \to (0, \hat{w}_i)$ of Lemma A.11 satisfies $F^i(f^i(w_j), w_j) = 0$ (blue); and the function $g^i : (0, \frac{1+\gamma_i}{2}) \to (0, \hat{w}_i)$ of Lemma A.12 satisfies $G^i(g^i(w_j), w_j) = 0$ (green) with the domain in dashed horizontal line in green. The parameters are $\gamma_i = 0.3$, $\gamma_j = 0.5$, $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$, $\lambda = 1/3$ and $\kappa = 10\%$

(b) *for any* $(w_i, w_j) \in \hat{\Delta}^i$,

$$G^i(w_i, w_j) \begin{cases} > 0 & \text{if } w_j \geq \frac{1+\gamma_i}{2}, & \text{(A.55a)} \\ > 0 & \text{if } w_j < \frac{1+\gamma_i}{2} \text{ and } w_i < g^i(w_j), & \text{(A.55b)} \\ < 0 & \text{if } w_j < \frac{1+\gamma_i}{2} \text{ and } w_i > g^i(w_j); & \text{(A.55c)} \end{cases}$$

(c) g^i *is differentiable and* $\frac{dg^i(w_j)}{dw_j} < 0$ *for all* $w_j \in (0, \frac{1+\gamma_i}{2})$;

(d) *for any* $w_j \in (0, \frac{1+\gamma_i}{2})$, *we have* $g^i(w_j) > \frac{1-\gamma_i}{2}$ *with* $\lim_{w_j \downarrow 0} g^i(w_j) = \hat{w}_i$ *and* $\lim_{w_j \uparrow \frac{1+\gamma_i}{2}} g^i(w_j) = \frac{1-\gamma_i}{2}$.

(ii) f^i *is strictly decreasing on* $(0, 1)$.

(iii) *Denote by* $f^{i,-1}$ *the inverse of* f^i. *For any* $(w_i, w_j) \in \hat{\Delta}^i$,

$$F^i(w_i, w_j) \begin{cases} > 0 & \text{if } w_j > f^{i,-1}(w_i), \\ < 0 & \text{if } w_j < f^{i,-1}(w_i). \end{cases}$$

Proof (i)(a) and (i)(b) Note that $\alpha_i < -\gamma_i$ if and only if $\hat{w}_i > \frac{1-\gamma_i}{2}$. A direct calculation reveals that

$$\lim_{w_i \downarrow 0} G^i(w_i, w_j) = \infty, \tag{A.56}$$

$$\lim_{w_i \uparrow \hat{w}_i} G^i(w_i, w_j) = (\gamma_i + \alpha_i)\big(\beta_i(1 - \gamma_i - w_i) + \gamma_i w_i\big)$$
$$< 0 \quad \text{if } w_j \in (0, 1 - \hat{w}_i), \tag{A.57}$$

$$\lim_{w_i \uparrow 1 - w_j} G^i(w_i, w_j) = \gamma_i(\alpha_i - \beta_i)(1 + \gamma_i - 2w_j)$$
$$< 0 \quad \text{if } w_j \in \left(1 - \hat{w}_i, \frac{1+\gamma_i}{2}\right), \tag{A.58}$$

$$\lim_{w_i \uparrow 1-w_j} G^i(w_i, w_j) = \gamma_i(\alpha_i - \beta_i)(1 + \gamma_i - 2w_j)$$

$$\geq 0 \quad \text{if } w_j \in \left[\frac{1+\gamma_i}{2}, 1\right), \tag{A.59}$$

and G^i has the partial derivative

$$\begin{aligned} G^i_{w_i}(w_i, w_j) = {} & (\alpha_i + \gamma_i)(\beta_i + \gamma_i) \\ & + (\gamma_i + \beta_i)\Big(\alpha_i - \gamma_i + (\beta_i - \alpha_i)\big(\gamma_i w_i + (1 - \gamma_i - w_i)\alpha_i\big)\Big) \\ & \times \left(\frac{(1-w_i)(1-w_j)}{w_i w_j}\right)^{\beta_i - \alpha_i} \\ < {} & 2\gamma_i(\alpha_i - \beta_i) < 0, \end{aligned} \tag{A.60}$$

where the inequality follows from $w_i < 1 - w_j$ and $\gamma_i w_i + (1 - \gamma_i - w_i)\alpha_i < 0$, which in turn follows from $w_i < \tilde{w}_i$.

If $w_j \geq (1 + \gamma_i)/2$, strict monotonicity by (A.60) and the limits (A.56) and (A.59) of equal sign imply (A.55a). If $w_j < (1 + w_i)/2$, the map $w_i \mapsto G^i(w_i, w_j)$ changes sign on $(0, \min(\hat{w}_i, 1 - w_j))$ due to (A.56) and (A.57) resp. (A.58). Strict monotonicity by (A.60) guarantees the existence of a unique $u_i = g^i(w_j)$ lying in $(0, \min(\hat{w}_i, 1 - w_j))$ such that $G^i(u_i, w_j) = 0$, and therefore also (A.55b) and (A.55c) must hold. This settles the proof of both (i)(b) and of (i)(a).

(i)(c) The differentiability follows by similar arguments as in the proof of Lemma A.11 (iii). To check monotonicity, note that the implicit function theorem implies for any $w_j \in (0, \frac{1+\gamma_i}{2})$ that

$$\frac{dg^i(w_j)}{dw_j} = -\frac{G^i_{w_j}(w_i, w_j)}{G^i_{w_i}(w_i, w_j)}\bigg|_{w_i = f^i(w_j)}.$$

Checking the partial derivative $G^i_{w_j}$ and recalling that $w_i < \hat{w}_i$, it follows that

$$\operatorname{sgn}\big(G^i_{w_j}(w_i, w_j)\big) = \operatorname{sgn}\big(\gamma_i w_i + (1 - \gamma_i - w_i)\alpha_i\big) < 0,$$

which together with (A.60) yields

$$\operatorname{sgn}\left(\frac{dg^i(w_j)}{dw_j}\right) = \operatorname{sgn}\left(-\frac{G^i_{w_j}(w_i, w_j)}{G^i_{w_i}(w_i, w_j)}\bigg|_{w_i = f^i(w_j)}\right) < 0.$$

(i)(d) First, $w_i = \frac{1-\gamma_i}{2}$ satisfies $w_j < 1 - w_i = \frac{1+\gamma_i}{2}$ which implies

$$\begin{aligned} G^i\left(\frac{1-\gamma_i}{2}, w_j\right) = {} & \frac{1-\gamma_i}{2}(\gamma_i + \alpha_i)(\gamma_i + \beta_i)\left(1 - \left(\frac{1+\gamma_i}{1-\gamma_i}\right)^{\beta_i - \alpha_i}\left(\frac{1-w_j}{w_j}\right)^{\beta_i - \alpha_i}\right) \\ > {} & 0. \end{aligned}$$

By the monotonicity of $w_i \mapsto G^i(w_i, w_j)$ (see (A.60)), it follows that $g^i(w_j) > \frac{1-\gamma_i}{2}$. The continuity of g^i and the inequality $g^i(w_j) < \min(\hat{w}, 1-w_j)$ from (i)(a) imply that $\lim_{w_j \uparrow \frac{1+\gamma_i}{2}} g^i(w_j) = \frac{1-\gamma_i}{2}$. Finally, for any $w_i \in (\frac{1-\gamma_i}{2}, \hat{w}_i)$, we have

$$\lim_{w_j \downarrow 0} G^i(w_i, w_j) = \infty,$$

$$\lim_{w_j \uparrow 1-w_i} G^i(w_i, w_j) = \gamma_i(\beta_i - \alpha_i)(1-\gamma_i-2w_i) < 0.$$

By the intermediate value theorem, there hence exists for any $w_i \in (\frac{1-\gamma_i}{2}, \hat{w}_i)$ some $u_j \in (0, 1-w_i) \subseteq (0, \frac{1+\gamma_i}{2})$ such that $G^i(w_i, u_j) = 0$. By the inclusion "$\subseteq$" in (i)(a), $g^i(u_j) = w_i$. Take a sequence $(w_i^n)_{n\in\mathbb{N}}$ in $(\frac{1-\gamma_i}{2}, \hat{w}_i)$ such that $w_i^n \uparrow \hat{w}_i$ as $n \to \infty$. Then there exists a sequence $(u_j^n)_{n\in\mathbb{N}}$ in $(0, \frac{1+\gamma_i}{2})$ such that $g^i(u_j^n) = w_i^n$, and by letting $n \to \infty$, $g^i(u_j^n) \to \hat{w}_i$. By (i)(a), g^i is strictly bounded from above by $\hat{w}_i$; hence $\sup_{w_j \in (0, \frac{1+\gamma_i}{2})} g^i(w_j) = \hat{w}_i$. By (i)(c), g^i is strictly decreasing; hence $\lim_{w_j \downarrow 0} g^i(w_j) = \hat{w}_i$.

(ii) The implicit function theorem yields

$$\frac{df^i(w_j)}{dw_j} = -\frac{F^i_{w_j}(w_i, w_j)}{F^i_{w_i}(w_i, w_j)}\bigg|_{w_i = f^i(w_j)}. \tag{A.61}$$

Hence (A.48), (A.51) and (A.61) yield

$$\operatorname{sgn}\left(\frac{df^i(w_j)}{dw_j}\right) = -\operatorname{sgn}\Big(G^i\big(f^i(w_j), w_j\big)\Big). \tag{A.62}$$

Next, to show that $f^i(w_j)$ is strictly decreasing, we distinguish two cases.

If $\alpha_i \geq -\gamma_i$, then by Lemma A.11 (i), the first term of G in (A.52) is positive as $\hat{w}_i < 1-\gamma_i$ and $\beta_i > 1-\gamma_i > 0$. Furthermore, because $\beta_i > 1-\gamma_i > 0$, also $\gamma_i + \beta_i > 0$, and since $w_i < \hat{w}_i$ (see (3.4) for the definition of $\hat{w}_i$), it follows that $\alpha_i(1-\gamma_i-w_i) + \gamma_i w_i < 0$; hence the second term of G, and thus also G, is positive. By Lemma A.11 (ii), the graph of f^i satisfies $\{(f^i(w_j), w_j) : w_j \in (0,1)\} \subseteq \hat{\Delta}^i$, and so it follows by (A.62) that $\frac{df^i(w_j)}{dw_j} < 0$ for all $w_j \in (0,1)$.

If $\alpha_i < -\gamma_i$, then $G^i(w_i, w_j) > 0$ on $\{(w_i, w_j) \in \hat{\Delta}^i : w_j \geq \frac{1+\gamma_i}{2}\}$ by (A.55a), and Lemma A.11 (ii) implies that

$$\left\{\big(f^i(w_j), w_j\big) : w_j \in \left[\frac{1+\gamma_i}{2}, 1\right)\right\} \subseteq \left\{(w_i, w_j) \in \hat{\Delta}^i : w_j \geq \frac{1+\gamma_i}{2}\right\}.$$

Hence by (A.62), $\frac{df^i(w_j)}{dw_j} < 0$ for any $w_j \in [\frac{1+\gamma_i}{2}, 1)$. It remains to check the monotonicity of f^i on the interval $(0, \frac{1+\gamma_i}{2})$ (which coincides with the whole domain of g^i).

We show that $z^i(w_j) := f^i(w_j) - g^i(w_j) \leq 0$ for all $w_j \in (0, \frac{1+\gamma_i}{2})$. Note that since $f^i(w_j) < 1 - w_j$ for any $w_j \in (0,1)$, we have $f^i(\frac{1+\gamma_i}{2}) < \frac{1-\gamma_i}{2}$. Then by (i)(d), it follows that

$$\lim_{w_j \uparrow \frac{1+\gamma_i}{2}} z^i(w_j) = f^i\left(\frac{1+\gamma_i}{2}\right) - \lim_{w_j \uparrow \frac{1+\gamma_i}{2}} g^i(w_j) < 0.$$

Suppose by contradiction that there exists $v_j \in (0, \frac{1+\gamma_i}{2})$ such that $z^i(v_j) > 0$. Then the intermediate value theorem implies that there exists $w_j \in (v_j, \frac{1+\gamma_i}{2})$ such that $z^i(w_j) = 0$. Let $w_j^\star$ be the first such point, i.e.,

$$w_j^\star = \inf\left\{w_j \in \left(v_j, \frac{1+\gamma_i}{2}\right) : z^i(w_j) = 0\right\}.$$

Note that this definition implies that

$$z^i(w_j) > 0 \qquad \text{for any } w_j \in (v_j, w_j^\star). \tag{A.63}$$

By the mean value theorem, there exists $u_j^\star \in (v_j, w_j^\star)$ such that

$$\left.\frac{dz^i(w_j)}{dw_j}\right|_{w_j=u_j^\star} = \left.\left(\frac{df^i(w_j)}{dw_j} - \frac{dg^i(w_j)}{dw_j}\right)\right|_{w_j=u_j^\star} < 0.$$

Then by (i)(c), it follows that

$$\left.\frac{df^i(w_j)}{dw_j}\right|_{w_j=u_j^\star} < \left.\frac{dg^i(w_j)}{dw_j}\right|_{w_j=u_j^\star} < 0,$$

and (A.62) implies $G^i(f(u_j^\star), u_j^\star) > 0$, which in turn by (A.55b) yields the inequality $f(u_j^\star) < g^i(u_j^\star)$, in contradiction to (A.63). Hence $z^i(w_j) \le 0$ for all $w_j \in (0, \frac{1+\gamma_i}{2})$.

Next, we show that $z^i(w_j) < 0$ almost everywhere (a.e.) on $(0, \frac{1+\gamma_i}{2})$. Suppose by contradiction that there exists an interval $(a, b) \subseteq (0, \frac{1+\gamma_i}{2})$ such that $z^i(w_j) = 0$ for any $w_j \in (a, b)$ or, equivalently,

$$f^i(w_j) = g^i(w_j) \qquad \text{on } (a, b). \tag{A.64}$$

By (i)(a), it follows that $G^i(f^i(w_j), w_j) = 0$ for any $w_j \in (a, b)$. Then (A.62) yields that f^i is constant on (a, b), which by (A.64) in turn implies that g^i is constant on (a, b), an impossibility due to (i)(c). Hence $f^i(w_j) < g^i(w_j)$ almost everywhere on $(0, \frac{1+\gamma_i}{2})$. By (A.55b) and (A.62), it follows that $\frac{df^i(w_j)}{dw_j} < 0$ a.e. also on $(0, \frac{1+\gamma_i}{2})$, completing the proof.

(iii) The strict monotonicity in (ii) shows that the inverse function $f^{i,-1}$ of f^i is well defined and differentiable. By Lemma A.11 (ii), $f^{i,-1}(w_i)$ is for any $w_i \in (0, \hat{w}_i)$ the unique point in $(0, 1 - w_i)$ such that $F^i(w_i, f^{i,-1}(w_i)) = 0$. Note that

$$\lim_{w_j \downarrow 0} F^i(w_i, w_j) = -\infty,$$

$$\lim_{w_j \uparrow 1-w_i} F^i(w_i, w_j) = -\gamma_i a_i b_i (a_i - b_i) \frac{w_i}{1-w_i}^{1-\gamma_i} > 0.$$

Suppose that there exists $u_j \in (0, f^{i,-1}(w_i))$ (resp. $u_j \in (f^{i,-1}(w_i), 1-w_i)$) such that $F^i(w_i, u_j) > 0$ (resp. $F^i(w_i, u_j) < 0$). Then the intermediate value theorem implies that there exists

$$v_j \in (0, u_j) \subseteq \big(0, f^{i,-1}(w_i)\big) \quad \Big(\text{resp. } v_j \in (u_j, 1-w_i) \subseteq \big(f^{i,-1}(w_i), 1-w_i\big)\Big)$$

such that $F^i(w_i, v_j) = 0$, contradicting the uniqueness of $f^{i,-1}(w_i)$ which is a zero of the map $w_j \mapsto F^i(w_i, w_j)$. Hence $F^i(w_i, w_j) < 0$ for any $w_j \in (0, f^{i,-1}(w_i))$, and $F^i(w_i, w_j) > 0$ for any $w_j \in (f^{i,-1}(w_i), 1-w_i)$. ☐

A.7 Proof of Proposition 4.1

First, by Lemma A.2 and Fubini's theorem,

$$\begin{aligned} J(x; A^a, A^b) &\le (1-p)\lambda(x_a + x_b)\mathbb{E}\Big[\int_0^\infty e^{-\lambda^\kappa t} Z_t dt\Big] \\ &= (1-p)\lambda(x_a + x_b)\int_0^\infty e^{-\lambda^\kappa t}\mathbb{E}[Z_t]dt, \end{aligned} \tag{A.65}$$

where

$$Z_t := \mathcal{E}\Big(\sum_{k\in\{a,b\}}\Big(\int_0^\cdot \mu_k r_k(Y_s^x)ds + \int_0^\cdot \sigma_k r_k(Y_s^x)dB_s^k\Big)\Big)_t,$$

and the inequality is strict if and only if $\Delta A_t^a \Delta A_t^b = 0$ a.s. for some $t \ge 0$. Moreover, since $\mathbb{E}[e^{\frac{1}{2}\int_0^t \sigma_k r_k(Y_s^x)ds}] < e^{\frac{1}{2}\int_0^t \sigma_k ds} < \infty$, Z satisfies the Novikov condition.

(i) As $\mu_a = \mu_b$, it follows that $\mathbb{E}[Z_t] = e^{\mu_a t}$ for any $(A^a, A^b) \in \mathcal{A}^2$. Hence equality holds in (A.65) and it follows that

$$J(x; A^{a,\prime}, A^{b,\prime}) = \sup_{(A^a, A^b)\in\mathcal{A}^2} J(x; A^a, A^b)$$

for any $A^{a,\prime}, A^{b,\prime}$ such that $\Delta A_t^{a,\prime} \Delta A_t^{b,\prime} = 0$ a.s. for all $t \ge 0$.

(ii) Consider first the case $\mu_a > \mu_b$. The assumption $A^a \ne 0$ implies that for some $s \ge 0$, $\mathbb{P}[A_s^a > 0] > 0$. The difference $Q := Y^a(A^a, 0) - Y^a(0, 0)$ satisfies

$$dQ_t = \mu_a Q_t dt + \sigma_a Q_t dB_t^a + Y_{t-}^S d\tilde{A}_t^a \qquad \text{with } Q_{0-} = 0.$$

By Jacod [23, Theorem 6.8], it follows that

$$Q_t = \mathcal{E}(\mu_a \cdot + \sigma_a B^b_\cdot)_t \int_{[0,t]} \mathcal{E}(\mu_a \cdot + \sigma_a B^b_\cdot)_u^{-1} Y^{S,x}_{u-} d\bar{A}^a_u,$$

where $\bar{A}^a_t = A^{a,c}_t + \sum_{0\le s\le t} \frac{\Delta \tilde{A}^a_s}{1+\Delta \tilde{A}^a_s}$ with $\bar{A}^a_{0-} = 0$. The expression of Q reveals that $Q_t \ge 0$ a.s. for all $t \ge 0$ and $\mathbb{P}[Q_t > 0] > 0$ for all $t > s$. Furthermore, the fact that Y^b does not depend on A^a implies that $r_a(Y^x_t(A^a, 0)) \ge r_a(Y^x_t(0,0))$ a.s. and

$$\mathbb{P}[r_a(Y^x_t(A^a, 0)) > r_a(Y^x_t(0,0))] > 0$$

for all $t > 0$. Therefore,

$$\begin{aligned}
&J\big((x_a, x_b); A^a, 0\big)\\
&= (1-p)\lambda(x_a + x_b)\mathbb{E}\bigg[\int_0^\infty e^{-\lambda^\kappa t}\mathcal{E}\bigg(\mu_b \cdot + (\mu_a - \mu_b)\int_0^\cdot r_a\big(Y^x_s(A^a,0)\big)ds\bigg)_t dt\bigg]\\
&> (1-p)\lambda(x_a + x_b)\mathbb{E}\bigg[\int_0^\infty e^{-\lambda^\kappa t}\mathcal{E}\bigg(\mu_b \cdot + (\mu_a - \mu_b)\int_0^\cdot r_a\big(Y^x_s(0,0)\big)ds\bigg)_t dt\bigg]\\
&= J\big((x_a, x_b); 0, 0\big),
\end{aligned}$$

establishing (4.2). For $\mu_a < \mu_b$, the last inequality is obviously reversed.

Next, the expectation of Z satisfies the ODE

$$d\mathbb{E}[Z_t] = \sum_{k\in\{a,b\}} \mu_k \mathbb{E}[r_k(Y^x_t)Z_t]dt = \mu_b\mathbb{E}[Z_t]dt + (\mu_a - \mu_b)\mathbb{E}[r_a(Y^x_t)Z_t]dt.$$

As $r_a(Y^x_t) < 1$ a.s. for all $t \ge 0$ and for any $(A^a, A^b) \in \mathcal{A}^2$, Gronwall's inequality yields $\mathbb{E}[Z_t] < e^{\mu_a t}$, whence we get the upper bound

$$J(x; A^a, A^b) < \frac{(1-p)\lambda(x_a + x_b)}{\lambda^\kappa - \mu_a} = J^S(x_a + x_b; A^a). \tag{A.66}$$

Let $A^{a,w}$ be the fraud process associated to the function $\Psi^{a,w}$ for some constant $w \in (0,1)$. Then Proposition A.10 yields

$$d\mathbb{E}[Z_t] \ge \mu_b\mathbb{E}[Z_t]dt + (\mu_a - \mu_b)w\mathbb{E}[Z_t]dt = \big(\mu_b + (\mu_a - \mu_b)w\big)\mathbb{E}[Z_t]dt,$$

and Gronwall's inequality implies that for all $t > 0$,

$$\mathbb{E}[Z_t] \ge e^{(\mu_b + (\mu_a - \mu_b)w)t},$$

which in turn yields

$$J(x; A^{a,w}, 0) \ge \frac{(1-p)\lambda(x_a + x_b)}{\lambda^\kappa - (\mu_b + (\mu_a - \mu_b)w)}. \tag{A.67}$$

Finally, consider a sequence $(w_n)_{n\in\mathbb{N}} \subseteq (0,1)$ with $\lim_{n\to\infty} w_n = 1$. As $J(x; A^{a,w_n})$ is bounded from above by the right-hand side of (A.66) and from below by the right-hand side of (A.67) and the bounds agree in the limit $n\to\infty$,

$$\lim_{n\to\infty} J(x; A^{a,w_n}, 0) = J^S(x_a + x_b; A^a) = \sup_{(A^a,A^b)\in\mathcal{A}^2} J\big((x_a,x_b); A^a, A^b\big),$$

which is not attainable by any $(A^a, A^b) \in \mathcal{A}^2$ due to (A.66). □

Appendix B: Derivation of the HJB equation

This section contains a heuristic derivation of the HJB equations (B.10)–(B.14). By the linear dependence of the SDE (2.2) on the fraud processes (the controls), we conjecture a singular-type Nash equilibrium in which each trader prevents the wealth process from leaving a region. For any $i \neq j \in \{a, b\}$, let $\mathcal{C}^j \subseteq \mathbb{R}^2_{++}$ be an open set and $\bar{\mathcal{C}}^j := \mathcal{C}^j \cup \partial\mathcal{C}^j$ its closure in $\mathbb{R}^2_{++}$. Let $\Psi^j \in \Lambda^j$ be such that for any $A^i \in \mathcal{A}$, the pair (A^j, Y^x) associated to Ψ^j (where A^j is the response given by (3.2)) is the unique pair satisfying a.s. for all $t \geq 0$ that

(i) $Y^x_t(A^i, A^j) \in \mathbb{R}^2_{++}\backslash\mathcal{C}^j$;

(ii) $\int_{\mathbb{R}_+} \mathbf{1}_{\{Y^x_t(A^i,A^j)\in\partial\mathcal{C}^j\}} dA^j_t = 0$.

In this way, trader j keeps the personal wealth inside the region $\mathbb{R}^2_{++}\backslash\mathcal{C}^j$ at any time $t \geq 0$ and for any trader i's fraud process $A^i \in \mathcal{A}$. Moreover, if $x \in \mathcal{C}^j$, then trader j cheats instantly so as to bring the wealth at time 0 to $\partial\mathcal{C}^j$, and if the wealth is at $\partial\mathcal{C}^j$, trader j cheats as little as necessary to keep the wealth process in the interior of $\mathbb{R}^2_{++}\backslash\mathcal{C}^j$. Hence $\bar{\mathcal{C}}^j$ is called the *fraud region* of trader j.

Suppose that in a Nash equilibrium, given trader j's strategy Ψ^j, the optimal fraud process $A^{i,\star} \in \mathcal{A}$ of trader i satisfies that

$$\Delta A^{i,\star}_t \Delta A^{j,\star}_t = 0 \qquad \text{a.s. for all } t \geq 0 \tag{B.1}$$

(i.e., the equilibrium fraud processes $A^{i,\star}$ and $A^{j,\star}$ do not jump simultaneously) and that the value function $x \mapsto V^i(x; A^{j,\star})$ is smooth on $\mathbb{R}^2_{++}$. Properties (i) and (ii) imply that for any $x = (x_a, x_b) \in \mathcal{C}^j$, $A^{j,\star}_0 > 0$ is such that $Y^x_0 \in \partial\mathcal{C}^j$. By (B.1) and Lemma 3.2 (ii), the game value satisfies for any $x \in \mathcal{C}^j$ and $0 \leq \alpha \leq A^{j,\star}_0$ that

$$\begin{aligned}
&V^i\big((x_i, x_j); A^{j,\star}\big)\\
&= e^{-\alpha}\mathbb{E}\left[\lambda\int_0^\infty e^{-\lambda^\kappa t - A^{S,\star}_t} U^i(Y^{i,x}_t)dt \,\middle|\, x = \big(x_i, x_j + (x_a + x_b)(e^\alpha - 1)\big)\right]\\
&= e^{-\alpha} V^i\Big(\big(x_i, x_j + (x_a + x_b)(e^\alpha - 1)\big); A^{j,\star}\Big),
\end{aligned}$$

where $A^{S,\star}_t := A^{a,\star}_t + A^{b,\star}_t$. Since

$$\lim_{\alpha\downarrow 0} \frac{e^{-\alpha} V^i((x_i, x_j + (x_a + x_b)(e^\alpha - 1)); A^{j,\star}) - V^i((x_i, x_j); A^{j,\star})}{\alpha} = 0,$$

it follows that

$$(x_a + x_b)V^i_{x_j} - V^i = 0 \qquad \text{on } \mathcal{C}^j. \tag{B.2}$$

Define the associated differential operator (this is the infinitesimal generator of the uncontrolled pre-bankruptcy process $Y^x(0,0)$)

$$\mathcal{L}\phi(x) = \sum_{k\in\{a,b\}} \mu_k x_k \partial_{x_k}\phi(x) + \frac{1}{2}\sum_{k\in\{a,b\}} \sigma_k^2 x_k^2 \partial^2_{x_k x_k}\phi(x)$$

for any $\phi \in C^2(\mathbb{R}^2_{++})$. For any $x \in \mathbb{R}^2_{++}\backslash\mathcal{C}^j$, the problem of trader i becomes an optimal (singular) control problem. Treating the triplet (A^i, A^j, Y^x) as the state process, the dynamic programming principle (see e.g. Fleming and Soner [17, Sect. VIII.2]) suggests the quasi-variational inequality

$$\max_{\mathbb{R}^2_{++}\backslash\mathcal{C}^j}\{\mathcal{L}V^i - \lambda^\kappa V^i + U^i, (x_a + x_b)V^i_{x_i} - V^i\} = 0, \tag{B.3}$$

and verification theorems (cf. Fleming and Soner [17, Chap. VIII, Theorem 4.1]) suggest that the set

$$\bar{\mathcal{C}}^i = \{x \in \mathbb{R}^2_{++}\backslash\mathcal{C}^j : (x_a + x_b)V^i_{x_i} - V^i = 0\} \tag{B.4}$$

corresponds to the fraud region of $A^{i,\star}$, so that the optimal cheating strategy for trader i is to only cheat in a region $\mathcal{C}^i \subseteq \mathbb{R}^2_{++}$ and prevent the wealth process from leaving the region $\mathbb{R}^2_{++}\backslash\mathcal{C}^i$ at any time $t \geq 0$. More formally, and similarly to $A^{j,\star}$, $A^{i,\star}$ is such that a.s. for all $t \geq 0$,

(i) $Y^x_t(A^{i,\star}, A^{j,\star}) \in \mathbb{R}^2_{++}\backslash(\mathcal{C}^j \cup \mathcal{C}^i)$;

(ii) $\int_{\mathbb{R}_+} \mathbf{1}_{\{Y^x_t(A^{i,\star},A^{j,\star})\in\partial\mathcal{C}^i\}} dA^{i,\star}_t = 0$.

Here $\mathbb{R}^2_{++}\backslash(\mathcal{C}^j \cup \mathcal{C}^i)$ is the common no-fraud region. Note that Condition (B.1) implies that $\mathcal{C}^i \cap \mathcal{C}^j = \emptyset$, that is, the traders' fraud regions do not intersect.

Substituting (B.4) into (B.3), it follows that for any $x \in \mathbb{R}^2_{++}\backslash\mathcal{C}^j$,

$$\mathcal{L}V^i - \lambda^\kappa V^i + U^i = 0 \qquad \text{on } \mathbb{R}^2_{++}\backslash(\mathcal{C}^i \cup \mathcal{C}^j), \tag{B.5}$$

$$\mathcal{L}V^i - \lambda^\kappa V^i + U^i < 0 \qquad \text{on } \mathcal{C}^i, \tag{B.6}$$

$$(x_a + x_b)V^i_{x_i} - V^i < 0 \qquad \text{on } \mathbb{R}^2_{++}\backslash(\mathcal{C}^i \cup \mathcal{C}^j). \tag{B.7}$$

Let $\mathcal{L}^i$ be the differential operator acting on $\varphi \in C^2(\mathbb{R}_{++})$ and given by

$$\begin{aligned}\mathcal{L}^i\varphi(w) = (1-\gamma_i)\Big(&\mu_i w + \mu_j(1-w) - \frac{\gamma_i}{2}\big(\sigma_i^2 w^2 + \sigma_j^2(1-w)^2\big)\Big)\varphi(w)\\ &+ \Big(\mu_i - \mu_j + \gamma_i\big(\sigma_j^2(1-w) - \sigma_i^2 w\big)\Big)w(1-w)\varphi_w(w)\\ &+ \frac{\sigma_i^2+\sigma_j^2}{2}w^2(1-w)^2\varphi_{ww}(w).\end{aligned}$$

Now, conjecture that for both traders $k \in \{a, b\}$, the fraud regions in a Nash equilibrium are of the form

$$\mathcal{C}^k = \{x \in \mathbb{R}^2_{++} : r_k(x) < m_k\}, \tag{B.8}$$

where $m_k \in (0, 1)$ is such that $0 < m_a + m_b < 1$. In other words, by cheating, traders prevent their fraction of wealth from going below their respective critical threshold m_k. Note that due to $r_i(x) + r_j(x) = 1$ for any $x \in \mathbb{R}^2_{++}$, the condition $m_a + m_b < 1$ is equivalent to $C^i \cap C^j = \{x \in \mathbb{R}^2_{++} : r_i(x) < m_i \text{ and } r_i(x) > 1 - m_j\} = \emptyset$. Hence the equilibrium fraud processes $(A^{a,\star}, A^{b,\star})$ and the pre-bankruptcy wealth process $Y^x(A^{a,\star}, A^{b,\star})$ are precisely the processes associated with the response maps Ψ^{k,m_k} which solve SP_{m_k+} for $k \in \{a, b\}$ (see Definition 3.6). The scale-invariance property from Lemma 3.2 (iii) is inherited by the value function, meaning that we have $V^i(cx; A^{j,\star}) = c^{1-\gamma_i} V^i(x; A^{j,\star})$ for any $c > 0$; so $\varphi^i(w) = V^i((w, 1 - w); A^{j,\star})$ for any w in $(0, 1)$. Thus Lemma 3.2 (ii) implies that V^i is of the form

$$V^i(x; A^{j,\star}) = \lambda(x_a + x_b)^{1-\gamma_i} \varphi^i\big(r_i(x)\big). \tag{B.9}$$

Let $r^i(x) = w$ and substitute (B.9) and (B.8) into (B.5), (B.6), (B.4), (B.7) and (B.2). This yields the HJB equations

$$\mathcal{L}^i \varphi^i(w) - \lambda^\kappa \varphi^i(w) + U^i(w) = 0 \qquad \text{on } (m_i, 1 - m_j), \tag{B.10}$$

$$\mathcal{L}^i \varphi^i(w) - \lambda^\kappa \varphi^i(w) + U^i(w) < 0 \qquad \text{on } (0, m_i), \tag{B.11}$$

$$(1 - w)\varphi^i_w(w) - \gamma_i \varphi^i(w) = 0 \qquad \text{on } (0, m_i), \tag{B.12}$$

$$(1 - w)\varphi^i_w(w) - \gamma_i \varphi^i(w) < 0 \qquad \text{on } (m_i, 1 - m_j), \tag{B.13}$$

$$w\varphi^i_w(w) + \gamma_i \varphi^i(w) = 0 \qquad \text{on } (1 - m_j, 1) \tag{B.14}$$

which are the starting point for the verification approach.

Appendix C: Proofs of the main results

The following result establishes the link between the HJB equations (B.12)–(B.14) and the optimisation problem.

Lemma C.1 *Let $(m_a, m_b) \in \Delta$. For any $i \neq j \in \{a, b\}$, let $\varphi^i \in C^1((0, 1))$ be an $\mathbb{R}_{++}$-valued function satisfying (B.12)–(B.14). For any $\alpha \geq 0$ and $w \in (0, 1)$, set*

$$\tilde{f}^j(\alpha, w) := \left(\ln\left(1 + \frac{1}{1 - m_j}\big(m_j e^\alpha - (1 - w)\big)\right)\right)^+ \tag{C.1}$$

and

$$\tilde{h}^i(\alpha, w) := e^{-\alpha - \tilde{f}^j(\alpha,w)}(e^\alpha + e^{\tilde{f}^j(\alpha,w)} - 1)^{1-\gamma_i} \varphi^i\left(\frac{e^\alpha - (1 - w)}{e^\alpha + e^{\tilde{f}^j(\alpha,w)} - 1}\right). \tag{C.2}$$

Then:

(i) $\tilde{f}^j(\alpha, w) > 0$ *if and only if one of the following two conditions holds*: (a) $w > 1 - m_j$, *or* (b) $w \leq 1 - m_j$ *and* $\alpha > \ln \frac{1-w}{m_j}$.

(ii) *If* (α, w) *is such that* $\tilde{f}^j(\alpha, w) > 0$, *then* $\partial_\alpha \tilde{h}^i(\alpha, w) < 0$.

(iii) *For any* $\alpha \geq 0$ *and all* $w \in (0, 1)$,

$$\tilde{h}^i(\alpha, w) - \varphi^i(1-w) \leq 0, \tag{C.3}$$

and equality in (C.3) *holds precisely in the following two cases*:

$$\alpha = 0 \quad \textit{and} \quad w \in (m_i, 1),$$

$$\alpha \leq \ln \frac{1-w}{1-m_i} \quad \textit{and} \quad w \in (0, m_i].$$

Proof (i) We show the equivalent statement that we have $\tilde{f}^j(\alpha, w) = 0$ if and only if $w \leq 1 - m_j$ and $\alpha \leq \ln \frac{1-w}{m_j}$. Note that $\tilde{f}^j(\alpha, w) = 0$ if and only if $e^\alpha \leq \frac{1-w}{m_j}$. If $e^\alpha \leq \frac{1-w}{m_j}$, then $\frac{1-w}{m_j} \geq 1$ because $e^\alpha \geq 1$, which together with $e^\alpha \leq \frac{1-w}{m_j}$ implies that $\alpha \leq \ln \frac{1-w}{m_j}$. The converse implication follows from the monotonicity of the exponential function, applied to $\alpha \leq \ln \frac{1-w}{m_j}$.

(ii) If $\tilde{f}^j(\alpha, w) > 0$, then

$$\frac{e^\alpha - (1-w)}{e^\alpha + e^{\tilde{f}^j(\alpha,w)} - 1} = 1 - m_j,$$

and thus $\tilde{h}^i$ (defined in (C.2)) simplifies to

$$\tilde{h}^i(\alpha, w) = \frac{1 - (1-w)e^{-\alpha}}{w + (e^\alpha - 1)m_j} \left(\frac{1 - m_j}{e^\alpha - (1-w)} \right)^{\gamma_i} \varphi^i(1 - m_j).$$

Differentiating $\tilde{h}^i$ with respect to α and recalling that φ^i is strictly positive, it follows that $\partial_\alpha \tilde{h}^i(\alpha, w)$ has the same sign as

$$g^i(\alpha, w) = (1-w)(w - m_j) - e^{2\alpha}(1+\gamma_i)m_j + e^\alpha\big(2(1-w)m_j - \gamma_i(w - m_j)\big).$$

As $\partial_\alpha g^i(\alpha, w) \leq e^\alpha(-\gamma_i(w + m_j) - 2wm_j) < 0$, it follows that

$$g^i(\alpha, w) \leq g^i(0, w) = -w\big(\gamma_i + m_j - (1-w)\big) < 0.$$

Hence $\partial_\alpha \tilde{h}(\alpha, w) < 0$.

(iii) All classical solutions of the linear ODEs (B.12) and (B.14) are of the form

$$w \mapsto C(1-w)^{-\gamma_i} \quad \text{and} \quad w \mapsto Dw^{-\gamma_i}, \qquad C, D \in \mathbb{R},$$

respectively. As φ^i is positive,

$$\varphi^i(w) = \begin{cases} C_0(1-w)^{-\gamma_i} & \text{for } w \in (0, m_i), \qquad \text{(C.4a)} \\ C_1 w^{-\gamma_i} & \text{for } w \in (1 - m_j, 1), \qquad \text{(C.4b)} \end{cases}$$

where $C_0 > 0$ and $C_1 > 0$. Distinguish now three cases:

(1) $w \in (1-m_j, 1)$: By (i), we have $\tilde{f}^i(\alpha, w) > 0$ for any $\alpha \geq 0$, and thus by (ii), it follows that $\tilde{h}^i(\alpha, w) \leq \tilde{h}^i(0, w)$, with equality if and only if $\alpha = 0$. Thus in conjunction with (C.4b), we get

$$\tilde{h}^i(\alpha, w) - \varphi^i(w) \leq \tilde{h}^i(0, w) - \varphi^i(w) = \left(\frac{1-m_j}{w}\right)^{\gamma_i} \varphi^i(1-m_j) - \varphi^i(w) = 0,$$

where equality holds if and only if $\alpha = 0$.

(2) $w \in (m_i, 1-m_j]$: If $\alpha > \ln \frac{1-w}{m_j}$, then $\tilde{f}^i(\alpha, w) > 0$ by part (i). So by (ii), we have $\partial_\alpha \tilde{h}^i(\alpha, w) < 0$ and thus $\tilde{h}^i(\alpha, w) < \tilde{h}^i(\ln \frac{1-w}{m_j}, w)$ for any $\alpha > \ln \frac{1-w}{m_j}$. If instead $\alpha \leq \ln \frac{1-w}{m_j}$, then $f^i(\alpha, w) = 0$ by (i) and $\tilde{h}^i$ reduces to

$$\tilde{h}^i(\alpha, w) = e^{-\gamma_i \alpha} \varphi^i\big(1-(1-w)e^{-\alpha}\big).$$

By (B.13),

$$\begin{aligned}
\partial_\alpha \tilde{h}^i(\alpha, w) &= -\gamma_i \tilde{h}^i(\alpha, w) + e^{-(1+\gamma_i)\alpha}(1-w)\varphi^i_w\big(1-(1-w)e^{-\alpha}\big) \\
&\leq -\gamma_i \tilde{h}^i(\alpha, w) + \gamma_i e^{-(1+\gamma_i)\alpha}(1-w)\varphi^i\big(1-(1-w)e^{-\alpha}\big) \\
&= \gamma_i e^{-\gamma_i \alpha}(e^{-\alpha}-1)\varphi^i\big(1-(1-w)e^{-\alpha}\big) \leq 0,
\end{aligned}$$

where equality holds if and only if $\alpha = 0$. Hence for any $\alpha \geq 0$,

$$\tilde{h}^i(\alpha, w) - \varphi^i(w) \leq \tilde{h}^i(0, w) - \varphi^i(w) = 0,$$

with equality if and only if $\alpha = 0$.

(3) $w \in (0, m_i]$: If $\alpha > \ln \frac{1-w}{m_j}$, then (i) and (ii) imply that

$$\tilde{h}^i(\alpha, w) < \tilde{h}^i\left(\ln \frac{1-w}{m_j}, w\right)$$

for $\alpha > \ln \frac{1-w}{m_j}$. If $\alpha \leq \ln \frac{1-w}{m_j}$, then property (i) implies that $f^i(\alpha, w) = 0$, and hence

$$\tilde{h}^i(\alpha, w) = e^{-\gamma_i \alpha} \varphi^i\big(1-(1-w)e^{-\alpha}\big).$$

If $\alpha \in (\ln \frac{1-w}{1-m_i}, \ln \frac{1-w}{m_j}]$, we get $1-(1-w)e^{-\alpha} \in [m_j, 1-m_i)$. Using (B.13), it analogously follows that $\partial_\alpha \tilde{h}^i(\alpha, w) < 0$ and thus

$$\tilde{h}^i\left(\ln \frac{1-w}{m_j}, w\right) < \tilde{h}^i\left(\ln \frac{1-w}{1-m_i}, w\right).$$

Finally, if $\alpha \leq \ln \frac{1-w}{1-m_i}$, then $1-(1-w)e^{-\alpha} \in [1-m_i, 1)$ and (C.4a) implies that

$$\begin{aligned}\tilde{h}^i(\alpha, w) - \varphi^i(w) &= e^{-\gamma_i \alpha}\varphi^i\big(1-(1-w)e^{-\alpha}\big) - \varphi^i(w) \\ &= C_0 e^{-\gamma_i \alpha}\big((1-w)e^{-\alpha}\big)^{-\gamma_i} - C_0(1-w)^{-\gamma_i} = 0. \qquad \square\end{aligned}$$

The following result verifies that the solutions to the HJB equations (B.10)–(B.14) are indeed the value functions (up to homogeneity in total firm's wealth).

Theorem C.2 *Let* $(m_a, m_b) \in \Delta$ *be fraud boundaries. For any* $i \neq j \in \{a, b\}$, *let the* $\mathbb{R}_{++}$*-valued function* $\varphi^i \in C^1([0,1]) \cap C^2((0, 1-m_j))$ *be such that* φ^i_w *is also Lipschitz-continuous on* $(0,1)$ *and satisfies the HJB equations* (B.10)–(B.14). *Define two functions by* $\phi^k(x) := \lambda(x_a + x_b)^{1-\gamma_k}\varphi^k(r_k(x))$ *for* $x \in \mathbb{R}^2_{++}$ *and* $k \in \{a, b\}$. *Then the pair* $(\Psi^{a,m_a}, \Psi^{b,m_b})$ *is a Nash equilibrium and* (ϕ^a, ϕ^b) *are the corresponding game values, i.e., for any* $i \neq j \in \{a, b\}$,

$$\phi^i(x) = V^i(x; A^{j,\star}).$$

Proof Let $i \neq j \in \{a, b\}$. We first show that

$$\phi^i(x) \geq \sup_{A^i \in \mathcal{A}} J^i(x; A^i, A^j), \qquad x \in \mathbb{R}^2_{++}, \tag{C.5}$$

where A^j satisfies (3.2) with $\Psi^j = \Psi^{j,m_j}$. To this end, extend φ^i to $\mathbb{R}$ by setting $\varphi^i(w) := \varphi^i(0)$ for $w < 0$ and $\varphi^i(w) := \varphi^i(1)$ for $w > 1$. Let $\xi \in C^\infty(\mathbb{R})$ be a nonnegative function, compactly supported in $[-1, 1]$ and such that $\int_{\mathbb{R}} \xi(x)dx = 1$. For any $m \geq 1$, let $\xi_m(w) := \frac{\xi(mw)}{m}$. By convolution, the function

$$\varphi^{i,m}(w) := \int_{\mathbb{R}} \varphi^i(y)\xi_m(w-y)dy$$

is infinitely differentiable. Since $\mathrm{supp}(\xi_m) \subseteq [-1/m, 1/m]$, $\varphi^{i,m}(w)$ depends only on the values of $\varphi^i(w)$ where $w \in [w_0 - 1/m, w_0 + 1/m]$. Since φ^i is continuous on $\mathbb{R}$, $\varphi^{i,m}$ converges to φ^i as $m \to \infty$ uniformly on any compact subset of $\mathbb{R}$. Moreover, as $\varphi^i_w \in C([0,1])$, also $\varphi^{i,m}_w$ converges to φ^i_w on any compact subset of $\mathbb{R}$ (cf. the argument in Fleming and Soner [17, Appendix C]). For $r > 0$, define the disk $D_r(x) := \{x \in \mathbb{R}^2 : |x| < r\}$ and set $\mathcal{R}_{m,r} := \mathcal{R}_m \cap D_r(0)$, where

$$\mathcal{R}_m := \left\{x \in \mathbb{R}^2_{++} : \min\{x_i, x_j\} > \frac{1}{m}\right\}.$$

Define the exit time $\tau_{m,r} := \inf\{t \geq 0 : Y^x_t \notin R_{m,r}\}$ and the function

$$\phi^{i,m}(x) := \lambda(x_a + x_b)^{1-\gamma_i}\varphi^{i,m}\big(r_i(x)\big).$$

Applying Itô's formula to $e^{-\lambda^\kappa(t\wedge\tau_{m,r})-A^S_{t\wedge\tau_{m,r}}}\phi^{i,m}(Y^x_{t\wedge\tau_{m,r}})$, we obtain upon taking expectations (using the abbreviation $Y^x_t = Y^x_t(A^i, A^j)$) that

$$\begin{aligned}
&\lambda^{-1}\phi^{i,m}(x)\\
&=\lambda^{-1}\mathbb{E}\Big[e^{-\lambda^\kappa(t\wedge\tau_{m,r})-A^S_{t\wedge\tau_{m,r}}}\phi^{i,m}(Y^x_{t\wedge\tau_{m,r}})\Big]\\
&\quad-\mathbb{E}\Big[\int_0^{t\wedge\tau_{m,r}} e^{-\lambda^\kappa s-A^S_s}(Y^{x,S}_s)^{1-\gamma_i}\big(\mathcal{L}^i\varphi^{i,m}(W^{i,w_i}_s)-\lambda^\kappa\varphi^{i,m}(W^{i,w_i}_s)\big)ds\Big]\\
&\quad-\mathbb{E}\Big[\int_0^{t\wedge\tau_{m,r}} e^{-\lambda^\kappa s-A^S_s}(Y^{x,S}_s)^{1-\gamma_i}\\
&\qquad\qquad\times\big((1-W^{i,w_i}_s)\varphi^{i,m}_w(W^{i,w_i}_s)-\gamma_i\varphi^{i,m}(W^{i,w_i}_s)\big)dA^{i,c}_s\Big]\\
&\quad+\mathbb{E}\Big[\int_0^{t\wedge\tau_{m,r}} e^{-\lambda^\kappa s-A^S_s}(Y^{x,S}_s)^{1-\gamma_i}\\
&\qquad\qquad\times\big(W^{i,w_i}_s\varphi^{i,m}_w(W^{i,w_i}_s)+\gamma_i\varphi^{i,m}(W^{i,w_i}_s)\big)dA^{j,c}_s\Big]\\
&\quad-\mathbb{E}\Big[\sum_{0\le s\le t\wedge\tau_{m,r}} e^{-\lambda^\kappa s-A^S_{s-}}(Y^{x,S}_{s-})^{1-\gamma_i}\\
&\qquad\qquad\times\Big(e^{-\Delta A^S_s}(e^{\Delta A^i_s}+e^{\Delta A^{j,\star}_s}-1)^{1-\gamma_i}\varphi^{i,m}(W^{i,w_i}_s)-\varphi^{i,m}(W^{i,w_i}_{s-})\Big)\Big],
\end{aligned}\tag{C.6}$$

where $W^{i,w_i}_t = r_i(Y^x_t)$ for any $t\ge 0$, with $W^{i,w_i}_{0-} = r_i(x) = w_i$.

Since Ψ^{j,m_j} solves $\mathrm{SP}^j_{m_j+}$, Proposition A.10 implies that $0 < W^{i,w_i}_t \le 1-m_j$ a.s. for all $t\ge 0$, and the continuity of $\mathcal{L}^i\varphi^i$ on $(0, 1-m_j)$ implies the convergence $\lim_{m\to\infty}\mathcal{L}^i\varphi^{i,m}(w) = \mathcal{L}^i\varphi^i(x)$ for any $w\in(0,1-m_j)$. Also, since φ^i_w is Lipschitz-continuous on $(0,1)$, there exists for any $r>0$ some $M>0$ such for any $m\in\mathbb{N}$ and $x\in\mathcal{R}_{m,r}$, we have

$$\big|(x_i+x_j)\mathcal{L}^i\varphi^{i,m}\big(r_i(x)\big)\big| < M.$$

As $\lim_{m\to\infty}\tau_{m,r} = \tau_r := \inf\{t\ge 0 : Y^x_t \notin \mathbb{R}^2_{++}\cap D_r(0)\}$, dominated convergence implies that

$$\begin{aligned}
&\lim_{m\to\infty}\int_0^{t\wedge\tau_{m,r}} e^{-\lambda^\kappa s-A^S_s}(Y^{x,S}_s)^{1-\gamma_i}\mathcal{L}^i\varphi^i_m(W^{i,w_i}_s)ds\\
&=\int_0^{t\wedge\tau_r} e^{-\lambda^\kappa s-A^S_s}(Y^{x,S}_s)^{1-\gamma_i}\mathcal{L}^i\varphi^i(W^{i,w_i}_s)ds
\end{aligned}$$

a.s. for all $t\ge 0$. Using (B.14) and the fact that $A^{j,c}$ increases only at $1-m_j$, letting $m\to\infty$ and $r\to\infty$ in (C.6) and noting that $\lim_{r\to\infty}\tau_r\to\infty$ a.s. because $\partial\mathbb{R}^2_{++}$ is unattainable for Y^x when $x \in \mathbb{R}^2_{++}$, we obtain

$$
\begin{aligned}
&\lambda^{-1}\phi^i(x)\\
&= \lambda^{-1}\mathbb{E}\Big[e^{-\lambda^\kappa t - A^S_t}\phi^i(Y^x_t)\Big]\\
&\quad - \mathbb{E}\Big[\int_0^t e^{-\lambda^\kappa s - A^S_s}(Y^{x,S}_s)^{1-\gamma_i}\big(\mathcal{L}^i\varphi^i(W^{i,w_i}_s) - \lambda^\kappa\varphi^i(W^{i,w_i}_s)\big)ds\Big] \qquad \text{(C.7)}\\
&\quad - \mathbb{E}\Big[\int_0^t e^{-\lambda^\kappa s - A^S_s}(Y^{x,S}_s)^{1-\gamma_i}\\
&\qquad\qquad \times \big((1-W^{i,w_i}_s)\varphi^i_w(W^{i,w_i}_s) - \gamma_i\varphi^i(W^{i,w_i}_s)\big)dA^{i,c}_s\Big] \qquad \text{(C.8)}\\
&\quad - \mathbb{E}\Big[\sum_{0\le s\le t} e^{-\lambda^\kappa s - A^S_{s-}}(Y^{x,S}_{s-})^{1-\gamma_i}\\
&\qquad\qquad \times \Big(e^{-\Delta A^S_s}\big(e^{\Delta A^i_s} + e^{\Delta A^{j,\star}_s} - 1\big)^{1-\gamma_i}\varphi^i(W^{i,w_i}_s) - \varphi^i(W^{i,w_i}_{s-})\Big)\Big]. \qquad \text{(C.9)}
\end{aligned}
$$

Note that by Lemma A.10 and (A.30), we have $\Delta A^j_t = \tilde{f}^j(\Delta A^i_t, W^{i,w_i}_{t-})$ a.s. for all $t \geq 0$, where $\tilde{f}^j$ is given by (C.1). Hence Lemma C.1 (iii) yields

$$
\begin{aligned}
&\tilde{h}^i\big(\Delta A^i_t, r_i(Y^x_t)\big) - \varphi^i\big(w_i(Y^x_{t-})\big)\\
&= e^{-\Delta A^S_t}(e^{\Delta A^i_t} + e^{\Delta A^j_t} - 1)^{1-\gamma_i}\varphi^i\big(r_i(Y^x_t)\big) - \varphi^i\big(r_i(Y^x_{t-})\big) \leq 0
\end{aligned}
$$

a.s. for all $t \geq 0$, where $\tilde{h}^i$ is given by (C.2). Together with the HJB equations (B.10)–(B.14) and the fact that $(x_i + x_j)^{1-\gamma_i}U^i(r_i(x)) = U^i(x_i)$ for $x \in \mathbb{R}^2_{++}$, it follows that for any $t \geq 0$,

$$
\begin{aligned}
\lambda^{-1}\phi^i(x) \geq{}& \mathbb{E}\big[e^{-\lambda^\kappa t - A^S_t}(Y^{x,S}_t)^{1-\gamma_i}\varphi^i\big(r_i(Y^x_t)\big)\big]\\
&+ \mathbb{E}\Big[\int_0^t e^{-\lambda^\kappa s - A^S_s}U^i(Y^{i,x}_s)ds\Big]. \qquad \text{(C.10)}
\end{aligned}
$$

Lemma A.3 implies $\mathbb{E}[e^{-A^S_t}(Y^{x,S}_t)^{1-\gamma_i}] = \mathbb{E}[(\mathbf{1}_{\{t<\tau_A\}}Y^{x,S}_t)^{1-\gamma_i}]$. Using the boundedness of φ^i, Lemma A.8, (A.17) and Jensen's inequality, we obtain

$$
\mathbb{E}\big[e^{-A^S_t}(Y^{x,S}_t)^{1-\gamma_i}\varphi^i\big(r_i(Y^x_t)\big)\big] \leq M\mathbb{E}[\mathbf{1}_{\{t<\tau_A\}}Y^{x,S}_t]^{1-\gamma_i} \leq M(x_i + x_j)e^{\mu_a\vee\mu_b(1-\gamma_i)t},
$$

where $M = \max_{0\leq w\leq 1}|\varphi^i(w)|$. Assumption 3.1 implies that

$$
\lim_{t\to\infty}\mathbb{E}\big[e^{-\lambda^\kappa t - A^S_t}(Y^{x,S}_t)^{1-\gamma_i}\varphi^i\big(r_i(Y^x_t)\big)\big] = 0.
$$

For $t \geq 0$, let $Z_t := \int_0^t e^{-\lambda^\kappa s - A^S_s}U^i(Y^{i,x}_s)ds \leq Z_\infty$ since the integrand is nonnegative a.s. Furthermore, Z_∞ is in L^1 due to Lemma 3.2 (i), whence $\lim_{t\to\infty}\mathbb{E}[Z_t] = \mathbb{E}[Z_\infty]$ by dominated convergence, which establishes (C.5).

Next, we show that equality holds in (C.5). If trader i employs the cheating strategy Ψ^{i,m_i}, then by Proposition A.10, $m_i < r_i(Y_t^x(A^{i,\star}, A^{j,\star})) < 1 - m_j$ a.s. for almost every $t \geq 0$, where $A_t^{i,\star} = \Psi^{i,m_i}(Y_{[0,t)}^{i,x}, Y_{[0,t)}^{j,x}, A_{[0,t)}^{i,\star})$ a.s. for all $t \geq 0$. The process $A^{i,c,\star}$ increases only when W^{i,w_i} is at m_i; hence the term (C.8) vanishes by (B.12). The jump $\Delta A_t^{i,\star} = \mathbf{1}_{\{t=0\}}(\ln \frac{1-w_i}{1-m_i})^+$ is nonzero only when $r_i(x) < m_i$, and such a jump brings W_0^{i,w_i} to m_i; thus by Lemma C.1 (iii), the term (C.9) vanishes. Therefore using (B.10) for the term (C.7) leads to equality in (C.10).

Finally, letting t converge to infinity yields

$$\phi^i(x) = J^i(x; A^{i,\star}, A^j) = \sup_{A^i \in \mathcal{A}} J^i(x; A^i, A^{j,\star}). \qquad \square$$

C.8 Proof of Theorem 3.9

Lemma C.3 (i) *The constants c_k^i $(k = 0, 1, 2, 3)$ in Theorem 3.9 are strictly positive.*

(ii) *Let $c > 0$, $w^\star \in (0, \hat{w}_i]$ and suppose $f^\star(w) := c(1-w)^{-\gamma_i}$ satisfies*

$$\mathcal{L}^i f^\star(w) - \lambda^\kappa f^\star(w) + U^i(w) = 0, \qquad w \in [w^\star, \hat{w}_i]. \tag{C.11}$$

Then $\mathcal{L}^i f^\star(w) - \lambda^\kappa f^\star(w) + U^i(w) < 0$ for any $w \in (0, w^\star)$.

Proof First, we show that $(\alpha_i + \beta_i - 1)w - \alpha_i\beta_i > 0$ for any $w \in (0, \hat{w}_i]$. Indeed, if $\alpha_i + \beta_i - 1 > 0$, then clearly $(\alpha_i + \beta_i - 1)w - \alpha_i\beta_i > 0$. On the other hand, if $\alpha_i + \beta_i - 1 < 0$, the inequalities $w < \hat{w}_i$ and $\beta_i > 1 - \gamma_i$ (see Lemma A.11 (i)) give

$$\begin{aligned}(\alpha_i + \beta_i - 1)w - \alpha_i\beta_i &> (\alpha_i + \beta_i - 1)\hat{w}_i - \alpha_i\beta_i \\ &= -\frac{\alpha_i(1-\alpha_i)(\beta_i - (1-\gamma_i))}{\gamma_i - \alpha_i} > 0.\end{aligned} \tag{C.12}$$

Since $\tilde{w}_i < \hat{w}_i$ by Lemma A.11 (v), it follows by (C.12) that c_0^i, c_1^i and c_2^i are strictly positive, which in turn implies $c_3^i > 0$. This finishes the proof of (i).

(ii) For any $w \in (0, w^\star)$, $\mathcal{L}^i f^\star(w) - \lambda^\kappa f^\star(w) + U^i(w)$ has the same sign as

$$\begin{aligned}\ell(w) &:= w^{1-\gamma_i}(1-w)^{\gamma_i} - c(1-\gamma_i)\bigg(p_i - \Big(\frac{\sigma^2}{2} - k_i\Big)w\bigg) \\ &= w^{1-\gamma_i}(1-w)^{\gamma_i} - \frac{\sigma^2}{2}c(1-\gamma_i)\big((\alpha_i + \beta_i - 1)w - \alpha_i\beta_i\big).\end{aligned} \tag{C.13}$$

The condition (C.11) implies $\ell(w^\star) = 0$, and we have

$$\lim_{w \downarrow 0} \ell(w) = \frac{1}{2}c(1-\gamma_i)\sigma^2\alpha_i\beta_i < 0, \tag{C.14}$$

$$\ell_{ww}(w) = -\gamma_i(1-\gamma_i)w^{-1-\gamma_i}(1-w)^{-2+\gamma_i} < 0. \tag{C.15}$$

Suppose for a contradiction that $\sup_{w \in (0,w^\star)} \ell(w) > \ell(w^\star) = 0$. By (C.14) and the strict concavity in (C.15), it follows that the maximum of ℓ is attained at some

$z \in (0, w^\star)$, i.e., $\sup_{w\in(0,w^\star)} \ell(w) = \ell(z)$. Thus

$$\ell_w(z) = z^{-\gamma_i}(1-z)^{\gamma_i-1}(1-\gamma_i-z) + \frac{\sigma^2}{2}c(1-\gamma_i)(1-\alpha_i-\beta_i) = 0. \tag{C.16}$$

As $z < \hat{w}_i < 1-\gamma_i$ by Lemma A.11 (i), plugging (C.16) into (C.13) yields

$$\begin{aligned}\ell(z) &= \frac{c(1-\gamma_i)\sigma^2}{2(1-\gamma_i-z)}\Big((\alpha_i-1)\gamma_i z + \big(\gamma_i z + (1-\gamma_i-z)\alpha_i\big)\beta_i\Big)\\ &< \frac{c(1-\gamma_i)\sigma^2}{2(1-\gamma_i-z)}(\alpha_i-1)\gamma_i z < 0,\end{aligned}$$

which contradicts $\ell(z) > 0$. Hence $\sup_{w\in(0,w^\star)} \ell(w) \le 0$, and since ℓ is a nonpositive strictly concave function with $\ell(w^\star) = 0$, it must be strictly negative on $(0, w^\star)$. □

Proof of Theorem 3.9 To establish the theorem, it suffices to prove that the conditions of Theorem C.2 are satisfied. By construction, for any $i \ne j \in \{a, b\}$, the function φ^i satisfies the ODEs (B.10), (B.12) and (B.14), as well as the smooth pasting conditions

$$\varphi^i(\tilde{w}_i-) = \varphi^i(\tilde{w}_i+), \tag{C.17}$$

$$\begin{aligned}\varphi^i_w(\tilde{w}_i-) &= \varphi^i_w(\tilde{w}_i+),\\ \varphi^i_{ww}(\tilde{w}_i-) &= \varphi^i_{ww}(\tilde{w}_i+).\end{aligned} \tag{C.18}$$

As $F^i(\tilde{w}_i, \tilde{w}_j) = 0$, we also have

$$\varphi^i\big((1-\tilde{w}_j)-\big) = \varphi^i\big((1-\tilde{w}_j)+\big), \tag{C.19}$$

$$\varphi^i_w\big((1-\tilde{w}_j)-\big) = \varphi^i_w\big((1-\tilde{w}_j)+\big). \tag{C.20}$$

By construction, φ^i is in $C^2((0,\tilde{w}_i)) \cap C^2((\tilde{w}_i, 1-\tilde{w}_j)) \cap C^2((1-\tilde{w}_j, 1))$. The equalities (C.17) and (C.18) therefore imply that $\varphi^i \in C^2(0, 1-\tilde{w}_j)$, and the equalities (C.19) and (C.20) yield $\varphi^i \in C^1(0,1)$. Because we have $\lim_{w\downarrow 0}\varphi^i(w) = c^i_0$, $\lim_{w\uparrow 1}\varphi^i(w) = c^i_3$, $\lim_{w\downarrow 0}\varphi^i_w(w) = \gamma_i c^i_0$ and $\lim_{w\uparrow 1}\varphi^i(w) = -\gamma_i c^i_3$, we may extend the function φ^i to an element in $C^1([0,1])$. Moreover, in view of the finite limits

$$\lim_{w\downarrow 0}\varphi^i_{ww}(w) = c^i_0\gamma_i(1+\gamma_i),$$

$$\lim_{w\uparrow 1}\varphi^i_{ww}(w) = c^i_3\gamma_i(1+\gamma_i),$$

$$\begin{aligned}\varphi^i_{ww}\big((1-\tilde{w}_j)+\big) &= c^i_3\gamma_i(1+\gamma_i)(1-\tilde{w}_j)^{-2-\gamma_i},\\ \varphi^i_{ww}\big((1-\tilde{w}_j)-\big) &= w^{-2}(1-w)^{-2}\\ &\quad\times\Big(c^i_1 w^{\alpha_i}(1-w)^{a_i}\big(\alpha_i^2-(1-2\gamma_i w)\alpha_i-(1-\gamma_i)\gamma_i w^2\big)\\ &\qquad + c^i_2 w^{\beta_i}(1-w)^{b_i}\big(\beta_i^2-(1-2\gamma_i w)\beta_i-(1-\gamma_i)\gamma_i w^2\big)\Big)\\ &\quad - \frac{\gamma_i}{q_i w^{1+\gamma_i}}\bigg|_{w=1-\tilde{w}_j}\end{aligned}$$

and the continuity of φ^i_{ww} on the intervals $(0, 1-\tilde{w}_j)$ and $(1-\tilde{w}_j, 1)$, it follows that $\sup_{w\in(0,1)} |\varphi^i_{ww}(w)| < \infty$, whence φ^i_w is Lipschitz-continuous.

Now (B.11) follows from $\tilde{w}_i < \hat{w}_i$ in conjunction with Lemma C.3 (i) and (ii). To check (B.13), first note that $(1-w)\varphi^i_w(w) - \gamma_i\varphi^i(w)$ has on the interval $(\tilde{w}_i, 1-\tilde{w}_j)$ the same sign as

$$(w_j - \gamma_i)F^i(\tilde{w}_i, w_j) + \gamma_i a_i b_i \ell^i(w_j) =: h^i(w_j),$$

where $w_j := 1 - w$ so that $w_j \in (\tilde{w}_j, 1-\tilde{w}_i)$, and

$$\begin{aligned}\ell^i(w_j) := &\big(\gamma_i\tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\beta_i\big)\left(\frac{w_j}{1-w_j}\right)^{-\alpha_i}\left(\frac{\tilde{w}_i}{1-\tilde{w}_i}\right)^{a_i}\\ &- \big(\gamma_i\tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\alpha_i\big)\left(\frac{1-w_j}{w_j}\right)^{\beta_i}\left(\frac{1-\tilde{w}_i}{\tilde{w}_i}\right)^{-b_i}.\end{aligned}$$

As $\tilde{w}_i < \hat{w}_i$ implies $\gamma_i\tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\alpha_i < 0$, it follows that $\ell^i(w_j) > 0$.

Next, Lemma A.11 (ii) and Lemma A.12 (ii) yield

$$(\tilde{w}_i, \tilde{w}_j) \in \big\{\big(w_i, f^{i,-1}(w_i)\big) : w_i \in (0, \hat{w}_i)\big\}.$$

It follows that $f^{i,-1}(\tilde{w}_i) = \tilde{w}_j$, and Lemma A.12 (iii) yields that

$$F^i(\tilde{w}_i, w_j) > 0 \qquad \text{for any } w_j \in (\tilde{w}_j, 1-\tilde{w}_i). \tag{C.21}$$

If $w_j \le \gamma_i$, then Lemma A.11 (i) and (C.21) imply that $h^i(w_j) < 0$. For $w_j > \gamma_i$, factoring out $(\frac{1-w_j}{w_j})^{1-\gamma_i}$ from h^i yields

$$\operatorname{sgn}\big(h^i(w_j)\big) = \operatorname{sgn}\big(\bar{h}^i(w_j)\big),$$

where

$$\begin{aligned}\bar{h}^i(w_j) := &\, a_i(w_j - b_i - \gamma_i)\big(\gamma_i\tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\alpha_i\big)\left(\frac{1-\tilde{w}_i}{\tilde{w}_i}\right)^{-b_i}\left(\frac{1-w_j}{w_j}\right)^{-b_i}\\ &+ b_i(a_i+\gamma_i-w_j)\big(\gamma_i\tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\beta_i\big)\left(\frac{\tilde{w}_i}{1-\tilde{w}_i}\right)^{a_i}\left(\frac{w_j}{1-w_j}\right)^{a_i}\\ &+ (a_i-b_i)(w_j-\gamma_i)\big(\tilde{w}_i(\alpha_i+\beta_i-1)-\alpha_i\big).\end{aligned}$$

It follows from $w_j > \gamma_i$ and Lemma A.11 (i) that

$$\begin{aligned} w_j - \gamma_i - b_i &> 0,\\ \gamma_i\tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\alpha_i &< 0,\\ a_i + \gamma_i - w_j > 1 - w_j &> 0.\end{aligned}$$

The inequalities $\frac{1-\tilde{w}_i}{w_j} > 1$ and $\frac{1-w_j}{\tilde{w}_i} > 1$ imply that

$$\begin{aligned}\bar{h}^i(w_j) &< a_i(w_j - b_i - \gamma_i)\big(\gamma_i \tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\alpha_i\big)\\ &\quad + b_i(a_i+\gamma_i-w_j)\big(\gamma_i\tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\beta_i\big)\\ &\quad + (a_i-b_i)(w_j-\gamma_i)\big(\tilde{w}_i(\alpha_i+\beta_i-1)-\alpha_i\beta_i\big)\\ &= a_ib_i(1-\tilde{w}_i-w_j)(\beta_i-\alpha_i) < 0.\end{aligned}$$

Therefore $(1-w)\varphi^i_w(w) - \gamma_i\varphi^i(w) < 0$ for $w \in (\tilde{w}_i, 1-\tilde{w}_j)$, and we get (B.13). □

C.9 Proof of Theorem 3.10

The proof of the following auxiliary statement is similar to (but shorter than) that of Lemma C.1 (iii). We skip its proof.

Lemma C.4 *For any $i \neq j \in \{a,b\}$ and any $m_i \in (0,1)$, let $\varphi^i \in C^1((0,1))$ be an $\mathbb{R}_{++}$-valued function satisfying* (B.12) *and* (B.13). *Then for any $\alpha \geq 0$ and $w \in (0,1)$, the function*

$$\hat{h}^i(\alpha, w) := e^{-\alpha\gamma_i}\varphi^i\big(1-e^{-\alpha}(1-w)\big)$$

satisfies

$$\hat{h}^i(\alpha, w) - \varphi^i(1-w) \leq 0,$$

where equality holds if and only if one of the following two conditions holds:

$$\begin{aligned}&\alpha = 0 \quad \textit{and} \quad w \in (m_i, 1),\\ &\alpha \leq \ln\frac{1-w}{1-m_i} \quad \textit{and} \quad w \in (0, m_i].\end{aligned}$$

Proof of Theorem 3.10 A direct calculation reveals that $\hat{\varphi}^i$ satisfies

$$\mathcal{L}^i\hat{\varphi}^i(w) - \lambda^\kappa\hat{\varphi}^i(w) + U^i(w) = 0, \qquad w \in (\hat{w}_i, 1), \tag{C.22}$$

$$(1-w)\hat{\varphi}^i_w(w) - \gamma_i\hat{\varphi}^i(w) = 0, \qquad w \in (0, \hat{w}_i), \tag{C.23}$$

and

$$\hat{\varphi}^i(\hat{w}_i-) = \hat{\varphi}^i(\hat{w}_i+), \tag{C.24}$$

$$\hat{\varphi}^i_w(\hat{w}_i-) = \hat{\varphi}^i_w(\hat{w}_i+), \tag{C.25}$$

$$\hat{\varphi}^i_{ww}(\hat{w}_i-) = \hat{\varphi}^i_{ww}(\hat{w}_i+). \tag{C.26}$$

As $\hat{\varphi}^i \in C^2((0, \hat{w}_i))$ and $\hat{\varphi}^i \in C^2((\hat{w}_i, 1))$, (C.24)–(C.26) imply the twice-differentiability across (0, 1), i.e., $\hat{\varphi}^i \in C^2((0, 1))$. Moreover, Lemma A.11 (i) implies that $s_0^i > 0$ and $s_1^i > 0$. Hence by Lemma C.3 (ii),

$$\mathcal{L}^i \hat{\varphi}^i(w) - \lambda^\kappa \hat{\varphi}^i(w) + U^i(w) < 0, \qquad w \in (0, \hat{w}_i). \tag{C.27}$$

Next, we prove that

$$(1-w)\hat{\varphi}^i_w(w) - \gamma_i \hat{\varphi}^i(w) < 0, \qquad w \in (\hat{w}_i, 1). \tag{C.28}$$

To this end, note that for any $w \in (\hat{w}_i, 1)$,

$$\operatorname{sgn}\big((1-w)\hat{\varphi}^i_w(w) - \gamma_i \hat{\varphi}^i(w)\big) = \operatorname{sgn}\big(\ell^i(w)\big),$$

where

$$\ell^i(w) = \frac{1-\gamma_i - w}{(1-\gamma_i)q_i} - s_1^i \left(\frac{1-w}{w}\right)^{a_i} (w - \alpha_i).$$

Also,

$$\lim_{w \downarrow \hat{w}_i} \ell^i(w) = \lim_{w \downarrow \hat{w}_i} \ell^i_w(w) = 0, \qquad \lim_{w \uparrow 1} \ell^i(w) = -\frac{\gamma_i}{(1-\gamma_i)q_i} < 0$$

and

$$\operatorname{sgn}\big(\ell^i_{ww}(w)\big) = \operatorname{sgn}\big((\gamma - \alpha_i)w + \alpha_i(1+a_i)\big).$$

As $w \in (\hat{w}_i, 1)$, it follows that

$$(\gamma - \alpha_i)w + \alpha_i(1+a_i) \in \big((1-\alpha_i)\alpha_i, (1-\alpha_i)(\gamma_i + \alpha_i)\big).$$

We now distinguish two cases. If $\gamma_i + \alpha_i \leq 0$, then $\ell^i_{ww}(w) < 0$ and hence

$$\ell^i_w(w) < \lim_{w \downarrow \hat{w}_i} \ell^i_w(w) = 0.$$

Therefore, an ODE comparison argument yields that $\ell^i < 0$ on $(\hat{w}_i, 1)$. If $\gamma_i + \alpha_i > 0$, then $\ell^i_{ww}(w) \leq 0$ on $(\hat{w}_i, \frac{-\alpha_i(1+a_i)}{\gamma_i - \alpha_i}]$ and it follows again by an ODE comparison argument that $\ell^i(w) < 0$ for any $w \in (\hat{w}_i, \frac{-\alpha_i(1+a_i)}{\gamma_i - \alpha_i}]$. As ℓ^i is strictly convex on the interval $(\frac{-\alpha_i(1+a_i)}{\gamma_i - \alpha_i}, 1)$ and below 0 at its boundaries, i.e., $\ell^i(\frac{-\alpha_i(1+a_i)}{\gamma_i - \alpha_i}) < 0$ and $\lim_{w \uparrow 1} \ell^i(w) < 0$, it follows that $\ell^i < 0$ on $(\frac{-\alpha_i(1+a_i)}{\gamma_i - \alpha_i}, 1)$.

In summary, this proves (C.28). We now apply Itô's formula to $e^{-\lambda^\kappa t - A^i_t}\hat{\varphi}^i(Y^x_t)$ and omit the dependence of Y^x and W^{i,w_i} on $(A^i, 0)$ for the sake of brevity. This yields, upon taking expectations, that

$$
\begin{aligned}
\lambda^{-1}\hat{\phi}^i(x) &= \lambda^{-1}\mathbb{E}\big[e^{-\lambda^\kappa t - A^i_t}\hat{\phi}^i(Y^x_t)\big] \\
&\quad - \mathbb{E}\bigg[\int_0^t e^{-\lambda^\kappa s - A^i_s}(Y^{x,S}_s)^{1-\gamma_i}\big(\mathcal{L}^i\hat{\varphi}^i(W^{i,w_i}_s) - \lambda^\kappa\hat{\varphi}^i(W^{i,w_i}_s)\big)ds\bigg] \\
&\quad - \mathbb{E}\bigg[a\int_0^t e^{-\lambda^\kappa s - A^i_s}(Y^{x,S}_s)^{1-\gamma_i} \\
&\qquad\qquad \times \big((1 - W^{i,w_i}_s)\hat{\varphi}^i_w(W^{i,w_i}_s) - \gamma_i\hat{\varphi}^i(W^{i,w_i}_s)\big)dA^{i,c}_s\bigg] \\
&\quad - \mathbb{E}\bigg[\sum_{0\le s\le t} e^{-\lambda^\kappa s - A^i_{s-}}(Y^{x,S}_{s-})^{1-\gamma_i} \\
&\qquad\qquad \times \big(e^{-\gamma_i\Delta A^i_s}\hat{\varphi}^i(W^{i,w_i}_s) - \hat{\varphi}^i(W^{i,w_i}_{s-})\big)\bigg]. \qquad (C.29)
\end{aligned}
$$

By Lemma C.4, as well as (C.23) and (C.28), the last expectation in (C.29) is non-negative. Using (C.22), (C.23) and (C.27), (C.28), similar arguments as in the proof of Theorem C.2 yield

$$\hat{\phi}^i(x) \ge \sup_{A^i\in\mathcal{A}} J^i(x; A^i, 0) \qquad \text{for } x \in \mathbb{R}^2_{++}. \qquad (C.30)$$

Finally, using the properties of $\Psi^{i,\hat{w}_i}$ in Proposition A.10, the equality in (C.30) follows. □

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Declarations

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