A Brezis-Oswald approach for mixed local and nonlocal operators

Published Version:
A Brezis-Oswald approach for mixed local and nonlocal operators / Biagi S.; Mugnai D.; Vecchi E.. - In: COMMUNICATIONS IN CONTEMPORARY MATHEMATICS. - ISSN 0219-1997. - STAMPA. - 26:2(2024), pp. 2250057.1-2250057.28. [10.1142/S0219199722500572]

Availability:
This version is available at: https://hdl.handle.net/11585/898551 since: 2022-11-01

Published:
DOI: http://doi.org/10.1142/S0219199722500572

A BREZIS-OSWALD APPROACH FOR MIXED LOCAL AND NONLOCAL OPERATORS

STEFANO BIAGI, DIMITRI MUGNAI, AND EUGENIO VECCHI

Abstract. In this paper we provide necessary and sufficient conditions for the existence of a unique positive weak solution for some sublinear Dirichlet problems driven by the sum of a quasilinear local and a nonlocal operator, i.e.,

$$\mathcal{L}_{p,s} = -\Delta_p + (-\Delta)^s_p.$$

Our main result is resemblant to the celebrated work by Brezis–Oswald [11]. In addition, we prove a regularity result of independent interest.

1. Introduction

In this paper we are concerned with quasilinear problems driven by the sum of a local and a nonlocal operator. More precisely, the leading operator is

$$\mathcal{L}_{p,s}u := -\Delta_p u + (-\Delta)^s_p u.$$

Here, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\,\nabla u)$ is the classical $p-$Laplacian operator and, for fixed $s \in (0,1)$ and up to a multiplicative positive constant, the fractional $p-$Laplacian is defined as

$$(-\Delta)^s_p u(x) := 2\,\text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}\,dy,$$

where P.V. denotes the Cauchy principal vale, namely

$$\begin{aligned} &\text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}\,dy \\ &= \lim_{\epsilon\to 0} \int_{\{y\in\mathbb{R}^n\,:\,|y-x|\geq\epsilon\}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}\,dy. \end{aligned}$$

Problems driven by operators like $\mathcal{L}_{p,s}$ have raised a certain interest in the last few years, both for the mathematical complications that the combination of two so different operators imply and for the wide range of applications, see for instance [5, 4, 6, 12, 13, 15, 16] and the references therein. A common feature of the aforementioned papers is to deal with *weak solutions*, in contrast with other results

2010 *Mathematics Subject Classification.* 35A01, 35R11.

Key words and phrases. Operators of mixed order, $p-$sublinear Dirichlet problems, existence and uniqueness of solutions, strong maximum principle, $L^\infty-$estimate.

The authors are members of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM). S. Biagi has been partially supported by the INdAM-GNAMPA project *Metodi topologici per problemi al contorno associati a certe classi di equazioni alle derivate parziali.* D. Mugnai has been supported by the FFABR "Fondo per il finanziamento delle attività base di ricerca" 2017 and by the INdAM-GNAMPA Project *Equazioni alle derivate parziali: problemi e modelli.* E. Vecchi has been partially supported by the INdAM-GNAMPA project *Convergenze variazionali per funzionali e operatori dipendenti da campi vettoriali.*

existing in the literature where viscosity solutions have been considered, see e.g. [2, 3].

The purpose of this paper is to prove an existence and uniqueness result in the spirit of the celebrated paper by Brezis-Oswald for the Laplacian, see [11]. So, let us consider the Dirichlet problem

$$\begin{cases} -\Delta_p u + (-\Delta)_p^s u = f(x,u) & \text{in } \Omega, \\ u \gneqq 0 & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded open set with C^1-smooth boundary. Under standard assumptions on f, we show that if u solves (1.1) (in some sense to be made precise later on), then $u > 0$ in Ω, and we give precise conditions under which such a solution exists and is unique. For this, as in [11] for the local case with $p = 2$, a crucial role is played by the monotonicity of the map

$$t \mapsto \frac{f(x,t)}{t^{p-1}}.$$

Indeed, in [11] the authors considered the problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u \gneqq 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

where $f : \Omega \times [0, \infty) \to \mathbb{R}$ satisfies suitable growth assumptions, and the map

$$t \mapsto \frac{f(x,t)}{t}$$

is decreasing in $(0, \infty)$. Under these conditions, the authors showed that (1.2) has at most one solution and that such a solution exists if and only if

$$\lambda_1(-\Delta - \tilde{a}_0(x)) < 0 \tag{1.3}$$

and

$$\lambda_1(-\Delta - \tilde{a}_\infty(x)) > 0, \tag{1.4}$$

where $\lambda_1(-\Delta - a(x))$ denotes the first eigenvalue of $-\Delta - a(x)$ with zero Dirichlet condition and

$$\tilde{a}_0(x) := \lim_{u \downarrow 0} \frac{f(x,u)}{u} \quad \text{and} \quad \tilde{a}_\infty(x) := \lim_{u \uparrow \infty} \frac{f(x,u)}{u}. \tag{1.5}$$

As already mentioned, in this paper we want to prove an analogous result in the quasilinear case given by problem (1.1), where f satisfies the following conditions:

- (f1) $f : \Omega \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function.
- (f2) $f(\cdot, t) \in L^\infty(\Omega)$ for every $t \geq 0$.
- (f3) There exists a constant $c_p > 0$ such that
$$|f(x,t)| \leq c_p(1 + t^{p-1}) \qquad \text{for a.e. } x \in \Omega \text{ and every } t \geq 0.$$
- (f4) For a.e. $x \in \Omega$, the function $t \mapsto \dfrac{f(x,t)}{t^{p-1}}$ is strictly decreasing in $(0, \infty)$.

We can then consider functions a_0 and a_∞ akin to those in (1.5), see (2.5) for the precise definition. Moreover, we denote respectively by

$$\lambda_1(\mathcal{L}_{p,s} - a_0) \qquad \text{and} \qquad \lambda_1(\mathcal{L}_{p,s} - a_\infty), \tag{1.6}$$

the smallest eigenvalues of $\mathcal{L}_{p,s} - a_0$ and $\mathcal{L}_{p,s} - a_\infty$, both in presence of nonlocal Dirichlet boundary condition (i.e. $u = 0$ in $\mathbb{R}^n \setminus \Omega$). Since the function a_0 can be *unbounded*, (notice that, on the other hand, a_∞ is bounded by (f3)), similarly to Brezis-Oswald [11], the precise definition of (1.6) is the following:

$$\begin{aligned} &\lambda_1(\mathcal{L}_{p,s} - a_0) := \inf_{\substack{u \in \mathbb{X}_p(\Omega) \\ \|u\|_{L^p(\Omega)}=1}} \Big\{ \mathcal{Q}_{p,s}(u) - \int_{\{u \neq 0\}} a_0 \, |u|^p \, dx \Big\}; \\ &\lambda_1(\mathcal{L}_{p,s} - a_\infty) := \inf_{\substack{u \in \mathbb{X}_p(\Omega) \\ \|u\|_{L^p(\Omega)}=1}} \Big\{ \mathcal{Q}_{p,s}(u) - \int_{\Omega} a_\infty \, |u|^p \, dx \Big\}, \end{aligned} \tag{1.7}$$

where $\mathbb{X}_p(\Omega)$ is defined in (2.1), and we have introduced the simplified notation

$$\mathcal{Q}_{p,s}(u) := \int_\Omega |\nabla u|^p \, dx + \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^p}{|x - y|^{n+ps}} \, dx \, dy. \tag{1.8}$$

In order to prove *uniqueness*, we shall add the following additional hypothesis:

(f5) there exists $\rho_f > 0$ such that

$$f(x,t) > 0 \quad \text{for a.e. } x \in \Omega \text{ and every } 0 < t < \rho_f. \tag{1.9}$$

We observe that, in the particular case of power-type nonlinearities $f(x,u) = u^\theta$ (with $0 \leq \theta \leq p-1$), assumption (f5) is trivially satisfied.

Remark 1.1. As a matter of fact, assumption (f5) is just a technical one (far from being optimal) which permits to overcome the lack of *boundary regularity* for $\mathcal{L}_{p,s}$. In fact, since we do not know the $C^{1,\alpha}$-regularity up to the boundary of weak solutions of (1.1), we do not have at our disposal a Hopf-type lemma for $\mathcal{L}_{p,s}$ and we cannot follow directly the approach by Brezis-Oswald to get the uniqueness of solutions for (1.1). We then need to exploit a suitable approximation argument (see Theorem 4.3 below), and for this approach assumption (f5) seems to be essential.

We are now ready to state our main result:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1-smooth boundary $\partial\Omega$. Assume that f satisfies* (f1)–(f5). *Then, the following assertions hold.*

(1) *There exists a unique positive solution to* (1.1) *if*

$$\lambda_1(\mathcal{L}_{p,s} - a_0) < 0 < \lambda_1(\mathcal{L}_{p,s} - a_\infty).$$

Moreover, if a solution to (1.1) *exists, then it is unique and*

$$\lambda_1(\mathcal{L}_{p,s} - a_0) < 0.$$

(2) *In the linear case $p = 2$, there exists a unique positive solution to* (1.1) if and only if

$$\lambda_1(\mathcal{L}_{2,s} - a_0) < 0 < \lambda_1(\mathcal{L}_{2,s} - a_\infty).$$

Let us now spend a few comments on the proof of Theorem 1.2. Despite the apparent simplicity in working with operators like $\mathcal{L}_{p,s}$, we have to face some difficulties related to the scarce literature available for such operators. First of all, we need to prove the validity of the *strong maximum principle* as stated in [29], namely: if u is a *nonnegative* weak solution of $\mathcal{L}_{p,s}u = f(x,u)$ (with zero-boundary conditions), then

$$\text{either } u \equiv 0 \text{ in } \Omega \quad \text{ or } \quad u > 0 \text{ a.e. in } \Omega,$$

see Theorem 3.1 for the precise statement. We believe that this preliminary result is of independent interest, and we stress that Theorem 3.1 cannot be deduced as a corollary of the maximum principles proved in [5] nor in [12].

A second delicate point concerns the uniqueness of the solution. Indeed, the lack of even a basic boundary regularity for $\mathcal{L}_{p,s}$ prevents from applying the original argument in [11]. For this reason, we have to exploit an approximation argument inspired by the one in [9], with the additional aid of a further assumption on f (see (1.9)).

Finally, we emphasize that in order to get a complete characterization of the existence and uniqueness of a positive weak solution, we must restrict ourselves to the linear case $p = 2$, see Proposition 6.4, since we cannot prove the inequality $\lambda_1(\mathcal{L}_{p,s} - a_\infty) > 0$ in the general case. Indeed, two fundamental tools would be needed to prove this fact: first, an L^∞ bound on solutions, and this is the content of Theorem 4.1; second, some nonlinear Green identities, used in [17] and in [19] for the local case. To the best of our knowledge, the nonlocal counterparts of such identities are still missing in the literature. Nevertheless, we think that the global boundedness result in Theorem 4.1, as well as the very recent results in [21], can be a useful tool for further investigations in the general case $p \neq 2$.

We conclude by noticing that, in the purely nonlocal case, the complete characterization is possible for any p since the precise behaviour of the solutions at the boundary is known, see [23]. Actually, even if an analogous result is not known in our mixed context, after the submission of this paper we found a way to bypass both the absence of appropriate nonlinear Green identities and the lack of boundary regularity for $\mathcal{L}_{p,s}$; thus, we can go full circle and obtain a complete characterization of the (unique) solvability of (1.1) for $p \neq 2$ as well, see [7].

We close this introduction with a plan of the paper: in Section 2 we introduce the relevant notation and we list the standing assumptions needed in the rest of the paper. Then, in Section 3 we prove the strong maximum principle for weak solutions of problem (1.1). Uniqueness and boundedness of positive solutions is proved in Section 4. In order to prove conditions analogous to those established in (1.3) and (1.4), in Section 5 we shall study the eigenvalue problem associated to $\mathcal{L}_{p,s}$ in presence of a bounded and indefinite weight. In fact, although the analogue of the functions defined in (1.5) could be unbounded, for the existence-uniqueness result we will reduce to study an eigenvalue problem in presence of a bounded weight, see Proposition 6.4. Finally, existence is proved in Section 6.

Acknowledgements We thank the anonymous referee for his/her careful reading of the manuscript.

2. Notation and preliminary results

In this first section, we introduce the main assumptions and notation which shall be used throughout the rest of the paper. Moreover, we state and prove some auxiliary results which shall be exploited in the next sections.

To begin with, we fix $p \in (1, +\infty)$ and we let $\Omega \subseteq \mathbb{R}^n$ be a connected and bounded open set with C^1-smooth boundary $\partial\Omega$. Accordingly, we define

$$\mathbb{X}_p(\Omega) := \big\{u \in W^{1,p}(\mathbb{R}^n) : \, u \equiv 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega\big\}. \tag{2.1}$$

In view of the regularity assumption on $\partial\Omega$, it is well-known that (see e.g. [10, Proposition 9.18]) $\mathbb{X}_p(\Omega)$ can be identified with the space $W_0^{1,p}(\Omega)$: more precisely, we have

$$u \in W_0^{1,p}(\Omega) \iff u \cdot \mathbf{1}_\Omega \in \mathbb{X}_p(\Omega), \tag{2.2}$$

where $\mathbf{1}_\Omega$ is the indicator function of Ω. From now on, we shall *tacitly identify* a function $u \in W_0^{1,p}(\Omega)$ with its 'zero-extension' $\hat{u} := u \cdot \mathbf{1}_\Omega \in \mathbb{X}_p(\Omega)$.

By the Poincaré inequality and (2.2), we get that the quantity

$$\|u\|_{\mathbb{X}_p} := \left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p}, \qquad u \in \mathbb{X}_p(\Omega),$$

endows $\mathbb{X}_p(\Omega)$ with a structure of (real) Banach space, which is actually isometric to $W_0^{1,p}(\Omega)$. In particular, the following properties hold true:

(1) $\mathbb{X}_p(\Omega)$ is separable and reflexive (since $p > 1$);
(2) $C_0^\infty(\Omega)$ is dense in $\mathbb{X}_p(\Omega)$.

Due to its relevance in the sequel, we also introduce an *ad-hoc* notation for the (convex) cone of the nonnegative functions in $\mathbb{X}_p(\Omega)$:

$$\mathbb{X}_p^+(\Omega) := \big\{u \in \mathbb{X}_p(\Omega) : \, u \geq 0 \text{ a.e. in } \Omega\big\}.$$

As anticipated in the Introduction, the aim of this paper is to provide necessary and sufficient conditions for solving the Dirichlet problem (1.1).

First of all, we give the definition of "solution" for (1.1).

Definition 2.1. Let the above assumptions and notation be in force. We say that a function $u \in \mathbb{X}_p(\Omega)$ is a *weak solution* of (1.1) if

(1) for every function $\varphi \in \mathbb{X}_p(\Omega)$ one has

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \varphi\rangle \, dx \\ &\qquad + \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+ps}} \, dx \, dy \\ &= \int_\Omega f(x,u)\varphi \, dx; \end{aligned} \tag{2.3}$$

(2) $u \geq 0$ a.e. in Ω and $|\{x \in \Omega : \, u(x) > 0\}| > 0$, $|A|$ denoting the Lebesgue measure of the set A.

Remark 2.2. A couple of remarks on Definition 2.1 are in order.

(1) We explicitly notice that the definition above is well-posed. Indeed, we know from [18, Proposition 2.2] that there exists $c_{n,s,p} > 0$ such that

$$\left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy\right)^{1/p} \leq c_{n,s,p}\|f\|_{W^{1,p}(\mathbb{R}^n)} \quad \forall\, f \in W^{1,p}(\mathbb{R}^n).$$

Thus, by using Hölder's inequality, we find that if $u, \varphi \in \mathbb{X}_p(\Omega)$, then

$$\begin{aligned}
&\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{p-1}|\varphi(x)-\varphi(y)|}{|x-y|^{n+ps}}\,dx\,dy \\
&\leq \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy\right)^{1-1/p}\left(\iint_{\mathbb{R}^{2n}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{n+ps}}\,dx\,dy\right)^{1/p} \\
&\leq c_{n,s,p}^2\|u\|_{W^{1,p}(\mathbb{R}^n)}^{p-1}\cdot\|\varphi\|_{W^{1,p}(\mathbb{R}^n)} < +\infty.
\end{aligned}$$

On the other hand, since $u \in \mathbb{X}_p(\Omega) \subseteq L^p(\Omega)$, by exploiting (f3), we have

$$\int_\Omega |f(x,u)\,\varphi|\,dx \leq c_p\left(\int_\Omega |\varphi|dx + \int_\Omega |u|^{p-1}|\varphi|\,dx\right) < +\infty.$$

(2) It is clear from (1) that the notion of solution similarly holds if we replace the growth assumption in (f3) with the more general one

$$|f(x,t)| \leq c_p(1+t^{q-1}) \qquad \text{for a.e. } x \in \Omega \text{ and every } t \geq 0, \tag{2.4}$$

where $q \in \left[p, \frac{pn}{n-p}\right]$ if $p < n$, or $q \in [p, +\infty)$ if $p \geq n$.

We conclude this section with some consequences of assumptions (f1)–(f4) which shall be useful in the sequel (see, e.g., [19] for related remarks). First, taking into account assumption (f4), we introduce the functions

$$a_0(x) := \lim_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} \qquad \text{and} \qquad a_\infty(x) := \lim_{t\to+\infty} \frac{f(x,t)}{t^{p-1}}, \tag{2.5}$$

noticing that the first one is allowed to be identically equal to $+\infty$.

Then, we notice that:

(1) by combining (f2) and (f4), we get that

$$\frac{f(x,t)}{t^{p-1}} \geq f(x,1) \geq -\|f(\cdot,1)\|_{L^\infty(\Omega)} =: -c_f > -\infty, \tag{2.6}$$

for a.e. $x \in \Omega$ and every $t \in (0,1]$. In particular, from (f1) and (2.6) we get

$$f(x,0) \geq 0 \text{ for a.e. } x \in \Omega. \tag{2.7}$$

(2) Using again assumption (f4), we have

$$a_0(x) \geq \frac{f(x,t)}{t^{p-1}} \geq a_\infty(x)$$

for a.e. $x \in \Omega$ and every $t > 0$. In particular, by (2.6) we get

$$a_0(x) \geq -c_f \geq a_\infty(x) \quad \text{for a.e. } x \in \Omega. \tag{2.8}$$

3. Strong maximum principle

While dealing with *nonnegative* weak solutions u of $\mathcal{L}_{p,s}u = f(x,u)$ (with zero-boundary conditions), it should be desirable to know that either

$$u \equiv 0 \text{ in } \Omega \qquad \text{or} \qquad u > 0 \text{ a.e. in } \Omega,$$

namely, that a *strong maximum principle* holds.

The next theorem shows that this is indeed true in our context.

Theorem 3.1. *Let f satisfy* (f1)–(f3) *and let $u \in \mathbb{X}_p^+(\Omega)$ satisfy identity* (2.3) *for every function $\varphi \in \mathbb{X}_p(\Omega)$. Then, either $u \equiv 0$ or $u > 0$ almost everywhere in Ω.*

Remark 3.2. Actually, as it will be clear from the proof, we prove a logarithmic inequality - inequality (3.8) - which implies, when u is not the trivial function, that the set $\{x \in \Omega : u(x) = 0\}$ has zero $W^{1,p}-$capacity, as in [25, Theorem 2.4].

Proof of Theorem 3.1. We suppose that there exists a set $\mathcal{Z} \subseteq \Omega$, with positive Lebesgue measure, such that $u \equiv 0$ a.e. on $\mathcal{Z}$. Then, we claim that

$$\exists\, x_0 \in \Omega,\ R > 0 \text{ such that } u \equiv 0 \text{ a.e. on } B(x_0,R) \Subset \Omega. \tag{3.1}$$

Taking this claim for granted for a moment, we now choose a nonnegative function $\varphi \in C_0^\infty(\Omega)$ satisfying the properties

$$\operatorname{supp}(\varphi) \subseteq B(x_0,R) \qquad \text{and} \qquad \int_{B(x_0,R)} \varphi\, dx = 1.$$

Using φ as a test function in (2.3), from (3.1) we get

$$\begin{aligned} \int_{B(x_0,R)} f(x,0)\varphi\, dx &= \int_\Omega f(x,u)\varphi\, dx \\ &= \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+ps}}\, dx\, dy \\ &= -2\int_{\Omega\setminus B(x_0,R)} \int_{B(x_0,R)} \frac{u(x)^{p-1}\varphi(y)}{|x-y|^{n+ps}}\, dx\, dy \\ &\le -\frac{2}{\operatorname{diam}(\Omega)^{n+ps}} \int_{\Omega\setminus B(x_0,R)} u(x)^{p-1}\, dx. \end{aligned} \tag{3.2}$$

On the other hand, since φ is nonnegative, by (2.7) we have

$$\int_{B(x_0,R)} f(x,0)\varphi\, dx \ge 0. \tag{3.3}$$

Gathering (3.2) and (3.3), we obtain

$$\int_{\Omega\setminus B(x_0,R)} u(x)^{p-1}\, dx = 0$$

and thus $u \equiv 0$ a.e. on $\Omega \setminus B(x_0,R)$ (remind that, by assumption, $u \in \mathbb{X}_p^+(\Omega)$). Owing to (3.1), we then conclude that $u \equiv 0$ a.e. in Ω, as desired.

To complete the proof, we are left to show the claim (3.1). First of all, since we are assuming that $\mathcal{Z} \subseteq \Omega$ has positive Lebesgue measure, it is possible to find a point $x_0 \in \Omega$ and some $R > 0$ such that

$$B(x_0,2R) \Subset \Omega \qquad \text{and} \qquad |\mathcal{Z} \cap B(x_0,R)| > 0.$$

Moreover, we choose a nonnegative function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ a.e. in $B(x_0, R)$ and $\mathrm{supp}(\varphi) \subseteq B(x_0, 2R)$. For every fixed $\varepsilon > 0$, we then set

$$\varphi_\varepsilon := \frac{\varphi^p}{(u+\varepsilon)^{p-1}}.$$

Since $\varphi \in C_0^\infty(\Omega)$ and $u \in \mathbb{X}_p^+(\Omega)$, it is easy to recognize that $\varphi_\varepsilon \in \mathbb{X}_p(\Omega)$ (see, e.g., [27, Lem. 2.3]); we can then use φ_ε as a test function in (2.3), obtaining

$$\begin{aligned} &(p-1)\int_\Omega \frac{|\nabla u|^p}{(u+\varepsilon)^p}\varphi^p\,dx \\ &\quad\leq \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi_\varepsilon(x)-\varphi_\varepsilon(y))}{|x-y|^{n+ps}}\,dx\,dy \\ &\qquad + p\int_\Omega \frac{|\nabla u|^{p-1}|\nabla\varphi|}{(u+\varepsilon)^{p-1}}\varphi^{p-1}\,dx - \int_\Omega f(x,u)\,\frac{\varphi^p}{(u+\varepsilon)^{p-1}}\,dx. \end{aligned} \tag{3.4}$$

We now turn to provide *ad-hoc* estimates for the three integrals in the right-hand side of (3.4). First of all, in the proof of [14, Lem. 1.3] it is showed that

$$\begin{aligned} &\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi_\varepsilon(x)-\varphi_\varepsilon(y))}{|x-y|^{n+ps}} \\ &\quad\leq -K\frac{1}{|x-y|^{n+ps}}\left|\log\left(\frac{u(x)+\varepsilon}{u(y)+\varepsilon}\right)\right|^p\varphi^p(y) + K\frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{n+ps}} \end{aligned}$$

for some positive constant $K = K_p > 0$. Hence, by integrating we find

$$\begin{aligned} &\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi_\varepsilon(x)-\varphi_\varepsilon(y))}{|x-y|^{n+ps}}\,dx\,dy \\ &\quad\leq K\iint_{\mathbb{R}^{2n}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{n+ps}}dx\,dy. \end{aligned} \tag{3.5}$$

Moreover, using the weighted Young inequality, for every $\varepsilon > 0$ one has

$$\begin{aligned} &p\int_\Omega \frac{|\nabla u|^{p-1}|\nabla\varphi|}{(u+\varepsilon)^{p-1}}\varphi^{p-1}\,dx \\ &\quad\leq \frac{p-1}{2}\int_\Omega \frac{|\nabla u|^p}{(u+\varepsilon)^p}\varphi^p\,dx + 2^{p-1}\int_\Omega |\nabla\varphi|^p\,dx. \end{aligned} \tag{3.6}$$

As for the remaining integral, we proceed essentially as in [27, Lem. 2.4]: by exploiting (2.6) and (2.7), we have the following chain of inequalities:

$$\begin{aligned} -\int_\Omega f(x,u)\,\frac{\varphi^p}{(u+\varepsilon)^{p-1}}\,dx &= -\int_{\Omega\cap\{u=0\}} f(x,0)\,\frac{\varphi^p}{\varepsilon^{p-1}}\,dx \\ &\quad - \int_{\Omega\cap\{0<u<1\}} \frac{f(x,u)\,\varphi^p}{(u+\varepsilon)^{p-1}}\,dx - \int_{\Omega\cap\{u\geq 1\}} \frac{f(x,u)\,\varphi^p}{(u+\varepsilon)^{p-1}}\,dx \\ &\leq c_f\int_{\Omega\cap\{0<u<1\}} \frac{u^{p-1}}{(u+\varepsilon)^{p-1}}\,\varphi^p\,dx + c_p\int_{\Omega\cap\{u\geq 1\}} \frac{1+u^{p-1}}{(u+\varepsilon)^{p-1}}\,\varphi^p\,dx \\ &\leq (c_f + 2c_p)\|\varphi\|^p_{L^p(\Omega)}. \end{aligned} \tag{3.7}$$

Gathering (3.4)–(3.7), we then obtain

$$\begin{aligned}\int_{B(x_0,R)} \Big|\nabla \log\Big(1+\frac{u}{\varepsilon}\Big)\Big|^p\,dx &= \int_{B(x_0,R)} \frac{|\nabla u|^p}{(u+\varepsilon)^p}\,dx \\ &\leq \int_{B(x_0,R)} \frac{|\nabla u|^p}{(u+\varepsilon)^p}\,\varphi^p\,dx \leq \kappa,\end{aligned} \tag{3.8}$$

where $\kappa = \kappa_\varphi > 0$ is a suitable constant *independent of* ε.

With (3.8) at hand, we are finally ready to prove (3.1). In fact, recalling that

$$E := \mathcal{Z} \cap B(x_0,R)$$

has positive Lebesgue measure and $u \equiv 0$ a.e. in E, by Chebyshev's inequality and the Poincaré inequality in [24, Theorem 13.27] for every $t > 0$ we have

$$\begin{aligned}\Big|\log\Big(1+\frac{t}{\varepsilon}\Big)\Big|^p \cdot \big|\{u \geq t\} \cap B(x_0,R)\big| &\leq \int_{B(x_0,R)} \Big|\log\Big(1+\frac{u}{\varepsilon}\Big)\Big|^p\,dx \\ &= \int_{B(x_0,R)} \Big|\log\Big(1+\frac{u}{\varepsilon}\Big) - m_E\Big|^p\,dx \\ &\leq C_P \int_{B(x_0,R)} \Big|\nabla \log\Big(1+\frac{u}{\varepsilon}\Big)\Big|^p\,dx \leq \kappa'.\end{aligned} \tag{3.9}$$

where m_E is the mean of $v := \log(1+u/\varepsilon) \in W^{1,p}(\mathbb{R}^n)$ on the set E, that is,

$$m_E := \frac{1}{|E|}\int_E \log\Big(1+\frac{u}{\varepsilon}\Big)\,dx = 0.$$

As a consequence, since identity (3.9) holds for every $\varepsilon > 0$ and the constant κ' is *independent of* ε, we readily infer that

$$\big|\{u \geq t\} \cap B(x_0,R)\big| = 0 \qquad \text{for every } t > 0.$$

This obviously implies that $u \equiv 0$ a.e. in $B(x_0,R)$, and the proof is complete. ☐

From Theorem 3.1, we immediately deduce the following result.

Corollary 3.3. *Let f satisfy* (f1)–(f3) *and let $u \in \mathbb{X}_p(\Omega)$ be a weak solution of* (1.1)*. Then,*

$$u > 0 \text{ a.e. in } \Omega.$$

Proof. Since u is a weak solution of (1.1), it follows from Definition 2.1 that

(a) $u \in \mathbb{X}_p^+(\Omega)$ (i.e., $u \geq 0$ a.e. in Ω);
(b) $|\{x \in \Omega : u(x) > 0\}| > 0$.

In particular, from (b) we get that u is not identically vanishing (a.e.) in Ω, and the conclusion follows immediately from Theorem 3.1. ☐

Remark 3.4. By carefully scrutinizing the proof of Theorem 3.1, one can easily check that the properties of f which have *actually* played a role are:

(1) $f(x,0) \geq 0$ for a.e. $x \in \Omega$;
(2) $f(x,t) \geq -c_f t^{p-1}$ for a.e. $x \in \Omega$ and every $0 < t < 1$;
(3) $f(x,t) \geq c_p(1+t^{p-1})$ for a.e. $x \in \Omega$ and every $t \geq 1$.

As a consequence, the strong maximum principle in Theorem 3.1 holds for weak solutions of every boundary-value problem of the type

$$\begin{cases} \mathcal{L}_{p,s}u = g(x,u) & \text{in } \Omega, \\ u \gneqq 0 & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.10}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying (1)–(3) and the growth condition in (2.4).

A remarkable example of a map g satisfying conditions (1)–(3) above is

$$g_\lambda(x,t) := \big(-a(x) + \lambda\big)|t|^{p-2}t, \tag{3.11}$$

where $\lambda \in \mathbb{R}$ and $a \in L^\infty(\Omega)$. The boundary-value problem associated with this function g_λ is the (Dirichlet) $\mathcal{L}_{p,s}$-eigenvalue problem

$$\begin{cases} \mathcal{L}_{p,s}u + a(x)|u|^{p-2}u = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

which shall be extensively studied in Section 5.

4. Uniqueness and boundedness of weak solutions

The aim of this section is to establish *uniqueness and boundedness* of weak solutions to problem (1.1).

We start by proving that weak solutions are globally bounded. We stress this "regularity result" requires f to satisfy *only* assumptions (f1)– (f3).

Theorem 4.1. *Let $u_0 \in \mathbb{X}_p(\Omega)$ be a nonnegative weak solution of* (1.1) *with f satisfying* (f1)–(f3). *Then $u_0 \in L^\infty(\Omega)$.*

Proof. To begin with, we arbitrarily fix $\delta \in (0,1)$ and we set

$$\tilde{u}_0 := \delta^{1/(p-1)}u_0.$$

Then $\tilde{u}_0$ solves

$$\begin{cases} -\Delta_p\tilde{u}_0 + (-\Delta)_p^s\tilde{u}_0 = \delta f(x,u_0) & \text{in } \Omega, \\ \tilde{u}_0 \gneqq 0 & \text{in } \Omega, \\ \tilde{u}_0 \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{4.1}$$

Now, for every $k \geq 0$, we define $C_k := 1 - 2^{-k}$ and

$$v_k := \tilde{u}_0 - C_k, \quad w_k := (v_k)_+ := \max\{v_k, 0\}, \quad U_k := \|w_k\|^p_{L^p(\Omega)}.$$

We explicitly point out that, in view of these definitions, one has

(a) $\|\tilde{u}_0\|^p_{L^p(\Omega)} = \delta^{p'}\,\|u_0\|^p_{L^p(\Omega)}$ (with $1/p' = 1 - 1/p$);
(b) $w_0 = v_0 = \tilde{u}_0$ (since $C_0 = 0$);
(c) $v_{k+1} \leq v_k$ and $w_{k+1} \leq w_k$ (since $C_k < C_{k+1}$).

We now observe that, since $u_0 \in \mathbb{X}_p(\Omega) \subseteq W^{1,p}(\mathbb{R}^n)$, we have $v_k \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$; furthermore, since $\tilde{u}_0 = u_0 \equiv 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, one also has

$$v_k = \tilde{u}_0 - C_k = -C_k < 0 \quad \text{on } \mathbb{R}^n \setminus \Omega,$$

and thus $w_k = (v_k)_+ \in \mathbb{X}_p(\Omega)$ (remind that Ω is bounded). We are then entitled to use the function w_k as a *test function* in (4.1), obtaining

$$\int_\Omega |\nabla \tilde{u}_0|^{p-2}\langle \nabla \tilde{u}_0, \nabla w_k\rangle\,dx + \iint_{\mathbb{R}^{2n}} \frac{J_p(\tilde{u}_0(x)-\tilde{u}_0(y))(w_k(x)-w_k(y))}{|x-y|^{n+2s}}\,dx\,dy = \delta\int_\Omega f(x,u_0)w_k\,dx. \tag{4.2}$$

To proceed further, we notice that for any measurable function z and for (almost every) couple of points $x, y \in \mathbb{R}^n$, one has

$$|z_+(x)-z_+(y)|^p \le |z(x)-z(y)|^{p-2}(z(x)-z(y))(z_+(x)-z_+(y))$$

see [20, Equation (14)] or [28, Equation (16)], so that, by choosing $z = v_k$, since

$$v_k(x) - v_k(y) = \tilde{u}_0(x) - \tilde{u}_0(y),$$

we find

$$|w_k(x)-w_k(y)|^p \le |\tilde{u}_0(x)-\tilde{u}_0(y)|^{p-2}(\tilde{u}_0(x)-\tilde{u}_0(y))(w_k(x)-w_k(y)). \tag{4.3}$$

Moreover, taking into account the definition of w_k, we get

$$\begin{aligned}\int_\Omega |\nabla \tilde{u}_0|^{p-2}\langle \nabla \tilde{u}_0, \nabla w_k\rangle\,dx &= \int_{\Omega\cap\{\tilde{u}_0 > C_k\}} |\nabla \tilde{u}_0|^{p-2}\langle \nabla \tilde{u}_0, \nabla v_k\rangle\,dx \\ &= \int_\Omega |\nabla w_k(x)|^p\,dx.\end{aligned} \tag{4.4}$$

Gathering (4.2)-(4.4) and assumption (f3), we obtain

$$\begin{aligned}\int_\Omega |\nabla w_k|^p\,dx &\le \delta\int_\Omega |f(x,u_0)|\,w_k\,dx \\ &\le c\int_\Omega (\delta + \delta u_0^{p-1})\,w_k\,dx \le c\int_\Omega (1+\tilde{u}_0^{p-1})w_k\,dx,\end{aligned} \tag{4.5}$$

since $\delta < 1$. We then recall that, for every $k \ge 1$, one has

$$\tilde{u}_0(x) < (2^k - 1)w_{k-1}(x) \quad \text{for} \quad x \in \{w_k > 0\}, \tag{4.6}$$

and the inclusions

$$\{w_k > 0\} = \{\tilde{u}_0 > C_k\} \subseteq \{w_{k-1} > 2^{-k}\} \tag{4.7}$$

hold true for every $k \ge 1$, see [20] or [28]. By combining (4.6) and (4.7) with (4.5), and taking into account that $w_k \le w_{k-1}$ a.e. in $\mathbb{R}^n$, for every $k \ge 1$, we get

$$\begin{aligned}\int_\Omega |\nabla w_k|^p\,dx &\le c\int_{\{w_k>0\}} (1+\tilde{u}_0^{p-1})w_k\,dx \\ &\le c\int_{\{w_k>0\}} \Big[w_{k-1} + (2^k-1)^{p-1}w_{k-1}^p\Big]\,dx \\ &\le c\int_{\{w_{k-1}>2^{-k}\}} \Big[2^{k(p-1)}w_{k-1}^p + (2^k-1)^{p-1}w_{k-1}^p\Big]\,dx \\ &\le c\,2^{kp}\int_{\{w_{k-1}>2^{-k}\}} w_{k-1}^p dx \\ &\le c\,2^{kp}\int_\Omega w_{k-1}^p dx = c\,2^{kp}U_{k-1}.\end{aligned} \tag{4.8}$$

We now estimate from below the term U_{k-1} in the right-hand side of (4.8). To this end we first observe that, as a consequence of (4.7), we obtain

$$
\begin{aligned}
U_{k-1} &= \int_\Omega w_{k-1}^p\,dx \geq \int_{\{w_{k-1}>2^{-k}\}} w_{k-1}^p\,dx \\
&\geq 2^{-kp}\,\big|\{w_{k-1}>2^{-k}\}\big| \geq 2^{-kp}\big|\{w_k>0\}\big|.
\end{aligned} \tag{4.9}
$$

Using the Hölder inequality (with exponents p^*/p and n/p), jointly with the Sobolev inequality, from (4.8)-(4.9) we obtain the following estimate:

$$
\begin{aligned}
U_k = \|w_k\|_{L^p(\Omega)}^p &\leq \Big(\int_\Omega w_k^{p^*}\,dx\Big)^{p/p^*}\,\big|\{w_k>0\}\big|^{p/n} \\
&\leq \mathbf{c}_S \int_\Omega |\nabla w_k|^p\,dx \cdot \big|\{w_k>0\}\big|^{p/n} \\
&\leq \mathbf{c}_S\,\big(c\,2^{kp}\,U_{k-1}\big)\,\big(2^{kp}U_{k-1}\big)^{p/n} \\
&= \mathbf{c}'\,\big(2^{p+p^2/n}\big)^{k-1}\,U_{k-1}^{1+p/n} \qquad (\text{with } \mathbf{c}' := c\,2^{p+p^2/n}\,\mathbf{c}_S),
\end{aligned} \tag{4.10}
$$

for every $k\geq 1$, where $\mathbf{c}_S$ is given by the Sobolev inequality.

Estimate (4.10) can be re-written as

$$U_k \leq \mathbf{c}'\eta^{k-1}U_{k-1}^{1+p/n},$$

where

$$\eta := 2^{p+p^2/n} > 1.$$

Hence, from [22, Lem. 7.1] we get that $U_k\to 0$ as $k\to\infty$, provided that

$$U_0 = \|\tilde{u}_0\|_{L^p(\Omega)}^p = \delta^{p'}\|u_0\|_{L^p(\Omega)}^p < (\mathbf{c}')^{-n/p}\,\eta^{-n^2/p^2}.$$

As a consequence, if $\delta>0$ is small enough, we obtain

$$0 = \lim_{k\to\infty} U_k = \lim_{k\to\infty}\int_\Omega (\tilde{u}_0 - C_k)_+^2\,dx = \int_\Omega (\tilde{u}_0-1)_+^2\,dx.$$

Bearing in mind that $\tilde{u}_0 = \delta^{1/(p-1)}u_0$ (and $u_0\geq 0$), we then get

$$0\leq u_0 \leq \frac{1}{\delta^{1/(p-1)}} \qquad \text{a.e. in } \Omega,$$

from which we conclude that $u_0\in L^\infty(\Omega)$. □

Remark 4.2. We notice that an analogous result still holds true, with suitable adaptations in the powers of u_0 or w_k in the right-hand sides of the inequalities in the above proof, also when f satisfies (f1), (f2) and (2.4). However, in view of the $p-$linear growth in the Brezis-Oswald theorem, we preferred to maintain such a case for the presentation of the proof.

We are now ready to state and prove the main result of this section.

Theorem 4.3. *Let f satisfy* (f1)–(f5). *Then there exists* at most *one weak solution $u\in\mathbb{X}_p(\Omega)$ of problem* (1.1).

In order to prove Theorem 4.3, we need the following elementary lemma.

Lemma 4.4. *Let $v, w \in \mathbb{R}^n$ and set*

$$\mathcal{A}_p(v,w) := |v|^p + (p-1)|w|^p - p|w|^{p-2}\langle v, w\rangle.$$

Then, $\mathcal{A}_p(v,w) \geq 0$.

Proof. We first notice that, if $v = 0$ or $w = 0$, the conclusion of the lemma is trivial. We then assume that $v, w \neq 0$, and we let $t > 0$ be such that

$$|w| = t|v|. \tag{4.11}$$

Using Cauchy-Schwarz's inequality and (4.11), we have

$$\begin{aligned} \mathcal{A}_p(v,w) &\geq |v|^p + (p-1)|w|^p - p|w|^{p-1}|v| \\ &= |v|^p\big(1 + (p-1)t^p - p\,t^{p-1}\big) =: |v|^p \cdot \ell_p(t). \end{aligned}$$

From this, since an elementary computation shows that

$$\ell_p(s) \geq \ell_p(1) = 0 \qquad \text{for every } s \geq 0,$$

we readily conclude that $\mathcal{A}_p(v,w) \geq 0$, as desired. □

Thanks to Lemma 4.4, we can proceed with the proof of Theorem 4.3.

Proof of Theorem 4.3. Let $u_1, u_2 \in \mathbb{X}_p(\Omega)$ be two solutions of (1.1). In order to show that $u_1 = u_2$ a.e. in Ω, we arbitrarily fix $\varepsilon > 0$ and we define

$$\varphi_{1,\varepsilon} := r_{1,\varepsilon} - u_1, \qquad \varphi_{2,\varepsilon} := r_{2,\varepsilon} - u_2,$$

where

$$r_{1,\varepsilon} := \frac{u_2^p}{(u_1+\varepsilon)^{p-1}}, \qquad r_{2,\varepsilon} := \frac{u_1^p}{(u_2+\varepsilon)^{p-1}}.$$

Taking into account that $u_1, u_2 \in \mathbb{X}_p(\Omega)$, $u_1, u_2 \geq 0$ a.e. in Ω and that u_1, u_2 are *globally bounded in* Ω (as it follows Theorem 4.1), we readily infer that $\varphi_{i,\varepsilon} \in \mathbb{X}_p(\Omega)$ for every $\varepsilon > 0$ and $i = 1, 2$. Hence, using $\varphi_{i,\varepsilon}$ as a test function in (2.3) for u_i and adding the resulting identities, we get

$$\begin{aligned} &\int_\Omega |\nabla u_1|^{p-2}\langle \nabla u_1, \nabla\varphi_{1,\varepsilon}\rangle\,dx + \int_\Omega |\nabla u_2|^{p-2}\langle \nabla u_2, \nabla\varphi_{2,\varepsilon}\rangle\,dx \\ &\qquad + \iint_{\mathbb{R}^{2n}} \frac{|u_1(x)-u_1(y)|^{p-2}(u_1(x)-u_1(y))(\varphi_{1,\varepsilon}(x)-\varphi_{1,\varepsilon}(y))}{|x-y|^{n+ps}}\,dx\,dy \\ &\qquad + \iint_{\mathbb{R}^{2n}} \frac{|u_2(x)-u_2(y)|^{p-2}(u_2(x)-u_2(y))(\varphi_{2,\varepsilon}(x)-\varphi_{2,\varepsilon}(y))}{|x-y|^{n+ps}}\,dx\,dy \\ &= \int_\Omega \big(f(x,u_1)\varphi_{1,\varepsilon} + f(x,u_2)\varphi_{2,\varepsilon}\big)\,dx. \end{aligned} \tag{4.12}$$

Now, a direct computation based on the very definition of $\varphi_{i,\varepsilon}$ gives

$$\begin{aligned} &\int_\Omega |\nabla u_1|^{p-2}\langle \nabla u_1, \nabla\varphi_{1,\varepsilon}\rangle\,dx + \int_\Omega |\nabla u_2|^{p-2}\langle \nabla u_2, \nabla\varphi_{2,\varepsilon}\rangle\,dx \\ &\qquad = -\int_\Omega \mathcal{A}_p\Big(\nabla u_1, \frac{u_1}{u_2+\varepsilon}\nabla u_2\Big)\,dx - \int_\Omega \mathcal{A}_p\Big(\nabla u_2, \frac{u_2}{u_1+\varepsilon}\nabla u_1\Big)\,dx, \end{aligned}$$

where $\mathcal{A}_p$ is as in Lemma 4.4; as a consequence, taking into account that $\mathcal{A}_p(\cdot,\cdot) \geq 0$ (as we know from Lemma 4.4), identity (4.12) boils down to

$$\begin{aligned}
&\int_\Omega \big(f(x,u_1)\varphi_{1,\varepsilon} + f(x,u_2)\varphi_{2,\varepsilon}\big)\,dx \\
&\quad\leq \iint_{\mathbb{R}^{2n}} \frac{J_p(u_1(x)-u_1(y))(r_{1,\varepsilon}(x)-r_{1,\varepsilon}(y))}{|x-y|^{n+ps}}\,dx\,dy \\
&\quad+ \iint_{\mathbb{R}^{2n}} \frac{J_p(u_2(x)-u_2(y))(r_{2,\varepsilon}(x)-r_{2,\varepsilon}(y))}{|x-y|^{n+ps}}\,dx\,dy \\
&\quad- \iint_{\mathbb{R}^{2n}} \frac{J_p(u_1(x)-u_1(y))(u_1(x)-u_1(y))}{|x-y|^{n+ps}}\,dx\,dy \\
&\quad- \iint_{\mathbb{R}^{2n}} \frac{J_p(u_2(x)-u_2(y))(u_2(x)-u_2(y))}{|x-y|^{n+ps}}\,dx\,dy \\
&\quad=: \mathrm{I}_{1,\varepsilon} + \mathrm{I}_{2,\varepsilon} - \mathrm{J}_1 - \mathrm{J}_2,
\end{aligned} \tag{4.13}$$

where we have introduced the standard notation

$$J_p(t) := |t|^{p-2}t \qquad (t \in \mathbb{R}).$$

We now aim at passing to the limit as $\varepsilon \to 0^+$ in the above (4.13).

To this end, we first remind the following discrete Picone inequality: *for every fixed $p \in (1,+\infty)$ and every $a,b,c,d \in [0,+\infty)$ with $a,b > 0$, one has*

$$J_p(a-b)\left(\frac{c^p}{a^{p-1}} - \frac{d^p}{b^{p-1}}\right) \leq |c-d|^p,$$

and equality holds if and only if

$$ad = bc$$

(for a proof of this inequality see, e.g., [8, Proposition 4.2] or [9, Proposition 2.2]). By using this inequality, we have

(i) $J_p(u_1(x)-u_1(y))(r_{1,\varepsilon}(x)-r_{1,\varepsilon}(y)) \leq |u_2(x)-u_2(y)|^p$;

(ii) $J_p((u_2(x)-u_2(y))(r_{2,\varepsilon}(x)-r_{2,\varepsilon}(y)) \leq |u_1(x)-u_1(y)|^p$.

Hence, we are entitled to apply the Fatou lemma for the integrals $\mathrm{I}_{1,\varepsilon}, \mathrm{I}_{2,\varepsilon}$, obtaining

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+} \big(\mathrm{I}_{1,\varepsilon} + \mathrm{I}_{2,\varepsilon} - \mathrm{J}_1 - \mathrm{J}_2\big) \\
&\quad\leq \iint_{\mathbb{R}^{2n}} \frac{J_p(u_1(x)-u_1(y))}{|x-y|^{n+ps}}\left(\frac{u_2^p}{u_1^{p-1}}(x) - \frac{u_2^p}{u_1^{p-1}}(y)\right)dx\,dy \\
&\quad+ \iint_{\mathbb{R}^{2n}} \frac{J_p(u_2(x)-u_2(y))}{|x-y|^{n+ps}}\left(\frac{u_1^p}{u_2^{p-1}}(x) - \frac{u_1^p}{u_2^{p-1}}(y)\right)dx\,dy \\
&\quad- \iint_{\mathbb{R}^{2n}} \frac{|u_1(x)-u_1(y)|^p}{|x-y|^{n+ps}}\,dx\,dy - \iint_{\mathbb{R}^{2n}} \frac{|u_2(x)-u_2(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \\
&\quad=: \kappa(u_1,u_2,p),
\end{aligned} \tag{4.14}$$

where $\kappa(u_1,u_2,p) \in [-\infty,0]$ again by the discrete Picone inequality (here, to give a meaning to the integrals when x or y are not in Ω, we have tacitly set $0/0 = 0$).

We now turn our attention to the left hand side of (4.13). Taking into account the very definition of $\varphi_{i,\varepsilon}$, we first write

$$\begin{aligned}\int_\Omega \big(f(x,u_1)\varphi_{1,\varepsilon} + f(x,u_2)\varphi_{2,\varepsilon}\big)\,dx &= \int_\Omega f(x,u_1)\,r_{1,\varepsilon}\,dx + \int_\Omega f(x,u_2)\,r_{2,\varepsilon}\,dx \\ &\quad - \int_\Omega f(x,u_1)u_1\,dx - \int_\Omega f(x,u_2)u_2\,dx \\ &=: \mathrm{A}_{1,\varepsilon} + \mathrm{A}_{2,\varepsilon} - \mathrm{B}_1 - \mathrm{B}_2.\end{aligned}$$

Moreover, recalling the value $\rho_f > 0$ in (1.9), we further split $\mathrm{A}_{i,\varepsilon}$ as

$$\mathrm{A}_{i,\varepsilon} = \int_{\{u_i<\rho_f\}} f(x,u_i)\,r_{i,\varepsilon}\,dx + \int_{\{u_i\geq\rho_f\}} f(x,u_i)\,r_{i,\varepsilon}\,dx =: \mathrm{A}'_{i,\varepsilon} + \mathrm{A}''_{i,\varepsilon}.$$

Now, by assumption (f3), for every $\varepsilon > 0$ we have

$$|f(x,u_1)\,r_{1,\varepsilon}|\cdot \mathbf{1}_{\{u_1\geq\rho_f\}} \leq c_p\big(1+\rho_f^{1-p}\big)\,u_2^p \equiv c_{p,f}\,u_2^p$$

and, analogously,

$$|f(x,u_2)\,r_{2,\varepsilon}|\cdot \mathbf{1}_{\{u_2\geq\rho_f\}} \leq c_{p,f}\,u_1^p.$$

Thus, we can then apply the Dominated Convergence theorem, obtaining

$$\begin{aligned}&\mathrm{A}''_1 := \lim_{\varepsilon\to0^+}\mathrm{A}''_{1,\varepsilon} = \int_{\{u_1\geq\rho_f\}} \frac{f(x,u_1)}{u_1^{p-1}}\,u_2^p\,dx \in \mathbb{R} \qquad \text{and} \\ &\mathrm{A}''_2 := \lim_{\varepsilon\to0^+}\mathrm{A}''_{2,\varepsilon} = \int_{\{u_2\geq\rho_f\}} \frac{f(x,u_2)}{u_2^{p-1}}\,u_1^p\,dx \in \mathbb{R}.\end{aligned} \tag{4.15}$$

Hence, it remains to study the behavior of $A'_{i,\varepsilon}$ when $\varepsilon \to 0^+$.

First of all, using (1.9) and the fact that $r_{i,\varepsilon}$ is nonnegative and monotone increasing with respect to ε, we can apply the Beppo Levi theorem, obtaining

$$\begin{aligned}&\mathrm{A}'_1 := \lim_{\varepsilon\to0^+}\mathrm{A}'_{1,\varepsilon} = \int_{\{u_1<\rho_f\}} \frac{f(x,u_1)}{u_1^{p-1}}\,u_2^p\,dx \in [0,+\infty] \qquad \text{and} \\ &\mathrm{A}'_2 := \lim_{\varepsilon\to0^+}\mathrm{A}'_{2,\varepsilon} = \int_{\{u_2<\rho_f\}} \frac{f(x,u_2)}{u_2^{p-1}}\,u_1^p\,dx \in [0,+\infty].\end{aligned} \tag{4.16}$$

On the other hand, going back to estimate (4.13) and taking into account the very definitions of the integrals $\mathrm{A}'_{i,\varepsilon}, \mathrm{A}''_{i,\varepsilon}, \mathrm{B}_i$, we get

$$\begin{aligned}0 \leq \mathrm{A}'_{1,\varepsilon}, \mathrm{A}'_{2,\varepsilon} &\leq \mathrm{A}'_{1,\varepsilon} + \mathrm{A}'_{2,\varepsilon} \\ &\leq \mathrm{I}_{1,\varepsilon} + \mathrm{I}_{2,\varepsilon} - \mathrm{J}_1 - \mathrm{J}_2 + \mathrm{B}_1 + \mathrm{B}_2 - \mathrm{A}''_{1,\varepsilon} - \mathrm{A}''_{2,\varepsilon}.\end{aligned}$$

Then, by letting $\varepsilon \to 0^+$ with the aid of (4.14)–(4.15), we obtain

$$0 \leq \mathrm{A}'_1, \mathrm{A}'_2 \leq \mathrm{A}'_1 + \mathrm{A}'_2 \leq \kappa(u_1,u_2,p) + \mathrm{B}_1 + \mathrm{B}_2 - \mathrm{A}''_1 - \mathrm{A}''_2,$$

from which we derive at once that

$$\kappa(u_1,u_2,p) > -\infty \qquad \text{and} \qquad \mathrm{A}'_1, \mathrm{A}'_2 < +\infty. \tag{4.17}$$

Gathering (4.15)–(4.16), and taking into account (4.17), we finally have

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\Big(\int_\Omega \big(f(x,u_1)\varphi_{1,\varepsilon}+f(x,u_2)\varphi_{2,\varepsilon}\big)\,dx\Big)\\
&\qquad= \lim_{\varepsilon\to 0^+}\big(\mathrm{A}'_{1,\varepsilon}+\mathrm{A}'_{2,\varepsilon}+\mathrm{A}''_{1,\varepsilon}+\mathrm{A}''_{2,\varepsilon}-\mathrm{B}_1-\mathrm{B}_2\big)\\
&\qquad= \int_\Omega\Big(\frac{f(x,u_1)}{u_1^{p-1}}u_2^p+\frac{f(x,u_2)}{u_2^{p-1}}u_1^p-f(x,u_1)u_1-f(x,u_2)u_2\Big)\,dx\\
&\qquad= -\int_\Omega\Big(\frac{f(x,u_1)}{u_1^{p-1}}-\frac{f(x,u_2)}{u_2^{p-1}}\Big)(u_1^p-u_2^p)\,dx.
\end{aligned} \tag{4.18}$$

With (4.14) and (4.18) at hand, we can easily conclude the proof of the theorem. Indeed, using these cited identities we can let $\varepsilon\to 0^+$ in (4.13), obtaining

$$-\int_\Omega\Big(\frac{f(x,u_1)}{u_1^{p-1}}-\frac{f(x,u_2)}{u_2^{p-1}}\Big)(u_1^p-u_2^p)\,dx\le \kappa(u_1,u_2,p)\le 0.$$

From this, by crucially exploiting assumption (f4) we conclude that

$$u_1\equiv u_2 \text{ a.e. in } \Omega,$$

and the proof is complete. □

5. The eigenvalue problem

As announced, here we consider the eigenvalue problem associated to $\mathcal{L}_{p,s}$ in presence of a weight $a\in L^\infty(\Omega)$, namely

$$\begin{cases} -\Delta_p u+(-\Delta)^s_p u+a(x)|u|^{p-2}u=\lambda|u|^{p-2}u & \text{in } \Omega,\\ u\not\equiv 0, & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega. \end{cases} \tag{5.1}$$

Proposition 5.1. *Let $a\in L^\infty(\Omega)$. Then, problem* (5.1) *admits a smallest eigenvalue $\lambda_1(\mathcal{L}_{p,s}+a)\in\mathbb{R}$ which is simple, and whose associated eigenfunctions do not change sign in $\mathbb{R}^n$. Moreover, every eigenfunction associated to an eigenvalue*

$$\lambda>\lambda_1(\mathcal{L}_{p,s}+a)$$

is nodal, i.e., sign changing.

Proof. Let $\gamma:\mathbb{X}_p(\Omega)\to\mathbb{R}$ be the C^1-functional defined as

$$\gamma(u)=\int_\Omega|\nabla u|^p dx+\iint_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy+\int_\Omega a(x)|u|^p\,dx$$

for all $u\in\mathbb{X}_p(\Omega)$, and let it be constrained on the C^1- Banach manifold

$$M:=\Big\{u\in\mathbb{X}_p(\Omega):\int_\Omega|u|^p\,dx=1\Big\}.$$

Define

$$\lambda_1(\mathcal{L}_{p,s}+a):=\inf\big\{\gamma(u):u\in M\big\}. \tag{5.2}$$

Let $\{u_n\}_{n\ge 1}\subseteq M$ be a minimizing sequence for (5.2). Since

$$\lambda_1(\mathcal{L}_{p,s}+a)\ge-\|a(x)\|_{L^\infty(\Omega)},$$

we immediately get that $\{u_n\}_{n\geq 1} \subseteq \mathbb{X}_p(\Omega)$ is bounded and so we may assume that there exists $e_1 \in M$ such that

$$u_n \rightharpoonup e_1 \ \text{ in } \mathbb{X}_p(\Omega) \qquad \text{as } n \to +\infty. \tag{5.3}$$

In particular, by the Rellich-Kondrachev embedding theorem, we know that

$$u_n \to e_1 \text{ in } L^p(\Omega). \tag{5.4}$$

By (5.3) and (5.4), we have

$$\begin{aligned}\gamma(e_1) &= \int_\Omega |\nabla e_1|^p dx + \iint_{\mathbb{R}^{2n}} \frac{|e_1(x) - e_1(y)|^p}{|x-y|^{n+ps}}\, dx\, dy + \int_\Omega a(x)|e_1|^p\, dx + \\ &\leq \liminf_{n\to+\infty} \gamma(u_n) = \lambda_1(\mathcal{L}_{p,s} + a).\end{aligned}$$

Since $e_1 \in M$, due to (5.4), by (5.2) we get

$$\gamma(e_1) = \lambda_1(\mathcal{L}_{p,s} + a).$$

By the Lagrange multiplier rule, we infer that $\lambda_1(\mathcal{L}_{p,s}+a)$ is the smallest eigenvalue for problem (5.1), with associated eigenfunction $e_1 \in \mathbb{X}_p(\Omega)$. Finally, notice that

$$\gamma(|u|) \leq \gamma(u) \ \text{ for all } u \in X^s_\beta,$$

and so we may assume that $e_1 \geq 0$ in $\mathbb{R}^n$. Since $\|e_1\|_{L^p(\Omega)} = 1$ by construction, we can then apply Theorem 3.1 and Remark 3.4 to conclude that

$$e_1(x) > 0, \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Now, we prove that e_1 is simple. To this end, let $u \in \mathbb{X}_p(\Omega)$ be another eigenfunction associated to $\lambda_1(\mathcal{L}_{p,s} + a)$. We first claim that u has *constant sign*: in fact, taking into account that the eigenfunctions associated to $\lambda_1(\mathcal{L}_{p,s} + a)$ are precisely the constrained minimizers of γ, we have

$$\gamma(u) = \lambda_1(\mathcal{L}_{p,s} + a)\|u\|^p_{L^p(\Omega)};$$

on the other hand, if both $\{u > 0\}$ and $\{u < 0\}$ have positive Lebesgue measure, by arguing exactly as in the proof of [30, Proposition 9], we have

$$\gamma(|u|) < \gamma(u) = \lambda_1(\mathcal{L}_{p,s} + a)\|u\|^p_{L^p(\Omega)},$$

which is clearly in contradiction with the fact that $\lambda_1(\mathcal{L}_{p,s} + a)$ is the minimum of γ. Hence, u has constant sign in Ω and we can assume that $u \geq 0$ a.e. in Ω; from this, using once again Theorem 3.1 and Remark 3.4, we obtain

$$u > 0 \text{ a.e. in } \Omega. \tag{5.5}$$

With (5.5) at hand, we now turn to prove that there exists $\alpha \geq 0$ such that

$$e_1 = \alpha u.$$

To this end we observe that, on account of Theorem 4.1 and (3.11) in Remark 3.4, we know that $e_1, u \in L^\infty(\Omega)$. Given any $\varepsilon > 0$, we then define

$$v_\varepsilon = \frac{u^p}{(e_1 + \varepsilon)^{p-1}}.$$

Since $v_\varepsilon \in \mathbb{X}_p(\Omega)$ (as the same is true of both e_1 and u), we are entitled to use v_ε as test function in the problem solved by e_1. Thus, using again the notation

$$J_p(t) := |t|^{p-2}t, \quad (t \in \mathbb{R}),$$

we obtain

$$
\begin{aligned}
&\int_\Omega |\nabla e_1|^{p-2}\langle \nabla e_1, \nabla v_\varepsilon\rangle dx \\
&\qquad + \iint_{\mathbb{R}^{2n}} \frac{J_p((e_1+\varepsilon)(x)-(e_1+\varepsilon)(y))(v_\varepsilon(x)-v_\varepsilon(y))}{|x-y|^{n+2s}}\, dx\, dy \\
&= \lambda_1(\mathcal{L}_{p,s}+a)\int_\Omega e_1^{p-1}\, v_\varepsilon\, dx - \int_\Omega a(x) e_1^{p-1}\, v_\varepsilon\, dx.
\end{aligned} \tag{5.6}
$$

By the already recalled discrete Picone inequality, we find

$$
J_p((e_1+\varepsilon)(x)-(e_1+\varepsilon)(y))(v_\varepsilon(x)-v_\varepsilon(y)) \leq |u(x)-u(y)|^p.
$$

Now, consider the function

$$
R(u, e_1+\varepsilon) = |\nabla u|^p - |\nabla e_1|^{p-2}\langle \nabla e_1, \nabla v_\varepsilon\rangle.
$$

As a consequence of the nonlinear Picone identity by Allegretto - Huang in [1] (see also [26, p. 244]), we have that $R(u, e_1+\varepsilon) \geq 0$. Then

$$
|\nabla e_1|^{p-2}\langle \nabla e_1, \nabla v_\varepsilon\rangle \leq |\nabla u|^p. \tag{5.7}
$$

Gathering these facts, we can pass to the limit as $\varepsilon \to 0$ in (5.6): by applying the Fatou lemma in the left had side of (5.6) and the Dominated Convergence theorem in the right hand side, we find

$$
\begin{aligned}
&\int_\Omega |\nabla e_1|^{p-2}\Big\langle \nabla e_1, \nabla\Big(\frac{u^p}{e_1^{p-1}}\Big)\Big\rangle\, dx \\
&\qquad + \iint_{\mathbb{R}^{2n}} \frac{J_p(e_1(x)-e_1(y))}{|x-y|^{n+2s}}\left(\frac{u^p(x)}{e_1^{p-1}(x)} - \frac{u^p(y)}{e_1^{p-1}(y)}\right) dx\, dy \\
&\geq \lambda_1(\mathcal{L}_{p,s}+a)\int_\Omega u^p\, dx - \int_\Omega a(x) u^p\, dx \\
&= \int_\Omega |\nabla u|^p\, dx + \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, dx\, dy.
\end{aligned} \tag{5.8}
$$

On the other hand, by using again inequality (5.7), we have the estimate

$$
\begin{aligned}
&\int_\Omega |\nabla e_1|^{p-2}\Big\langle \nabla e_1, \nabla\Big(\frac{u^p}{e_1^{p-1}}\Big)\Big\rangle\, dx \\
&\qquad + \iint_{\mathbb{R}^{2n}} \frac{J_p(e_1(x)-e_1(y))}{|x-y|^{n+2s}}\left(\frac{u^p(x)}{e_1^{p-1}(x)} - \frac{u^p(y)}{e_1^{p-1}(y)}\right) dx\, dy \\
&\leq \int_\Omega |\nabla u|^p\, dx + \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, dx\, dy.
\end{aligned} \tag{5.9}
$$

Hence, all the inequalities in (5.8) and (5.9) are actually identities. In particular, the discrete Picone inequality implies that

$$
\frac{e_1(x)}{e_1(y)} = \frac{u(x)}{u(y)} \text{ in } \mathbb{R}^{2n},
$$

and so we can conclude that there exists $\alpha \geq 0$ such that

$$
e_1 = \alpha u \text{ in } \mathbb{R}^n.
$$

Now, suppose that $\lambda > \lambda_1(\mathcal{L}_{p,s}+a)$ is another eigenvalue of (5.1) with associated L^p-normalized eigenfunction $u \in \mathbb{X}_p(\Omega)$, and assume by contradiction that u has constant sign, say $u \geq 0$. By Theorem 3.1 we have $u > 0$.

Then, starting from the equation solved by u and using

$$\frac{e_1^p}{(u+\varepsilon)^{p-1}}$$

as test function, by arguing exactly as for reaching (5.8)-(5.9), we get

$$\int_\Omega |\nabla e_1|^p\,dx + \iint_{\mathbb{R}^{2n}} \frac{|e_1(x)-e_1(y)|^p}{|x-y|^{n+ps}}\,dx\,dy = \lambda - \int_\Omega a(x)e_1^p\,dx.$$

On the other hand, e_1 being a solution to (5.1) with λ_1, we have

$$\int_\Omega |\nabla e_1|^p\,dx + \iint_{\mathbb{R}^{2n}} \frac{|e_1(x)-e_1(y)|^p}{|x-y|^{n+ps}}\,dx\,dy + \int_\Omega a(x)e_1^p\,dx = \lambda_1(\mathcal{L}_{p,s}+a).$$

Since $\lambda > \lambda_1(\mathcal{L}_{p,s}+a)$, we get a contradiction, and thus u must change sign. □

6. Existence

In this last section we combine all the results established so far in order to give the proof of Theorem 1.2. Throughout what follows, we tacitly adopt all the notation introduced in Sections 2-5: in particular,

- $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with C^1 boundary;
- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (f1)–(f5);
- a_0 and a_∞ are the functions defined in (2.5);
- $\lambda_1(\mathcal{L}_{p,s}-a_0)$ and $\lambda_1(\mathcal{L}_{p,s}-a_\infty)$ are defined in (1.7).

Remark 6.1. As already pointed out in the Introduction, the 'sign assumption' (f5) is needed only to prove the uniqueness part of Theorem 1.2, since it allows us to invoke Theorem 4.3; all the other results we are going to establish in this section actually hold under assumptions (f1)–(f4) solely.

To begin with, we set

$$F(x,u) = \int_0^u f(x,t)\,dt,$$

and we consider the functional $E : \mathbb{X}_p(\Omega) \to \mathbb{R}$ defined as follows:

$$E(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{p}\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy - \int_\Omega F(x,u)\,dx. \tag{6.1}$$

The functional E is well-defined, differentiable and its critical points are weak solutions of problem (1.1).

Proposition 6.2. *Let E be the functional defined in* (6.1)*, and assume that*

$$\lambda_1(\mathcal{L}_{p,s}-a_0) < 0 < \lambda_1(\mathcal{L}_{p,s}-a_\infty).$$

Then, the following hold:

(a) *E is coercive on $\mathbb{X}_p(\Omega)$.*
(b) *E is weakly l.s.c. in $\mathbb{X}_p(\Omega)$, so it has a minimum $v \in \mathbb{X}_p(\Omega)$.*
(c) *There exists $\phi \in \mathbb{X}_p(\Omega)$ such that $E(\phi) < 0$, so that*

$$\min_{u\in\mathbb{X}_p(\Omega)} E(u) < 0,$$

and $u = |v|$ is a solution to (1.1).

Proof. (a) It is sufficient to note that, by its very definition,

$$E(u) \geq J(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega F(x,u)\,dx, \tag{6.2}$$

for every $u \in \mathbb{X}_p(\Omega)$. Since we can identify $\mathbb{X}_p(\Omega)$ with $W_0^{1,p}(\Omega)$, the functional J is precisely the one considered in [17] and therefore coercive, see [11] for the details in the linear case $p = 2$. For completeness, we recall that the condition

$$\lambda_1(\mathcal{L}_{p,s} - a_\infty) > 0$$

is used at this stage.

(b) Let $u \in \mathbb{X}_p(\Omega)$ be fixed, and let $\{u_n\}_n$ be a sequence in $\mathbb{X}_p(\Omega)$ which weakly converges to u as $n \to +\infty$. By (f3), we have

$$|F(x,u)| \leq c_p(|u| + |u|^p);$$

hence, by the Rellich-Kondrachev theorem we get

$$\lim_{n\to+\infty}\int_\Omega F(x,u_n)\,dx = \int_\Omega F(x,u)\,dx,$$

which immediately implies the claim.

(c) To prove this assertion, we can follow the argument originally presented in [11]. Since $\lambda_1(\mathcal{L}_{p,s} - a_0) < 0$, there exists $\phi \in \mathbb{X}_p(\Omega)$ such that $\|\phi\|_{L^p(\Omega)} = 1$ and

$$\int_\Omega |\nabla\phi|^p\,dx + \iint_{\mathbb{R}^{2n}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}}\,dx\,dy < \int_{\{\phi\neq 0\}} a_0\,|\phi|^p dx. \tag{6.3}$$

We then claim that it is not restrictive to assume that $\phi \geq 0$ and $\phi \in L^\infty(\mathbb{R}^n)$. In fact, since $||x| - |y|| \leq |x-y|$ for every $x, y \in \mathbb{R}$, from (6.3) we find

$$\begin{aligned}&\int_\Omega |\nabla|\phi||^p\,dx + \iint_{\mathbb{R}^{2n}} \frac{||\phi(x)|-|\phi(y)||^p}{|x-y|^{n+sp}}\,dx\,dy \\ &\qquad \leq \int_\Omega |\nabla\phi|^p\,dx + \iint_{\mathbb{R}^{2n}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}}\,dx\,dy < \int_{\{\phi\neq 0\}} a_0\,|\phi|^p dx,\end{aligned}$$

so that we can assume $\phi \geq 0$. As for the assumption $\phi \in L^\infty(\mathbb{R}^n)$, we define

$$\phi_M = \min\{\phi, M\} \qquad (\text{for } M > 0).$$

As usual, $\phi_M \in \mathbb{X}_p(\Omega)$; moreover, since a direct computation gives

$$|\phi_M(x) - \phi_M(y)| \leq |\phi(x) - \phi(y)|,$$

from (6.3) we obtain

$$\begin{aligned}&\int_\Omega |\nabla\phi_M|^p\,dx + \iint_{\mathbb{R}^{2n}} \frac{|\phi_M(x)-\phi_M(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \\ &\qquad \leq \int_\Omega |\nabla\phi|^p\,dx + \iint_{\mathbb{R}^{2n}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}}\,dx\,dy < \int_{\{\phi\neq 0\}} a_0\,|\phi|^p dx.\end{aligned}$$

On the other hand, since a_0 is bounded from below (see (2.8)), we have

$$\int_\Omega a_0\phi^p \leq \liminf_{M\to+\infty}\int_\Omega a_0\phi_M^p;$$

as a consequence, can find $M > 0$ large enough so that

$$\int_\Omega |\nabla \phi_M|^p \, dx + \iint_{\mathbb{R}^{2n}} \frac{|\phi_M(x) - \phi_M(y)|^p}{|x-y|^{n+sp}} \, dx \, dy < \int_{\{\phi \neq 0\}} a_0 \, |\phi_M|^p \, dx.$$

Summing up, by replacing ϕ with $|\phi_M|$, we can choose $\phi \geq 0$ and bounded.

Now, we have that

$$\liminf_{u \to 0} \frac{F(x,u)}{u^p} \geq \frac{a_0(x)}{p}$$

and proceeding as in [11, Proof of (15)] we get

$$\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^n} \frac{F(x, \varepsilon\phi)}{\varepsilon^p} \geq \frac{1}{p} \int_{\{\phi \neq 0\}} a_0 \phi^p.$$

Therefore using (6.3) we conclude that

$$\int_\Omega |\nabla \phi|^p \, dx + \iint_{\mathbb{R}^{2n}} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{n+sp}} \, dx \, dy - p \int_{\mathbb{R}^n} \frac{F(x, \varepsilon\phi)}{\varepsilon^p} < 0$$

for any $\varepsilon > 0$ small enough. Clearly, the latter can be rewritten as

$$E(\varepsilon\phi) < 0,$$

and this closes the proof. □

Concerning the "necessity" part, we start from the next result.

Lemma 6.3. *Let $u \in \mathbb{X}_p(\Omega)$ be a solution of* (1.1)*. Then*

$$\lambda_1(\mathcal{L}_{p,s} - a_0) < 0.$$

Proof. On one hand, by the very definition of $\lambda_1(\mathcal{L}_{p,s} - a_0)$, we have

$$\lambda_1(\mathcal{L}_{p,s} - a_0) \leq \frac{\mathcal{Q}_{p,s}(u) - \int_{\{u \neq 0\}} a_0 \, |u|^p}{\|u\|_{L^p(\Omega)}},$$

where $\mathcal{Q}_{p,s}$ is as in (1.8). On the other hand, since u solves (1.1), we have that

$$\int_\Omega |\nabla u|^p \, dx + \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}} = \mathcal{Q}_{p,s}(u) = \int_\Omega f(x,u) u \, dx.$$

By the strong maximum principle, $u > 0$ in Ω. Therefore, by definition of a_0 and by assumption (f4), we get that

$$\frac{f(x,u)}{u^{p-1}} < a_0(x) \text{ a.e. in } \Omega,$$

so that

$$\int_\Omega f(x,u) u \, dx < \int_\Omega a_0 u^p dx$$

and the conclusion follows. □

Although up to now we have been able to treat the general case, we are now led to focus on the semilinear case.

Proposition 6.4. *Assume that $p = 2$, and let $u \in \mathbb{X}_2(\Omega)$ be a nonnegative solution of problem* (1.1)*. Then*

$$\lambda_1(\mathcal{L}_{2,s} - a_\infty) > 0.$$

Proof. First of all we observe that, in view of Theorem 4.1, we know that

$$u \in L^\infty(\Omega).$$

Hence, as in [11], we define the bounded and indefinite weight

$$\overline{a}(x) := \frac{f(x, \|u\|_{L^\infty(\Omega)} + 1)}{\|u\|_{L^\infty(\Omega)} + 1}.$$

Notice that $\overline{a} \in L^\infty(\Omega)$ by (f2). Then, we consider the auxiliary eigenvalue problem

$$\begin{cases} \mathcal{L}_{2,s}\psi - \overline{a}(x)\psi = \mu\psi & \text{in } \Omega, \\ \psi \geq 0 & \text{in } \Omega, \\ \psi = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{6.4}$$

By Proposition 5.1 we get the existence of a principal eigenvalue with associated bounded and nonnegative eigenfuction $\psi \in \widetilde{H}$. We can therefore use such a ψ as test function for (1.1), finding

$$\int_\Omega u\psi\,(\overline{a} + \mu)\,dx = \int_\Omega f(x,u)\psi\,dx. \tag{6.5}$$

Clearly,

$$\int_\Omega u\psi\,(\overline{a} + \mu)\,dx = \int_{\Omega\cap\{u>0\}} u\psi\,(\overline{a} + \mu)\,dx,$$

and on $\Omega \cap \{u > 0\}$ we can exploit condition (f4), which yields

$$\int_{\Omega\cap\{u>0\}} f(x,u)\psi\,dx > \int_{\Omega\cap\{u>0\}} \overline{a}(x)u\psi\,dx = \int_\Omega \overline{a}(x)u\psi\,dx.$$

Therefore, we find that

$$\mu \int_\Omega u\psi\,dx > 0,$$

and then conclude as in [11]. □

By combining the results in this section, we can finally prove Theorem 1.2.

Proof of Theorem 1.2. As for the uniqueness, it is a consequence of Lemma 6.3, together with Theorem 4.3. Moreover, the strict positivity is contained in Corollary 3.3.

The existence part of assertion (1) is exactly the content of Proposition 6.2. As for assertion (2), it follows from Lemma 6.3 and Proposition 6.4. □

References

[1] W. Allegretto, Y.X. Huang, *A Picone's identity for the p−Laplacian and applicdations*, Nonlinear Anal. **32** (1998), 819–830.

[2] G. Barles, C. Imbert, *Second-order elliptic integro-differential equations: viscosity solutions' theory revisited*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), no. 3, 567–585.

[3] G. Barles, E. Chasseigne, A. Ciomaga, C. Imbert, *Lipschitz regularity of solutions for mixed integro-differential equations*, J. Differential Equations **252** (2012), no. 11, 6012–6060.

[4] S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, *Semilinear elliptic equations involving mixed local and nonlocal operators*, Proc. Roy. Soc. Edinburgh Sect. A **151** (2021), no. 5, 1611–1641

[5] S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, *Mixed local and nonlocal elliptic operators: regularity and maximum principles*, Comm. Partial Differential Equations **47**(3), (2022), 585–629.

[6] S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, *A quantitative Faber-Krahn inequality for some mixed local and nonlocal operators*, to appear in J. Anal. Math.

[7] S. Biagi, D. Mugnai and E. Vecchi, *Necessary condition in a Brezis-Oswald-type problem for mixed local and nonlocal operators*, Appl. Math. Lett. **132**, (2022), 108177.
[8] L. Brasco, G. Franzina, *Convexity properties of Dirichlet integrals and Picone-type inequalities*, Kodai Math. J. **37** (2014), 769–799.
[9] L. Brasco, M. Squassina, *Optimal solvability for a nonlocal problem at critical growth.* J. Differential Equations **264** (2018), 2242–2269.
[10] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.
[11] H. Brezis, L. Oswald, *Remarks on sublinear elliptic equations*, Nonlinear Anal. **10** (1986), 55–64.
[12] S. Buccheri, J. V. da Silva, L. H. de Miranda, *A System of Local/Nonlocal p-Laplacians: The Eigenvalue Problem and Its Asymptotic Limit as $p \to \infty$*, Asymptot. Anal. **128**(2), (2022), 149–181.
[13] J.V. da Silva, A.M. Salort, *A limiting problem for local/nonlocal p-Laplacians with concave-convex nonlinearities*, Z. Angew. Math. Phys. **71** (2020), Paper No. 191, 27pp.
[14] A. Di Castro, T. Kuusi, G. Palatucci, *Local behavior of fractional p-minimizers*, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), no. 5, 1279–1299.
[15] S. Dipierro, E. Proietti Lippi, E. Valdinoci, *Linear theory for a mixed operator with Neumann conditions*, Asymptot. Anal. **128**(4), (2022), 571–594.
[16] S. Dipierro, E. Proietti Lippi, E. Valdinoci, *(Non)local logistic equations with Neumann conditions*, to appear in Ann. Inst. H. Poincaré Anal. Non Linéaire.
[17] J.I. Díaz, J.E. Saá, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 12, 521–524.
[18] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.
[19] G. Fragnelli, D. Mugnai, N. Papageorgiou, *The Brezis–Oswald result for quasilinear Robin problems*, Adv. Nonlinear Stud. **16** (2016), no. 3, 603–622.
[20] G. Franzina, G. Palatucci, *Fractional $p-$eigenvalues*, Riv. Math. Univ. Parma (N.S.) **5** (2014), 373–386.
[21] P. Garain, J. Kinnunen, *On the regularity theory for mixed local and nonlocal quasilinear elliptic equations*, to appear in Trans. Amer. Math. Soc.
[22] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
[23] A. Iannizzotto, D. Mugnai, *Optimal solvability for the fractional p-Laplacian with Dirichlet conditions*, arXiv:2206.08685v2.
[24] G. Leoni, *A first course in Sobolev space. Second edition*, Graduate Studies in Mathematics, 181. American Mathematical Society, Providence, RI, 2017.
[25] M. Lucia, S. Prashanth, *Simplicity of principal eigenvalue for $p-$Laplace operator with singular indefinite weight*, Arch. Math. (Basel) **86** (2006), 79–89.
[26] D. Motreanu, V. V. Motreanu, N.S. Papageorgiou, *Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems*, Springer, New York (2014).
[27] D. Mugnai, A. Pinamonti, E. Vecchi, *Towards a Brezis-Oswald-type result for fractional problems with Robin boundary conditions*, Calc. Var. Partial Differential Equations **59** (2020), no. 2.
[28] D. Mugnai, E. Proietti Lippi, *Neumann fractional p-Laplacian: Eigenvalues and existence results*, Nonlinear Anal. **188** (2019), 455–474.
[29] P. Pucci, J. Serrin, *The maximum principle*, Progress in Nonlinear Differential Equations and their Applications, 73. Birkhäuser Verlag, Basel, 2007.
[30] R. Servadei, E. Valdinoci, *Variational methods for non-local operators of elliptic*, type. Discrete Contin. Dyn. Syst. **33**(5), (2013), 2105–2137.

(S. Biagi) Dipartimento di Matematica
Politecnico di Milano
Via Bonardi 9, 20133 Milano, Italy
Email address: stefano.biagi@polimi.it

(D. Mugnai) Dipartimento di Ecologia e Biologia (DEB)
Università della Tuscia
Largo dell'Università, 01100 Viterbo, Italy
Email address: dimitri.mugnai@unitus.it

(E. Vecchi) Dipartimento di Matematica
Università degli Studi di Bologna
Piazza di Porta San Donato 5, 40126 Bologna , Italy
Email address: eugenio.vecchi2@unibo.it