Research Article

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Very large solutions for the fractional Laplacian: Towards a fractional Keller–Osserman condition

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Abstract: We look for solutions of $(-\Delta)^s u + f(u) = 0$ in a bounded smooth domain Ω, $s \in (0, 1)$, with a strong singularity at the boundary. In particular, we are interested in solutions which are $L^1(\Omega)$ and higher order with respect to $\operatorname{dist}(x, \partial\Omega)^{s-1}$. We provide sufficient conditions for the existence of such a solution. Roughly speaking, these functions are the real fractional counterpart of *large solutions* in the classical setting.

Keywords: Fractional Laplacian, large solutions, Keller–Osserman condition

MSC 2010: Primary 35A01, 45K05, 35B40; secondary 35S15

1 Introduction

In the theory of semilinear elliptic equations, functions solving

$$-\Delta u + f(u) = 0 \quad \text{in } \Omega \subseteq \mathbb{R}^N, \tag{1.1}$$

where Ω is open and bounded, coupled with the boundary condition

$$\lim_{x \to \partial\Omega} u(x) = +\infty$$

are known as *boundary blow-up solutions* or *large solutions*. There is a huge amount of bibliography dealing with this problem which dates back to the seminal work of Bieberbach [3], for $N = 2$ and $f(u) = e^u$. Keller [18] and Osserman [22] independently established a sufficient and necessary condition on the nonlinear term f for the existence of a boundary blow-up solution which takes the form

$$\int^{+\infty} \frac{dt}{\sqrt{F(t)}} < +\infty, \quad \text{where } F' = f \geq 0, \tag{1.2}$$

and it is known as the *Keller–Osserman condition*. One can find these solutions with singular behaviour at the boundary in a number of applications. For example, Loewner and Nirenberg [21] studied the case $f(u) = u^{(N+2)/(N-2)}$, $N \geq 3$, which is strictly related to the *singular Yamabe problem* in conformal Geometry, while Labutin [19] completely characterized the class of sets Ω that admit a large solution for $f(u) = u^q$, $q > 1$, with capacitary methods inspired by the theory of *spatial branching processes*, that are particular stochastic processes. See also the purely probabilistic works by Le Gall [20], and Dhersin and Le Gall [9] dealing with the particular case $q = 2$.

In this paper we tackle equation (1.1) when the Laplacian operator is replaced by one of its fractional powers. The fractional Laplacian $(-\Delta)^s$, $s \in (0, 1)$, is an integral nonlocal operator of fractional order which

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admits different equivalent definitions, see, e.g., [10]. We will use the following:

$$(-\Delta)^s u(x) = \mathcal{A}(N,s)\,\mathrm{PV}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy = \mathcal{A}(N,s)\lim_{\varepsilon\downarrow 0}\int_{\{|y-x|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy, \tag{1.3}$$

where $\mathcal{A}(N,s)$ is a renormalizing positive constant. This operator generates[1] a Wiener process subordinated in time with an s-stable Lévy process. The Dirichlet problem related to $(-\Delta)^s$ is of the form

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega,\\ u = g & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases}$$

because the data have to take into account the nonlocal character of the operator. Nevertheless, in [1] the author showed how this problem is ill-posed in a weak L^1 sense, of Stampacchia's sort, unless a singular trace is prescribed at the boundary. A well-posed Dirichlet problem needs to deal with two conditions at the same time. Namely, if d denotes the distance to the boundary $\partial\Omega$, it looks like

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega,\\ u = g & \text{in } \mathbb{R}^N\setminus\Omega,\\ d^{1-s}u = h & \text{on } \partial\Omega, \end{cases}$$

where the data satisfy the following assumptions:

$$\int_\Omega |f|\,d^s < +\infty, \qquad \int_{\mathbb{R}^N\setminus\Omega} |g| d^{-s}\min\{1, d^{-N-s}\} < +\infty, \qquad \|h\|_{L^\infty(\partial\Omega)} < +\infty.$$

Further references in this direction are the recent works by Grubb [16, 17], where also the regularity up to the boundary is investigated. This means in particular that in the context of fractional Dirichlet problems there are solutions with an explosive behaviour at the boundary as a result of a linear phenomenon. For instance, the solutions to

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1,\\ u(x) = (|x|^2-1)^{-s/2} & \text{in } \mathbb{R}^N\setminus B_1,\\ d^{1-s}u = 0 & \text{on } \partial B_1 \end{cases} \qquad \text{and} \qquad \begin{cases} (-\Delta)^s u = 0 & \text{in } B_1,\\ u = 0 & \text{in } \mathbb{R}^N\setminus B_1,\\ d^{1-s}u = 1 & \text{on } \partial B_1 \end{cases}$$

are of the order of $O(d^{-s/2})$ and $O(d^{s-1})$, respectively, at ∂B_1, see [1]. The existence of harmonic functions of this sort can therefore be used to prove, via a sub- and supersolution argument, the existence of boundary blow-up solutions to nonlinear problems of the form

$$\begin{cases} (-\Delta)^s u = -f(x,u) & \text{in } \Omega,\\ u = g & \text{in } \mathbb{R}^N\setminus\Omega,\\ d^{1-s}u = h & \text{on } \partial\Omega \end{cases}$$

with $f(x,u)\geq 0$. Anyhow, this singular behaviour is driven by a linear phenomenon rather than a compensation between the nonlinearity and the explosion (as in the classical case). Indeed no growth condition on f arises except when $h\not\equiv 0$, where one needs

$$\int_\Omega f(x, d(x)^{s-1})\,d(x)^s\,dx < \infty,$$

in order to make sense of the weak L^1 definition.

For this reason we address here the question of the existence of solutions to problems of the form

$$\begin{cases} (-\Delta)^s u = -f(u) & \text{in } \Omega,\\ u = g & \text{in } \mathbb{R}^N\setminus\Omega,\\ d^{1-s}u = +\infty & \text{on } \partial\Omega, \end{cases} \qquad g\geq 0, \qquad \int_{\mathbb{R}^N\setminus\Omega} g d^{-s}\min\{1, d^{-N-s}\} = +\infty,$$

providing sufficient conditions for the solvability. In doing so, we extend the results by Felmer and Quaas [13],

1 Recall that $-\Delta$ is the infinitesimal generator of the Wiener process, modelling the Brownian motion.

and Chen, Felmer and Quaas [6] for $f(u) = u^p$, which is the only reference available on the topic, and we also clarify the notion of *large solution* in this setting. The results listed in Theorems 1.3 and 1.5 below can be applied to a particular case of the *fractional singular Yamabe problem*, see, e.g., [15].

1.1 Hypotheses and main results

We work with the following set of assumptions:
- Ω is a bounded open domain of class C^2.
- f is an increasing C^1 function with $f(0) = 0$.
- F is the antiderivative of f vanishing in 0, that is,

$$F(t) := \int_0^t f(\tau)\, d\tau. \tag{1.4}$$

- There exist $0 < m < M$ such that

$$1 + m \le \frac{tf'(t)}{f(t)} \le 1 + M, \tag{1.5}$$

  and thus f satisfies (1.2), because by integrating the lower inequality, one gets

$$f(t) \ge f(1)t^{1+m} \quad \text{and} \quad F(t) \ge \frac{f(1)}{2+m} t^{2+m}.$$

  We can therefore define the function

$$\phi(u) := \int_u^{+\infty} \frac{dt}{\sqrt{F(t)}}. \tag{1.6}$$

- The function ϕ satisfies

$$\int_1^{+\infty} \phi(t)^{1/s}\, dt < +\infty. \tag{1.7}$$

  In what follows we will use the expression $g \asymp h$, where $g, h\colon (0, +\infty) \to (0, +\infty)$, to shorten the following condition:

$$\text{“there exists } C > 0 \text{ such that } \frac{h(t)}{C} \le g(t) \le Ch(t) \text{ for any } t > 0.\text{”}$$

Remark 1.1. The function $\phi\colon (0, +\infty) \to (0, +\infty)$ is monotone decreasing and

$$\lim_{t\downarrow 0} \phi(t) = +\infty, \qquad \lim_{t\uparrow +\infty} \phi(t) = 0.$$

Moreover,

$$\phi'(u) = -\frac{1}{\sqrt{F(u)}}$$

is of the same order as $-(u\,f(u))^{-1/2}$ since for $t > 0$ and some $\tau \in (0, t)$, by the Cauchy theorem, we have

$$\frac{F(t)}{tf(t)} = \frac{f(\tau)}{f(\tau) + \tau f'(\tau)} \begin{cases} \ge \dfrac{1}{2+M}, \\ \le \dfrac{1}{2+m}. \end{cases}$$

This entails that the order of $\phi(u)$ is the same as $(u/f(u))^{1/2}$. Indeed, for $u > 0$ and some $t \in (u, +\infty)$,

$$\frac{\sqrt{\frac{u}{f(u)}}}{\phi(u)} = \frac{\frac{1}{2}\sqrt{\frac{f(t)}{t}} \cdot \frac{f(t)-tf'(t)}{f(t)^2}}{\phi'(t)} \asymp \frac{f(t) - tf'(t)}{-f(t)} = \frac{tf'(t)}{f(t)} - 1,$$

which belongs to (m, M) by hypothesis (1.5). Note that hypothesis (1.7) is therefore equivalent to

$$\int_1^{+\infty} \left(\frac{t}{f(t)}\right)^{1/(2s)} dt < +\infty. \tag{1.8}$$

Remark 1.2. In [18] and [22] condition (1.2) is proven to be necessary and sufficient for the existence of a solution of

$$\begin{cases} -\Delta u = -f(u) & \text{in } \Omega, \\ \lim\limits_{x\to\partial\Omega} u(x) = +\infty. \end{cases}$$

Note that if we set $s = 1$ in (1.7), then

$$+\infty > \int_u^{+\infty} \phi(t)\,dt \asymp \int_u^{+\infty} \sqrt{\frac{t}{f(t)}}\,dt \asymp \int_u^{+\infty} \frac{t}{\sqrt{F(t)}}\,dt,$$

and we get the condition to force the classical solution u to be $L^1(\Omega)$. Indeed, in [11, Theorem 1.6] it was proved that a solution u satisfies

$$\lim_{x\to\partial\Omega} \frac{\phi(u(x))}{d(x)} = 1. \tag{1.9}$$

This yields that $u \in L^1(\Omega)$ if and only if ϕ^{-1}, the inverse function of ϕ (recall it is monotone decreasing), is integrable in a neighbourhood of 0, i.e., with a change of integration variable

$$+\infty > \int_0^{\eta} \phi^{-1}(r)\,dr = \int_{\phi^{-1}(\eta)}^{+\infty} t|\phi'(t)|\,dt = \int_{t_0}^{+\infty} \frac{t}{\sqrt{F(t)}}\,dt.$$

Our results can be summarised as follows.

Theorem 1.3. *Suppose that the nonlinear term f satisfies hypotheses* (1.5) *and* (1.7) *above and*

$$\int_{t_0}^{+\infty} f(t)t^{-2/(1-s)}\,dt < +\infty. \tag{1.10}$$

Then, the following problem:

$$\begin{cases} (-\Delta)^s u = -f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ d^{1-s}u = +\infty & \text{on } \partial\Omega \end{cases} \tag{1.11}$$

admits a solution $u \in L^1(\Omega)$. Moreover, there exists $c > 0$ for which

$$\phi(u(x)) \geq c\,d(x)^s \quad \text{in } \Omega. \tag{1.12}$$

Remark 1.4. The condition $u \in L^1(\Omega)$ is necessary to make sense of the fractional Laplacian, see equation (1.3). Also, compare the boundary behaviour in this setting expressed by equation (1.12) with the classical one in equation (1.9).

Theorem 1.5. *Suppose that the nonlinear term f satisfies hypotheses* (1.5) *and* (1.7) *above and*

$$\begin{aligned} g\colon \mathbb{R}^N \setminus \Omega \to [0,+\infty), &\quad g \in L^1(\mathbb{R}^N \setminus \Omega) \\ \phi(g(x)) \geq d(x)^s, &\quad \text{near } \partial\Omega. \end{aligned} \tag{1.13}$$

Then, the following problem:

$$\begin{cases} (-\Delta)^s u = -f(u) & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.14}$$

admits a solution $u \in L^1(\Omega)$. Moreover, there exists $c > 0$ for which

$$\phi(u(x)) \geq c\,d(x)^s \quad \text{near } \partial\Omega.$$

Remark 1.6. Note that in problem (1.14) we do not prescribe the singular trace at $\partial\Omega$.

Remark 1.7. The hypotheses in Theorem 1.3, when considering $f(u) = u^p$, reduce to

$$p \in \left(1 + 2s, 1 + \frac{2s}{1-s}\right),$$

see Theorem 1.10. Note that this range of exponents does not converge, letting $s \uparrow 1$, to the set of admissible exponents for $-\Delta$, which is given by (1.2) and simply reads as $p \in (1, +\infty)$. Indeed, we only have $1 + 2s \to 3$ as $s \uparrow 1$. This is not discouraging though. In this fractional setting we need $u \in L^1(\Omega)$ to make sense of the operator. This is an additional (natural) restriction we do not have in the classical problem, so it is reasonable to get smaller ranges for p. Moreover, the classical solution to the large problem is known to behave like (cf. equation (1.9))

$$u \asymp d^{-2/(p-1)},$$

and such a u is in $L^1(\Omega)$ when $p > 3$. In this sense, we actually have the asymptotic convergence of the admissible ranges of exponents. Compare this also with Remark 1.2.

Remark 1.8. As it will be clear in the following proofs, hypothesis (1.5) is technical and not structural. We conjecture that it is not necessary to establish existence results. But let us mention how a similar assumption arises naturally even in the classical framework when dealing with the computation of the asymptotic behaviour of the solution, see [2, equations (B) and (B)′].

The strategy to prove the existence result in Theorem 1.3 is to build the sequence $\{u_k\}_{k\in\mathbb{N}}$ of solutions to the following problem[2]:

$$\begin{cases} (-\Delta)^s u_k = -f(u_k) & \text{in } \Omega, \\ u_k = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ E\,u_k = k & \text{on } \partial\Omega, \end{cases} \tag{1.15}$$

and then let $k \uparrow +\infty$. In case u_k admits a limit, then we will need to prove that this is the solution we were looking for. This might also be called the *minimal large solution*, by borrowing the expression used in the classical theory.

We can also provide a partial nonexistence result.

Theorem 1.9. *Suppose there exist $a, b > 0$ for which*

$$f(t) \le a + bt \quad \text{for any } t \in (0, +\infty). \tag{1.16}$$

Then, there exists $\alpha > 0$ such that

$$u_k(x) \uparrow +\infty \quad \text{as } k \uparrow +\infty, \text{ whenever } d(x) < \alpha.$$

In the case of power-like nonlinearities we can prove the following.

Theorem 1.10. *Let $f(t) = t^p$, $p > 0$. Then, the following hold:*
(1) *If $p \in [1 + \frac{2s}{1-s}, +\infty)$, then the approximating sequence $\{u_k\}_{k\in\mathbb{N}}$ does not exist.*
(2) *If $p \in (1 + 2s, 1 + \frac{2s}{1-s})$, then the approximating sequence converges to a solution u of* (1.11) *and*

$$u \le C d^{-2s/(p-1)}.$$

(3) *If $p \in (1, 1 + s)$, then the approximating sequence exits $L^1(\Omega)$, meaning $\|u_k\|_{L^1(\Omega)} \uparrow +\infty$ as $k \uparrow +\infty$.*
(4) *If $p \in (0, 1]$, then the approximating sequence blows-up uniformly in some open strip near the boundary.*

1.2 Notations

In the following we will always denote by $\complement E = \mathbb{R}^N \setminus E$ for any $E \subset \mathbb{R}^N$.

2 The operator E denotes the singular trace operator defined in [1].

Hypothesis (1.5) implies that $f(t)t^{-1-M}$ is monotone decreasing and $f(t)t^{-1-m}$ is monotone increasing, since

$$\frac{d}{dt}\frac{f(t)}{t^{1+M}} = \frac{1}{t^{1+M}}\Big(f'(t) - (1+M)\frac{f(t)}{t}\Big) \le 0 \quad \text{and} \quad \frac{d}{dt}\frac{f(t)}{t^{1+m}} = \frac{1}{t^{1+m}}\Big(f'(t) - (1+m)\frac{f(t)}{t}\Big) \ge 0.$$

We write this monotonicity conditions as

$$c^{1+m} f(t) \le f(ct) \le c^{1+M} f(t), \qquad c > 1,\ t > 0. \tag{1.17}$$

The function F satisfies, similar to (1.5), the following inequalities:

$$2 + m \le \frac{t f(t)}{F(t)} \le 2 + M. \tag{1.18}$$

Indeed, by integrating (1.5), we deduce

$$(1+m)F(t) \le \int_0^t \tau f'(\tau)\, d\tau = tf(t) - F(t).$$

Let $\psi = \phi^{-1}$ be the inverse of ϕ, so that

$$v = \int_{\psi(v)}^{+\infty} \frac{dt}{\sqrt{F(t)}}, \qquad v \ge 0. \tag{1.19}$$

The function ψ is decreasing and $\psi(v) \uparrow +\infty$ as $v \downarrow 0$. Moreover, by Remark 1.1 and (1.18), for $u > 0$ and some $y \in (u, +\infty)$, we have

$$\frac{\phi(u)}{u|\phi'(u)|} = \frac{\sqrt{F(u)}}{u} \int_u^{+\infty} \frac{dt}{\sqrt{F(t)}} = \frac{-\frac{1}{\sqrt{F(y)}}}{\frac{1}{\sqrt{F(y)}} - \frac{yf(y)}{2F(y)^{3/2}}} = \frac{1}{\frac{yf(y)}{2F(y)} - 1} \begin{cases} \ge \dfrac{2}{M}, \\ \le \dfrac{2}{m}, \end{cases}$$

which in turn, by setting $v = \phi(u)$, implies that

$$\frac{2}{M} \le \frac{v|\psi'(v)|}{\psi(v)} \le \frac{2}{m}. \tag{1.20}$$

One can prove also

$$\psi(cv) \le c^{-2/M} \psi(v), \qquad c \in (0,1),\ v > 0, \tag{1.21}$$

as we have done for (1.17) above. Also, by (1.18) and (1.20), we have

$$\frac{v^2 \psi''(v)}{\psi(v)} = \frac{v^2 f(\psi(v))}{2\psi(v)} \asymp \frac{v^2 F(\psi(v))}{\psi(v)^2} = \frac{v^2 \psi'(v)^2}{\psi(v)^2} \asymp 1. \tag{1.22}$$

1.3 Construction of a supersolution

In this paragraph we prove the key point for the proof of Theorems 1.3 and 1.5, that is, we build a supersolution to both problems by handling the function U defined in (1.23) below.

Since by assumption $\partial\Omega \in C^2$, the function $\operatorname{dist}(x, \partial\Omega)$ is C^2 in an open strip around the boundary, except on $\partial\Omega$ itself. Consider a positive function $\delta(x)$ which is obtained by extending $\operatorname{dist}(x, \partial\Omega)$ smoothly to $\mathbb{R}^N \setminus \partial\Omega$. Define

$$U(x) = \psi(\delta(x)^s), \qquad x \in \mathbb{R}^N. \tag{1.23}$$

Lemma 1.11. *The function U defined in* (1.23) *is in* $L^1(\Omega)$.

Proof. Since both ψ and δ^s are continuous in Ω, we have that $U \in L^1_{\mathrm{loc}}(\Omega)$. Fix $\delta_0 > 0$ small and consider $\Omega_0 = \{x \in \Omega : \delta(x) < \delta_0\}$. We have (using once the coarea formula)

$$\int_{\Omega_0} \psi(\delta(x)^s)\, dx \le C \int_0^{\delta_0} \psi(t^s)\, dt.$$

Apply now the transformation $\psi(t^s) = \eta$ to get

$$\int_{\Omega_0} U(x)\,dx \le C \int_{\eta_0}^{+\infty} \eta\,\phi(\eta)^{(1-s)/s} |\phi'(\eta)|\,d\eta,$$

where, by Remark 1.1,

$$|\phi'(\eta)| \asymp \frac{1}{\sqrt{\eta f(\eta)}} \quad \text{and} \quad \phi(\eta) \asymp \sqrt{\frac{\eta}{f(\eta)}}.$$

Therefore,

$$\int_{\Omega_0} U(x)\,dx \le C \int_{\eta_0}^{+\infty} \Big(\frac{\eta}{f(\eta)}\Big)^{1/(2s)} d\eta,$$

which is finite by (1.8). □

The following Proposition shows that U is a good starting point to build a supersolution. The proof is technical but this is the key step for the following.

Proposition 1.12. *For some $C, \delta_0 > 0$, the function U defined in (1.23) satisfies*

$$(-\Delta)^s U \ge -Cf(U) \quad \text{in } \Omega_{\delta_0} = \{x \in \Omega : \delta(x) < \delta_0\}. \tag{1.24}$$

Before giving the proof, we prove a preliminary lemma.

Lemma 1.13. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain with compact boundary $\partial\Omega$. Cover $\partial\Omega$ by a finite number of open portions $\Gamma_j \subset \partial\Omega$, $j = 1, \dots, n$. For any $\eta \in \partial\Omega$, there exists $i(\eta) \in \{1, \dots, n\}$ such that $\eta \in \Gamma_{i(\eta)}$ for which*

$$\operatorname{dist}(\eta, \partial\Omega \setminus \Gamma_{i(\eta)}) \ge c \tag{1.25}$$

for some constant $c > 0$ independent of $\eta \in \partial\Omega$.

Proof. For any $j = 1, \dots, n$, the function $\eta \mapsto \operatorname{dist}(\eta, \partial\Omega \setminus \Gamma_j)$ is continuous on $\partial\Omega$ and so is the function $\eta \mapsto \max_j \operatorname{dist}(\eta, \partial\Omega \setminus \Gamma_j)$. There exists a point $\eta_* \in \partial\Omega$, where $\eta \mapsto \max_j \operatorname{dist}(\eta, \partial\Omega \setminus \Gamma_j)$ attains its minimum. Such a minimum cannot be 0 because η_* belongs at least to one of the Γ_j. This implies that for any $\eta \in \partial\Omega$, there exists $i(\eta) \in \{1, \dots, n\}$ such that

$$\max_j \operatorname{dist}(\eta_*, \partial\Omega \setminus \Gamma_j) \le \max_j \operatorname{dist}(\eta, \partial\Omega \setminus \Gamma_j) = \operatorname{dist}(\eta, \partial\Omega \setminus \Gamma_{i(\eta)}).$$ □

Proof of Proposition 1.12. We start by writing, for $x \in \Omega$,

$$\frac{(-\Delta)^s U(x)}{\mathcal{A}(N,s)} = \mathrm{PV}\int_{\Omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x-y|^{N+2s}}\,dy + \int_{\mathcal{C}\Omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x-y|^{N+2s}}\,dy. \tag{1.26}$$

Let us begin with an estimate for

$$\mathrm{PV}\int_{\Omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x-y|^{N+2s}}\,dy.$$

Split the integral into

$$\int_{\Omega_1} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x-y|^{N+2s}}\,dy + \mathrm{PV}\int_{\Omega_2} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x-y|^{N+2s}}\,dy + \int_{\Omega_3} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x-y|^{N+2s}}\,dy,$$

where we have set

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$$

with

$$\Omega_1 = \Big\{y \in \Omega : \delta(y) > \frac{3}{2}\delta(x)\Big\},$$
$$\Omega_2 = \Big\{y \in \Omega : \frac{1}{2}\delta(x) \le \delta(y) \le \frac{3}{2}\delta(x)\Big\},$$
$$\Omega_3 = \Big\{y \in \Omega : \delta(y) < \frac{1}{2}\delta(x)\Big\}.$$

In Ω_1 we have in particular $\delta(y) > \delta(x)$ so that, since ψ is a decreasing function, the first integral contributes by a positive quantity. Now, let us turn to the integrals on Ω_2 and Ω_3. Set $x = \theta + \delta(x)\nabla\delta(x)$, $\theta \in \partial\Omega$. Up to a rotation and a translation, we can suppose that $\theta = 0$ and $\nabla\delta(x) = e_N$.

Let $\{\Gamma_j\}_{j=1}^n$ be a finite open covering of $\partial\Omega$ and let $\Gamma := \Gamma_{i(0)}$ (in the notations of the last lemma) be a neighbourhood of 0 on $\partial\Omega$ chosen from $\{\Gamma_j\}_{j=1}^n$ and for which (1.25) is fulfilled. Let also

$$\omega = \{y \in \mathbb{R}^N : y = \eta + \delta(y)\nabla\delta(y),\ \eta \in \Gamma\}.$$

The set $\Gamma \subset \partial\Omega$ can be described as the graph of the following C^2 function:

$$\gamma\colon B'_r(0) \subseteq \mathbb{R}^{N-1} \to \mathbb{R}, \quad \eta' \mapsto \gamma(\eta') \quad \text{such that } \eta = (\eta', \gamma(\eta')) \in \Gamma,$$

satisfying $\gamma(0) = |\nabla\gamma(0)| = 0$.

The integration on $(\Omega_2 \cup \Omega_3) \setminus \omega$ is of lower order with respect to the one on $(\Omega_2 \cup \Omega_3) \cap \omega$, since in the latter we have the singularity in x to deal with, while in the former $|x - y|$ is a quantity bounded below independently on x. Indeed, when $y \in (\Omega_2 \cup \Omega_3) \setminus \omega$, we have

$$|x - y| \geq |\eta + \delta(y)\nabla\delta(y)| - \delta(x) \geq |\eta| - \delta(y) - \delta(x) \geq \operatorname{dist}(0, \partial\Omega \setminus \Gamma) - \frac{5}{2}\delta(x),$$

where $\delta(x)$ is small and the first addend is bounded uniformly in x by (1.25).

We are left with

$$C \cdot \mathrm{PV} \int_{\Omega_2\cap\omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}}\,dy + C \int_{\Omega_3\cap\omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}}\,dy.$$

Let us split the remainder of the estimate in steps.

First step: The distance between x and y. We claim that there exists $c > 0$ such that

$$\begin{cases} |x - y|^2 \geq c(|\delta(x) - \delta(y)|^2 + |\eta'|^2), & y \in (\Omega_2 \cup \Omega_3) \cap \omega, \\ \quad y = \eta + \delta(y)\nabla\delta(y), & \eta = (\eta', \gamma(\eta')). \end{cases} \tag{1.27}$$

Since in our set of coordinates $x = \delta(x)e_N$, we can write

$$\begin{aligned} |x - y|^2 &= |\delta(x)e_N - \delta(y)e_N + \delta(y)e_N - y_N e_N - y'|^2 \\ &\geq |\delta(x) - \delta(y)|^2 - 2|\delta(x) - \delta(y)| \cdot |\delta(y) - y_N| + |\delta(y) - y_N|^2 + |y'|^2. \end{aligned}$$

We concentrate our attention on $|\delta(y) - y_N|$. The idea is to show that this is a small quantity. Indeed, in the particular case when Γ lies on the hyperplane $y_N = 0$, this quantity is actually zero. As in the definition of ω, we let $y = \eta + \delta(y)\nabla\delta(y)$ and $\eta = (\eta', \gamma(\eta')) \in \Gamma$. Thus, $y_N = \gamma(\eta') + \delta(y)\langle\nabla\delta(y), e_N\rangle$, where $\nabla\delta(y)$ is the inward unit normal to $\partial\Omega$ at the point η, so that

$$\nabla\delta(y) = \frac{(-\nabla\gamma(\eta'), 1)}{\sqrt{|\nabla\gamma(\eta')|^2 + 1}},$$

$$y' = \eta' - \frac{\delta(y)\nabla\gamma(\eta')}{\sqrt{|\nabla\gamma(\eta')|^2 + 1}} \quad \text{and} \quad y_N = \gamma(\eta') + \frac{\delta(y)}{\sqrt{|\nabla\gamma(\eta')|^2 + 1}}. \tag{1.28}$$

Now, since $y \in \Omega_2 \cup \Omega_3$, we have $|\delta(x) - \delta(y)| \leq \delta(x)$ and

$$|\delta(y) - y_N| \leq |\gamma(\eta')| + \delta(y)\left(1 - \frac{1}{\sqrt{|\nabla\gamma(\eta')|^2 + 1}}\right) \leq C|\eta'|^2 + 2C\delta(x)|\eta'|^2,$$

where, in this case, $C = \|\gamma\|_{C^2(B_r)}$ depends only on the geometry of $\partial\Omega$ and not on x. By (1.28), we have

$$|\eta'|^2 \leq 2|y'|^2 + \frac{2\delta(y)^2\,|\nabla\gamma(\eta')|^2}{|\nabla\gamma(\eta')|^2 + 1} \leq 2|y'|^2 + 2C\delta(y)^2\,|\eta'|^2 \leq 2|y'|^2 + C\delta(x)^2\,|\eta'|^2,$$

so that $|\eta'|^2 \le C|y'|^2$ when $\delta(x)$ is small enough. Finally,

$$\begin{aligned}|x-y|^2 &\ge |\delta(x)-\delta(y)|^2 + |y'|^2 - 2|\delta(x)-\delta(y)|\cdot|\delta(y)-y_N|\\ &\ge |\delta(x)-\delta(y)|^2 + c|\eta'|^2 - 2C\delta(x)|\eta'|^2,\end{aligned}$$

where, again, $C = \|\gamma\|_{C^2(B_r)}$ and (1.27) is proved provided x is close enough to $\partial\Omega$.

Second step: Integration on $\Omega_2 \cap \omega$. Using the regularity of ψ and δ we write

$$\psi(\delta(x)^s) - \psi(\delta(y)^s) \ge \nabla(\psi\circ\delta^s)(x)\cdot(x-y) - \|D^2(\psi\circ\delta^s)\|_{L^\infty(\Omega_2\cap\omega)}|x-y|^2,$$

where

$$D^2(\psi\circ\delta^s) = \frac{s\psi'(\delta^s)}{\delta^{1-s}}D^2\delta + \frac{s^2\psi''(\delta^s)}{\delta^{2-2s}}\nabla\delta\otimes\nabla\delta + \frac{s(s-1)\psi'(\delta^s)}{\delta^{2-s}}\nabla\delta\otimes\nabla\delta,$$

so that

$$\|D^2(\psi\circ\delta^s)\|_{L^\infty(\Omega_2\cap\omega)} \le C\left\|\frac{\psi'(\delta^s)}{\delta^{1-s}}\right\|_{L^\infty(\Omega_2\cap\omega)} + C\left\|\frac{\psi''(\delta^s)}{\delta^{2-2s}}\right\|_{L^\infty(\Omega_2\cap\omega)} + C\left\|\frac{\psi'(\delta^s)}{\delta^{2-s}}\right\|_{L^\infty(\Omega_2\cap\omega)}.$$

By the definition of Ω_2 and by (1.21), we can control the sup-norm by the value at x, i.e.,

$$\|D^2(\psi\circ\delta^s)\|_{L^\infty(\Omega_2\cap\omega)} \le C\frac{|\psi'(\delta(x)^s)|}{\delta(x)^{1-s}} + C\frac{\psi''(\delta(x)^s)}{\delta(x)^{2-2s}} + C\frac{|\psi'(\delta(x)^s)|}{\delta(x)^{2-s}} \le C\frac{\psi''(\delta(x)^s)}{\delta(x)^{2-2s}} + C\frac{|\psi'(\delta(x)^s)|}{\delta(x)^{2-s}},$$

and using equation (1.22), we finally get

$$\|D^2(\psi\circ\delta^s)\|_{L^\infty(\Omega_2\cap\omega)} \le C\frac{\psi''(\delta(x)^s)}{\delta(x)^{2-2s}}.$$

If we now retrieve the whole integral and exploit (1.27), we have

$$\begin{aligned}\mathrm{PV}\int_{\Omega_2\cap\omega}\frac{\psi(\delta(x)^s)-\psi(\delta(y)^s)}{|x-y|^{N+2s}}\,dy &\ge -C\frac{\psi''(\delta(x)^s)}{\delta(x)^{2-2s}}\int_{\Omega_2\cap\omega}\frac{dy}{|x-y|^{N+2s-2}}\\ &\ge -C\frac{\psi''(\delta(x)^s)}{\delta(x)^{2-2s}}\int_{\Omega_2\cap\omega}\frac{dy}{(|\delta(x)-\delta(y)|^2+|\eta|^2)^{(N+2s-2)/2}}.\end{aligned}$$

We focus our attention on the integral on the right-hand side. By the coarea formula,

$$\begin{aligned}\int_{\Omega_2\cap\omega}\frac{dy}{(|\delta(x)-\delta(y)|^2+|\eta|^2)^{(N+2s-2)/2}} &= \int_{\delta(x)/2}^{3\delta(x)/2}dt\int_{\{\delta(y)=t\}\cap\omega}\frac{d\sigma(\eta)}{(|\delta(x)-t|^2+|\eta|^2)^{(N+2s-2)/2}}\\ &\le C\int_{\delta(x)/2}^{3\delta(x)/2}dt\int_{B_r}\frac{d\eta'}{(|\delta(x)-t|^2+|\eta'|^2)^{(N+2s-2)/2}}\\ &\le C\int_{\delta(x)/2}^{3\delta(x)/2}dt\int_0^r\frac{\rho^{N-2}}{(|\delta(x)-t|^2+\rho^2)^{(N+2s-2)/2}}\,d\rho\\ &\le C\int_{\delta(x)/2}^{3\delta(x)/2}dt\int_0^r\frac{\rho}{(|\delta(x)-t|^2+\rho^2)^{(2s+1)/2}}\,d\rho\\ &\le C\int_{\delta(x)/2}^{3\delta(x)/2}\frac{dt}{|t-\delta(x)|^{2s-1}}.\end{aligned}$$

We can retrieve now the chain of inequalities we stopped above:

$$\int_{\Omega_3\cap\omega}\frac{\psi(\delta(x)^s)-\psi(\delta(y)^s)}{|x-y|^{N+2s}}\,dy \ge -C\frac{\psi''(\delta(x)^s)}{\delta(x)^{2-2s}}\int_{\delta(x)/2}^{3\delta(x)/2}\frac{dt}{|\delta(x)-t|^{-1+2s}} \ge -C\,\psi''(\delta(x)^s).$$

Third step: Integration on $\Omega_3 \cap \omega$. Using (1.27) once again, we have

$$\begin{aligned}\int_{\Omega_3\cap\omega} \frac{\psi(\delta(x)^s)-\psi(\delta(y)^s)}{|x-y|^{N+2s}}\,dy &\geq -\int_{\Omega_3\cap\omega} \frac{\psi(\delta(y)^s)}{|x-y|^{N+2s}}\,dy \\ &\geq -C\int_{\Omega_3\cap\omega} \frac{\psi(\delta(y)^s)}{(|\delta(x)-\delta(y)|^2+|\eta'|^2)^{\frac{N+2s}{2}}}\,dy \\ &\geq -C\int_0^{\delta(x)/2} \frac{\psi(t^s)}{(\delta(x)-t)^{1+2s}}\,dt \\ &\geq -\frac{C}{\delta(x)^{1+2s}}\int_0^{\delta(x)/2} \psi(t^s)\,dt.\end{aligned}$$

The term we have obtained is of the same order of $\delta(x)^{-2s}\psi(\delta(x)^s)$, thus by (1.20),

$$\int_0^{\delta(x)/2} \psi(t^s)\,dt \asymp \int_0^{\delta(x)/2} t^s\psi'(t^s)\,dt = \frac{\delta(x)}{2s}\psi\Big(\frac{\delta(x)^s}{2^s}\Big) - \frac{1}{s}\int_0^{\delta(x)/2} \psi(t^s)\,dt,$$

so that

$$\int_0^{\delta(x)/2} \psi(t^s)\,dt \asymp \delta(x)\psi(\delta(x)^s) = \delta(x)^{1+2s}\cdot\frac{\psi(\delta(x)^s)}{\delta(x)^{2s}}. \tag{1.29}$$

Recall now that $\psi(\delta(x)^s)\delta(x)^{-2s}$ is in turn of the same size of $\psi''(\delta(x)^s)$ by (1.22).

Fourth step: The outside integral in (1.26). We focus now our attention on

$$\int_{\mathcal{C}\Omega} \frac{\psi(\delta(y)^s)-\psi(\delta(x)^s)}{|x-y|^{N+2s}}\,dy.$$

First, by using the monotonicity of ψ, we write

$$\int_{\mathcal{C}\Omega} \frac{\psi(\delta(y)^s)-\psi(\delta(x)^s)}{|x-y|^{N+2s}}\,dy \leq \int_{\{y\in\mathcal{C}\Omega:\delta(y)<\delta(x)\}\cap\omega} \frac{\psi(\delta(y)^s)-\psi(\delta(x)^s)}{|x-y|^{N+2s}}\,dy + \int_{\{y\in\mathcal{C}\Omega:\delta(y)<\delta(x)\}\setminus\omega} \frac{\psi(\delta(y)^s)-\psi(\delta(x)^s)}{|x-y|^{N+2s}}\,dy.$$

The second integral gives

$$\int_{\{y\in\mathcal{C}\Omega:\delta(y)<\delta(x)\}\setminus\omega} \frac{\psi(\delta(y)^s)-\psi(\delta(x)^s)}{|x-y|^{N+2s}}\,dy \leq C\|\psi(\delta^s)\|_{L^1(\mathbb{R}^N)},$$

because the distance between x and y is bounded there. Again we point out that

$$\begin{aligned}\int_{\{\delta(y)<\delta(x)\}\cap\omega} \frac{\psi(\delta(y)^s)-\psi(\delta(x)^s)}{|x-y|^{N+2s}}\,dy &\leq C\int_0^{\delta(x)} \frac{\psi(t^s)-\psi(\delta(x)^s)}{|\delta(x)+t|^{1+2s}}\,dt \\ &\leq C\int_0^{\delta(x)/2} \frac{\psi(t^s)}{|\delta(x)+t|^{1+2s}}\,dt + C\int_{\delta(x)/2}^{\delta(x)} \frac{\psi(t^s)}{|\delta(x)+t|^{1+2s}}\,dt \\ &\leq C\delta(x)^{-1-2s}\int_0^{\delta(x)/2} \psi(t^s)\,dt + C\psi\Big(\frac{\delta(x)^s}{2^s}\Big)\int_{\delta(x)/2}^{\delta(x)} (\delta(x)+t)^{-1-2s}\,dt \\ &\leq C\delta(x)^{-1-2s}\int_0^{\delta(x)} \psi(t^s)\,dt + C\psi(\delta(x)^s)\delta(x)^{-2s},\end{aligned}$$

which is of the order of $\psi''(\delta(x)^s)$, by (1.29) and (1.22).

In conclusion, we have proved that for $\delta(x)$ sufficiently small,

$$(-\Delta)^s U(x) \geq -C\psi''(\delta(x)^s).$$

Recall now that $\psi''(\delta^s) = f(\psi \circ \delta^s)$ and $U = \psi \circ \delta^s$ in Ω, so that

$$(-\Delta)^s U \geq -Cf(U)$$

holds in a neighbourhood of $\partial\Omega$. □

Starting from U, it is possible to build a full supersolution in view of the following lemma.

Lemma 1.14. *Let $v\colon \mathbb{R}^N \to \mathbb{R}$ a function which satisfies $(-\Delta)^s v \in C(\Omega)$. If there exist $C, \delta_0 > 0$ such that*

$$(-\Delta)^s v \geq -Cf(v) \quad \text{in } \Omega_{\delta_0} := \{x \in \Omega : \delta(x) < \delta_0\},$$

then there exists $\overline{u} \geq v$ such that $(-\Delta)^s \overline{u} \geq -f(\overline{u})$ throughout Ω.

Proof. Define $\xi\colon \mathbb{R}^N \to \mathbb{R}$ as the solution to

$$\begin{cases} (-\Delta)^s \xi = 1 & \text{in } \Omega, \\ \xi = 0 & \text{in } \mathcal{C}\Omega, \\ E\xi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.30}$$

and consider $\overline{u} = \mu v + \lambda\xi$, where $\mu, \lambda \geq 1$. If $C \in (0, 1]$, then $(-\Delta)^s v \geq -f(v)$ in Ω_{δ_0}, so choose $\mu = 1$. If $C > 1$, then choose $\mu = C^{1/M} > 1$, in order to have in Ω_{δ_0},

$$(-\Delta)^s \overline{u} + f(\overline{u}) = (-\Delta)^s(\mu v + \lambda\xi) + f(\mu v + \lambda\xi) \geq -\mu\, Cf(v) + f(\mu v) \geq (-\mu\, C + \mu^{1+M})f(v) = 0,$$

where we have heavily used the positivity of ξ and (1.17). Now, since $(-\Delta)^s v \in C(\overline{\Omega \setminus \Omega_{\delta_0}})$, we can choose $\lambda = \mu\|(-\Delta)^s v\|_{L^\infty(\Omega\setminus\Omega_{\delta_0})}$ so that, also in $\Omega \setminus \Omega_{\delta_0}$,

$$(-\Delta)^s \overline{u} = (-\Delta)^s(\mu v + \lambda\xi) = \mu(-\Delta)^s v + \lambda \geq 0 \geq -f(\overline{u}).$$ □

2 Existence

Lemma 2.1. *If the nonlinear term f satisfies the growth condition* (1.10), *then the function U defined in* (1.23) *satisfies*

$$\lim_{x\to\partial\Omega} \delta(x)^{1-s} U(x) = +\infty.$$

Proof. Write

$$\liminf_{x\to\partial\Omega} \delta(x)^{1-s}\psi(\delta(x)^s) = \liminf_{u\uparrow+\infty} u\,\phi(u)^{(1-s)/s}.$$

Such a limit is $+\infty$ if and only if

$$\liminf_{u\uparrow+\infty} u^{s/(1-s)} \int_u^{+\infty} \frac{dt}{\sqrt{2F(t)}} = +\infty.$$

If we use L'Hôpital's rule to

$$\frac{\int_u^{+\infty} \frac{dt}{\sqrt{2F(t)}}}{u^{-s/(1-s)}},$$

we get the ratio $u^{1/(1-s)}/\sqrt{2F(u)}$, and applying once again L'Hôpital's rule, this time to $u^{2/(1-s)}/F(u)$, we get $u^{(1+s)/(1-s)}/f(u)$, which diverges by hypothesis (1.10). Indeed, since f is increasing,

$$u^{-(1+s)/(1-s)} f(u) = f(u) \cdot \frac{1-s}{1+s} \int_u^{+\infty} t^{-2/(1-s)}\,dt \leq \int_u^{+\infty} f(t)t^{-2/(1-s)}\,dt \xrightarrow[u\uparrow+\infty]{} 0.$$ □

Collecting the information so far, we have that Lemmas 1.11, 1.14 and 2.1 fully prove the following theorems.

Theorem 2.2. *If the nonlinear term f satisfies the growth condition* (2.1)*, then there exists a function $\overline{u}$ which is a supersolution to* (1.11)*. Moreover,*

$$\overline{u} = \mu\psi(\delta^s) + \lambda\xi \quad \text{in } \Omega,$$

where ξ is the solution of (1.30)*, $\lambda > 0$ and $\mu = \max\{1, C^{1/M}\}$, where $C > 0$ is the constant in* (1.24) *and $M > 0$ the one in* (1.5).

Theorem 2.3. *There exists a function $\overline{u}$ which is a supersolution to* (1.14)*. Moreover,*

$$\overline{u} = \mu\psi(\delta^s) + \lambda\xi \quad \text{in } \Omega,$$

where ξ is the solution of (1.30)*, $\lambda > 0$ and $\mu = \max\{1, C^{1/M}\}$, where $C > 0$ is the constant in* (1.24) *and $M > 0$ the one in* (1.5).

2.1 Proof of Theorem 1.3

Build the sequence of solutions to problems of the following form:

$$\begin{cases} (-\Delta)^s u_k = -f(u_k) & \text{in } \Omega, \\ u_k = 0 & \text{in } \mathcal{C}\Omega, \\ E\, u_k = k & \text{on } \partial\Omega,\ k \in \mathbb{N}. \end{cases} \tag{2.1}$$

The existence of any u_k can be proved as in [1, Theorem 1.2.12], in view of hypothesis (1.10), since it implies

$$\int_0^{\delta_0} f(\delta^{s-1})\delta^s \, d\delta < +\infty.$$

The first tool we need is a Comparison Principle.

Lemma 2.4 (Comparison principle). *Let $v, w \in C(\Omega) \cap L^1(\Omega)$ solve pointwisely*

$$\begin{cases} (-\Delta)^s v \le -f(v) & \text{in } \Omega, \\ v \le 0 & \text{in } \mathcal{C}\Omega \end{cases} \quad \text{and} \quad \begin{cases} (-\Delta)^s w \ge -f(w) & \text{in } \Omega, \\ w \ge 0 & \text{in } \mathcal{C}\Omega. \end{cases}$$

If $v \le w$ in $U_\alpha := \{x \in \Omega : \delta(x) < \alpha\}$ for some $\alpha > 0$, then $v \le w$ in the whole Ω.

Proof. Consider $\Omega^+ = \{v > w\} \subset (\Omega \setminus U_\alpha)$. The difference $v - w$ achieves its (global) maximum at some point $x^* \in \Omega^+$. So

$$0 < (-\Delta)^s(v - w)(x^*) \le f(w(x^*)) - f(v(x^*)) \le 0,$$

in view of the monotonicity of f. Thus, Ω^+ must be empty. □

Step 1: $\{u_k\}_{k\in\mathbb{N}}$ has a pointwise limit. Any u_k solves the equation in a pointwise sense, as Lemma 4.3 below implies. The sequence $\{u_k\}_{k\in\mathbb{N}}$ is increasing with k by the comparison principle (Lemma 2.4). Moreover, any u_k lies below $\overline{u}$. Indeed, since $E(\overline{u} - u_k) = +\infty$, then $u_k \le \overline{u}$ holds close to $\partial\Omega$ and another application of the comparison principle yields $u_k \le \overline{u}$ in Ω.

Finally, $\{u_k\}_{k\in\mathbb{N}}$ is increasing and pointwisely bounded by $\overline{u}$ throughout Ω. This entails that

$$u(x) := \lim_{k\uparrow+\infty} u_k(x)$$

is well defined and finite for any $x \in \Omega$. Also, $0 \le u \le \overline{u}$ in Ω and since $\overline{u} \in L^1(\Omega)$, by Lemma 1.11 we have that $u \in L^1(\Omega)$.

Step 2: $u \in C(\Omega)$. Fix any compact $D \subset \Omega$ and choose $c > 0$ such that $\delta(x) > 2c$ for any $x \in D$. Let

$$\widetilde{D} := \{y \in \Omega : \delta(y) > c\}.$$

For any $k,\ j \in \mathbb{N}$, we have

$$(-\Delta)^s(u_{k+j} - u_k) = f(u_k) - f(u_{k+j}) \le 0 \quad \text{in } \widetilde{D},$$

and therefore

$$0 \le u_{k+j}(x) - u_k(x) \le \int_{\complement\widetilde{D}} P_{\widetilde{D}}(x,y)[u_{k+j}(y) - u_k(y)]\,dy,$$

where $P_{\widetilde{D}}(x,y)$ is the Poisson kernel associated to $\widetilde{D}$, which satisfies (see [8, Theorem 2.10])

$$P_{\widetilde{D}}(x,y) \le \frac{C\,\delta_{\widetilde{D}}(x)^s}{\delta_{\widetilde{D}}(y)^s|x-y|^N}, \quad x \in \widetilde{D},\ y \in \complement\widetilde{D}.$$

When $x \in D \subset \widetilde{D}$ one has $|x-y| > c$ for any $y \in \complement\widetilde{D}$, and therefore

$$0 \le u_{k+j}(x) - u_k(x) \le C\int_{\complement\widetilde{D}} \frac{u_{k+j}(y) - u_k(y)}{\delta_{\widetilde{D}}(y)^s}\,dy \le C\int_{\complement\widetilde{D}} \frac{u(y) - u_k(y)}{\delta_{\widetilde{D}}(y)^s}\,dy,$$

where the last integral converges by monotone convergence to 0 independently on x. This means the convergence $u_k \to u$ is uniform on compact subsets and since $\{u_k\}_{k\in\mathbb{N}} \subset C(\Omega)$ (cf. [1, Theorem 1.2.12]), we have also that $u \in C(\Omega)$.

Step 3: $u \in C^2(\Omega)$. This is a standard bootstrap argument using the elliptic regularity in [25, Propositions 2.8 and 2.9].

Step 4: u solves (1.11) *in a pointwise sense.* The function $(-\Delta)^s u(x)$ is well defined for any $x \in \Omega$ because $u \in C^2(\Omega) \cap L^1(\mathbb{R}^N)$. Using the regularity results in [25, Propositions 2.8 and 2.9], we have

$$(-\Delta)^s u = \lim_{k\uparrow+\infty} (-\Delta)^s u_k = -\lim_{k\uparrow+\infty} f(u_k) = -f(u).$$

Also, $\delta^{1-s}u \ge \delta^{1-s}u_k$ holds in Ω for any $k \in \mathbb{N}$. Therefore, for any $k \in \mathbb{N}$,

$$\liminf_{x\to\partial\Omega} \delta(x)^{1-s}u(x) \ge \lim_{x\to\partial\Omega} \delta(x)^{1-s}u_k(x) \ge \lambda E\,u_k = \lambda k$$

for some constant $\lambda > 0$ depending on Ω and not on k. This entails

$$\lim_{x\to\partial\Omega} \delta(x)^{1-s}u(x) = +\infty$$

and completes the proof of Theorem 1.3.

Remark 2.5. The proof of Theorem 1.5 is similar. Indeed, in the same way, the sequence of solutions to the following problem:

$$\begin{cases} (-\Delta)^s u_k = -f(u_k) & \text{in } \Omega, \\ u_k = g_k := \min\{k, g\} & \text{in } \complement\Omega,\ k \in \mathbb{N}, \\ E\,u_k = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

approaches a solution of problem (1.14), which lies below the supersolution provided by Theorem 2.3.

2.2 Proof of Theorem 1.9

Following [1], we write the Green representation for u_k:

$$u_k(x) = k\int_{\partial\Omega} M_\Omega(x,\theta)\,d\sigma(\theta) - \int_\Omega G_\Omega(x,y)f(u_k(y))\,dy, \quad x \in \Omega.$$

Denoting simply

$$h_1(x) := \int_{\partial\Omega} M_\Omega(x,\theta)\,d\sigma(\theta) \quad \text{and} \quad \xi(x) := \int_\Omega G_\Omega(x,y)\,dy,$$

we get

$$u_k(x) \ge kh_1(x) - a\xi(x) - b\int_\Omega G_\Omega(x,y)\,u_k(y)\,dy \ge kh_1(x) - a\xi(x) - bk\int_\Omega G_\Omega(x,y)\,h_1(y)\,dy. \tag{2.3}$$

Recall that $\xi \asymp \delta^s$ and $h_1 \asymp \delta^{s-1}$. Applying [1, Proposition 3], we see that

$$h_1(x) - b \int_\Omega G_\Omega(x,y)\, h_1(y)\, dy > 0$$

holds when x is taken close enough to $\partial\Omega$. This concludes the proof.

3 The power case: Proof of Theorem 1.10

Proof of (1). We show how the following problem:

$$\begin{cases} (-\Delta)^s u_1 = -u_1^p & \text{in } \Omega, \\ u_1 = 0 & \text{in } \complement\Omega, \\ E\, u_1 = 1 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

does not admit any weak or pointwise solution.

In both cases the solution would satisfy $u_1 \geq c\delta^{s-1}$ in Ω for some $c > 0$. If u_1 was a weak solution then for any $\phi \in \mathcal{T}(\Omega)$,

$$\int_\Omega u_1\, (-\Delta)^s \phi + \int_\Omega u_1^p \phi = \int_{\partial\Omega} D_s \phi,$$

where

$$\int_\Omega u_1^p \phi \geq C \int_\Omega \delta^{p(s-1)} \delta^s = +\infty,$$

because (1.10) does not hold, a contradiction.

If u_1 was a pointwise solution, then by Lemma 4.4 it would be a weak solution on any subdomain $D \subset \overline{D} \subset \Omega$. Therefore,

$$u_1(x) = -\int_D G_D(x,y)\, u_1(y)^p\, dy + \int_{\complement D} P_D(x,y)\, u_1(y)\, dy.$$

If u_0 denotes the s-harmonic function induced by $E\,u = 1$, then $u_1 \leq u_0$ in Ω and

$$u_1(x) \leq -\int_D G_D(x,y)\, u_1(y)^p\, dy + \int_{\complement D} P_D(x,y)\, u_0(y)\, dy = -\int_D G_D(x,y) u_1(y)^p\, dy + u_0(x).$$

Fix $x \in \Omega$. Letting now $D \nearrow \Omega$ we have that $G_D(x,y) \uparrow G_\Omega(x,y)$ and

$$\int_\Omega G_\Omega(x,y)\, u_1(y)^p\, dy \geq c\delta(x)^s \int_{\{2\delta(y)<\delta(x)\}} \delta(y)^s\, u_1(y)^p\, dy = +\infty,$$

because (1.10) does not hold, a contradiction. □

Proof of (2). We apply Theorem 1.3 when $f(t) = t^p$. In this case,

$$\frac{tf'(t)}{f(t)} = p > 1,$$

so that hypothesis (1.5) is fulfilled. The function ϕ reads as (cf. equation (1.6))

$$\phi(u) = \int_u^{+\infty} \sqrt{\frac{p+1}{2}}\, t^{-(p+1)/2}\, dt = \sqrt{\frac{2(p+1)}{p-1}}\, u^{(1-p)/2}.$$

Hypothesis (1.7) can then be written as

$$\int_u^{+\infty} \eta^{(1-p)/(2s)}\, d\eta < +\infty,$$

which holds if and only if $p > 1 + 2s$. On the other hand, hypothesis (1.10) becomes

$$p - \frac{2}{1-s} < -1, \quad \text{i.e.,} \quad p < \frac{1+s}{1-s} = 1 + \frac{2s}{1-s}. \qquad \square$$

Remark 3.1. We retrieve in this case some of the results in [6, Theorem 1.1, equations (1.6) and (1.7)] and we obtain the explicit value of the parameter denoted[3] by $\tau_0(\alpha) \in (-1, 0)$, which is $\tau_0(\alpha) = \alpha - 1$.

Proof of (3). Following [1], we write the Green representation for u_k:

$$u_k(x) = k \int_{\partial\Omega} M_\Omega(x,\theta)\, d\sigma(\theta) - \int_\Omega G_\Omega(x,y)\, u_k(y)^p\, dy, \quad x \in \Omega.$$

Denoting simply

$$h_1(x) := \int_{\partial\Omega} M_\Omega(x,\theta)\, d\sigma(\theta),$$

we have $u_k \le k h_1$ in Ω and

$$u_k(x) \ge k h_1(x) - k^s \int_\Omega G_\Omega(x,y)\, u_k(y)^{p-s}\, h_1(y)^s\, dy. \tag{3.2}$$

Define

$$\xi(x) := \int_\Omega G_\Omega(x,y)\, dy,$$

and recall that $\xi \asymp \delta^s$, while $h_1 \asymp \delta^{s-1}$. By (3.2) we deduce

$$\int_\Omega u_k \ge k \int_\Omega h_1 - \int_\Omega u_k^{p-s} h_1^s \xi.$$

Since $p \in (1, 1+s)$, we have that $p - s \in (1-s, 1)$ and $u_k^{p-s} \le u_k$. Thus, there exists a constant $C > 0$ such that

$$\int_\Omega u_k + C \int_\Omega u_k \delta^{s(s-1)+s} \ge k \int_\Omega h_1,$$

where $s(s-1) + s > 0$, so (modifying C if necessary)

$$(C+1) \int_\Omega u_k \ge k \int_\Omega h_1,$$

which concludes the proof. $\square$

Proof of (4). This is a straightforward consequence of Theorem 1.9. $\square$

4 Remarks and comments

In this section we would like to point out some elements that may be unclear if left implicit. In the first subsection we discuss the relation between pointwise solutions and weak L^1 solutions. The second one studies the equivalence of variational weak solutions and weak L^1 solutions in some particular cases. The third and last subsection deals with the definition of weak L^1 solution given by Chen and Véron [7], which amounts to be equivalent to the one given in [1].

3 In the notations of [6], $\alpha \in (0, 1)$ is the power of the Laplacian, which corresponds to s in our notations.

4.1 Pointwise solutions vs weak L^1 solutions

For the sake of clarity we recall here the definitions involved. In the following, Ω will always be a bounded open subset of $\mathbb{R}^N$ with C^2 boundary.

Definition 4.1. Given three measurable functions

$$f\colon \Omega \to \mathbb{R}, \quad g\colon \mathcal{C}\Omega \to \mathbb{R} \quad \text{and} \quad h\colon \partial\Omega \to \mathbb{R},$$

a function $u\colon \mathbb{R}^N \to \mathbb{R}$ is said to be a pointwise solution of

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = g & \text{in } \mathcal{C}\Omega, \\ E\,u = h & \text{on } \partial\Omega, \end{cases}$$

provided that the following hold:

(i) $u \in L^1(\Omega)$.

(ii) For any $x \in \mathcal{C}\Omega$, $u(x) = g(x)$.

(iii) The principal value

$$\mathrm{PV}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy$$

converges for any $x \in \Omega$ and

$$\mathcal{A}(N,s)\,\mathrm{PV}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy = f(x) \quad \text{for any } x \in \Omega.$$

(iv) For any $\theta \in \partial\Omega$ the limit $\lim_{x\to\theta} \delta(x)^{1-s}u(x)$ exists and the renormalized limit $E\,u$ satisfies $E\,u(\theta) = h(\theta)$.

Definition 4.2. Given three measurable functions

$$f\colon \Omega \to \mathbb{R}, \quad g\colon \mathcal{C}\Omega \to \mathbb{R} \quad \text{and} \quad h\colon \partial\Omega \to \mathbb{R},$$

a function $u\colon \mathbb{R}^N \to \mathbb{R}$ is said to be a weak L^1 solution of

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = g & \text{in } \mathcal{C}\Omega, \\ E\,u = h & \text{on } \partial\Omega, \end{cases}$$

provided $u \in L^1(\Omega)$ and for any

$$\phi \in \mathcal{T}(\Omega) = \{\phi \in C^s(\mathbb{R}^N) : (-\Delta)^s\phi|_\Omega \in C_c^\infty(\Omega),\ \phi = 0 \text{ in } \mathcal{C}\Omega\}$$

the following holds:

$$\int_\Omega u(-\Delta)^s\phi = \int_\Omega f\phi - \int_{\mathcal{C}\Omega} g(-\Delta)^s\phi + \int_{\partial\Omega} hD_s\phi.$$

For further details and notation, we refer to [1].

Lemma 4.3. *Take $f \in C^\alpha_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$ with*

$$\int_\Omega |f|\delta^s < +\infty,$$

$g\colon \mathcal{C}\Omega \to \mathbb{R}$ measurable with

$$\int_{\mathcal{C}\Omega} |g|\delta^{-s}\min\{1, \delta^{-N-s}\} < +\infty,$$

$h \in C(\partial\Omega)$ and $u\colon \mathbb{R}^N \to \mathbb{R}$ a weak L^1 solution to

$$\begin{cases} (-\Delta)^s u = f & \textit{in } \Omega, \\ u = g & \textit{in } \mathcal{C}\Omega, \\ E\,u = h & \textit{on } \partial\Omega. \end{cases}$$

Then, u is also a pointwise solution.

Proof. We can write u as the sum

$$u(x) = \int_\Omega G_\Omega(x,y)f(y)\,dy + u_0(x),$$

where u_0 is the s-harmonic function induced in Ω by the data g and h. For any $x \in \Omega$, in a pointwise sense we have that $(-\Delta)^s u(x) = f(x)$, in view of the regularity of f and the construction of the Green kernel. Then, to completely prove the lemma, it suffices to prove

$$\lim_{x\to\partial\Omega} \delta(x)^{1-s} \int_\Omega G_\Omega(x,y)\,f(y)\,dy = 0.$$

This is proved in Lemma 4.5 below. □

Lemma 4.4. *Take $f \in C^\alpha_{loc}(\Omega)$ for some $\alpha \in (0,1)$, $h \in C(\partial\Omega)$ and $u\colon \mathbb{R}^N \to \mathbb{R}$ a pointwise solution to*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = g & \text{in } \mathcal{C}\Omega, \\ Eu = h & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

If

$$\int_\Omega |f|\delta^s < +\infty, \qquad \int_{\mathcal{C}\Omega} |g|\delta^{-s}\min\{1,\delta^{-N-s}\} < +\infty \quad \text{and} \quad h \in C(\partial\Omega),$$

then u is also a weak L^1 solution to the same problem.

Proof. We refer to [1, Theorem 1.2.8] for the existence and uniqueness of a weak L^1 solution v to problem (4.1). By Lemma 4.3, v is also a pointwise solution. Thus,

$$\begin{cases} (-\Delta)^s (u-v) = 0 & \text{in } \Omega, \\ u - v = 0 & \text{in } \mathcal{C}\Omega, \\ E(u-v) = 0 & \text{on } \partial\Omega \end{cases}$$

in a pointwise sense. In particular, $u - v \in C(\Omega)$, since harmonic functions are smooth. Define

$$\Omega^+ := \{x \in \Omega : u(x) > v(x)\},$$

in which $u - v$ is a nonnegative s-harmonic function and, by [4, Lemma 5 and Theorem 1], it decomposes into the sum of the s-harmonic function induced by the $E_{\Omega^+}(u-v)$ trace and the one by its values on $\mathcal{C}\Omega^+$. But, on the one hand $E_{\Omega^+}(u-v) = 0$ on $\partial\Omega^+$ as it is implied by the singular trace datum in (4.1) and the continuity on $\partial\Omega^+ \cap \Omega$ while, on the other $u - v \le 0$ in $\mathcal{C}\Omega^+$. This yields $\Omega^+ = \emptyset$ and $v \ge u$ in Ω. Repeating the argument, we deduce also $u \le v$ and this completes the proof of the lemma. □

Lemma 4.5. *Let $f\colon \Omega \to \mathbb{R}$ be a continuous function such that*

$$\int_\Omega |f|\delta^s < +\infty. \tag{4.2}$$

Then,

$$\lim_{\eta\downarrow 0}\left(\frac{1}{\eta}\int_{\{\delta(x)<\eta\}} \delta(x)^{1-s}\int_\Omega G_\Omega(x,y)f(y)\,dy\,dx\right) = 0. \tag{4.3}$$

Proof. Equation (4.3) expresses a notion of weak trace at the boundary introduced by Ponce [23, Proposition 3.5]. Choose $\eta > 0$ small and consider the integral

$$\frac{1}{\eta}\int_\Omega \delta(x)^{1-s}\chi_{(0,\eta)}(\delta(x))\int_\Omega G_\Omega(x,y)f(y)\,dy\,dx. \tag{4.4}$$

We are going to show that it converges to 0 as $\eta \downarrow 0$. By splitting f into its positive and negative part, we can assume, without loss of generality, that $f \ge 0$. Fix $\sigma \in (0,s)$ and exchange the order of integration in (4.4).

Our claim is that

$$\int_\Omega G_\Omega(x,y)\delta(x)^{1-s}\chi_{(0,\eta)}(\delta(x))\,dx \le \begin{cases} C\eta^{1+\sigma}\delta(y)^{s-\sigma} & \text{if } \delta(y)\ge\eta, \\ C\eta\delta(y)^s & \text{if } \delta(y)<\eta. \end{cases} \tag{4.5}$$

This would prove

$$\frac{1}{\eta}\int_\Omega f(y)\int_\Omega G_\Omega(x,y)\delta(x)^{1-s}\chi_{(0,\eta)}(\delta(x))\,dx\,dy \le C\eta^\sigma \int_{\{\delta(y)\ge\eta\}\cap\Omega} f(y)\delta(y)^{s-\sigma}\,dy + C\int_{\{\delta(y)<\eta\}\cap\Omega} f(y)\delta(y)^s\,dy,$$

where the second addend converges to 0 as $\eta\downarrow 0$ by (4.2). For the first addend, we have that $\eta^\sigma f(y)\delta(y)^{s-\sigma}$ converges pointwisely to zero for any $y\in\Omega$ and $\eta^\sigma f(y)\delta(y)^{s-\sigma}\le f(y)\delta(y)^s$ if $y\in\Omega\cap\{\delta(y)>\eta\}$. Therefore, we have the convergence to 0 by dominated convergence.

We turn now to the proof of (4.5). For any $y\in\Omega$ one has

$$\int_\Omega G_\Omega(x,y)\delta(x)^{1-s}\chi_{(0,\eta)}(\delta(x))\,dx \le \eta^{1+\sigma}\int_\Omega G_\Omega(x,y)\delta(x)^{-s-\sigma}\,dx \le C\eta^{1+\sigma}\delta(y)^{s-\sigma}, \tag{4.6}$$

where we have used the regularity at the boundary in [1, Proposition 1.2.9]. In particular, (4.6) holds when $\delta(y)>\eta$.

To prove the other part of (4.5), we write (dropping from now on multiplicative constants depending on N, Ω and s)

$$\int_\Omega G_\Omega(x,y)\,\delta(x)^{1-s}\chi_{(0,\eta)}(\delta(x))\,dx \le \eta^{1-s}\int_{\{\delta(x)<\eta\}\cap\Omega} \frac{(\delta(x)\delta(y)\wedge|x-y|^2)^s}{|x-y|^N}\delta(x)^{1-s}\,dx,$$

and we are allowed to perform the computations only in the case where $\partial\Omega$ is locally flat where the above reads as

$$\int_0^\eta\int_B \frac{[x_Ny_N\wedge(|x'-y'|^2+|x_N-y_N|^2)]^s}{(|x'-y'|^2+|x_N-y_N|^2)^{N/2}}\cdot x_N^{1-s}\,dx'\,dx_N,$$

where $x=(x',x_N)\in\mathbb{R}^{N-1}\times\mathbb{R}$ and $y=(y',y_N)\in\mathbb{R}^{N-1}\times\mathbb{R}$. First note that we can assume, without loss of generality, that $y'=0$ and $a\wedge b\le 2ab/(a+b)$ when $a,b>0$. With the change of variable $x_N=y_Nt$ and $x'=y_N\xi$, we reduce to

$$y_N^{1+s}\int_0^{\eta/y_N}\int_{B_{1/y_N}} \frac{t}{(|\xi|^2+|t-1|^2)^{N/2-s}}\cdot\frac{d\xi}{(|\xi|^2+|t-1|^2+t)^s}\,dt$$

and, passing to polar coordinates in the ξ variable,

$$\begin{aligned} y_N^{1+s}&\int_0^{\eta/y_N}\int_0^{1/y_N} \frac{t\rho^{N-2}}{(\rho^2+|t-1|^2)^{N/2-s}}\cdot\frac{d\rho}{(\rho^2+|t-1|^2+t)^s}\,dt \\ &\le y_N^{1+s}\int_0^{\eta/y_N}\int_0^{1/y_N} \frac{t\rho}{(\rho^2+|t-1|^2)^{3/2-s}}\cdot\frac{d\rho}{(\rho^2+|t-1|^2+t)^s}\,dt. \end{aligned}$$

We deal first with the integral in the ρ variable. We have[4]

$$\begin{aligned} t&\int_0^{1/y_N} \frac{\rho}{(\rho^2+|t-1|^2)^{3/2-s}}\cdot\frac{d\rho}{(\rho^2+|t-1|^2+t)^s} \\ &\le \frac{t}{(|t-1|^2+t)^s}\int_0^1 \frac{\rho}{(\rho^2+|t-1|^2)^{3/2-s}}\,d\rho + t\int_1^{1/y_N}\frac{\rho}{(\rho^2+|t-1|^2)^{3/2}}\,d\rho \\ &\le \frac{t}{(|t-1|^2+t)^s}\cdot\frac{(\rho^2+|t-1|^2)^{-1/2+s}}{2s-1}\bigg|_{\rho=0}^1 + \frac{t}{(1+|t-1|^2)^{1/2}}. \end{aligned}$$

4 The computation which follows is not valid in the particular case $s=1/2$, but with some minor natural modifications the same idea will work.

Then,

$$t\int_0^{1/y_N}\frac{\rho}{(\rho^2+|t-1|^2)^{3/2-s}}\cdot\frac{d\rho}{(\rho^2+|t-1|^2+t)^s}\le\begin{cases}\dfrac{t}{(|t-1|^2+t)^s|t-1|^{1-2s}}+\dfrac{t}{(1+|t-1|^2)^{1/2}}, & s\in(0,1/2),\\[2mm] \dfrac{t(1+|t-1|^2)^{s-1/2}}{(|t-1|^2+t)^s}+\dfrac{t}{(1+|t-1|^2)^{1/2}}, & s\in(1/2,1).\end{cases}$$

The two quantities are both integrable in $t=1$ and converge to a positive constant as $t\uparrow+\infty$, therefore

$$y_N^{1+s}\int_0^{\eta/y_N}\int_0^{1/y_N}\frac{t\rho^{N-2}}{(\rho^2+|t-1|^2)^{N/2-s}}\cdot\frac{d\rho}{(\rho^2+|t-1|^2+t)^s}\,dt\le\eta y_N^s=\eta\delta(y)^s,$$

which completes the proof of (4.5). □

4.2 Variational weak solutions vs weak L^1 solutions

In this subsection we are going to prove the equivalence – for some class of Dirichlet problems – between the definition of weak L^1 solution and the more standard one of variational weak solution.

Definition 4.6. Given $f\in L^\infty(\Omega)$, a variational weak solution of

$$\begin{cases}(-\Delta)^s u=f & \text{in }\Omega,\\ u=0 & \text{in }\mathcal{C}\Omega\end{cases}\tag{4.7}$$

is a function $u\in H^s(\mathbb{R}^N)$ such that $u\equiv0$ in $\mathcal{C}\Omega$ and for any other $v\in H^s(\mathbb{R}^N)$ such that $v\equiv0$ in $\mathcal{C}\Omega$, we have

$$\int_{\mathbb{R}^N}(-\Delta)^{s/2}u\,(-\Delta)^{s/2}v=\int_\Omega f\,v.$$

Lemma 4.7. *Recall the definition of the space $\mathcal{T}(\Omega)$ given in Definition 4.2. We have that $\mathcal{T}(\Omega)\subset H^s(\mathbb{R}^N)$.*

Proof. Consider $\phi\in\mathcal{T}(\Omega)$. The fractional Laplacian $(-\Delta)^{s/2}\phi$ is a continuous function decaying like $|x|^{-N-s}$ at infinity. So $\|(-\Delta)^{s/2}\phi\|_{L^2(\mathbb{R}^N)}<\infty$ and we can apply [10, Proposition 3.6]. □

Proposition 4.8. *Let $f\in L^\infty(\Omega)$. Let u be a variational weak solution of (4.7). Then, it is also a weak L^1 solution to the problem*

$$\begin{cases}(-\Delta)^s u=f & \textit{in }\Omega,\\ u=0 & \textit{in }\mathcal{C}\Omega,\\ Eu=0 & \textit{on }\partial\Omega.\end{cases}$$

Proof. Consider $\phi\in\mathcal{T}(\Omega)$. Then,

$$\int_\Omega u(-\Delta)^s\phi=\int_{\mathbb{R}^N}(-\Delta)^{s/2}u(-\Delta)^{s/2}\phi=\int_\Omega f\phi,$$

where we have used Lemma 4.7 on ϕ. □

Proposition 4.9. *Let $f\in L^\infty(\Omega)$. Let u be a weak L^1 solution to the problem*

$$\begin{cases}(-\Delta)^s u=f & \textit{in }\Omega,\\ u=0 & \textit{in }\mathcal{C}\Omega,\\ Eu=0 & \textit{on }\partial\Omega.\end{cases}$$

Then, u is also a variational weak solution of (4.7).

Proof. Call w the variational weak solution of (4.7). By the previous Lemma, w is also a weak L^1 solution. We thus conclude $u=w$, by the uniqueness of a weak L^1 solution. □

4.3 The test function space

In [7] the following definition of a weak solution is given.

Definition 4.10. Given a Radon measure ν such that $\delta^s \in L^1(\Omega, d\nu)$, a function $u \in L^1(\Omega)$ is a weak solution of

$$\begin{cases} (-\Delta)^s u + f(u) = \nu & \text{in } \Omega, \\ u = 0 & \text{in } \mathcal{C}\Omega, \end{cases}$$

if $f(u) \in L^1(\Omega, \delta^s\, dx)$ and

$$\int_\Omega u(-\Delta)^s \xi + \int_\Omega f(u)\xi = \int_\Omega \xi\, d\nu$$

for any $\xi \in \mathbb{X}_s \subset C(\mathbb{R}^N)$, i.e., the following hold:
(1) $\operatorname{supp} \xi \subseteq \overline{\Omega}$.
(2) $(-\Delta)^s \xi(x)$ is pointwisely defined for any $x \in \Omega$ and $\|(-\Delta)^s \xi\|_{L^\infty(\Omega)} < +\infty$.
(3) There exist a positive $\phi \in L^1(\Omega, \delta^s dx)$ and $\varepsilon_0 > 0$ such that

$$|(-\Delta)^s_\varepsilon \xi(x)| = \bigg| \int_{\mathcal{C}B_\varepsilon(x)} \frac{\xi(x) - \xi(y)}{|x-y|^{N+2s}}\, dy \bigg| \le \phi(x) \quad \text{for all } \varepsilon \in (0, \varepsilon_0].$$

The test space $\mathbb{X}_s$ in Definition 4.10 is quite different from the space $\mathcal{T}(\Omega)$ which is used in Definition 4.2. Still, testing a Dirichlet problem against one or the other does not yield two different solutions, i.e., the two notions of weak L^1 solutions are equivalent. We split the proof of this fact into two lemmas.

Lemma 4.11. *We have that* $\mathcal{T}(\Omega) \subset \mathbb{X}_s$.

Proof. Pick $\phi \in \mathcal{T}(\Omega)$. Properties (1) and (2) of Definition 4.10 are satisfied by construction. In order to prove (3), write for $\delta(x) < 2\varepsilon$,

$$\begin{aligned} (-\Delta)^s_\varepsilon \phi(x) &= \psi(x) - \mathrm{PV} \int_{B_\varepsilon(x)} \frac{\phi(x) - \phi(y)}{|x-y|^{N+2s}}\, dy \\ &= \psi(x) - \mathrm{PV} \int_{B_{\delta(x)/2}(x)} \frac{\phi(x) - \phi(y)}{|x-y|^{N+2s}}\, dy - \int_{B_\varepsilon(x) \setminus B_{\delta(x)/2}(x)} \frac{\phi(x) - \phi(y)}{|x-y|^{N+2s}}\, dy \end{aligned} \tag{4.8}$$

with $\psi := (-\Delta)^s \phi|_\Omega \in C^\infty_c(\Omega)$. Consider $\alpha \in (0, s)$. For the first integral,

$$\begin{aligned} \bigg| \mathrm{PV} \int_{B_{\delta(x)/2}(x)} \frac{\phi(x) - \phi(y)}{|x-y|^{N+2s}}\, dy \bigg| &\le \|\phi\|_{C^{2s+\alpha}(B_{\delta(x)/2}(x))} \int_{B_{\delta(x)/2}(x)} \frac{dy}{|x-y|^{N-\alpha}} \\ &= \|\phi\|_{C^{2s+\alpha}(B_{\delta(x)/2}(x))} \frac{\omega_{N-1}}{\alpha} \Big(\frac{\delta(x)}{2} \Big)^\alpha, \end{aligned}$$

where, by [25, Proposition 2.8],

$$\begin{aligned} \|\phi\|_{C^{2s+\alpha}(B_{\delta(x)/2}(x))} &= 2^{2s+\alpha} \delta(x)^{-2s-\alpha} \Big\| \phi\Big(x + \frac{\delta(x)}{2} \cdot \Big) \Big\|_{C^{2s+\alpha}(B)} \\ &\le C\delta(x)^{-2s-\alpha} \Big(\Big\| \phi\Big(x + \frac{\delta(x)}{2} \cdot \Big) \Big\|_{L^\infty(B)} + \Big\| \psi\Big(x + \frac{\delta(x)}{2} \cdot \Big) \Big\|_{C^\alpha(B)} \Big) \\ &\le C\delta(x)^{-2s-\alpha} (\|\phi\|_{L^\infty(B_{\delta(x)/2}(x))} + \delta(x)^\alpha \|\psi\|_{C^\alpha(B_{\delta(x)/2}(x))}) \\ &\le C\delta(x)^{-2s-\alpha} (\|\psi\|_{L^\infty(\mathbb{R}^N)} \delta(x)^s + \delta(x)^\alpha \|\psi\|_{C^\alpha(\mathbb{R}^N)}) \\ &\le C\|\psi\|_{C^\alpha(\mathbb{R}^N)} \delta(x)^{-2s}. \end{aligned}$$

The integration far from x gives

$$\begin{aligned}\Bigg|\int_{B_\varepsilon(x)\setminus B_{\delta(x)/2}(x)} \frac{\phi(x)-\phi(y)}{|x-y|^{N+2s}}\,dy\Bigg| &\le \|\phi\|_{C^s(\mathbb{R}^N)} \int_{B_\varepsilon(x)\setminus B_{\delta(x)/2}(x)} \frac{dy}{|x-y|^{N+s}}\\ &\le \|\phi\|_{C^s(\mathbb{R}^N)} \int_{\mathbb{R}^N\setminus B_{\delta(x)/2}} \frac{dz}{|z|^{N+s}}\\ &\le \|\phi\|_{C^s(\mathbb{R}^N)} \frac{\omega_{N-1}}{s}\Big(\frac{2}{\delta(x)}\Big)^s.\end{aligned}$$

All this entails

$$\delta(x)^s|(-\Delta)^s_\varepsilon\phi(x)| \le \delta(x)^s|\psi(x)| + C\|\psi\|_{C^\alpha(\mathbb{R}^N)}\delta(x)^{\alpha-s} + C\|\phi\|_{C^s(\mathbb{R}^N)} \quad \text{when } \delta(x) < 2\varepsilon.$$

For $\delta(x) \ge 2\varepsilon$ one does not have the second integral on the right-hand side of (4.8) whereas the first one is computed on the ball of radius ε, where the same computations can be carried out. This proves the statement of the lemma. □

Lemma 4.12. *Given a Radon measure $\nu \in \mathcal{M}(\Omega)$ such that $\delta^s \in L^1(\Omega, d\nu)$, if a function $u \in L^1(\Omega)$ satisfies*

$$\int_\Omega u(-\Delta)^s\xi = \int_\Omega \xi\,d\nu \quad \text{for any } \xi \in \mathcal{T}(\Omega), \tag{4.9}$$

then the same holds true for any $\xi \in \mathbb{X}_s$.

Proof. Pick $\xi \in \mathbb{X}_s$. By definition, $\zeta := (-\Delta)^s\xi \in L^\infty(\Omega)$. Consider the standard mollifier $\eta \in C^\infty_c(\mathbb{R}^N)$ and $\eta_\varepsilon(x) := \varepsilon^{-N}\eta(x/\varepsilon)$. Then,

$$\zeta_\varepsilon := \zeta\chi_\Omega * \eta_\varepsilon \in C^\infty(\mathbb{R}^N) \quad \text{and} \quad \|\zeta_\varepsilon\|_{L^\infty(\Omega)} \le \|\zeta\|_{L^\infty(\Omega)}. \tag{4.10}$$

Define ξ_ε as the solution to

$$\begin{cases} (-\Delta)^s\xi_\varepsilon = \zeta_\varepsilon & \text{in } \Omega,\\ \xi_\varepsilon = 0 & \text{in } \mathcal{C}\Omega,\\ E\,\xi_\varepsilon = 0 & \text{on } \partial\Omega.\end{cases}$$

Also, for $\rho > 0$ small, consider

$$\Omega_\rho := \{x \in \Omega : \delta(x) > \rho\}$$

and a bump function $b_\rho \in C^\infty_c(\mathbb{R}^N)$ such that

$$b_\rho \equiv 1 \quad \text{in } \Omega_{2\rho}, \qquad b_\rho \equiv 0 \quad \text{in } \mathbb{R}^N\setminus\Omega_\rho, \qquad 0 \le b_\rho \le 1 \quad \text{in } \mathbb{R}^N.$$

Then, $\zeta_{\varepsilon,\rho} := b_\rho\zeta_\varepsilon \in C^\infty_0(\Omega)$. Let $\xi_{\varepsilon,\rho} \in \mathcal{T}(\Omega)$ be the function induced by $\zeta_{\varepsilon,\rho}$. By (4.9),

$$\int_\Omega u\zeta_{\varepsilon,\rho} = \int_\Omega \xi_{\varepsilon,\rho}\,d\nu. \tag{4.11}$$

We have $\zeta_{\varepsilon,\rho} \to \zeta_\varepsilon$ as $\rho \downarrow 0$ with $\|\zeta_{\varepsilon,\rho}\|_{L^\infty(\Omega)} \le \|\zeta_\varepsilon\|_{L^\infty(\Omega)}$ and

$$|\xi_{\varepsilon,\rho}(x)| \le C\|\zeta_{\varepsilon,\rho}\|_{L^\infty(\Omega)}\delta(x)^s \le \|\zeta_\varepsilon\|_{L^\infty(\Omega)}\delta(x)^s,$$

so that we can push equality (4.11) to the limit to deduce, by dominated convergence,

$$\int_\Omega u\zeta_\varepsilon = \int_\Omega \xi_\varepsilon\,d\nu. \tag{4.12}$$

Similarly, since $\|\zeta_\varepsilon\|_{L^\infty(\Omega)} \le \|\zeta\|_{L^\infty(\Omega)}$, letting $\varepsilon \downarrow 0$, yields

$$\int_\Omega u\zeta = \int_\Omega \xi\,d\nu.$$ □

5 Examples

In the next two examples we look at the two critical cases in the power-like nonlinearity, adding a logarithmic weight.

Example 5.1 (Lower critical case for powers). We consider here

$$f(t) = t^{1+2s}\ln^{\alpha}(1+t), \qquad \alpha > 0.$$

In this case,

$$\frac{tf'(t)}{f(t)} = \frac{(1+2s)f(t) + \frac{\alpha t f(t)}{(1+t)\ln(1+t)}}{f(t)} = 1 + 2s + \frac{\alpha t}{(1+t)\ln(1+t)}.$$

Condition (1.8) turns into

$$\int_u^{+\infty} \left(\frac{t}{t^{1+2s}\ln^{\alpha}(1+t)}\right)^{1/(2s)} dt = \int_u^{+\infty} \frac{dt}{t\ln^{\alpha/(2s)}(1+t)} < +\infty,$$

which is fulfilled only for $\alpha > 2s$. Also, hypothesis (1.10) becomes

$$\int_{t_0}^{+\infty} t^{1+2s-2/(1-s)}\ln^{\alpha}(1+t)\,dt < +\infty,$$

which is satisfied by any $\alpha > 0$ since $(1+2s)(1-s) - 2 < s - 1$.

Example 5.2 (Upper critical case for powers). We consider here $f(t) = t^{(1+s)/(1-s)}\ln^{-\beta}(1+t)$, $\beta > 0$. In this case

$$\frac{tf'(t)}{f(t)} = \frac{\frac{1+s}{1-s}f(t) - \frac{\beta t f(t)}{(1+t)\ln t}}{f(t)} = \frac{1+s}{1-s} - \frac{\beta t}{(1+t)\ln(1+t)}.$$

Hypothesis (1.8) turns into

$$\int_u^{+\infty} \left(\frac{t\ln^{\beta}(1+t)}{t^{(1+s)/(1-s)}}\right)^{1/(2s)} dt = \int_u^{+\infty} \frac{\ln^{\beta/(2s)}(1+t)}{t^{1/(1-s)}}\,dt, < +\infty$$

which is fulfilled for any $\beta > 0$. Also, hypothesis (1.10) becomes

$$\int_{t_0}^{+\infty} t^{-1}\ln^{-\beta}(1+t)\,dt < +\infty,$$

which is satisfied by any $\beta > 1$.

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