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Maximal regularity, analytic semigroups, and dynamic and general Wentzell boundary conditions with a diffusion term on the boundary

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Abstract

We show maximal regularity results concerning parabolic systems with dynamic boundary conditions and a diffusion theorem on the boundary in the framework of L^p spaces, $1 < p < \infty$. Analyticity results can be derived for the semigroups generated by suitable classes of uniformly elliptic operators with general Wentzell boundary conditions having diffusion terms on the boundary.

1 Introduction

The main aim of this paper is the study of parabolic systems with dynamic boundary conditions in the form

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x) + f(t,x), & (t,x) \in (0,T) \times \Omega, \\ D_t \gamma u(t,\cdot) = L\gamma u(t,\cdot) + \gamma E u(t,\cdot) + h(t,\cdot), & t \in (0,T) \\ u(0,x) = u_0(x), & x \in \Omega, \\ (\gamma u)(0) = v_0. \end{cases} \tag{1.1}$$

Here $\mathcal{A}$ is a linear, strongly elliptic, second order differential operator in the open bounded subset Ω of $\mathbb{R}^n$, L is a second order strongly elliptic operator in $\partial\Omega$, E is a first order differential operator and γ is the trace operator in $\partial\Omega$. A typical example of (1.1) is

$$\begin{cases} D_t u(t,x) = \alpha(x)\Delta u(t,x) + f(t,x), & (t,x) \in (0,T) \times \Omega, \\ D_t u(t,x') - a(x')\Delta_{LB} u(t,x') + b(x')\frac{\partial u}{\partial \nu}(t,x') - c(x')u(t,x') = h(t,x'), & (t,x') \in (0,T) \times \partial\Omega, \\ u(t,x) = u_0(x), & (t,x) \in \Omega \end{cases} \tag{1.2}$$

where we have indicated with Δ_{LB} the Laplace-Beltrami operator in $\partial\Omega$, with $\frac{\partial \cdot}{\partial \nu}$ the unit normal derivative, pointing outside Ω, and α and a are positively valued. Strictly connected with (1.1) and (1.2) are, respectively,

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x) + f(t,x), & (t,x) \in (0,T) \times \Omega, \\ \mathcal{A}u(t,x') - L\gamma u(t,\cdot) - \gamma E u(t,\cdot) = h(t,\cdot), & t \in (0,T) \\ u(0,x) = u_0(x), & x \in \Omega, \\ (\gamma u)(0) = v_0. \end{cases} \tag{1.3}$$

and

$$
\begin{cases}
D_t u(t,x) = \alpha(x)\Delta u(t,x) + f(t,x), & (t,x) \in (0,T)\times\Omega, \\
\alpha(x')\Delta u(t,x') - a(x')\Delta_{LB} u(t,x') + b(x')\frac{\partial u}{\partial \nu}(t,x') - c(x')u(t,x') = h(t,x'), & (t,x') \in (0,T)\times\partial\Omega, \\
u(t,x) = u_0(x), & (t,x) \in \Omega
\end{cases}
\tag{1.4}
$$

in the framework of L^p spaces, both in Ω and in $\partial\Omega$. Here Ω is an open, bounded subset of $\mathbb{R}^n$, with suitably smooth boundary $\partial\Omega$, α and a are positively valued and Δ_{LB} is the Laplace-Beltrami operator in $\partial\Omega$. We shall call boundary conditions in the form of (1.3) general Wentzell boundary conditions.

In our knowledge problems (1.2) and (1.4) seem to have been introduced and discussed (from the physical point of view) in [18]. These systems contain a diffusion term of the boundary, given by a strongly elliptic operator in $\partial\Omega$ (for example, the Laplace-Beltrami operator). Similar systems without this term were studied, in different functional settings, in [13], [6], [1], [9], [4], [14], [10], [12].

Systems in the form (1.1)-(1.2) seem to have been considered only recently. The first paper where a problem in the form (1.2) is really studied seems to be [7]. In it, it was considered the system

$$
\begin{cases}
D_t u(t,x) = Au(t,x) = \nabla\cdot(a(x)\nabla u)(t,x), & (t,x) \in [0,T]\times\Omega, \\
Au(t,x') + \beta(x')D_{\nu_A}u(t,x') + \gamma(x') - q\Delta_{LB}u(t,x') = 0, & (t,x') \in [0,T]\times\partial\Omega, \\
u(0,x) = u_0(x), & x \in \Omega,
\end{cases}
\tag{1.5}
$$

with A strongly elliptic in divergence form, $\beta(x') > 0$ in $\partial\Omega$, D_{ν_A} conormal derivative, $q \in [0,\infty)$. It is proved that, if $1 \le p \le \infty$, then the closure of a suitable realization of the problem in the space $L^p(\Omega\times\partial\Omega)$ $(1 \le p \le \infty)$ gives rise to an analytic semigroup (not strongly continuous if $p = \infty$). The continuous dependence on the coefficients had already been considered in [3]. The case of a non symmetric elliptic operator has been recently discussed in [8].

In [23] the author considered the case of a domain Ω with merely Lipschitz boundary, with a strongly elliptic operator A (independent of t). It was shown that a realisation of A with the general boundary condition $(Au)_{|\partial\Omega} - \gamma\Delta_{LB}u + D_{\nu_A}u + \beta u = g$ in $\partial\Omega$ generates a strongly continuous compact semigroup in $C(\overline{\Omega})$.

In the paper [22] the authors treated (2.6) in the particular case $A(t,x,D_x) = \Delta_x$, $f \equiv 0$, $h \equiv 0$, $L(t) = l\Delta_{LB}$ with $l > 0$ and $B(t,x',D_x) = kD_\nu$, where k may be negative (in contrast with the previously quoted literature). They showed that, if the initial datum u_0 is in $H^1(\Omega)$ and $u_{0|\partial\Omega} \in H^1(\partial\Omega)$, then (2.6) has a unique solution u in $C([0,\infty);H^1(\Omega)) \cap C^1((0,\infty);H^1(\Omega)) \cap C((0,\infty);H^3(\Omega))$, with $u_{|\partial\Omega}$ in $C([0,\infty);H^1(\partial\Omega)) \cap C^1((0,\infty);H^1(\partial\Omega)) \cap C((0,\infty);H^3(\partial\Omega))$.

In [11] (1.1) and (1.2) are studied in the setting of spaces of Hölder continuous functions. Results of maximal regularity are proved. Here also the operator E may be essentially arbitrary in the class of linear partial differential operators of order not exceeding one (apart some regularity of the coefficients).

Finally, we discuss some content of [4]. In this paper the authors prove maximal regularity results for very general classes of mixed parabolic problems. Even systems in the form (1.1) are considered. In this particular case, they find necessary and sufficient conditions in order that there exists a unique soluzione (u,ρ), with $\rho = \gamma u$, with $u \in W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$, $\rho \in W^{\frac{3}{2}-\frac{1}{2p},p}(0,T;L^p(\partial\Omega)) \cap L^p(0,T;W^{3-\frac{1}{p},p}(\partial\Omega))$.

In the present paper we discuss (1.1) from several points of view. We begin (Section 2) by considering the strongly elliptic problem depending on the complex parameter λ

$$\lambda g - Lg = h$$

in a compact smooth manifold Γ (without boundary) and the corresponding parabolic problem

$$
\begin{cases}
D_t v(t,x') = Lv(t,x') + h(t,x'), \\
v(0,x') = v_0(x')
\end{cases}
$$

We find necessary and sufficient conditions on h and v_0, in order that there exists a unique solution v in $W^{1,p}(0,T;L^p(\Gamma)) \cap L^p(0,T;W^{2,p}(\Gamma))$ $(p \in (1,\infty))$. These results are essentially well known, but we are not aware of an exposition of them fitting our needs.

In Section 3 we prove a theorem of maximal regularity for (1.1), giving necessary and sufficient conditions in order that there exists a unique solution u in $W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$, with γu in $W^{1,p}(0,T;L^p(\partial\Omega)) \cap L^p(0,T;W^{2,p}(\partial\Omega))(p \in (1,\infty) \setminus \{\frac{3}{2}\})$. So we prove a maximal regularity result in a class of functions which is larger that the one considered in [4]. As in [11], E is essentially an arbitrary linear partial differential operator of order not exceeding one. The argument of the proof is quite simple: we begin by studying the case $E = 0$ and employ the results of Section 2, together with classical results for mixed parabolic problems with Dirichlet boundary conditions (see [16]). The general case can be treated by a perturbation argument.

In Section 4 we show that, for any p in $(1,\infty)$, the unbounded operator G_p defined as follows:

$$\begin{cases} D(G_p) := \{(u,\gamma u) : u \in W^{2,p}(\Omega), \gamma u \in W^{2,p}(\partial\Omega)\}, \\ \\ G_p(u,\gamma u) := (\mathcal{A}u, L\gamma u + \gamma E u). \end{cases}$$

is the infinitesimal generator of an analytic semigroup in $L^p(\Omega) \times L^p(\partial\Omega)$.

Finally, in Section 5 we establish the following precise relation between problems (1.1) and (1.3). We introduce the operator M_p defined as follows:

$$\begin{cases} D(M_p) := \{(u,\gamma u) : u \in C^2(\overline{\Omega}), \gamma\mathcal{A}u - L\gamma u - \gamma E u = 0\}, \\ \\ \qquad M_p(u,\gamma u) = (\mathcal{A}u, \gamma\mathcal{A}u) = (Au, L\gamma u + \gamma E u). \end{cases}$$

and show that, if the coefficients and the boundary of Ω are suitably regular, M_p is closable in $X_p = L^p(\Omega) \times L^p(\partial\Omega)$ and its closure coincides with G_p. The closure of M_p is precisely the main operator studied in [7] and [8], as we explain more in detail in Section 5.

In conclusion of this introduction, we precise some notation. $\mathbb{N}$ will indicate the set pf positive integers; $BC(A)$ is the class of complex valued continuous and bounded functions with domain A; if $A \subseteq \mathbb{R}^n$, $BUC(A)$ will be the class of complex valued uniformly continuous and bounded functions with domain A.

Given the Banach spaces X_0, X_1, X, with $X_1 \hookrightarrow X \hookrightarrow X_0$, and $\alpha \in (0,1)$, we shall write $X \in J^\alpha(X_0, X_1)$ to indicate that there exists M positive, such that, for any x in X_1,

$$\|x\|_X \le M\|x\|_{X_0}^{1-\alpha}\|x\|_{X_1}^{\alpha}.$$

The symbol γ will be employed to indicate the trace operator.

2 Elliptic problems depending on a parameter and parabolic problems in a differentiable manifold.

We introduce the following assumptions:

(A1) Γ *is a compact, smooth differentiable manifold of class* C^2 *and dimension* m *(*$m \in \mathbb{N}$*).*

(A2) L *is a second order, partial differential operator in* Γ*. More precisely: for every local chart* (U,Φ)*, with* U *open in* Γ *and* Φ C^2- *diffeomorphism between* U *and* $\Phi(U)$*, with* $\Phi(U)$ *open in* $\mathbb{R}^m$*, for any* $v \in C^2(\Gamma)$*, if* $x' \in U$*,*

$$Lv(x') = \sum_{|\alpha|\le 2} l_{\alpha,\Phi}(x')D_y^\alpha(v \circ \Phi^{-1})(\Phi(x')); \tag{2.6}$$

we suppose, moreover, that, if $|\alpha| \le 2$*,* $l_{\alpha,\Phi} \in L^\infty_{loc}(U)$*, if* $|\alpha| = 2$*,* $l_{\alpha,\Phi} \in C(U)$ *and is real valued, for any* $x' \in U$ *there exists* $\nu(x') > 0$ *such that,* $\forall \eta \in \mathbb{R}^m$*,*

$$\sum_{|\alpha|=2} l_{\alpha,\Phi}(x')\eta^\alpha \ge \nu(x')|\eta|^2.$$

We consider the elliptic system depending on the parameter $\lambda \in \mathbb{C}$

$$\lambda g(x') - Lg(x') = h(x'), \quad x' \in \Gamma. \tag{2.7}$$

We prove the following

Theorem 2.1. *Suppose that (A1) and (A2) hold. Let $p \in (1, \infty)$. Then:*

(I) there exists ω in $\mathbb{R}$ such that, if $\lambda \in \mathbb{C}$, $Re(\lambda) \geq \omega$ and $h \in L^p(\Gamma)$, (2.7) has a unique solution g in $W^{2,p}(\Gamma)$; moreover, there exists $C_0 > 0$ such that

$$|\lambda| \|g\|_{L^p(\Gamma)} + \|g\|_{W^{2,p}(\Gamma)} \leq C_0 \|h\|_{L^p(\Gamma)}.$$

(II) As a consequence, the operator $L_p : W^{2,p}(\Gamma) \to L^p(\Gamma)$, $L_p u = Lu$ is the infinitesimal generator of an analytic semigroup in $L^p(\Gamma)$.

Proof. We follow the argument in [11], proof of Theorem 2.1.

We take an arbitrary $x^0 \in \Gamma$ and consider a local chart (U, Φ) around x^0, with U open subset of Γ and Φ diffeomorphism between U and $\Phi(U)$, open subset in $\mathbb{R}^m$. We introduce in $\Phi(U)$ the strongly elliptic operator $L^\sharp$,

$$L^\sharp v(y) := L(v \circ \Phi)(\Phi^{-1}(y)), \quad y \in \Phi(U). \tag{2.8}$$

By shrinking U (if necessary), we may assume that the coefficients of $L^\sharp$ are in $BC(\overline{\Phi(U)})$ and are extensible to elements l_β in $BUC(\mathbb{R}^m)$, in such a way that the operator which we continue to call $L^\sharp = \sum_{|\alpha| \leq 2} l_\beta(y) D_y^\beta$ is uniformly strongly elliptic in $\mathbb{R}^m$. Now we consider the problem

$$\lambda v(y) - L^\sharp v(y) = k(y), \quad y \in \mathbb{R}^m, \tag{2.9}$$

with $k \in L^p(\mathbb{R}^m)$. Then, (see [17, Chapter 3.1.2]), there exists $\omega(x^0) \in \mathbb{R}$, such that, if $\lambda \in \mathbb{C}$ and $Re(\lambda) \geq \omega(x^0)$, then (2.9) has a unique solution v in $W^{2,p}(\mathbb{R}^m)$; moreover, there exists $C(x^0) > 0$ such that

$$\sum_{j=0}^{2} |\lambda|^{1-j/2} \|v\|_{W^{j,p}(\mathbb{R}^m)} \leq C(x^0) \|k\|_{L^p(\mathbb{R}^m)}.$$

Now we fix U_1 open subset of U, with $\overline{U_1}$ contained in U, $x^0 \in U_1$ and $\phi \in C^2(\Gamma)$, with compact support in U, $\phi(x) = 1$ for any $x \in U_1$. Given $h \in L^p(\Gamma)$, we indicate with k the trivial extension of $(\phi h) \circ \Phi^{-1}$ to $\mathbb{R}^m$. If λ is such that (2.9) is uniquely solvable for every k in $L^p(\mathbb{R}^m)$, we set

$$[S(x^0, \lambda)h](x) := \phi(x) v(\Phi(x)), \quad x \in \Gamma, \tag{2.10}$$

with v solving (2.9). We observe that

(α_1) $S(x^0, \lambda)h \in W^{2,p}(\Gamma)$;

(α_2)

$$\sum_{j=0}^{2} |\lambda|^{1-j/2} \|S(x^0, \lambda)h\|_{W^{j,p}(\Gamma)} \leq C_1(x^0) \|h\|_{L^p(\Gamma)};$$

(α_3) $(\lambda - L)S(x^0, \lambda)h = h$ in U_1;

(α_4): if (2.7) is satisfied, for $h \in L^p(\Gamma)$, by some $g \in W^{2,p}(\Gamma)$ and g vanishes outside U_1, then $g = S(x^0, \lambda)h$;

in fact, the trivial extension of $g \circ \Phi^{-1}$ solves (2.9), with k trivial extension of $h \circ \Phi^{-1}$.

Now we fix, for every $x \in \Gamma$, neighbourhoods $U(x)$, $U_1(x)$ of x as before. As Γ is compact, there exist $x_1, \dots, x_N$ in Γ such that $\Gamma = \cup_{j=1}^N U_1(x_j)$.

Let $\lambda \in \mathbb{C}$. We show that, if $g \in W^{2,p}(\Gamma)$, it solves (2.7) with $h \equiv 0$ and $Re(\lambda)$ sufficiently large, then $g \equiv 0$. In fact, let $(\phi_j)_{j=1}^N$ be a C^2- partition of unity in Γ, with $supp(\phi_j) \subseteq U_1(x_j)$, for each $j \in \{1, \dots, N\}$. Observe that

$$(\lambda - L)(\phi_j g) = [\phi_j; L]g,$$

where we have indicated with $[\phi_j; L]$ the commutator $\phi_j L - L(\phi_j \cdot)$, which is a differential operator of order one. As $(\phi_j g)(x) = 0$ outside $U_1(x_j)$, we deduce from (α_4), if $Re(\lambda)$ is sufficiently large,

$$\phi_j g = S(x_j, \lambda)([\phi_j; L]g).$$

So, from (α_2),

$$\|g\|_{W^{1,p}(\Gamma)} \leq \sum_{j=1}^{N} \|\phi_j g\|_{W^{1,p}(\Gamma)} \leq C_1 |\lambda|^{-1/2} \sum_{j=1}^{N} \|[\phi_j; L]g\|_{L^p(\Gamma)} \leq C_2 |\lambda|^{-1/2} \|g\|_{W^{1,p}(\Gamma)},$$

implying $g \equiv 0$ if $Re(\lambda)$ is sufficiently large.

Next, we show that, if $|\lambda|$ is large enough,then (2.7) is solvable for every $h \in L^p(\Gamma)$. This time we fix, for each $j \in \{1, \dots, N\}$, $\psi_j \in C^2(\Gamma)$, vanishing outside $U_1(x_j)$ and such that $\sum_{j=1}^{N} \psi_j(x)^2 = 1$ for any x in Γ. We look for g in the form

$$g = \sum_{j=1}^{N} \psi_j S(x_j, \lambda)(\psi_j \tilde{h}),$$

for some $\tilde{h} \in L^p(\Gamma)$. Again observing that $\psi_j S(x_j, \lambda)(\psi_j \tilde{h})$ vanishes outside $U_1(x_j)$ and that

$$(\lambda - L)[\psi_j S(x_j, \lambda)(\psi_j \tilde{h})] = \psi_j^2 \tilde{h} + [\psi_j; L][S(x_j, \lambda)(\psi_j \tilde{h})],$$

we deduce

$$(\lambda - L)g = \tilde{h} + \sum_{j=1}^{N} [\psi_j; L][S(x_j, \lambda)(\psi_j \tilde{h})].$$

So, we have to choose $\tilde{h}$ in such a way that

$$\tilde{h} + \sum_{j=1}^{N} [\psi_j; L][S(x_j, \lambda)(\psi_j \tilde{h})] = h. \tag{2.11}$$

This is uniquely possible if $Re(\lambda)$ is sufficiently large, because

$$\| \sum_{j=1}^{N} [\psi_j; L][S(x_j, \lambda)(\psi_j \tilde{h})] \|_{L^p(\Gamma)} \leq C_0 \sum_{j=1}^{N} \|S(x_j, \lambda)(\psi_j \tilde{h})]\|_{W^{1,p}(\Gamma)} \leq C_1 |\lambda|^{-1/2} \|\tilde{h}\|_{L^p(\Gamma)}.$$

So, if $C_1 |\lambda|^{-1/2} \leq \frac{1}{2}$, we deduce from (2.11)

$$\|\tilde{h}\|_{L^p(\Gamma)} \leq 2 \|h\|_{L^p(\Gamma)},$$

which, together with (α_2), implies (I).

(II) follows from (I). Observe also that, as $W^{2,p}(\Gamma)$ is dense in $L^p(\Gamma)$, the domain of L_p is dense in $L^p(\Gamma)$.

□

Corollary 2.2. *Suppose that (A1)-(A2) are satisfied. Let* $1 < p < \infty$, $\epsilon \in \mathbb{R}^+$, $g_0 \in W^{2,p}(\Gamma)$, $T \in \mathbb{R}^+$, $f \in C^\epsilon([0, T]; L^p(\Gamma))$. *Then the problem*

$$\begin{cases} u'(t) - L_p u(t) = f(t), & t \in [0, T], \\ u(0) = g_0. \end{cases} \tag{2.12}$$

has a unique solution u *in* $C^1([0, T]; L^p(\Gamma)) \cap C([0, T]; W^{2,p}(\Gamma))$ *and*

$$u(t) = e^{tL_p} u_0 + \int_0^t e^{(t-s)L_p} f(s) ds, \tag{2.13}$$

with $(e^{tL_p})_{t \geq 0}$ *analytic semigroup generated by* L_p.

The following "maximal regularity" result holds also:

Proposition 2.3. *Let $p \in (1, \infty)$. Consider the problem (2.12). Then the following conditions are necessary and sufficient in order that there exists a unique solution u in $W^{1,p}(0,T; L^p(\Gamma)) \cap L^p(0,T; W^{2,p}(\Gamma))$:*

(a) $f \in L^p(0,T; L^p(\Gamma))$;

(b) $g_0 \in W^{2-2/p,p}(\Gamma)$

If (a)-(b) hold, this unique solution is given by (2.13).

Proof. (a) is obviously necessary. The necessity of (b) follows from the fact that

$$\{v(0) : v \in W^{1,p}(0,T; L^p(\Gamma)) \cap L^p(0,T; W^{2,p}(\Gamma))\} = (L^p(\Gamma), W^{2,p}(\Gamma))_{1-1/p,p} = W^{2-2/p,p}(\Gamma) \qquad (2.14)$$

(see [17], Chapter 2.2.1 and Theorem 3.2.3).

On the other hand, suppose that (a)-(b) hold. It is well known that the only possible solution of (2.12) is (2.13). So the solution with the desired properties is, if it exists, unique. It is known that, if $v(t) = e^{tL_p}u_0$, $v \in W^{1,p}(0,T; L^p(\Gamma)) \cap L^p(0,T; W^{2,p}(\Gamma))$ (see [17], Chapter 2.2.1). Assume that $u_0 = 0$. In this case, we deduce, for any $t \in [0,T]$, as $W^{1,p}(\Gamma) \in J^{1/2}(L^p(\Gamma); W^{2,p}(\Gamma))$, if u is given by (2.13),

$$\|u(t)\|_{W^{1,p}(\Gamma)} \le C_0 \int_0^t (t-s)^{-1/2} \|f(s)\|_{L^p(\Gamma)} ds$$

so that, by Young's inequality,

$$\|u\|_{L^p(0,T;W^{1,p}(\Gamma))} \le C_1 \|f\|_{L^p(0,T;L^p(\Gamma))}. \qquad (2.15)$$

Suppose now that $f \in C^\epsilon([0,T]; L^p(\Gamma))$. Then u really solves (2.12) (by Corollary 2.2). We fix a local chart (U, Φ) and take $\phi \in C^2(\Gamma))$, with support in U. Then, if

$$u_\phi(t,x) := \phi(x) u(t,x),$$

we get

$$\begin{cases} D_t(u_\phi)(t,x) - L_p(u_\phi)(t,x) = \phi(x) f(t,x) + ([\phi; L_p]u)(t,x), & (t,x) \in [0,T] \times \Gamma, \\ u_\phi(0,x) = 0, & x \in \Gamma. \end{cases}$$

Setting

$$v(t,y) := u_\phi(t, \Phi^{-1}(y)), \quad (t,y) \in [0,T] \times \Phi(U),$$

and identifying v with its trivial extension to $[0,T] \times \mathbb{R}^m$, we get

$$\begin{cases} D_t v(t,y) - L^\sharp v(t,y) = \phi(\Phi^{-1}(y)) f(t, \Phi^{-1}(y)) + ([\phi; L_p]u)(t, \Phi^{-1}(y)), & (t,y) \in [0,T] \times \mathbb{R}^m, \\ v(0,y) = 0, & y \in \mathbb{R}^m, \end{cases}$$

where we have employed again the operator $L^\sharp$ introduced in (2.8). From well known maximal regularity results in $\mathbb{R}^m$ (which can be deduced, for example, from [15], Theorem 6.8), we obtain

$$\|u_\phi\|_{W^{1,p}(0,T;L^p(\Gamma))} + \|u_\phi\|_{L^p(0,T;W^{2,p}(\Gamma))}$$

$$\le C_1(\|v\|_{W^{1,p}(0,T;L^p(\mathbb{R}^m))} + \|v\|_{L^p(0,T;W^{2,p}(\mathbb{R}^m))})$$

$$\le C_2(\|f\|_{L^p(0,T;L^p(\Gamma)} + \|u\|_{L^p(0,T;W^{1,p}(\Gamma))})$$

$$\le C_3 \|f\|_{L^p(0,T;L^p(\Gamma))},$$

by (2.15). From this estimate, it follows immediately that

$$\|u\|_{W^{1,p}(0,T;L^p(\Gamma))} + \|u\|_{L^p(0,T;W^{2,p}(\Gamma))} \le C \|f\|_{L^p(0,T;L^p(\Gamma))}.$$

This implies the conclusion, taking a sequence $(f_k)_{k\in\mathbb{N}}$ in (say) $C^1([0,T]; L^p(\Gamma))$ and converging to f in $L^p(0,T; L^p(\Gamma))$.

□

Example 2.4. We show an example of an operator fulfilling conditions (A1)-(A2). Let Γ be a smooth compact Riemannian manifold with dimension m and class C^2. For every x in Γ, we indicate with $T_x(\Gamma)$ the tangent space and with $T_x(\Gamma) + iT_x(\Gamma)$ its complexification. The real scalar product $(\cdot, \cdot)_x$ in $T_x(\Gamma)$ can be extended in a natural way to a complex scalar product, which we continue to indicate with $(\cdot, \cdot)_x$ (for these elementary facts, see [19], Chapter 6.5). We shall indicate with $T(\Gamma) + iT(\Gamma)$ the disjoint union of the spaces $T_x(\Gamma) + iT_x(\Gamma)$ ($x \in \Gamma$), which is naturally equipped with a structure of $m-$dimensional complex vector bundle on Γ.

If $f : \Gamma \to \mathbb{C}$ is of class C^1, we indicate with $\nabla f(x)$ the gradient of f in x, which belongs to $T_x(\Gamma) + iT_x(\Gamma)$. ∇ is a first order differential operator, mapping smooth complex valued functions defined in Γ into sections of $T(\Gamma) + iT(\Gamma)$. We recall that $\nabla f(x)$ is the element of $T_x(\Gamma) + iT_x(\Gamma)$ such that, for every $v \in T_x(\Gamma)$,

$$(\nabla f(x), v)_x = v(f)$$

(see, for example, [2], Chapter V). Suppose that we fix a local chart (U, Φ) in Γ. We indicate with $\frac{\partial}{\partial x_j}$ ($1 \le j \le m$) the field in U such that

$$\frac{\partial f}{\partial x_j}(x) = \frac{\partial(f \circ \Phi^{-1})}{\partial y_j}(\Phi(x)), \quad x \in U,$$

where we have indicated by $y_1, \dots, y_m$ the standard coordinates in $\mathbb{R}^m$. Moreover, we set

$$g(x) = ((\frac{\partial}{\partial x_i}(x), \frac{\partial}{\partial x_j}(x))_x)_{1 \le i,j \le m}.$$

It is easily seen that the matrix $g(x)$ is symmetric and positive definite. We introduce also its inverse

$$G(x) := g(x)^{-1},$$

again symmetric and positive definite. Then it is not difficult to check that, in local coordinates,

$$\nabla f(x) = \sum_{i=1}^{m} \sum_{j=1}^{m} G_{ij}(x) \frac{\partial f}{\partial x_j}(x) \frac{\partial}{\partial x_i}(x). \tag{2.16}$$

Now we assume that, for any $x \in \Gamma$, $B(x)$ is a linear operator from $T_x(\Gamma)$ into itself, Hermitian and positive definite with respect to $(\cdot, \cdot)_x$, that is, $\forall \xi, \eta \in T_x(\Gamma)$,

$$(B(x)\xi, \eta)_x = (\xi, B(x)\eta)_x$$

and, if $v \in T_x(\Gamma) \setminus \{0\}$,

$$(B(x)v, v)_x > 0.$$

We suppose also that $B(x)$ depends smoothly on x. This is equivalent to prescribe that, for every local chart (U, Φ),h the following conditions are satisfied:

(a) for each $i \in \{1, \dots, m\}$, $B(x)(\frac{\partial}{\partial x_i}(x)) = \sum_{j=1}^{m} B_{ij}(x) \frac{\partial}{\partial x_j}(x)$, with $B_{ij} \in C^1(U)$;

(b) if we set, for any x in U, $\mathcal{B}(x) := (B_{ij}(x))_{1 \le i,j \le m}$, the product $\mathcal{B}(x)g(x)$ is symmetric and positive definite.

Observe that (a)-(b) imply that ,for any x in U, even $G(x)\mathcal{B}(x)$ is symmetric and positive definite. In fact,

$$(G(x)\mathcal{B}(x))^T = \mathcal{B}(x)^T G(x) = G(x)(g(x)\mathcal{B}(x)^T)G(x)$$

$$= G(x)(\mathcal{B}(x)g(x))^T G(x) = G(x)\mathcal{B}(x)g(x)G(x) = G(x)\mathcal{B}(x).$$

Moreover, if $\xi \in \mathbb{R}^m \setminus \{0\}$,

$$(G(x)\mathcal{B}(x)\xi) \cdot \xi = (\mathcal{B}(x)g(x)G(x)\xi) \cdot G(x)\xi > 0.$$

We indicate by σ the measure induced by the Riemannian metric in Γ and by $-div$ the adjoint operator of ∇. So, if $u : \Gamma \to \mathbb{C}$ and v is a smooth vector field,

$$\int_\Gamma (\nabla u(x), v(x))_x d\sigma = - \int_\Gamma u(x)\overline{div(v)(x)} d\sigma.$$

It is not difficult to check that, if (U, ϕ) is the usual chart, and if $\rho : \Phi(U) \to \mathbb{R}^+$ is such that, for every measurable subset A of U

$$\sigma(A) = \int_{\Phi(A)} \rho(y)dy,$$

for every smooth vector field $X = \sum_{k=1}^m X_k \frac{\partial}{\partial x_k}$ in U, one has

$$div(X)(x) = (\nabla \cdot X)(x) = \sum_{k=1}^m \frac{\partial}{\partial x_k}((\rho \circ \Phi)X_k)(x). \tag{2.17}$$

We introduce now the operator

$$Lu(x) := div(B(x)\nabla_x u) \tag{2.18}$$

Observe that, if $B(x) = I_{T_x(\Gamma)}$ for any x in Γ, $\mathcal{B}$ is nothing but the Laplace-Beltrami operator. We show that it satisfies the conditions (A1)-(A2). In fact, if $f : U \to \mathbb{C}$ is sufficiently smooth and $x \in U$, we have, on account of (2.16),

$$B(x)\nabla f(x) = \sum_{i=1}^m \sum_{j=1}^m G_{ij}(x)\tfrac{\partial f}{\partial x_j}(x)B(x)(\tfrac{\partial}{\partial x_i}(x))$$

$$= \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m G_{ij}(x)\tfrac{\partial f}{\partial x_j}(x)B_{ik}(x)\tfrac{\partial}{\partial x_k}(x) = \sum_{j=1}^m \sum_{k=1}^m (G(x)\mathcal{B}(x))_{jk}\tfrac{\partial f}{\partial x_j}(x)\tfrac{\partial}{\partial x_k}(x),$$

so that, by (2.18),

$$Lf(x) = \sum_{k=1}^m \sum_{j=1}^m \frac{\partial}{\partial x_k}[(\rho \circ \Phi)(x)(G(x)\mathcal{B}(x))_{jk}\frac{\partial f}{\partial x_j}(x)].$$

or

$$Lf(x) = \sum_{k=1}^m \sum_{j=1}^m \frac{\partial}{\partial y_k}[\rho(y)(G\mathcal{B})_{jk}(\Phi^{-1}(y))\frac{\partial(f \circ \Phi^{-1})}{\partial y_j}(y)](\Phi(x)).$$

The principal part of the operator is

$$\sum_{k=1}^m \sum_{j=1}^m \rho(\Phi(x))(G\mathcal{B})_{jk}(x)\frac{\partial^2(f \circ \Phi^{-1})}{\partial y_k \partial y_j}(\Phi(x)).$$

and the matrix $\rho(\Phi(x))(G\mathcal{B})(x)$ is symmetric and positive definite.

So L, defined in (2.18), satisfies the conditions (A1)-(A2).

3 Maximal regularity

Now we consider the following classical Cauchy-Dirichlet parabolic problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x) + f(t,x), & (t,x) \in (0,T) \times \Omega, \\ \gamma u(t,\cdot) = g(t,\cdot), & t \in (0,T), \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

with the following conditions:

(B1) Ω *is an open, bounded subset of* $\mathbb{R}^n$*, lying on one side of its boundary* Γ*, which is a submanifold of* $\mathbb{R}^n$ *of class* C^2.

(B2) $\mathcal{A} = \sum_{i,j=1}^n a_{ij}(x)D_{x_i x_j} + \sum_{j=1}^n b_j(x)D_{x_j} + c(x)$*, with* $a_{ij}, b_j, c \in C(\overline{\Omega})$ $(1 \le i,j \le n)$*; the functions* a_{ij} *are real valued and there exists* $\nu \in \mathbb{R}^+$ *such that* $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2$*, for any* $x \in \overline{\Omega}$*,* $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

The following classical result holds (see [16], Theorem 9.1):

Theorem 3.1. *Suppose that (B1)-(B2) hold. Let $p \in (1,\infty)\setminus\{\frac{3}{2}\}$. Then the following conditions are necessary and sufficient, in order that (3.1) has a unique solution u in $W^{1,p}(0,T;L^p(\Omega))\cap L^p(0,T;W^{2,p}(\Omega))$:*
(I) $f \in L^p(0,T;L^p(\Omega))$;
(II) $g \in W^{1-1/(2p),p}(0,T;L^p(\Gamma)) \cap L^p(0,T;W^{2-1/p,p}(\Gamma))$;
(III) $u_0 \in W^{2-2/p,p}(\Omega)$;
(IV) in case $p > \frac{3}{2}$, $\gamma u_0 = g(0)$.

Remark 3.2. Observe that, as $u \in L^p(0,T;W^{2,p}(\Omega))$, the second equation in (3.1) is assumed to be satisfied only almost everywhere in $(0,T)$.

However, the identity (2.14) and the analogous identity obtained by replacing Γ with Ω imply that

$$W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;L^p(\Omega)) \subseteq C([0,T];W^{2-2/p,p}(\Omega)),$$

$$W^{1,p}(0,T;L^p(\Gamma)) \cap L^p(0,T;L^p(\Gamma)) \subseteq C([0,T];W^{2-2/p,p}(\Gamma)).$$

If $p > \frac{3}{2}$, $2 - \frac{2}{p} > \frac{1}{p}$, so that $\gamma u \in C([0,T];L^p(\Gamma))$ then the second equation in (3.1) can be assumed to be satisfied for every $t \in [0,T]$. This explains the necessity of (IV) in this case. Observe also that, as $1 - \frac{1}{2p} > \frac{1}{p}$, (II) implies that $g \in C([0,T];L^p(\Gamma))$.

Now we consider the problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x) + f(t,x), & (t,x) \in (0,T)\times\Omega, \\ D_t \gamma u(t,\cdot) = L\gamma u(t,\cdot) + h(t,\cdot), & t \in (0,T), \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{3.2}$$

with L as in (2.6).

We consider first the case $p > \frac{3}{2}$:

Proposition 3.3. *Let $p \in (\frac{3}{2},\infty)$. Consider problem (3.2). Suppose that (B1)-(B2) hold and L is as in (2.6). Then the following conditions are necessary and sufficient in order that (3.2) has a unique solution u in $W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ with $\gamma u \in W^{1,p}(0,T;L^p(\Gamma)) \cap L^p(0,T;W^{2,p}(\Gamma))$:*
(I) $f \in L^p(0,T;L^p(\Omega))$;
(II) $h \in L^p(0,T;L^p(\Gamma))$;
(III) $u_0 \in W^{2-2/p,p}(\Omega)$, $\gamma u_0 \in W^{2-2/p,p}(\Gamma)$.

Proof. (I)-(II) are obviously necessary. The belonging of u_0 to $W^{2-2/p,p}(\Omega)$ follows from Theorem 3.1. From what we have observed in Remark **??**, if we set $v := \gamma u$, the identity $v(t) = \gamma[u(t)]$ can be intended pointwise. We deduce that $v(0)$ must coincide with γu_0. So from Proposition 2.3 we deduce the necessity of (III).

On the other hand, suppose that (I)-(III) hold. We consider the system

$$\begin{cases} D_t v(t,\cdot) = Lv(t,\cdot) + h(t,\cdot), & t \in (0,T) \\ v(0,\cdot) = \gamma u_0. \end{cases} \tag{3.3}$$

Then, by Proposition 2.3, (3.3) has a unique solution v in $W^{1,p}(0,T;L^p(\Gamma)) \cap L^p(0,T;W^{2,p}(\Gamma))$. Now we consider the solution u to

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x) + f(t,x), & (t,x) \in (0,T)\times\Omega, \\ \gamma u(t,\cdot) = v(t,\cdot), & t \in (0,T), \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

By Theorem 3.1, such u is the unique solution to (3.2).

□

Now we consider the case $p < \frac{3}{2}$. In this case, (3.2) is underdetermined. It is more convenient to consider the problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}u(t,x) + f(t,x), & (t,x) \in (0,T) \times \Omega, \\ D_t \gamma u(t,\cdot) = L\gamma u(t,\cdot) + h(t,\cdot), & t \in (0,T) \\ u(0,x) = u_0(x), & x \in \Omega, \\ (\gamma u)(0) = v_0. \end{cases} \tag{3.4}$$

The following result holds:

Proposition 3.4. *Let $p \in (1, \frac{3}{2})$. Consider problem (3.4). Suppose that (B1)-(B2) hold and L is as in (2.6). Then the following conditions are necessary and sufficient in order that (3.4) has a unique solution u in $W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ with $\gamma u \in W^{1,p}(0,T;L^p(\Gamma)) \cap L^p(0,T;W^{2,p}(\Gamma))$:*
(I) $f \in L^p(0,T;L^p(\Omega))$;
(II) $h \in L^p(0,T;L^p(\Gamma))$;
(III) $u_0 \in W^{2-2/p,p}(\Omega)$, $v_0 \in W^{2-2/p,p}(\Gamma)$.

Proof. The necessity of (I)-(III) follows immediately from Proposition 2.3 and Theorem 3.1. The proof of the sufficiency is the same as in Proposition 3.3. □

Remark 3.5. As already observed in Remark **??**, if $v(t) = \gamma u(t)$, the identity should be intended to be satisfied only for almost every t. In our case v should be extensible to an element of $C([0,T];L^p(\Gamma))$, but $v(0)$ should not necessarily coincide with γu_0; by the way, as $u_0 \in W^{2-2/p,p}(\Omega)$ and $2 - \frac{2}{p} < \frac{1}{p}$ if $p < \frac{3}{2}$, u_0 does not necessarily admit a trace on Γ.

It is convenient to reformulate together the results of Propositions 3.3 and 3.4:

Proposition 3.6. *Let $p \in (1,\infty) \setminus \{\frac{3}{2}\}$. Consider problem (3.4). Suppose that (B1)-(B2) hold and L is as in (2.6). Then the following conditions are necessary and sufficient in order that (3.4) has a unique solution u in $W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ with $\gamma u \in W^{1,p}(0,T;\Gamma) \cap L^p(0,T;W^{2,p}(\Gamma))$:*
(I) $f \in L^p(0,T;L^p(\Omega))$;
(II) $h \in L^p(0,T;L^p(\Gamma))$;
(III) $u_0 \in W^{2-2/p,p}(\Omega)$, $v_0 \in W^{2-2/p,p}(\Gamma)$ and, in case $p > \frac{3}{2}$, $\gamma u_0 = v_0$.

We proceed with some useful estimates.

Lemma 3.7. *Consider problem (3.4). Suppose that (B1)-(B2) hold and L is as in (2.6). Let $p \in (1,\infty) \setminus \{\frac{3}{2}\}$, $T_0 \in \mathbb{R}^+$, $0 < T \le T_0$. Suppose that $f \in L^p(0,T;L^p(\Omega))$, $h \in L^p(0,T;L^p(\Gamma))$, $u_0 \in W^{2-2/p,p}(\Omega)$, $v_0 \in W^{2-2/p,p}(\Gamma)$ and, in case $p > \frac{3}{2}$, $\gamma u_0 = v_0$. Then there exists $C(T_0)$ in $\mathbb{R}^+$ such that*

$$\begin{aligned} &\|D_t u\|_{L^p(0,T;L^p(\Omega))} + \|u\|_{L^p(0,T;W^{2,p}(\Omega))} + \|D_t \gamma u\|_{L^p(0,T;L^p(\Gamma))} + \|\gamma u\|_{L^p(0,T;W^{2,p}(\Gamma))} \\ &\le C(T_0)(\|f\|_{L^p(0,T;L^p(\Omega))} + \|h\|_{L^p(0,T;L^p(\Gamma))} + \|u_0\|_{W^{2-2/p,p}(\Omega)} + \|v_0\|_{W^{2-2/p,p}(\Gamma)}). \end{aligned}$$

Proof. We set, for $t \in (0,T_0)$,

$$F(t,\cdot) = \begin{cases} f(t,\cdot) & \text{if} \quad t \in (0,T), \\ 0 & \text{if} \quad t \in [T,T_0), \end{cases}$$

$$H(t,\cdot) = \begin{cases} h(t,\cdot) & \text{if} \quad t \in (0,T), \\ 0 & \text{if} \quad t \in [T,T_0), \end{cases}$$

and consider the problem

$$
\begin{cases}
D_t U(t,x) = \mathcal{A}U(t,x) + F(t,x), & (t,x) \in (0,T_0) \times \Omega, \\
D_t \gamma U(t,\cdot) = L\gamma U(t,\cdot) + H(t,\cdot), & t \in (0,T_0) \\
U(0,x) = u_0(x), & x \in \Omega, \\
(\gamma U)(0) = v_0.
\end{cases}
\tag{3.5}
$$

By Proposition 3.6, (3.5) has a unique solution U in $W^{1,p}(0,T_0;L^p(\Omega)) \cap L^p(0,T_0;W^{2,p}(\Omega))$ with $\gamma U \in W^{1,p}(0,T_0;L^p(\Gamma)) \cap L^p(0,T_0;W^{2,p}(\Gamma))$, which is clearly an extension of u. We deduce

$$
\begin{aligned}
&\|D_t u\|_{L^p(0,T;L^p(\Omega))} + \|u\|_{L^p(0,T;W^{2,p}(\Omega))} + \|D_t\gamma u\|_{L^p(0,T;L^p(\Gamma))} + \|\gamma u\|_{L^p(0,T;W^{2,p}(\Gamma))} \\
&\leq \|D_t U\|_{L^p(0,T_0;L^p(\Omega))} + \|U\|_{L^p(0,T_0;W^{2,p}(\Omega))} + \|D_t\gamma U\|_{L^p(0,T_0;L^p(\Gamma))} + \|\gamma U\|_{L^p(0,T_0;W^{2,p}(\Gamma))} \\
&\leq C(T_0)(\|F\|_{L^p(0,T;L^p(\Omega))} + \|H\|_{L^p(0,T;L^p(\Gamma))} + \|u_0\|_{W^{2-2/p,p}(\Omega)} + \|v_0\|_{W^{2-2/p,p}(\Gamma)}) \\
&= C(T_0)(\|f\|_{L^p(0,T;L^p(\Omega))} + \|h\|_{L^p(0,T;L^p(\Gamma))} + \|u_0\|_{W^{2-2/p,p}(\Omega)} + \|v_0\|_{W^{2-2/p,p}(\Gamma)}).
\end{aligned}
$$

□

Lemma 3.8. *Suppose that the assumptions of Lemma 3.7 are fulfilled. Suppose that $u_0 = 0$ and let $\theta \in [0,2]$. Then there exists $C(T_0,\theta) > 0$ such that*

$$
\begin{aligned}
&\|u\|_{L^p(0,T;W^{\theta,p}(\Omega))} \\
&\leq C(T_0)T^{1-\theta/2}(\|f\|_{L^p(0,T;L^p(\Omega))} + \|h\|_{L^p(0,T;L^p(\Gamma))} + \|v_0\|_{W^{2-2/p,p}(\Gamma)}).
\end{aligned}
$$

Proof. Consider first the case $\theta = 0$. Then, as $u_0 = 0$, $u = 1 * D_t u$. It follows from Young's inequality and Lemma 3.7 that

$$
\begin{aligned}
&\|u\|_{L^p(0,T;L^p(\Omega))} \leq T\|D_t u\|_{L^p(0,T;L^p(\Omega))} \\
&\leq C(T_0)T(\|f\|_{L^p(0,T;L^p(\Omega))} + \|h\|_{L^p(0,T;L^p(\Gamma))} + \|v_0\|_{W^{2-2/p,p}(\Gamma)}).
\end{aligned}
$$

In general, there exists $C(\theta) >$ such that, for any $z \in W^{2,p}(\Omega)$,

$$
\|z\|_{W^{\theta,p}(\Omega))} \leq C(\theta)\|z\|_{L^p(\Omega))}^{1-\theta/2}\|z\|_{W^{2,p}(\Omega))}^{\theta/2}.
$$

As $W^{\theta,p}(\Omega)$ coincides with the real interpolation space $(L^p(\Omega), W^{2,p}(\Omega))_{\theta/2,p}$ in case $\theta \neq 1$, with the complex interpolation space $(L^p(\Omega), W^{2,p}(\Omega))_{[\frac{1}{2}]}$ in case $\theta = \frac{1}{2}$ (see [21]), we deduce that

$$
\begin{aligned}
&\|u\|_{L^p(0,T;W^{\theta,p}(\Omega))} \leq C(\theta)(\textstyle\int_0^T \|u(t)\|_{L^p(\Omega)}^{p(1-\theta/2)}\|u(t)\|_{W^{2,p}(\Omega)}^{p\theta/2}dt)^{1/p} \\
&\leq C(\theta)\|u\|_{L^p(0,T;L^p(\Omega))}^{1-\theta/2}\|u\|_{L^p(0,T;W^{2,p}(\Omega))}^{\theta/2}.
\end{aligned}
$$

So the conclusion follows from the case $\theta = 0$ and Lemma 3.7. □

Now we introduce an operator E of order not exceeding one, with coefficients in $C^1(\overline{\Omega})$:

$$
Eu(x) = \sum_{j=1}^{n} e_j(x)D_{x_j}u(x) + e_0(x)u(x) \tag{3.6}
$$

and the following system:

$$
\begin{cases}
D_t u(t,x) = \mathcal{A}u(t,x) + f(t,x), & (t,x) \in (0,T) \times \Omega, \\
D_t \gamma u(t,\cdot) = L\gamma u(t,\cdot) + \gamma E u(t,\cdot) + h(t,\cdot), & t \in (0,T) \\
u(0,x) = u_0(x), & x \in \Omega, \\
(\gamma u)(0) = v_0.
\end{cases}
\tag{3.7}
$$

We show the following

Theorem 3.9. *Let $p \in (1,\infty) \setminus \{\frac{3}{2}\}$. Consider problem (3.7). Suppose that (B1)-(B2) hold, L is as in (2.6) and E is as in (3.6) with coefficients in $C^1(\overline{\Omega})$. Then the following conditions are necessary and sufficient in order that (3.7) has a unique solution u in $W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ with $\gamma u \in W^{1,p}(0,T;L^p(\Gamma)) \cap L^p(0,T;W^{2,p}(\Gamma))$:*
(I) $f \in L^p(0,T;L^p(\Omega))$;
(II) $h \in L^p(0,T;L^p(\Gamma))$;
(III) $u_0 \in W^{2-2/p,p}(\Omega)$, $v_0 \in W^{2-2/p,p}(\Gamma)$ and, in case $p > \frac{3}{2}$, $\gamma u_0 = v_0$.

Proof. The fact that (I)-(III) are necessary can be shown with the same arguments as in the proofs of Propositions 3.3 and 3.4.

We show that they are also sufficient. We fix $\theta \in (1 + \frac{1}{p}, 2)$. We observe that, by classical trace theorems, $u \to \gamma E u$ belongs to $\mathcal{L}(W^{\theta,p}(\Omega), L^p(\Gamma))$. We take $\tau \in (0,T]$ and consider the system

$$
\begin{cases}
D_t u(t,x) = \mathcal{A}u(t,x) + f(t,x), & (t,x) \in (0,\tau) \times \Omega, \\
D_t \gamma u(t,\cdot) = L\gamma u(t,\cdot) + \gamma E U(t,\cdot) + h(t,\cdot), & t \in (0,\tau), \\
u(0,x) = u_0(x), & x \in \Omega, \\
(\gamma u)(0) = v_0.
\end{cases}
\tag{3.8}
$$

with $U \in L^p(0,\tau;W^{\theta,p}(\Omega))$. By Proposition 3.6, (3.8) has a unique solution $u = S(U)$ in $W^{1,p}(0,\tau;L^p(\Omega)) \cap L^p(0,\tau;W^{2,p}(\Omega))$ with $\gamma u \in W^{1,p}(0,\tau;\Gamma) \cap L^p(0,\tau;W^{2,p}(\Gamma))$. If $U_j \in L^p(0,\tau;W^{\theta,p}(\Omega))$ ($j \in \{1,2\}$), we set $u_j := S(U_j)$. Then $u_1 - u_2$ solves the system

$$
\begin{cases}
D_t(u_1 - u_2)(t,x) = \mathcal{A}(u_1 - u_2)(t,x), & (t,x) \in (0,\tau) \times \Omega, \\
D_t \gamma(u_1 - u_2)(t,\cdot) = L\gamma(u_1 - u_2)(t,\cdot) + \gamma E(U_1 - U_2)(t,\cdot), & t \in (0,T), \\
(u_1 - u_2)(0,x) = 0, & x \in \Omega, \\
\gamma(u_1 - u_2)(0) = 0.
\end{cases}
\tag{3.9}
$$

We deduce from Lemma 3.8 the estimate

$$
\|u_1 - u_2\|_{L^p(0,T;W^{\theta,p}(\Omega))}
$$
$$
\leq C(T)\tau^{1-\theta/2}\|\gamma E(U_1 - U_2)\|_{L^p(0,\tau;L^p(\Gamma))} \leq C_1(T)\tau^{1-\theta/2}\|U_1 - U_2\|_{L^p(0,\tau;W^{\theta,p}(\Omega))}.
$$

So, if we choose τ so small that $C_1(T)\tau^{1-\theta/2} < 1$, S has a unique fixed point in $L^p(0,\tau;W^{\theta,p}(\Omega))$. We deduce that (3.8) has a unique solution u in $W^{1,p}(0,\tau;L^p(\Omega)) \cap L^p(0,\tau;W^{2,p}(\Omega))$ with γu in $W^{1,p}(0,\tau;L^p(\Gamma)) \cap L^p(0,\tau;W^{2,p}(\Gamma))$. Observe that τ can be chosen independently of f, h, u_0, v_0.

Now we show that, in case $f \equiv 0$, $h \equiv 0$, $u_0 = 0$, $v_0 = 0$, the unique solution u in $W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ with γu in $W^{1,p}(0,T;L^p(\Gamma)) \cap L^p(0,T;W^{2,p}(\Gamma))$ is $u \equiv 0$. This is true (by the uniqueness of the fixed point for S), if we replace T by τ sufficiently small. Assume that there exists a nontrivial solution u in $(0,T)$. We set

$$
\sigma := \inf\{t \in [0,T] : u(t,\cdot) \neq 0\}.
$$

As $u \in C([0,T]; W^{2-2/p,p}(\Omega))$ and $u(0,\cdot) = 0$, $\sigma \in [0,T)$ and $u(\sigma,\cdot) = 0$. Moreover, $\gamma u(t,\cdot) = 0$ for almost every t in $[0,\sigma)$. As $\gamma u \in C([0,T]; W^{2-2/p,p}(\Gamma))$ we deduce that $(\gamma u)(\sigma,\cdot) = 0$. So, if $\tau > 0$, and $\sigma + \tau \leq T$, $w(t) := u(\sigma + t)$ solves the system

$$\begin{cases} D_t w(t,x) = \mathcal{A} w(t,x), & (t,x) \in (0,\tau) \times \Omega, \\ D_t \gamma w(t,\cdot) = L\gamma w(t,\cdot) + \gamma E w(t,\cdot), & t \in (0,\tau), \\ w(0,x) = 0, & x \in \Omega, \\ (\gamma w)(0) = 0. \end{cases}$$

If τ is sufficiently small, we deduce $w(t,\cdot) = 0$ for any $t \in [0,\tau]$, so that $u(t,\cdot) = 0$ for any $t \in [0,\sigma+\tau]$, in contradiction with the definition of σ.

Finally, we show the existence of a global solution. We have already proved the existence of a solution z in some interval $[0,\tau]$, independent of the data. Suppose that $\tau < T$. We extend the solution to $[0,(2\tau) \wedge T]$. We have that $z(\tau,\cdot) \in W^{2-2/p,p}(\Omega)$, $(\gamma z)(\tau) \in W^{2-2/p,p}(\Gamma)$. In case $p > \frac{3}{2}$ we have also

$$\gamma[z(\tau)] = (\gamma z)(\tau).$$

So we consider the system

$$\begin{cases} D_t w(t,x) = \mathcal{A} w(t,x) + f(\tau + t, x), & (t,x) \in (0,\tau \wedge (T-\tau)) \times \Omega, \\ D_t \gamma w(t,\cdot) = L\gamma w(t,\cdot) + \gamma E w(t,\cdot) + h(\tau+t,\cdot), & t \in (0,\tau \wedge (T-\tau)) \\ w(0,x) = z(\tau,x), & x \in \Omega, \\ (\gamma w)(0) = (\gamma z)(\tau). \end{cases} \tag{3.10}$$

(3.10) has a unique solution w in $W^{1,p}(0,\tau \wedge (T-\tau); L^p(\Omega)) \cap L^p(0,\tau \wedge (T-\tau); W^{2,p}(\Omega))$, with γw in $W^{1,p}(0,\tau \wedge (T-\tau); L^p(\Gamma)) \cap L^p(0,\tau \wedge (T-\tau); W^{2,p}(\Gamma))$. If we set

$$u(t,\cdot) := \begin{cases} z(t,\cdot) & \text{if} \quad t \in (0,\tau], \\ w(t-\tau,\cdot) & \text{if} \quad t \in (\tau, \tau \wedge (T-\tau)], \end{cases}$$

it is easily seen that $u \in W^{1,p}(0,(2\tau) \wedge T; L^p(\Omega)) \cap L^p(0,(2\tau) \wedge T; W^{2,p}(\Omega))$, with γu in $W^{1,p}(0,(2\tau) \wedge T; L^p(\Gamma)) \cap L^p(0,(2\tau) \wedge T; W^{2,p}(\Gamma))$. and solves (3.7) , if we replace T with $(2\tau) \wedge T$. In case $2\tau < T$, we iterate the argument extending the solution to $(3\tau) \wedge T$. It is clear that in a finite number of steps we reach the conclusion. $\square$

Remark 3.10. It is easily seen that the conclusion of Theorem 3.9 still holds if we replace γE with an arbitrary operator F which is bounded from $W^{\theta,p}(\Omega)$ to $L^p(\Gamma)$, for some θ in $[0,2)$.

4 Generation of an analytic semigroup

Now we prove a result of generation of an analytic semigroup.

Theorem 4.1. *Suppose that the conditions (B1)-(B2) hold, L is as in (2.6) and E is as in (3.6), with coefficients e_j in $C^1(\overline{\Omega})$ $(0 \leq j \leq n)$. Let $p \in (1,\infty)$. Consider the space $X_p := L^p(\Omega) \times L^p(\Gamma)$ and define the following operator G_p acting on X_p:*

$$\begin{cases} D(G_p) := \{(u,\gamma u) : u \in W^{2,p}(\Omega), \gamma u \in W^{2,p}(\Gamma)\}, \\ G_p(u,\gamma u) := (\mathcal{A}u, L\gamma u + \gamma E u). \end{cases} \tag{4.1}$$

Then G_p is the infinitesimal generator of an analytic semigroup in X_p.

In the proof we shall employ the following

Lemma 4.2. *For any $p \in [1,\infty]$ there exists a linear operator $P : W^{2,p}(\Gamma) \to W^{2,p}(\Omega)$ such that $\gamma P g = g$ for any $g \in W^{2,p}(\Gamma)$ and, for some $C > 0$, independent of g,*

$$\|Pg\|_{L^p(\Omega)} \leq C\|g\|_{L^p(\Gamma)}, \quad \|Pg\|_{W^{2,p}(\Omega)} \leq C\|g\|_{W^{2,p}(\Gamma)}.$$

Proof. Firstly, P can be constructed in the particular case $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}^+$, $\Gamma = \mathbb{R}^{n-1} \times \{0\}$, setting, for $g \in W^{2,p}(\Gamma)$,

$$Pg(x', x_n) := g(x', 0)\phi(x_n),$$

with $\phi \in C^2([0,\infty))$, $\phi(t) = 1$ if $0 \leq t \leq 1$, $\phi(t) = 0$ if $t \geq 2$. The general case can be reduced to this one, employing partitions of unity and changes of variable. □

Remark 4.3. It can be easily seen that P can be extended to a linear bounded operator from $L^p(\Gamma)$ to $L^p(\Omega)$, for any p in $[1,\infty]$, and from $C^\alpha(\Gamma)$ to $C^\alpha(\overline{\Omega})$ for any α in $[0, 2]$.

Proof of Theorem 4.1. Let $\lambda \in \mathbb{C}$, $Re(\lambda) \geq 0$. We shall show that the problem

$$\lambda(u, \gamma u) - G_p(u, \gamma u) = (f, h) \tag{4.2}$$

has a unique solution $(u, \gamma u)$ in $D(G_p)$ if $|\lambda|$ is sufficiently large. Moreover, there exists $C > 0$, independent of λ and (f, h), such that

$$\|(u, \gamma u)\|_{X_p} \leq C|\lambda|^{-1}\|(f, h)\|_{X_p}.$$

Observe that (4.2) is equivalent to

$$\begin{cases} \lambda u(x) - Au(x) = f(x), \quad x \in \Omega, \\ \\ \lambda\gamma u(x') - L\gamma u(x') - \gamma Eu(x') = h(x'), \quad x' \in \Gamma. \end{cases} \tag{4.3}$$

We begin by considering the particular case $E = 0$, that is,

$$\begin{cases} \lambda u(x) - Au(x) = f(x), \quad x \in \Omega, \\ \\ \lambda\gamma u(x') - L\gamma u(x') = h(x'), \quad x' \in \Gamma. \end{cases} \tag{4.4}$$

By Theorem 2.1, there exists R_1 positive such that, if $|\lambda| \geq R_1$, the equation

$$\lambda v(x') - Lv(x') = h(x'), \quad x' \in \Gamma$$

has a unique solution v in $W^{2,p}(\Gamma)$. Moreover, for some C_1 positive, independent of λ and h,

$$|\lambda|\|v\|_{L^p(\Gamma)} + \|v\|_{W^{2,p}(\Gamma)} \leq C_1\|h\|_{L^p(\Gamma)}.$$

Now we consider the system

$$\begin{cases} \lambda u(x) - Au(x) = f(x), \quad x \in \Omega, \\ \\ \gamma u(x') = v(x'), \quad x' \in \Gamma. \end{cases} \tag{4.5}$$

By [20], Chapter 3.8, there exists $R \geq R_1$ such that (4.5) has a unique solution u in $W^{2,p}(\Omega)$. Moreover, for some $C_2 > 0$ independent of λ and f, for any $V \in W^{2,p}(\Omega)$ such that $\gamma V = v$,

$$|\lambda|\|u\|_{L^p(\Omega)} + \|u\|_{W^{2,p}(\Omega)} \leq C_2(\|f\|_{L^p(\Omega)} + \|V\|_{W^{2,p}(\Omega)} + |\lambda|\|V\|_{L^p(\Omega)}).$$

Choosing $V = Pv$, with P as in Lemma 4.2, we deduce

$$\begin{aligned} |\lambda|\|u\|_{L^p(\Omega)} + \|u\|_{W^{2,p}(\Omega)} &\leq C_2(\|f\|_{L^p(\Omega)} + \|Pv\|_{W^{2,p}(\Omega)} + |\lambda|\|Pv\|_{L^p(\Omega)}) \\ &\leq C_3(\|f\|_{L^p(\Omega)} + \|v\|_{W^{2,p}(\Gamma)} + |\lambda|\|v\|_{L^p(\Gamma)}) \\ &\leq C_4(\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Gamma)}). \end{aligned} \tag{4.6}$$

Now we consider the general case $E \neq 0$. For any $\theta \in [0,2]$, it follows from (4.6) that

$$\|u\|_{W^{\theta,p}(\Omega)} \leq C(\theta)|\lambda|^{\theta/2-1}(\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Gamma)}). \tag{4.7}$$

Now we fix $\theta \in (1+\frac{1}{p}, 2)$ and, for $U \in W^{\theta,p}(\Omega)$, we consider the system

$$\begin{cases} \lambda u(x) - Au(x) = f(x), \quad x \in \Omega, \\ \lambda\gamma u(x') - L\gamma u(x') = \gamma EU(x') + h(x'), \quad x' \in \Gamma. \end{cases} \tag{4.8}$$

If $|\lambda|$ is sufficiently large, there exists a unique solution $u = u(U)$ in $W^{2,p}(\Omega)$. We shall think of $U \to u(U)$ as an operator from $W^{\theta,p}(\Omega)$ into itself. If $u_j = u(U_j)$ ($j \in \{1,2\}$), we have

$$\begin{cases} \lambda(u_1 - u_2)(x) - A(u_1 - u_2)(x) = 0, \quad x \in \Omega, \\ \lambda\gamma(u_1 - u_2)(x') - L\gamma(u_1 - u_2)(x') = \gamma E(U_1 - U_2)(x'), \quad x' \in \Gamma, \end{cases}$$

so that, by (4.7),

$$\|u_1 - u_2\|_{W^{\theta,p}(\Omega)} \leq C(\theta)|\lambda|^{\theta/2-1}\|\gamma E(U_1 - U_2)\|_{L^p(\Gamma)}) \leq C_1(\theta)|\lambda|^{\theta/2-1}\|U_1 - U_2\|_{W^{\theta,p}(\Omega)}.$$

We deduce that $U \to u(U)$ is a contraction if $|\lambda|$ is sufficiently large. We conclude that, for such choice of λ, (4.3) has a unique solution u. Moreover, from (4.7),

$$\begin{aligned} \|u\|_{W^{\theta,p}(\Omega)} &\leq C(\theta)|\lambda|^{\theta/2-1}(\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Gamma)} + \|\gamma Eu\|_{L^p(\Gamma)}) \\ &\leq C_1|\lambda|^{\theta/2-1}(\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Gamma)} + \|u\|_{W^{\theta,p}(\Omega)}), \end{aligned}$$

implying

$$\|u\|_{W^{\theta,p}(\Omega)} \leq C_2(\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Gamma)})$$

if $|\lambda|$ is sufficiently large. We deduce that

$$\begin{aligned} &|\lambda|\|u\|_{L^p(\Omega)} + \|u\|_{W^{2,p}(\Omega)} + |\lambda|\|\gamma u\|_{L^p(\Gamma)} + \|\gamma u\|_{W^{2,p}(\Gamma)} \\ &\leq C_3(\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Gamma)} + \|u\|_{W^{\theta,p}(\Omega)}) \\ &\leq C_4(\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Gamma)}). \end{aligned}$$

The proof is complete.

□

Remark 4.4. Here also the assertion of Theorem 4.1 holds replacing γE with any operator F which is bounded from $W^{\theta,p}(\Omega)$ to $L^p(\Omega)$, for some θ in $[0,2)$.

Remark 4.5. We have chosen to prove Theorem 4.1 estimating directly the resolvent $(\lambda - G_p)^{-1}$. In fact, the result can be obtained quite quickly, applying Theorem 3.1 together with a nice theorem by G. Dore (see [5]).

5 General Wentzell boundary conditions

In [7] and [8] the authors considered the problem

$$\begin{cases} D_t u(t,x) = Mu(t,x), \quad (t,x) \in (0,T) \times \Omega, \\ Mu(t,x') + \beta(x')\partial_\nu^a u(t,x') - q\beta(x')L_\partial\gamma u(t,x') + q\tilde{a}(x') \cdot \nabla_\tau\gamma u(t,x') + \tilde{r}(x')\gamma u(t,x') = 0, \\ (t,x') \in (0,T) \times \Gamma, \\ u(0,x) = u_0(x), \quad x \in \Omega. \end{cases} \tag{5.1}$$

Here

$$Mu = \sum_{i,j=1}^{n} \partial_i(a_{ij}(\cdot)\partial_j u) + \sum_{i=1}^{n} c_i \partial_i u + ru,$$

with a_{ij} real valued, $a_{ij} = a_{ji}$, $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha_0|\xi|^2$ for any $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$ for some α_0 positive, β positive, $\partial_\nu^a = \sum_{i,j=1}^{n} a_{ij}(\cdot)\nu_i\partial_j u$, $q \in \mathbb{R}^+$, a_{ij}, c_i, r defined and sufficiently regular on $\overline{\Omega}$, $\tilde{a}$ and $\tilde{r}$ defined and sufficiently regular on Γ. ∇_τ stands for the gradient operator in Γ and

$$L_\partial \gamma u = div(B(x)\nabla_\tau \gamma u)$$

is an operator of the form considered in Example 2.4. Of course the Riemannian structure in Γ is that inherited as an embedded submanifold of $\mathbb{R}^n$. The open set Ω is not assumed to be bounded. System (5.1) is studied in the following way: it is introduced the following operator M_p:

$$\begin{cases} D(\tilde{M}_p) := \{(u, \gamma u) : u \in C_c^2(\overline{\Omega}), \gamma Mu + \beta\partial_\nu^a u - q\beta L_\partial \gamma u + q\tilde{a} \cdot \nabla_\tau u + \tilde{r}\gamma u = 0\}, \\ \\ M_p(u, \gamma u) = (Mu, \gamma Mu) = (Au, -\beta\partial_\nu^a u + q\beta L_\partial \gamma u - q\tilde{a} \cdot \nabla_\tau u - \tilde{r}\gamma u). \end{cases} \tag{5.2}$$

Then it is proved that the closure of $\hat{M}_p$ in $L^p(\Omega) \times L^p(\Gamma)$ generates an analytic semigroup. It follows that, for every u_0 belonging to the domain of $\hat{M}_p$, (5.1) has a solution (in some generalized sense).

Following this idea, we can consider the problem

$$\begin{cases} D_t u(t, x) = \mathcal{A}u(t, x), & (t, x) \in (0, T) \times \Omega, \\ \\ \gamma\mathcal{A}u(t, \cdot) - L\gamma u(t, \cdot) - \gamma Eu(t, \cdot) = 0, & t \in (0, T) \\ \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases} \tag{5.3}$$

with the assumptions of Theorem 3.9: we introduce the following operator M_p, for $p \in (1, \infty)$:

$$\begin{cases} D(M_p) := \{(u, \gamma u) : u \in C^2(\overline{\Omega}), \gamma\mathcal{A}u - L\gamma u - \gamma Eu = 0\}, \\ \\ M_p(u, \gamma u) = (\mathcal{A}u, \gamma\mathcal{A}u) = (Au, L\gamma u + \gamma Eu). \end{cases} \tag{5.4}$$

We show the following

Theorem 5.1. *Suppose that (B1)-(B2) hold, L is as in (2.6) and E is as in (3.6) with coefficients in $C^1(\overline{\Omega})$. Moreover,*

(a) $\Gamma = \partial\Omega$ is of class $C^{2+\alpha}$, for some $\alpha \in (0, 1)$;

(b) the coefficients a_{ij}, b_j, c of $\mathcal{A}$ $(1 \leq i, j \leq n)$ are of class $C^\alpha(\overline{\Omega})$;

(c) the coefficients $l_{\alpha,\Phi}$ in (2.6) are in $C^\alpha(U)$;

(d) the coefficients e_j $(0 \leq j \leq n)$ of E (see (3.6) are in $C^\alpha(\overline{\Omega})$).

Then, if $1 < p < \infty$, M_p is closable in $X_p = L^p(\Omega) \times L^p(\partial\Omega)$ and its closure coincides with G_p (defined in (4.1)).

Proof. We have to prove the following:

$\forall (u, \gamma u) \in D(G_p)$ *there exists a sequence* $((u_k, \gamma u_k))_{k\in\mathbb{N}}$ *in* $D(M_p)$ *such that*

$$\|(u_k, \gamma u_k) - (u, \gamma u)\|_{X_p} + \|M_p(u_k, \gamma u_k) - G_p(u, \gamma u)\|_{X_p} \to 0 \quad (k \to \infty).$$

We start by proving three steps.

Step 1: Let $(u, \gamma u) \in D(G_p)$ be such that, for some $\lambda \in \mathbb{C}$, $(\lambda - G_p)(u, \gamma u) \in C^\alpha(\overline{\Omega}) \times C^\alpha(\Gamma)$. Then $(u, \gamma u) \in C^{2+\alpha}(\overline{\Omega}) \times C^{2+\alpha}(\Gamma)$

We start by considering the case $E = 0$. Then, $\lambda\gamma u - L\gamma u = h \in C^\alpha(\Gamma)$ and so $\gamma u \in C^{2+\alpha}(\Gamma)$ (see [11], Theorem 2.1). So $u \in W^{2,p}(\Omega)$ and solves the system

$$\begin{cases} (\lambda - \mathcal{A})u = f \in C^\alpha(\overline{\Omega}), \\ \\ \gamma u \in C^{2+\alpha}(\Gamma), \end{cases}$$

again implying $u \in C^{2+\alpha}(\overline{\Omega})$.

Now we consider the case $E \neq 0$, employing a bootstrap argument. Suppose that we have shown that $(u, \gamma u) \in W^{2,q}(\Omega) \times W^{2,q}(\Gamma)$ for some $q \geq p$. Then $\gamma Eu \in W^{1-1/q,q}(\Gamma)$. Assume that

$$q \geq \frac{n}{1-\alpha}.$$

Then $W^{1-1/q,q}(\Gamma) \hookrightarrow C^\alpha(\Gamma)$, so that $(\lambda u - \mathcal{A}u, \lambda\gamma u - L\gamma u) \in C^\alpha(\overline{\Omega}) \times C^\alpha(\Gamma)$ and the conclusion follows.

Suppose $q > n$. Then $\gamma Eu \in C^{\alpha'}(\Gamma)$, for some $\alpha' \in (0,1)$. It follows that $(\lambda u - \mathcal{A}u, \lambda\gamma u - L\gamma u) \in C^{\alpha'}(\overline{\Omega}) \times C^{\alpha'}(\Gamma)$. This implies $u \in C^{2+\alpha'}(\overline{\Omega})$, so that $\gamma Eu \in C^{1+\alpha'}(\Gamma) \hookrightarrow C^\alpha(\Gamma)$ and we have again the conclusion.

Suppose $q < n$. Then $\gamma Eu \in W^{1-1/q,q}(\Gamma) \hookrightarrow L^{\frac{n-1}{n-q}q}(\Gamma)$. We deduce $(\lambda u - \mathcal{A}u, \lambda\gamma u - L\gamma u) \in C^\alpha(\overline{\Omega}) \times L^{\frac{n-1}{n-q}q}(\Gamma)$, implying $(u, \gamma u) \in W^{2,q_1}(\Omega) \times W^{2,q_1}(\Gamma)$, with $q_1 = \frac{n-1}{n-q}q > q$. If $q_1 > n$, we can conclude. Otherwise, we deduce that $(u, \gamma u) \in W^{2,q_2}(\Omega) \times W^{2,q_2}(\Gamma)$, with $q_2 = \frac{n-1}{n-q_1}q_1 > q_1$. We can iterate the process until we get the belonging of $(u, \gamma u)$ to $W^{2,r}(\Omega) \times W^{2,r}(\Gamma)$ for some $r > n$. This can be necessarily achieved in a finite number of steps. Otherwise, we should obtain the belonging of $(u, \gamma u)$ to $W^{2,q_k}(\Omega) \times W^{2,q_k}(\Gamma)$ with $q < q_1 < \cdots < q_k < q_{k+1} \leq \cdots < n$ for a certain sequence $(q_k)_{k\in\mathbb{N}}$. But this is not possible, because

$$q_k = \frac{n-1}{n-q_{k-1}} q_{k-1} \geq (\frac{n-1}{n-q})^k q \to \infty \quad (k \to \infty),$$

a contradiction.

Step 2: Let $(u, \gamma u) \in D(G_p)$ be such that, for some $\lambda \in \mathbb{C}$, $(\lambda - G_p)(u, \gamma u) = (f, h) \in C^\alpha(\overline{\Omega}) \times C^\alpha(\Gamma)$, with $\alpha \in (0,1)$ and $h = \gamma f$. Then $(u, \gamma u) \in D(M_p)$.

In fact, by Step 1, $(u, \gamma u) \in C^{2+\alpha}(\overline{\Omega}) \times C^{2+\alpha}(\Gamma)$. Moreover,

$$\gamma\mathcal{A}u - L\gamma u - \gamma Eu = \lambda\gamma u - \gamma f - \lambda\gamma u + h = 0.$$

Step 3: $\{(\psi, \gamma\psi) : \psi \in C^\alpha(\overline{\Omega})\}$ is dense in X_p.

In fact, let $(f, h) \in X_p$. We begin by considering a sequence $(h_k)_{k\in\mathbb{N}}$ with values in $C^\alpha(\Gamma)$, such that $\|h_k - h\|_{L^p(\Gamma)} \to 0$ $(k \to \infty)$. Let P be the extension operator described in Lemma 4.2. By Remark 4.3, it can be extended to a linear bounded operator from $C^\alpha(\Gamma)$ to $C^\alpha(\overline{\Omega})$ and from $L^p(\Gamma)$ to $L^p(\Omega)$. So $Ph_k \in C^\alpha(\overline{\Omega})$ for every $k \in \mathbb{N}$ and $(Ph_k)_{k\in\mathbb{N}}$ converges to Ph in $L^p(\Omega)$. Now we consider a sequence $(\phi_k)_{k\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ converging to $f - Ph$ in $L^p(\Omega)$. We set $\psi_k := Ph_k + \phi_k$. Then $\psi_k \in C^\alpha(\overline{\Omega})$, $(\psi_k)_{k\in\mathbb{N}}$ converges to f in $L^p(\Omega)$ and $(\gamma\psi_k)_{k\in\mathbb{N}} = (h_k)_{k\in\mathbb{N}}$ converges to h in $L^p(\Gamma)$.

Now let us consider $(u, \gamma u) \in D(G_p)$. We fix $\lambda \in \rho(G_p)$ and set $(f, h) := \lambda(u, \gamma u) - G_p((u, \gamma u)) \in X_p$. We take a sequence $((\psi_k, \gamma\psi_k))_{k\in\mathbb{N}}$ with $\psi_k \in C^\alpha(\overline{\Omega})$, converging to (f, h) in X_p. We set $(u_k, \gamma u_k) := (\lambda - G_p)^{-1}(\psi_k, \gamma\psi_k)$ $(k \in \mathbb{N})$. Then $(u_k, \gamma u_k) \in D(M_p)$, the sequence $((u_k, \gamma u_k))_{k\in\mathbb{N}}$ converges to $(u, \gamma u)$ in $W^{2,p}(\Omega) \times W^{2,p}(\Gamma)$, so that $(M_p(u_k, \gamma u_k))_{k\in\mathbb{N}}$ converges to $G_p(u, \gamma u)$ in X_p.

□

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