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A TRACE THEOREM FOR MARTINET–TYPE VECTOR FIELDS

DANIELE GEROSA, ROBERTO MONTI, AND DANIELE MORBIDELLI

Abstract. In $\mathbb{R}^3$ we consider the vector fields

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y} + |x|^\alpha \frac{\partial}{\partial z},$$

where $\alpha \in [1, +\infty[$. Let $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ be the (closed) upper half-space and let $f \in C^1(\mathbb{R}^3_+)$ be a function such that $X_1 f, X_2 f \in L^p(\mathbb{R}^3_+)$ for some $p > 1$. In this paper, we prove that the restriction of f to the plane $z = 0$ belongs to a suitable Besov space that is defined using the Carnot-Carathéodory metric associated with X_1 and X_2 and the related perimeter measure.

1. Introduction

By a classical result due to Gagliardo [Gag57], for any $p > 1$ and any bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary there is a constant $C > 0$ such that for any function $f \in C^1(\bar{\Omega})$ the following *trace estimate* holds:

$$\int_{\partial\Omega\times\partial\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+ps}} d\mathscr{H}^{n-1}(x) d\mathscr{H}^{n-1}(y) \leq C \int_\Omega |\nabla f(x)|^p dx, \tag{1.1}$$

where $s = 1-1/p$. The inequality extends to Sobolev functions, showing that traces of $W^{1,p}$-functions are well defined and have a fractional order of differentiability $1 - 1/p$ at the boundary $\partial\Omega$.

In this paper, we prove a similar trace estimate in a setting where the gradient of f in the right-hand side of (1.1) is replaced by a subelliptic gradient that, at some point of the boundary, may be "tangential". In $\mathbb{R}^3$ we consider the vector fields

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y} + |x|^\alpha \frac{\partial}{\partial z},$$

where $\alpha \in [1, +\infty[$ is a real parameter. When $\alpha = 2$ the distribution of planes spanned by X_1 and X_2 is known as Martinet-distribution. We denote the X-gradient of a function $f \in C^1(\mathbb{R}^3)$ by $Xf := (X_1 f, X_2 f)$.

Let $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ be the closed upper-halfspace and $\Sigma = \mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ its boundary. The plane Σ is characteristic at all points where $x = 0$, in the sense that both the vector fields X_1 and X_2 are tangent to Σ, here.

According to a general procedure introduced in [GN96] and studied in [MSC01], the vector fields X_1 and X_2 induce on Σ a natural surface measure, known as X-perimeter

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measure. In the present setting, this X-perimeter measure is

$$\mu = |x|^\alpha \mathscr{L}^2, \tag{1.2}$$

where $\mathscr{L}^2$ is the Lebesgue measure in the plane.

We denote by d the Carnot-Carathéodory metric on $\mathbb{R}^3$ induced by X_1, X_2 and by $B(q,r)$ the metric ball centered at $q \in \mathbb{R}^3$ with radius $r > 0$. With abuse of notation, we identify $u \in \mathbb{R}^2$ with $(u,0) \in \mathbb{R}^3$.

Theorem 1.1. *Let $\alpha \in [1,+\infty[$, $p \in]1,+\infty[$ and $s = 1-1/p$. There exists a constant $C > 0$ depending on α and p such that any function $f \in C^1(\mathbb{R}^3_+)$ satisfies*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|f(u,0)-f(v,0)|^p}{d(u,v)^{ps}\mu(B(u,d(u,v)))}\, d\mu(u)\, d\mu(v) \le C \int_{\mathbb{R}^3_+} |Xf(x,y,z)|^p dxdydz. \tag{1.3}$$

The Besov seminorm in the left-hand side is defined in terms of the metric d and of the measure μ. When d is the standard metric and μ is the Lebesgue measure, the seminorm reduces to the one in the left-hand side of (1.1).

This seminorm was first introduced by Danielli, Garofalo and Nhieu in [DGN06], where a metric approach to the problem is developed. The authors prove trace and lifting theorems for (ε,δ)-domains with *Ahlfors regular* boundary. The (ε,δ)-property is in general difficult to check because of the presence of boundary characteristic points. For systems of Hörmander vector fields with step 3, it may fail even for "flat" or analytic boundaries, see [MM05]. In a companion paper [MM18], we are able to show the (ε,δ)-property for a different family of vector fields related to generalized Siegel domains. The Ahlfors regularity of the measure μ in (1.2) will be studied in Section 3.

The classical proof of (1.1) by Gagliardo relies on an elegant construction of families of curves transversal to the surface $\partial\Omega$ and connecting pairs of points on the boundary. The estimate is achieved by an integration of the gradient of the function along such curves. This technique can be extended to the subelliptic setting if $\partial\Omega$ does not contain *characteristic points*. Indeed, in the noncharacteristic case the construction of transversal *horizontal curves* is easy because at any noncharacteristic boundary point there is at least one vector field transversal to the tangent space to the boundary. Trace inequalities in this setting are proved by Berhanu and Pesenson in [BP99], by Bahoury, Chemin and Xu in [BCX05] for vector fields of step 2 and by the authors for general Hörmander vector fields in [MM02].

In the characteristic case, the construction of horizontal curves entering the domain from boundary points is much more delicate. Some trace theorems are known also in this case, mainly in two classes of examples. The first one is the Heisenberg group, see the contribution by Bahouri, Chemin and Xu [BCX09] for some characteristic surfaces. A second class of examples is that of *diagonal* vector fields, i.e., a system of n vector fields in $\mathbb{R}^n$ of the form $X_i = w_i(x)\frac{\partial}{\partial x_i}$, $i = 1,\dots,n$, with suitable weights w_i. See the results of Franchi [Fra86] and the authors [MM02].

In this paper, we are able to deal with the following three difficulties:

– the plane $z = 0$ contains characteristic points and actully a whole line, the y-axis;
– the vector fields can have arbitrarily large step, depending on $\alpha \geq 1$;
– the vector fields are not of diagonal type.

In a future work, we plan to generalize our results to more general surfaces and to more general families of vector fields.

In our proof of (1.3), it is enough to estimate the difference $f(u, 0) - f(v, 0)$ for noncharacteristic points. However, both $(u, 0)$ and $(v, 0)$ may be arbitrarily close to the characteristic line. The choice of the curves connecting them is rather delicate and must take into account "how much" close to the characteristic set the points are. Once the correct construction is devised, the trace estimate is obtained by integrating the subelliptic gradient along such curves and using the Minkowski and Hardy integral inequalities. The correct estimate of the Besov seminorm must be split in several sub-cases and each of them requires a separate effort.

The argument requires a precise description of the size of the Carnot-Carathéodory balls of the distance d associated with the vector fields X_1, X_2. Since α can be non-integer, we cannot use the ball-box theorems of Nagel, Stein and Wainger [NSW85]. For this reason, in Section 2 we give a self contained proof of the ball-box estimate for d, which has an independent interest.

Notation. By $C_\alpha > 0$ we denote a constant depending on $\alpha \geq 1$ that may change from line to line. By $C_{\alpha,p} > 0$ we denote a constant depending $\alpha \geq 1$ and $p > 1$ that may change from line to line. For $a, b > 0$, we use the standard notation $a \simeq b$ meaning that $a \leq Cb$ and $b \leq Ca$ for an absolute constant C that may depend on α and/or p.

2. Structure of the metric

Let d be the Carnot-Carathéodory distance associated with the vector fields $X_1 = \partial_x$ and $X_2 = \partial_y + |x|^\alpha \partial_z$. The construction of d is well-known and can by found in [NSW85].

When $\alpha = 2$, the vector fields X_1 and X_2 span a distribution of 2-planes in $\mathbb{R}^3$ known as *Martinet*-distribution. When α is an even number, the vector fields satisfy the Hörmander condition with step $\alpha + 1$ and the structure of metric balls follows from [NSW85].

When α is not even, the results of [NSW85] cannot be used. For this reason, we give here a self-contained proof of the relevant estimates. The case $\alpha = 1$ of the familiar Heisenberg group is not included in our discussion. However, with some minor adaptations, the results of this section hold verbatim for vector fields of the form $X_1 = \partial_x$ and $X_2 = \partial_y + |x|^{\alpha-1} x \partial_z$, including the Heisenberg vector fields in the limit case $\alpha = 1$.

By the particular structure of the vector fields, the distance d possesses the following invariance properties

$$d((x,y,z),(x',y',z',)) = d((x,y+\eta,z+\zeta),(x',y'+\eta,z'+\zeta)), \tag{2.1a}$$

$$d((rx,xy,r^{\alpha+1}z),(rx',ry',r^{\alpha+1}z')) = rd((x,y,z),(x',y',z')), \tag{2.1b}$$

$$d((-x,y,z),(-x',y',z')) = d((x,y,z),(x',y',z')) \tag{2.1c}$$

for all $(x,y,z),(x',y',z') \in \mathbb{R}^3$, $\eta,\zeta \in \mathbb{R}$ and $r \geq 0$. In this section, we describe the structure of d in terms of an equivalent function defined by algebraic functions.

For $\alpha \geq 1$, we define the function $\delta : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$

$$\delta((x,y,z),(x',y',z')) := |x'-x| + |y'-y| + \min\left\{|\zeta|^{1/(\alpha+1)}, \frac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}}\right\},$$

where we let $\zeta = z - z' + |x|^\alpha(y'-y)$. In the definition above, we agree that the minimum is $|\zeta|^{1/(\alpha+1)}$ if $x=0$, and is 0 if $\zeta = 0$.

Theorem 2.1. *For $\alpha \geq 1$, let d be the Carnot-Carathéodory metric induced on $\mathbb{R}^3$ by the vector-fields X_1, X_2. There exists a constant $C_0 > 0$, depending on α, such that for all $p,q \in \mathbb{R}^3$ we have*

$$C_0^{-1}\delta(p,q) \leq d(p,q) \leq C_0\delta(p,q). \tag{2.2}$$

Proof. For $\alpha \geq 1$, we will use the equivalence

$$|u^\alpha - v^\alpha| \geq C_\alpha(|u|+|v|)^{\alpha-1}|u-v|, \quad \text{for all } u,v \in [0,+\infty[\,. \tag{2.3}$$

By the translation invariance (2.1a), we can assume that $p = (x,y,z) = (x,0,0)$. We also let $q = (x',y',z')$.

Step 1. We first show the estimate $\delta \leq C_0 d$. Let $\gamma : [0,T] \to \mathbb{R}^3$ be a horizontal curve with $\gamma(0) = p$, $\gamma(T) = q$ and $\dot\gamma = h_1(t)X_1(\gamma) + h_2(t)X_2(\gamma)$ with $|(h_1,h_2)| \leq 1$ a.e. The functions

$$x(t) = x + \int_0^t h_1(s)ds =: x + \widehat{x}(t), \qquad y(t) = \int_0^t h_2(s)ds,$$

satisfy the estimates $|\widehat{x}(t)| \leq t$ and $|y(t)| \leq t$, and thus

$$|x'-x| = |x(T)-x(0)| \leq T \qquad \text{and} \qquad |y'| = |y(T)-y(0)| \leq T. \tag{2.4}$$

The quantity $\zeta = z - z' + |x|^\alpha(y'-y) = |x|^\alpha y' - z'$ satisfies

$$|\zeta| = \Big|\int_0^T (|x(s)|^\alpha - |x|^\alpha)\dot y(s)ds\Big| \leq C_\alpha \int_0^T (|x|^{\alpha-1} + s^{\alpha-1})sds \leq C_\alpha(|x|^{\alpha-1}T^2 + T^{\alpha+1}),$$

and this implies that either $|\zeta| \leq C_\alpha|x|^{\alpha-1}T^2$ or $|\zeta| \leq C_\alpha T^{\alpha+1}$. This is equivalent to

$$\min\left\{\frac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}}, |\zeta|^{1/(\alpha+1)}\right\} \leq C_\alpha T,$$

and this estimate together with (2.4) concludes the proof of *Step 1*.

Step 2. We prove the estimate $d \le C_0\delta$ in the case when points are one above the other. Namely, we claim that $d((x,y,z),(x,y,z')) \le C_0\delta((x,y,z),(x,y,z'))$ for all $x,y,z,z' \in \mathbb{R}$. As above, we can assume that $y = z = 0$.

We prove the claim for $x \ge 0$ and $z > 0$. The cases $x < 0$ and $z < 0$ are analogous. For $u > 0$, let $\kappa : [0,4u] \to \mathbb{R}^2$ be the plane curve with unit speed which connects the points $(x,0)$, $(x+u,0)$, $(x+u,u)$, (x,u) and $(x,0)$, and let $\kappa(t) = (x(t),y(t))$. This path encloses a square which we denote by R_u. Let $t \mapsto \gamma(t) = (x(t),y(t),z(t))$ be the horizontal lift of κ starting from $z(0) = 0$. By Stokes' theorem

$$z(4u) = \int_\kappa \xi^\alpha d\eta = \int_{R_u} \alpha\xi^{\alpha-1} d\xi d\eta = u((x+u)^\alpha - x^\alpha) \ge C_\alpha u^2(x^{\alpha-1} + u^{\alpha-1}), \tag{2.5}$$

where we used (2.3). By the definition of d, we have

$$d((x,0,0),(x,0,z')) \le 4\min\{u > 0 : z(4u) = z'\},$$

and, by (2.5), the number u realizing the minimum satisfies

$$u \le C_\alpha \min\Big\{\frac{|z'|^{1/2}}{|x|^{(\alpha-1)/2}}, |z'|^{1/(\alpha+1)}\Big\}.$$

This proves the claim.

Step 3. We prove the estimate $d \le C_0\delta$ for arbitrary points $p = (x,0,0)$ and $q = (x',y',z')$. By the triangle inequality we have

$$\begin{aligned} d(p,q) &\le d\big(p, \mathrm{e}^{(x'-x)X_1+y'X_2}(p)\big) + d\big(\mathrm{e}^{(x'-x)X_1+y'X_2}(p), q\big) \\ &\le |x-x'| + |y'| + d\big(\mathrm{e}^{(x'-x)X_1+y'X_2}(p), q\big) \\ &\le \delta(p,q) + d\big(\mathrm{e}^{(x'-x)X_1+y'X_2}(p), q\big), \end{aligned} \tag{2.6}$$

where we adopt the standard notation $e^Z(p)$ or $\exp(Z)(p)$ to denote the value at time 1 of the integral curve of the vector field Z starting from p at $t = 0$. An easy computation shows that

$$\mathrm{e}^{(x'-x)X_1+y'X_2}(p) = \Big(x', y', y'\int_0^1 |x + s(x'-x)|^\alpha ds\Big),$$

i.e., the point is above q. By the *Step 2*, we have

$$d\big(\mathrm{e}^{(x'-x)X_1+y'X_2}(p), q\big) \le C_0\delta\big(\mathrm{e}^{(x'-x)X_1+y'X_2}(p), q\big).$$

Now, letting

$$\zeta = z' - |x|^\alpha y' \qquad \text{and} \qquad \zeta' = z' - y'\int_0^1 |x + s(x'-x)|^\alpha ds,$$

to conclude the estimate it suffices to show that

$$\min\Big\{|\zeta'|^{1/(\alpha+1)}, \frac{|\zeta'|^{1/2}}{|x'|^{(\alpha-1)/2}}\Big\} \le C_\alpha\Big(|x-x'| + |y'| + \min\Big\{\ |\zeta|^{1/(\alpha+1)}, \frac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}}\Big\}\Big). \tag{2.7}$$

First of all we have

$$\begin{aligned}|\zeta'| &= \Big|z' - y'\int_0^1 |x+s(x'-x)|^\alpha ds\Big| \\ &\le |\zeta| + |y'|\Big|\int_0^1 \Big(|x+s(x'-x)|^\alpha - |x|^\alpha\Big)ds\Big| \le C_\alpha(|\zeta|+\omega),\end{aligned} \tag{2.8}$$

where we let $\omega = |y'|(|x|+|x'|)^{\alpha-1}|x'-x|$. To prove (2.7) we distinguish the following two cases:

Case A: $|\zeta|^{1/(\alpha+1)} \le \dfrac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}}$, or, equivalently, $|x| \le |\zeta|^{1/(\alpha+1)}$.

Case B: $|\zeta|^{1/(\alpha+1)} \ge \dfrac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}}$, or, equivalently, $|x| \ge |\zeta|^{1/(\alpha+1)}$.

In the *Case A*, the claim (2.7) is implied by

$$\min\Big\{(|\zeta|+\omega)^{1/(\alpha+1)}, \frac{(|\zeta|+\omega)^{1/2}}{|x'|^{(\alpha-1)/2}}\Big\} \le C_\alpha\Big(|x-x'|+|y'|+|\zeta|^{1/(\alpha+1)}\Big).$$

The estimate of $|\zeta|^{1/(\alpha+1)}$ is trivial. The quantity ω is estimated in the following way:

$$\begin{aligned}\omega &\le C_\alpha|y'|(|x|+|x-x'|)^{\alpha-1}|x-x'| \le C_\alpha\big(|y'|^{\alpha+1}+|x|^{\alpha+1}+|x-x'|^{\alpha+1}\big) \\ &\le C_\alpha\big(|y'|^{\alpha+1}+|\zeta|+|x-x'|^{\alpha+1}\big),\end{aligned}$$

and the claim follows.

In the *Case B*, the claim (2.7) is implied by

$$\min\Big\{(|\zeta|+\omega)^{1/(\alpha+1)}, \frac{(|\zeta|+\omega)^{1/2}}{|x'|^{(\alpha-1)/2}}\Big\} \le C_\alpha\Big(|x-x'|+|y'|+|\zeta|^{1/2}/|x|^{(\alpha-1)/2}\Big).$$

Sub-case B1: $|x'-x| \le \frac{1}{2}|x|$. In this sub-case, we have $|x'| \simeq |x|$ and thus the term with ζ is easily estimated, because $\frac{|\zeta|^{1/2}}{|x'|^{(\alpha-1)/2}} \simeq \frac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}}$.

We estimate the term with ω. From $\omega \simeq |x|^{\alpha-1}|y'|\,|x-x'|$ we deduce that

$$\frac{\omega^{1/2}}{|x'|^{(\alpha-1)/2}} \simeq \frac{\big(|x|^{\alpha-1}|y'|\,|x-x'|\big)^{1/2}}{|x|^{(\alpha-1)/2}} \simeq |y'|^{1/2}|x-x'|^{1/2},$$

which is smaller than $|x-x'|+|y'|$, as required.

Sub-case B2: $|x-x'| > \frac{1}{2}|x|$. We claim that

$$|\zeta|^{1/(\alpha+1)} \le C_\alpha\Big(|x-x'| + \frac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}}\Big).$$

Indeed, the function $h(s) = s + \frac{|\zeta|^{1/2}}{s^{(\alpha-1)/2}}$ attains the minimum on $(0,\infty)$ at the point $s_{\min} \simeq |\zeta|^{1/(\alpha+1)}$.

To end the discussion of the *Sub-case B2*, we estimate the term with ω:

$$\omega \le |y'|(2|x|+|x-x'|)^{\alpha-1}|x-x'| \le C|y'|\,|x-x'|^\alpha,$$

and the estimate $\omega^{1/(\alpha+1)} \le C_\alpha(|x-x'|+|y'|)$ follows.

This concludes the proof of Theorem 2.1. □

Remark 2.2. For all points $(x,y),(x',y')\in\mathbb{R}^2$ we have the equivalence

$$d((x,y,0),(x',y',0))\simeq|x-x'|+|y-y'|+|x|^{1/2}|y'-y|^{1/2}. \tag{2.9}$$

To prove this, we start from

$$\delta((x,y,0),(x',y',0))=|x-x'|+|y-y'|+\min\left\{|x|^{\alpha/(\alpha+1)}|y'-y|^{1/(\alpha+1)},|x|^{1/2}|y'-y|^{1/2}\right\},$$

and we observe that the minimum is equivalent to the second term, because

$$|x|^{1/2}|y'-y|^{1/2}\le C_\alpha\big(|y-y'|+|x|^{\alpha/(\alpha+1)}|y'-y|^{1/(\alpha+1)}\big).$$

□

We rephrase the estimates in Theorem 2.1 as a ball-box theorem. For a fixed point $p=(x,y,z)\in\mathbb{R}^3$, define the mappings $\Phi_1(p;\cdot),\Phi_2(p;\cdot):\mathbb{R}^3\to\mathbb{R}^3$:

$$\Phi_1(p;u)=\Phi_1(u)=\Big(x+u_1,y+u_2,z+|x|^\alpha u_2+|x|^{\alpha-1}u_3\Big),$$
$$\Phi_2(p;u)=\Phi_2(u)=\Big(x+u_1,y+u_2,z+|x|^\alpha u_2+u_3\Big).$$

We let $\|u\|_{1,1,2}=\max\{|u_1|,|u_2|,|u_3|^{1/2}\}$ and $\|u\|_{1,1,\alpha+1}=\max\{|u_1|,|u_2|,|u_3|^{1/(\alpha+1)}\}$, and we define the boxes

$$B_1(p,r)=\{\Phi_1(p;u):\|u\|_{1,1,2}<r\}\quad\text{and}\quad B_2(p,r)=\{\Phi_2(p;u):\|u\|_{1,1,\alpha+1}<r\}.$$

Let $C_0>1$ be a constant such that $C_0^{-1}\delta\le d\le C_0\delta$ globally.

Corollary 2.3. *Let $\eta>0$. There are constants $b_1(\eta)$ and $b_2(\eta)$ such that for all $p=(x,y,z)\in\mathbb{R}^3$ and $r>0$ we have:*

i) if $|x|\ge\eta r$ then

$$B_1\big(p,C_0^{-1}r\big)\subset B(p,r)\subset B_1\big(p,b_1(\eta)r\big); \tag{2.10}$$

ii) if $r\ge\eta|x|$, then

$$B_2\big(p,C_0^{-1}r\big)\subset B(p,r)\subset B_2\big(p,b_2(\eta)r\big). \tag{2.11}$$

Proof. Step 1. We claim that for all p and r we have:

$$B_1(p,C_0^{-1}r)\cup B_2(p,C_0^{-1}r)\subset B(p,r)\subset B_1(p,C_0r)\cup B_2(p,C_0r).$$

Indeed, letting $\zeta=z-z'+|x|^\alpha(y'-y)$, we have

$$(x',y',z')\in B_1(p,r)\quad\Leftrightarrow\quad\max\left\{|x-x'|,|y-y'|,\Big(\frac{|\zeta|}{|x|^{\alpha-1}}\Big)^{1/2}\right\}<r,$$
$$(x',y',z')\in B_2(p,r)\quad\Leftrightarrow\quad\max\left\{|x-x'|,|y-y'|,|\zeta|^{1/(\alpha+1)}\right\}<r.$$

This means that $(x',y',z')\in(B_1\cup B_2)(p,r)$ if and only if $\delta((x,y,z),(x',y',z'))<r$. Then *Step 1* is concluded thanks to Theorem 2.1. The argument also proves the inclusions in the left-hand side of (2.10) and (2.11).

Step 2. We prove the inclusion in the right-hand side of (2.10). Let $|x| > \eta r$ and let $(x', y', z') \in B(p, r)$. By *Step 1* we have $(x', y', z') \in B_1(p, C_0 r) \cup B_2(p, C_0 r)$. To conclude the proof it suffices to show that there is a constant $b_1(\eta) > 0$ so that the following implication holds:

$$\begin{cases} |x| > \eta r \\ \min\left\{\dfrac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}}, |\zeta|^{1/(\alpha+1)}\right\} \le C_0 r \quad (*) \end{cases} \quad \Rightarrow \quad \frac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}} \le b_1(\eta) r. \tag{2.12}$$

If $\frac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}} \le |\zeta|^{1/(\alpha+1)}$, there is nothing to prove and we can choose $b_1(\eta) = C_0$. In the case $\frac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}} \ge |\zeta|^{1/(\alpha+1)}$, inequality $(*)$ reads $|\zeta|^{1/(\alpha+1)} \le C_0 r$ and we have:

$$\frac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}} \le \frac{|\zeta|^{1/2}}{\eta^{(\alpha-1)/2} r^{(\alpha-1)/2}} \le \frac{C_0^{(\alpha+1)/2}}{\eta^{(\alpha-1)/2}}\, r.$$

The proof of *Step 2* is concluded, with $b_1(\eta) = \max\left\{C_0, \frac{C_0^{(\alpha+1)/2}}{\eta^{(\alpha-1)/2}}\right\}$.

Step 3. We prove the inclusion in the right-hand side of (2.11). As in the *Step 2*, it suffices to show the implication

$$\begin{cases} r > \eta |x| \\ \min\left\{\dfrac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}}, |\zeta|^{1/(\alpha+1)}\right\} \le C_0 r \end{cases} \quad \Rightarrow \quad |\zeta|^{1/(\alpha+1)} \le b_2(\eta) r. \tag{2.13}$$

If the minimum is $|\zeta|^{1/(\alpha+1)}$, we trivially get the implication with $b_2(\eta) = C_0$. Otherwise, we have

$$C_0 r \ge \min\left\{\frac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}}, |\zeta|^{1/(\alpha+1)}\right\} = \frac{|\zeta|^{1/2}}{|x|^{(\alpha-1)/2}} \ge |\zeta|^{1/2} \frac{\eta^{(\alpha-1)/2}}{r^{(\alpha-1)/2}},$$

which is equivalent to $|\zeta|^{1/(\alpha+1)} \le \left(\frac{C_0}{\eta^{(\alpha-1)/2}}\right)^{2/(\alpha+1)} r$, as required. Therefore, implication (2.13) holds with $b_2(\eta) = \max\left\{C_0, \left(\frac{C_0}{\eta^{(\alpha-1)/2}}\right)^{2/(\alpha+1)}\right\}$. □

Using the previous corollary, it is immediate to get the following estimates of the Lebesgue measure of the balls $B(p, r)$.

Corollary 2.4. *Let $\eta > 0$. For all $p = (x, y, z) \in \mathbb{R}^3$ and $r \in]0, +\infty[$ we have:*

i) if $|x| \ge \eta r$ then $\mathscr{L}^3(B(p, r)) \simeq r^4 |x|^{\alpha-1}$;
ii) if $|x| \le \eta r$ then $\mathscr{L}^3(B(p, r)) \simeq r^{\alpha+3}$.

The equivalence constants depend on α and η.

We omit the proof, which is trivially based on Corollary 2.3.

3. Ahlfors' property

The boundary of the half-space $\mathbb{R}^3_+$ is the plane $\Sigma = \mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. According to the general construction of [GN96] and [MSC01], the vector-fields X_1, X_2 induce on Σ a Borel measure known as X-perimeter measure. We denote this measure by μ. The integral-geometric formula for this measure is the following:

$$\mu(B) = \int_B \sqrt{\langle X_1, N\rangle^2 + \langle X_2, N\rangle^2} dxdy, \qquad B \subset \mathbb{R}^2 \text{ Borel set.}$$

Above, $N = (0, 0, -1)$ is the exterior normal to the boundary of $\mathbb{R}^3_+$ and $\langle \cdot, \cdot\rangle$ denotes the standard scalar product of $\mathbb{R}^3$. In fact, the measure μ is simply

$$\mu = |x|^\alpha \mathscr{L}^2 \quad \text{ on } \Sigma = \mathbb{R}^2. \tag{3.1}$$

The metric d and the balls $B(p, r)$ can be restricted to Σ. With abuse of notation we let $\mu(B(p, r)) = \mu(B(p, r)) \cap \Sigma)$. The measure μ is Ahlfors regular in the following sense (see [DGN06]).

Proposition 3.1. *There is a constant $C_\alpha > 0$ such that for any $p \in \Sigma$ and for all $r > 0$ we have*

$$C_\alpha^{-1} \frac{\mathscr{L}^3(B(p, r))}{r} \le \mu(B(p, r)) \le C_\alpha \frac{\mathscr{L}^3(B(p, r))}{r}. \tag{3.2}$$

Proof. Let $p = (\bar{x}, \bar{y}, 0)$ and $r > 0$ be such that $|\bar{x}| \ge r$. The section of the ball $B_1(p, r)$ with the plane Σ is

$$\begin{aligned} &B_1(p, r) \cap \Sigma \\ &= \Big\{\big(\bar{x} + u_1, \bar{y} + u_2, |\bar{x}|^\alpha u_2 + |\bar{x}|^{\alpha-1} u_3\big) \in \mathbb{R}^3 : \|u\|_{1,1,2} < r, |\bar{x}|^\alpha u_2 + |\bar{x}|^{\alpha-1} u_3 = 0\Big\} \\ &= \big[\bar{x} - r, \bar{x} + r\big] \times \big[\bar{y} - \min\{r, r^2/|\bar{x}|\}, \bar{y} + \min\{r, r^2/|\bar{x}|\}\big]. \end{aligned}$$

Then, from

$$\mu(B_1(p, r)) = 2 \min\{r, r^2/|\bar{x}|\} \int_{\bar{x}-r}^{\bar{x}+r} |x|^\alpha dx$$

and from Corollary 2.3 we deduce that when $|\bar{x}| \ge r$ we have

$$\mu(B(p, r)) \simeq \mu(B_1(p, r)) = \frac{2r^2}{|\bar{x}|} \int_{\bar{x}-r}^{\bar{x}+r} |x|^\alpha dx \simeq r^3 |\bar{x}|^{\alpha-1}.$$

On the other hand, the section of the ball $B_2(p, r)$ with the plane Σ is

$$\begin{aligned} B_2(p, r) \cap \Sigma &= \Big\{\big(\bar{x} + u_1, \bar{y} + u_2, |\bar{x}|^\alpha u_2 + u_3\big) \in \mathbb{R}^3 : \|u\|_{1,1,\alpha+1} < r, |\bar{x}|^\alpha u_2 + u_3 = 0\Big\} \\ &= \big[\bar{x} - r, \bar{x} + r\big] \times \big[\bar{y} - \min\{r, r^{\alpha+1}/|\bar{x}|^\alpha\}, \bar{y} + \min\{r, r^{\alpha+1}/|\bar{x}|^\alpha\}\big], \end{aligned}$$

and thus

$$\mu(B_2(p, r)) = 2 \min\{r, r^{\alpha+1}/|\bar{x}|^\alpha\} \int_{\bar{x}-r}^{\bar{x}+r} |x|^\alpha dx.$$

When $|\bar{x}| \le r$, from Corollary 2.3 we deduce that

$$\mu(B(p, r)) \simeq \mu(B_2(p, r)) = 2r \int_{\bar{x}-r}^{\bar{x}+r} |x|^\alpha dx \simeq r^{\alpha+1}.$$

Now the claim (3.2) is a consequence of Corollary 2.4. □

4. Schema of the proof

In this section, we outline the scheme of the proof of Theorem 1.1. We have the points $u = (x, y, 0)$, $v = (x', y', 0)$ and their distance $d = d(u, v)$. Since $B(u, d) \subset B(v, 2d) \subset B(u, 3d)$, the integral kernel

$$\frac{|f(u,0) - f(v,0)|^p}{d(u,v)^{ps}\mu(B(u, d(u,v)))}$$

is "almost" symmetric in u and v and we can assume that

$$y' \geq y \quad \text{and} \tag{4.1a}$$

$$0 < x' < \infty. \tag{4.1b}$$

Assumption (4.1b) can be made without loss of generality in view of the invariance property (2.1c).

We will connect the points u and v by a number of integral curves of the vector fields $\pm X_1$, $\pm X_2$, or of their sum $\pm(X_1 + X_2)$. The correct choice depends on the following cases.

Let $\varepsilon_0 \in]0, 1[$ be a small parameter that will be fixed along Section 6. We have the following cases:

1) $d \geq \varepsilon_0|x|$ and $d \geq \varepsilon_0|x'|$. We call this the *characteristic case*.
2) $d < \varepsilon_0|x|$ and $d < \varepsilon_0|x'|$. We call this the *noncharacteristic case*.
3) $d < \varepsilon_0|x|$ and $d \geq \varepsilon_0|x'|$, or viceversa.

In the third case, we have $|x'| \geq |x| - |x - x'| \geq \varepsilon_0^{-1}d - d = (\varepsilon_0^{-1} - 1)d$ because $|x - x'| \leq d$. So this case is essentially contained in the second one.

Theorem 1.1 is then reduced to the proof of (1.3) when the integration domain $\mathbb{R}^2 \times \mathbb{R}^2$ is replaced by the case 1) and 2), separately, along with the two conditions (4.1). The proof for the characteristic case is in Section 5. The proof for the noncharacteristic case is in Section 6.

5. Characteristic case

We connect the points $u = (x, y, 0)$ and $v = (x', y', 0)$ with integral curves of the vector fields $\pm X_1$ and $\pm X_2$. Our first task is to fix the order in the sequence of these vector fields. We start by discussing a first subcase of (4.1b). Namely, we assume that

$$0 < |x| \leq x'. \tag{5.1}$$

We will explain in Remark 5.2 (see page 15) how to deal with the second sub-case case $|x| > x' > 0$. In the following, for $u, v \in \mathbb{R}^2$ we let $d = d(u, v)$. We define the set

$$A = \big\{(u, v) \in \mathbb{R}^2 \times \mathbb{R}^2 : y \leq y',\, 0 < |x| \leq x' \leq d/\varepsilon_0\big\}.$$

Starting from u, we introduce certain intermediate points interpolating u and v. Let $\tau = \tau(u,v) > 0$ be the number

$$\tau = \frac{|2x|^\alpha}{(2x')^\alpha - |x|^\alpha}\,(y'-y). \tag{5.2}$$

By (5.1) we have $\tau \le \frac{2^\alpha}{2^\alpha-1}(y'-y)$ and in particular $\tau \le C_\alpha d$. Then we define the points:

$$\begin{aligned}
&u_0 = u = (x,y,0),\\
&u_1 = \exp\big((y'-y)X_2\big)(u_0) = \big(x,y',|x|^\alpha(y'-y)\big),\\
&u_2 = \exp\Big(\Big(\frac{|x|}{2}-x\Big)X_1\Big)(u_1) = \Big(\frac{|x|}{2},y',|x|^\alpha(y'-y)\Big),\\
&u_3 = \exp(\tau X_2)(u_2) = \Big(\frac{|x|}{2},y'+\tau,|x|^\alpha\Big(y'-y+\frac{\tau}{2^\alpha}\Big)\Big),\\
&u_4 = \exp\Big(\Big(x'-\frac{|x|}{2}\Big)X_1\Big)(u_3) = \Big(x',y'+\tau,|x|^\alpha\Big(y'-y+\frac{\tau}{2^\alpha}\Big)\Big),\\
&u_5 = \exp(-\tau X_2)(u_4) = \Big(x',y',|x|^\alpha\Big(y'-y+\frac{\tau}{2^\alpha}\Big)-x'^\alpha\tau\Big) = (x',y',0) = v.
\end{aligned} \tag{5.3}$$

The last identity is due to (5.2). Let $\gamma_j : [0,T_j] \to \mathbb{R}^3_+$ be the integral curve connecting u_{j-1} and u_j, where $T_j > 0$ are such that $\gamma_j(T_j) = u_j$. The support of γ_j is contained in $\mathbb{R}^3_+$ for all $j = 1,\dots,5$.

If $(u,v) \in A$, by (2.9) we have $d(u,v) \simeq |x-x'| + |y-y'|$. Furthermore, Proposition 3.1 and Corollary 2.4 give $\mu(B(u,d)) \simeq d^{\alpha+2}$ and so we have

$$d^{ps}\mu(B(u,d)) = d^{p-1}\mu(B(u,d)) \simeq d^{\alpha+p+1}. \tag{5.4}$$

Finally, by (3.1) we have $d\mu(u) = |x|^\alpha dxdy$ and $d\mu(v) = |x'|^\alpha dx'dy'$.

Using these estimates and starting from the inequality

$$|f(u)-f(v)| \le \sum_{j=1}^{5} |f(u_j)-f(u_{j-1})|,$$

we obtain

$$\int_A \frac{|f(u)-f(v)|^p}{d^{ps}\mu(B(u,d))}\,d\mu(u)\,d\mu(v) \le C_{\alpha,p}\sum_{j=1}^{5} I_j,$$

where

$$I_j := \int_A \frac{|x|^\alpha x'^\alpha}{d^{\alpha+p+1}}\Big(\int_0^{T_j} |Xf(\gamma_j(t))|dt\Big)^p dxdydx'dy', \quad j = 1,\dots,5. \tag{5.5}$$

We claim that for all $j = 1,\dots,5$ we have

$$I_j \le C_{\alpha,p}\int_{\mathbb{R}^3_+} |Xf(x,y,z)|^p dxdydz. \tag{5.6}$$

Estimate of I_1. The curve connecting u_0 and u_1 is $\gamma_1(t) = \mathrm{e}^{tX_2}(x, y, 0)$ with $t \in [0, y' - y]$. The corresponding integral is

$$I_1 = \int_A \frac{|x|^\alpha x'^\alpha}{d^{\alpha+p+1}} \Big(\int_0^{y'-y} |Xf(x, y+t, |x|^\alpha t)| dt \Big)^p dx dy dx' dy'.$$

Using $x' \le d/\varepsilon_0$ and $0 \le y' - y \le d$ we obtain

$$\begin{aligned} I_1 &\le C_\alpha \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|x|^\alpha}{d^{p+1}} \Big(\int_0^d |Xf(x, y+t, |x|^\alpha t)| dt \Big)^p dx dy\, dx' dy' \\ &\le C_\alpha \int_{\mathbb{R}^2} \int_0^\infty \int_{\{d=r\}} \frac{|x|^\alpha}{d^{p+1}} \Big(\int_0^d |Xf(x, y+t, |x|^\alpha t)| dt \Big)^p d\mathscr{H}^1(x', y')\, dr\, dx dy \\ &\le C_\alpha \int_0^\infty \int_{\mathbb{R}^2} \Big(\int_0^r |Xf(x, y+t, |x|^\alpha t)| dt \Big)^p |x|^\alpha dx dy\, \frac{dr}{r^p}. \end{aligned}$$

We used the coarea formula with $|\nabla d| \simeq 1$. Now, by the Minkowski inequality we obtain

$$I_1 \le C_\alpha \int_0^\infty \Big(\frac{1}{r} \int_0^r \Big[\int_{\mathbb{R}^2} |Xf(x, y+t, |x|^\alpha t)|^p |x|^\alpha dx dy \Big]^{1/p} dt \Big)^p dr,$$

and after the change of variable $y \mapsto y + t = \eta$ we can use the Hardy inequality to get

$$I_1 \le C_{\alpha,p} \int_0^\infty \int_{\mathbb{R}^2} |Xf(x, \eta, |x|^\alpha r)|^p |x|^\alpha dx d\eta\, dr \le C_{\alpha,p} \int_{\mathbb{R}^3_+} |Xf(x, y, z)|^p dx dy dz,$$

by the change of variable $r \mapsto z := |x|^\alpha r$. This proves (5.6) for $j = 1$.

Estimate of I_2. We connect the points u_1 and u_2 using the integral curve of X_1, i.e., the curve $\gamma_2(t) = (x - t\,\mathrm{sgn}(x), y', |x|^\alpha(y' - y))$ with $t \in [0, |x| - x/2]$. The corresponding integral is

$$\begin{aligned} I_2 &= \int_A \frac{|x|^\alpha x'^\alpha}{d^{\alpha+p+1}} \Big| \int_0^{|x|-x/2} |Xf(x - t\,\mathrm{sgn}(x), y', |x|^\alpha(y' - y))| dt \Big|^p dx dy dx' dy' \\ &\le C_\alpha \int_A \frac{1}{d^{p+1}} \Big(\int_0^{Cd} |Xf(x - t\,\mathrm{sgn}(x), y', |x|^\alpha(y' - y))| dt \Big)^p |x|^\alpha\, dx dy dx' dy', \end{aligned}$$

where $d \simeq \max\{|x - x'|, |y - y'|\}$. We used $x' \le d/\varepsilon_0$ and $|x| - x/2 \le Cd$. We perform the change of variable in time $x - t\,\mathrm{sgn}\, x = s$ with $|s| \le |x| + |t| \le Cd$. Then, we pass from the variables (x', y) to the variables $\zeta = (\xi, \eta) = (x' - x, y' - y)$ with $d\zeta = dx' dy$ and $|\zeta| \simeq d$. Finally, we use the Minkowski inequality to interchange integration in ds and $dx dy'$ and we obtain the estimate

$$I_2 \le C_\alpha \int_{\mathbb{R}^2_+} \Big(\int_{|s|\le C|\zeta|} \Big[\int_{A_\zeta} |X_1 f(s, y', \eta|x|^\alpha))|^p |x|^\alpha\, dx dy' \Big]^{1/p} ds \Big)^p \frac{d\zeta}{|\zeta|^{p+1}},$$

where we let $A_\zeta = \{(x, y') \in \mathbb{R}^2 : |x| \le C|\zeta|\}$. By symmetry in the variable x, it suffices to estimate the last integral when $x > 0$.

We perform the change of variable $x \mapsto z = \eta x^\alpha$ with $dz \simeq \eta x^{\alpha-1}dx$, that is equivalent to

$$x^\alpha dx \simeq \frac{z^{1/\alpha}}{\eta^{(\alpha+1)/\alpha}}dz. \tag{5.7}$$

In order to apply the coarea formula in the ζ variable for fixed z and y', we need the following estimate.

Lemma 5.1. *There exists a constant $C > 0$ such that for any $r > 0$ and $z > 0$ we have*

$$J_r(z) := \int_{D_r(z)} \frac{z^{1/\alpha}}{\eta^{(\alpha+1)/\alpha}} d\mathscr{H}^1(\zeta) \le rC, \tag{5.8}$$

where $D_r(z) = \{\zeta = (\xi, \eta) \in \mathbb{R} \times \mathbb{R}^+ : |\zeta| = r,\ 0 < z \le \eta|\zeta|^\alpha\}$.

Proof. We use the max-definition $|\zeta| = \max\{|\xi|, |\eta|\}$. The estimate is obvious when $D_r(z) = \emptyset$. Assume this is not the case, i.e., $0 < z < r^{\alpha+1}$. Then by direct calculation

$$J_r(z) = 2\int_{z/r^\alpha}^r \frac{z^{1/\alpha}}{\eta^{(\alpha+1)/\alpha}} d\eta + \int_{-r}^r \frac{z^{1/\alpha}}{\eta^{(\alpha+1)/\alpha}} d\xi \simeq (z^{1/\alpha} r^{-1/\alpha} + r) \le Cr.$$

□

We finish the estimate for I_2 in the following way. Let $E_r = \{(y', z) \in \mathbb{R}^2_+ : 0 < z \le \eta r^\alpha\}$. Using (5.7), the coarea formula, the Minkowski inequality and (5.8):

$$\begin{aligned}
I_2 &\le C_\alpha \int_0^\infty \frac{1}{r^{p+1}} \int_{|\zeta|=r} \Big(\int_{|s|\le Cr} \Big(\int_{E_r} |Xf(s, y', z))|^p \frac{z^{1/\alpha}}{\eta^{(\alpha+1)/\alpha}} dz dy' \Big)^{1/p} ds \Big)^p d\mathscr{H}^1(\zeta) dr \\
&\le C_\alpha \int_0^\infty \frac{dr}{r^{p+1}} \Big(\int_{|s|\le Cr} \Big[\int_{\mathbb{R}^2_+} \int_{D_r(z)} |Xf(s, y', z)|^p \frac{z^{1/\alpha}}{\eta^{(\alpha+1)/\alpha}} d\mathscr{H}^1(\zeta) dy' dz \Big]^{1/p} ds \Big)^p \\
&\le C_\alpha \int_0^\infty \frac{1}{r^{p+1}} \Big(\int_{|s|\le Cr} \Big(\int_{\mathbb{R}^2_+} |Xf(s, y', sz))|^p J_r(z) dz dy' \Big)^{1/p} ds \Big)^p dr \\
&\le C_\alpha \int_0^\infty \Big(\int_{|s|\le Cr} \Big(\int_{\mathbb{R}^2_+} |Xf(s, y', z))|^p dz dy' \Big)^{1/p} ds \Big)^p \frac{dr}{r^p} \\
&\le C_{\alpha,p} \int_{\mathbb{R}^3_+} |Xf(x, y, z))|^p dx dy dz.
\end{aligned}$$

In the last line we used again the Hardy inequality. This proves (5.6) when $j = 2$.

Estimate of I_3. Let $\gamma_3(t) = \mathrm{e}^{tX_2}\Big(\frac{|x|}{2}, y', |x|^\alpha(y' - y)\Big)$ be the integral curve of X_2 connecting u_2 and u_3, with $0 \le t \le \tau$. Recall that the number τ in (5.2) satisfies $\tau \le Cd$. Also using $0 < x' \le d/\varepsilon_0$, we obtain

$$I_3 \le C_\alpha \int_A \frac{|x|^\alpha}{d^{p+1}} \Big(\int_0^{Cd} \Big| Xf\Big(\frac{|x|}{2}, y' + t, |x|^\alpha\Big(y' - y + \frac{t}{2^\alpha}\Big)\Big)\Big| dt \Big)^p dx dy dx' dy'.$$

We perform the change of variable $x' \mapsto \xi = x' - x$ and $y \mapsto \eta = y' - y$, so that $|\zeta| = |(\xi, \eta)| \simeq d$, and then the change of variable in time $t \mapsto s = \eta + \frac{t}{2^\alpha}$, so that

$0 \le s \le C|\zeta|$. We get

$$\begin{aligned} I_3 &\le C_\alpha \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Big(\int_0^{C|\zeta|} \Big| Xf\Big(\frac{|x|}{2}, y'+t, |x|^\alpha\Big(\eta + \frac{t}{2^\alpha}\Big)\Big)\Big| dt\Big)^p |x|^\alpha dx dy' \frac{d\zeta}{|\zeta|^{p+1}} \\ &\le C_\alpha \int_0^\infty \int_{|\zeta|=r}\int_{\mathbb{R}^2} \Big(\int_0^{C|\zeta|} \big| Xf(\cdots)\big| ds\Big)^p |x|^\alpha dx dy' d\mathscr{H}^1(\zeta) \frac{dr}{r^{p+1}}, \end{aligned}$$

where

$$Xf(\cdots) = Xf\Big(\frac{|x|}{2}, y' + 2^\alpha(s-\eta), |x|^\alpha s\Big).$$

Next we apply the Minkowski inequality and, after that, we change variable from y' to $w = y' + 2^\alpha(s-\eta)$. We obtain

$$\begin{aligned} I_3 &\le C_\alpha \int_0^\infty \int_{|\zeta|=r} \Big(\int_0^{Cr} \Big[\int_{\mathbb{R}^2} \Big| Xf\Big(\frac{|x|}{2}, w, |x|^\alpha s\Big)\Big|^p |x|^\alpha dx dw\Big]^{\frac{1}{p}} ds\Big)^p d\mathscr{H}^1(\zeta) \frac{dr}{r^{p+1}} \\ &\le C_\alpha \int_0^\infty \Big(\int_0^{Cr} \Big[\int_{\mathbb{R}^2} \Big| Xf\Big(\frac{|x|}{2}, w, |x|^\alpha s\Big)\Big|^p |x|^\alpha dx dw\Big]^{\frac{1}{p}} ds\Big)^p \frac{dr}{r^p} \\ &\le C_{\alpha,p} \int_0^\infty \int_{\mathbb{R}^2} |Xf(|x|/2, w, |x|^\alpha s)|^p |x|^\alpha dx dw dr, \end{aligned}$$

by the Hardy inequality.

Estimate of I_4. Let $\gamma_4(t) = \mathrm{e}^{tX_1}\Big(\frac{|x|}{2}, y'+\tau, |x|^\alpha(y'-y+\tau/2^\alpha)\Big)$ be the curve connecting u_3 and u_4, with $0 \le t \le x' - \frac{|x|}{2} \le Cd$. The corresponding integral is

$$\begin{aligned} I_4 &= \int_A \frac{|x|^\alpha x'^\alpha}{d^{p+\alpha+1}} \Big(\int_0^{x'-\frac{|x|}{2}} \Big| Xf\Big(\frac{|x|}{2}+t, y'+\tau, |x|^\alpha\Big(y'-y+\frac{\tau}{2^\alpha}\Big)\Big)\Big| dt\Big)^p dx dy dx' dy' \\ &\le C_\alpha \int_A \frac{|x|^\alpha}{d^{p+1}} \Big(\int_0^{Cd} \Big| Xf\Big(s, y'+\tau, |x|^\alpha\Big(y'-y+\frac{\tau}{2^\alpha}\Big)\Big)\Big| ds\Big)^p dx dy dx' dy', \end{aligned}$$

where we used $x' \le Cd$ and we changed variable $t \mapsto s = |x|/2 + t$ using the estimate $0 \le s \le Cd$.

Next we pass to the variables $y \mapsto \eta = y' - y$ and $x' \mapsto \xi = x' - x$, where $\xi \ge |x| - x$ is nonnegative and observe that $|\zeta| := |(\xi,\eta)| \simeq d$. We use the following rule for changing integration variables and order

$$\int_{-\infty}^{+\infty}\int_{|x|}^{+\infty} \cdots dx' dx = \int_{-\infty}^{+\infty}\int_{|x|-x}^{+\infty} \cdots d\xi dx = \int_0^{+\infty}\int_{-\xi/2}^{+\infty} \cdots dx d\xi.$$

Recall also that $|x| \le Cd \simeq C|\zeta|$. Letting $E_\zeta = \big\{(x,y') \in \mathbb{R}^2 : |x| \le C|\zeta|,\, x > -\xi/2\big\}$, we obtain the following estimate

$$\begin{aligned} I_4 &\le C_\alpha \int_{\mathbb{R}^+\times\mathbb{R}^+} \int_{E_\zeta} \Big(\int_0^{C|\zeta|} \Big| Xf\Big(s, y'+\tau, |x|^\alpha\Big(\eta + \frac{\tau}{2^\alpha}\Big)\Big)\Big| ds\Big)^p |x|^\alpha dx dy' \frac{d\zeta}{|\zeta|^{p+1}} \\ &\le C_\alpha \int_{\mathbb{R}^+\times\mathbb{R}^+} \Big(\int_0^{C|\zeta|} \Big[\int_{E_\zeta} \Big| Xf\Big(s, y'+\tau, |x|^\alpha\Big(\eta + \frac{\tau}{2^\alpha}\Big)\Big)\Big|^p |x|^\alpha dx dy' \Big]^{\frac{1}{p}} ds\Big)^p \frac{d\zeta}{|\zeta|^{p+1}} \\ &\le C_\alpha \int_{\mathbb{R}^+\times\mathbb{R}^+} \Big(\int_0^{C|\zeta|} \Big[\int_{E_\zeta} \Big| Xf\Big(s, u, |x|^\alpha\Big(\eta + \frac{\tau}{2^\alpha}\Big)\Big)\Big|^p |x|^\alpha dx du \Big]^{\frac{1}{p}} ds\Big)^p \frac{d\zeta}{|\zeta|^{p+1}}. \end{aligned}$$

In the change of variable $y' \mapsto u = y' + \tau$ we used the fact that τ is independent of y' after letting $\eta = y' - y$, by (5.2). The next step is the change of variable

$$z = |x|^\alpha\Big(\eta + \frac{\tau}{2^\alpha}\Big) = \frac{2^\alpha (x+\xi)^\alpha |x|^\alpha}{2^\alpha (x+\xi)^\alpha - |x|^\alpha}\eta. \tag{5.9}$$

Observe that $|x|^\alpha \eta \le z \le \frac{2^\alpha}{2^\alpha - 1}|x|^\alpha \eta$, for all $\xi > 0$ and $x \in \,]-\xi/2, +\infty[$. Note that if x gets too close to $-\frac{2}{3}\xi$, then the estimate fails. Furthermore, we have

$$\frac{dz}{dx} = \alpha 2^\alpha \eta \frac{(x+\xi)^{\alpha-1}|x|^{\alpha-1}}{\big(2^\alpha (x+\xi)^\alpha - |x|^\alpha\big)^2}\big(- |x|^{\alpha+1} + 2^\alpha (x+\xi)^{\alpha+1}\operatorname{sign}(x)\big) \simeq \eta |x|^{\alpha-1}\operatorname{sign}(x).$$

To proceed, we split the integration on E_ζ into the integration on the following two sets

$$E_\zeta^+ = \big\{(x,y') \in \mathbb{R}^2 : 0 \le x \le C|\zeta|\big\} \quad \text{and} \quad E_\zeta^- = \big\{(x,y') \in \mathbb{R}^2 : -\xi/2 < x < 0\big\}.$$

We denote the corresponding integrals I_4^+ and I_4^-, respectively.

We estimate I_4^+. With the change of variable (5.9), by the previous discussion we get $x^\alpha dx \simeq \frac{z^{1/\alpha}}{\eta^{(\alpha+1)/\alpha}} dz$ and thus

$$I_4^+ \le C_\alpha \int_{\mathbb{R}^+\times\mathbb{R}^+} \Big(\int_0^{C|\zeta|} \Big[\int_{0<z<C\eta|\zeta|^\alpha} |Xf(s,u,z))|^p \frac{z^{1/\alpha}}{\eta^{(\alpha+1)/\alpha}} dz du \Big]^{\frac{1}{p}} ds\Big)^p \frac{d\zeta}{|\zeta|^{p+1}}$$

and we conclude using (5.8). The estimate of I_4^- is analogous.

Estimate of I_5. The curve γ_5 connecting u_4 to $u_5 = v$, in a backward parametrization, gives the following estimate for the integral I_5:

$$I_5 \le C_\alpha \int_A \Big(\int_0^\tau |Xf(x', y'+t, x'^\alpha t)| dt\Big)^p \frac{|x|^\alpha x'^\alpha}{d^{\alpha+p+1}} dx dy dx' dy',$$

and using $\tau \le Cd$ the evaluation of this integral is identical to the one for I_1.

Remark 5.2. In this section, we proved the integral estimate (1.3) starting from a couple of points $u = (x, y, 0)$ and $v = (x', y', 0)$ satisfying (4.1a) and (4.1b). Since (4.1a) can be always assumed, we briefly discuss the case when (4.1b) fails. This can happen in two situations: either $y' > y$ and $x > x' > 0$, or $y' > y$ and $x < -x' < 0$.

In the first case, it suffices to add to the points $u = (x, y, 0)$ and $v = u' = (x', y', 0)$ a third point $u'' = (x'', y'', 0) =: (2x - x', 2y' - y)$. Both the ordered pairs of points u, u''

and u', u'' satisfy (4.1b) and (4.1a). Then it suffices to use the triangle inequality and to recognize that the kernels appearing in the Besov seminorm related to the three pairs of points (u, u'), (u, u'') and (u', u'') are mutally equivalent.

In the second case, we add a third point $u'' = (x'', y'') := (-x, 2y' - y)$. Both the ordered pairs (u, u') and (u, u'') satisfy (4.1a) and (4.1b), and again, since $d(u, u') \simeq d(u, u'') \simeq d(u', u'')$ and the kernels appearing in the Besov norm related to the three pairs of points (u, u'), (u, u'') and (u', u'') are all equivalent.

6. Noncharacteristic case

We are in the case $d \le \varepsilon_0 |x|$ and $d \le \varepsilon_0 |x'|$, where the constant $\varepsilon_0 > 0$ will be fixed along the proof. Since $|x - x'| \le d$ we can assume that $x', x \ge d/\varepsilon_0$, i.e., they are both positive. Without loss of generality, we assume that

$$y < y' \qquad \text{and} \qquad x > x' > 0. \tag{6.1}$$

The case $y < y'$ and $0 < x < x'$ is discussed in Remark 6.3. If x, x' are both negative, it suffices to apply the transformation $(x, y, t) \mapsto (-x, y, t)$, see (2.1c).

With the notation $u = (x, y, 0)$, $v = (x', y', 0)$ and $d = d(u, v)$, we consider the integration domain

$$B = \{(u, v) \in \mathbb{R}^2 \times \mathbb{R}^2 : y' \ge y,\ x > x' \ge d/\varepsilon_0\}. \tag{6.2}$$

Notice that for ε_0 sufficiently small we may also assume that $x \simeq x'$.

Starting from u, we introduce certain intermediate points interpolating u and v. Consecutive points are connected by integral curves of the vector fields $\pm X_2$ and $\pm Z$ with $Z = X_1 + X_2$.

Let $\sigma = \sigma(u, v) > 0$ be the number $\sigma := y' - y + x - x'$. We define the points

$$\begin{aligned}
u_0 &= u = (x, y, 0)\\
u_1 &= \exp(\sigma X_2)(u_0) = (x, y' + x - x', x^\alpha \sigma)\\
u_2 &= \exp\big((x' - x)Z\big)(u_1) = \left(x', y', \sigma x^\alpha + \frac{x'^{\alpha+1} - x^{\alpha+1}}{\alpha + 1}\right).
\end{aligned}$$

Notice that $u_2 \in \mathbb{R}^3_+$, because

$$z' := \sigma x^\alpha - \frac{x^{\alpha+1} - x'^{\alpha+1}}{\alpha + 1} = (y' - y)x^\alpha + \int_{x'}^{x} (x^\alpha - t^\alpha)dt > 0, \tag{6.3}$$

as soon as $y' > y$ or $x > x' > 0$. This inequality may fail if $x < x'$, but see Remark 6.3. Observe also that

$$z' \le \sigma x^\alpha \le C d\, x^\alpha \le C\varepsilon_0 x'^{\alpha+1}, \tag{6.4}$$

because $x' \ge d/\varepsilon_0$ and $x \simeq x'$.

To reach v starting from u_2 we follow for a positive time an approximation of the commutator $[X_2, Z] = [X_2, X_1 + X_2] = -\alpha x^{\alpha-1} \frac{\partial}{\partial z}$. In a standard way, we approximate the flow along this commutator with a composition of flows of the vector fields $\pm Z$ and $\pm X_2$.

Let $\tau = \tau(u,v)$ be the positive solution of the equation the equation $z' + \tau x'^{\alpha} - \tau(x'+\tau)^{\alpha} = 0$, that reads

$$\tau(x'+\tau)^{\alpha} - \tau x'^{\alpha} = \big(y' - y + x - x'\big)x^{\alpha} - \frac{x^{\alpha+1} - x'^{\alpha+1}}{\alpha+1}. \tag{6.5}$$

This equation has a unique positive solution $\tau \geq 0$. By (2.3) we have

$$\tau \simeq \min\Big\{\sqrt{\frac{z'}{x'^{\alpha-1}}}, z'^{1/(\alpha+1)}\Big\} \simeq d((x',y',z'),(x',y',0)) \leq Cd(u,v), \tag{6.6}$$

where we used Theorem 2.1 and the triangle inequality.

Finally, we define the following further points:

$u_3 = \exp(\tau X_2)(u_2) = \big(x', y'+\tau, z' + x'^{\alpha}\tau\big)$

$u_4 = \exp(\tau Z)(u_3) = \Big(x'+\tau, y'+2\tau, z' + x'^{\alpha}\tau + \dfrac{(x'+\tau)^{\alpha+1} - x'^{\alpha+1}}{\alpha+1}\Big)$

$u_5 = \exp(-\tau X_2)(u_4) = \Big(x'+\tau, y'+\tau, z' + x'^{\alpha}\tau + \dfrac{(x'+\tau)^{\alpha+1} - x'^{\alpha+1}}{\alpha+1} - (x'+\tau)^{\alpha}\tau\Big)$

$u_6 = \exp(-\tau Z)(u_5) = \big(x', y', z' + x'^{\alpha}\tau - (x'+\tau)^{\alpha}\tau\big) = (x', y', 0).$

In the last identity we used (6.5). For $i = 1, \dots, 6$, we denote by $\gamma_i : [0, T_i] \to \mathbb{R}^3_+$ the curve connecting u_{i-1} and u_i, where T_i is such that $\gamma_i(T_i) = u_i$.

According to Proposition 3.1 and Corollary 2.4, for points $(u,v) \in B$ the kernel in (6.8) satisfies

$$d^{ps}\mu(B(u,d)) \simeq d^{p+2}x^{\alpha-1}, \tag{6.7}$$

and by (2.9), the distance function has the structure

$$d \simeq |x-x'| + |y-y'| + \sqrt{|y-y'|x} \simeq \max\{|x-x'|, \sqrt{|y-y'|x}\}.$$

The last equivalence follows from $0 \leq |y'-y| \leq \sqrt{|y'-y|\,d} \leq \sqrt{\varepsilon_0}\sqrt{|y'-y|\,|x|}$.

By the triangle inequality we obtain

$$\int_B \frac{|f(u)-f(v)|^p}{d^{ps}\mu(B(u,d))}\,d\mu(u)\,d\mu(v) \leq C_{\alpha,p}\sum_{i=1}^{6} J_i, \tag{6.8}$$

where

$$J_i := \int_B \Big(\int_0^{T_i} |Xf(\gamma_i(t)|dt\Big)^p \frac{x^{\alpha}\,x'^{\alpha}}{d^{p+2}x^{\alpha-1}} dxdydx'dy', \qquad \text{for } i = 1, \dots, 6.$$

We claim that the integrals J_i satisfy

$$J_i \leq C_{\alpha,p}\int_{\mathbb{R}^3_+} |Xf(x,y,z))|^p dxdydz.$$

Estimate of J_1. Starting from the point $u_0 = (x,y,0)$, we follow the vector field X_2 for a positive time $\sigma = y' - y + x - x' \leq d$. Using the estimate in (6.7), we arrive at the inequality

$$J_1 \leq \int_B \frac{1}{d^{p+2}x^{\alpha-1}}\Big(\int_0^d |Xf(x, y+t, tx^{\alpha})|\,dt\Big)^p x^{\alpha}x'^{\alpha}dxdydx'dy'. \tag{6.9}$$

We use the coarea formula along with the following lemma.

Lemma 6.1. *There exists a constant $C > 0$ such that for any $x, y \in \mathbb{R}$ with $x \geq r/\varepsilon_0 > 0$ we have*

$$\int_{D_r(x,y)} \frac{|x'|^\alpha}{|\nabla d(x',y')|} d\mathscr{H}^1(x',y') \leq C r^2 x^{\alpha-1}, \tag{6.10}$$

where $D_r(x,y) = \{(x',y') \in \mathbb{R}^2 : d = r\}$ with $d = \max\{|x-x'|, \sqrt{|y-y'|x}\}$.

Proof. The set $D_r(x,y)$ is the boundary of the rectangle $[x-r, x+r] \times [y - \frac{r^2}{x}, y + \frac{r^2}{x}]$. When in the max-definition of d we have $d = \sqrt{|y-y'|x}$, then, on $\{d = r\}$, the gradient of d satisfies $|\nabla d(x',y')| = \frac{\sqrt{x}}{2\sqrt{|y-y'|}} = \frac{x}{2r}$. In the corresponding part of the integral (6.10), the function $|x'|^\alpha$ is integrated on the interval $(x-r, x+r)$.

On the set where $d = |x-x'|$ we have $|\nabla d| = 1$ and, in (6.10), the constant $|x'|^\alpha = |x \pm r|^\alpha \simeq |x|^\alpha = x^\alpha$ is integrated for $y' \in (y - r^2/x, y + r^2/x)$. In both cases the claim follows. □

Starting from (6.9), by the coarea formula and inequality (6.10), by the Minkowski and Hardy inequalities we obtain

$$\begin{aligned}
J_1 &\leq \int_{x>0} \int_0^{\varepsilon_0 x} \int_{D_r(x,y)} \frac{x'^\alpha}{|\nabla d|} \Big(\int_0^d |Xf(x, y+t, x^\alpha t)|\, dt \Big)^p d\mathscr{H}^1(x',y') \frac{dr}{r^{p+2}} x dx dy \\
&\leq C_\alpha \int_0^\infty \int_{x>0} \Big(\int_0^r |Xf(x, y+t, x^\alpha t)|\, dt \Big)^p x^\alpha dx dy \frac{dr}{r^p} \\
&\leq C_\alpha \int_0^\infty \Big(\int_0^{Cr} \Big[\int_{x>0} |Xf(x, y+t, x^\alpha t)|^p x^\alpha dx dy \Big]^{1/p} dt \Big)^p \frac{dr}{r^p} \\
&\leq C_{\alpha,p} \int_0^\infty \int_{x>0} |Xf(x, y+r, r x^\alpha)|^p x^\alpha dx dy dr \\
&\leq C_{\alpha,p} \int_{\mathbb{R}^3_+} |Xf(x,y,z))|^p dx dy dz,
\end{aligned}$$

as required.

Estimate of J_2. The integral curve connecting u_1 and u_2 is

$$\gamma_2(t) = \Big(x - t, y + \sigma - t, \sigma x^\alpha - \frac{x^{\alpha+1} - (x-t)^{\alpha+1}}{\alpha+1} \Big), \qquad \text{for } 0 \leq t \leq x - x'.$$

where $\sigma = y' - y + x - x'$. Using (6.7) and $0 \leq x - x' \leq d$, we start from the estimate

$$J_2 \leq \int_B \Big(\int_0^{x-x'} |Xf(\gamma_2(t))| dt \Big)^p \frac{x^\alpha x'^\alpha dx dy dx' dy'}{d^{p+2} x^{\alpha-1}}.$$

We perform the change of variable from x', y' to $h = (h_1, h_2)$

$$h = (h_1, h_2) = (x - x', \sqrt{(y'-y)x}) \in]0, +\infty[\times]0, +\infty[\,. \tag{6.11}$$

Note that $|h| \simeq d$. The Jacobian satisfies $dx'dy' \leq C\frac{|h|}{x}dh$ and so we obtain

$$J_2 \leq C \int_{\widehat{B}} \frac{1}{|h|^{p+1}} \Big(\int_0^{h_1} \Big| Xf\Big(x-t, y+\widehat{\sigma}-t, \widehat{\sigma}x^\alpha - \frac{x^{\alpha+1}-(x-t)^{\alpha+1})}{\alpha+1}\Big| dt\Big)^p x^\alpha dxdydh,$$

where $\widehat{B} = \{(x,y,h) \in \mathbb{R}^4 : h_1, h_2 > 0, \quad x \geq |h|/\varepsilon_0\}$ and $\widehat{\sigma} := h_2^2/x + h_1 \leq C|h|$. Now we perform the change of variable in time

$$s = \varphi_{x,h}(t) = \frac{\widehat{\sigma}x^\alpha + ((x-t)^{\alpha+1} - x^{\alpha+1})/(\alpha+1)}{(x-t)^\alpha}.$$

By (6.3), on the integration set we have $s \geq 0$. Moreover, it is $0 \leq t \leq C|h|$ and $x \geq d/\varepsilon_0 \simeq |h|/\varepsilon_0$, with constants independent of ε_0. Then, choosing $\varepsilon_0 > 0$ small enough, we have $x - t \geq x/2$. Since $\widehat{\sigma} = \frac{h_2^2}{x} + |h_1| \leq Cd \simeq |h|$, we conclude that $0 \leq s \leq C|h|$. An easy computation also shows that, for $0 \leq t \leq C|h|$,

$$|\varphi'_{x,h}(t) + 1| \leq C\frac{|h|}{|x|} \leq \frac{1}{2}, \tag{6.12}$$

for $|x| \geq |h|/\varepsilon_0$ and ε_0 small enough. Letting $\widehat{t} = \varphi_{x,h}^{-1}(s) \in [0, h_1]$ we arrive at the estimate

$$J_2 \leq C_\alpha \int_{\widehat{B}} \Big(\int_0^{C|h|} |Xf(x-\widehat{t}, y+\widehat{\sigma}-\widehat{t}, s(x-\widehat{t})^\alpha)| ds\Big)^p x^\alpha dxdy \frac{dh}{|h|^{p+1}}.$$

Next we use the Minkowski inequality to interchange integration in ds with integration in $dxdy$:

$$J_2 \leq C_\alpha \int_{\mathbb{R}^2} \Big(\int_0^{C|h|} \Big[\int_{\{x \geq |h|/\varepsilon_0\}} |Xf(x-\widehat{t}, y+\widehat{\sigma}-\widehat{t}, s(x-\widehat{t})^\alpha)|^p x^\alpha dxdy\Big]^{1/p} ds\Big)^p \frac{dh}{|h|^{p+1}}.$$

The change of variable $\overline{y} = y + \widehat{\sigma} - \widehat{t}$ in the inner integral is elementary because $\widehat{\sigma}$ and $\widehat{t}$ do not depend on y. Let us consider the transformation $x \mapsto \overline{x}$ defined by

$$\overline{x} = x - \widehat{t} = x - \varphi_{x,h}^{-1}(s). \tag{6.13}$$

Note first that $\overline{x} \in [x - h_1, x]$, because $s \in \varphi_{x,h}([0, h_1])$. Using the definition of $\varphi_{x,h}$ and $\widehat{\sigma}$, we see that (6.13) can be written in the form

$$F_s(\overline{x}) := \frac{\overline{x}^{\alpha+1}}{\alpha+1} - s\overline{x}^\alpha = \frac{x^{\alpha+1}}{\alpha+1} - h_2^2 x^{\alpha-1} - h_1 x^\alpha =: G_h(x). \tag{6.14}$$

It is easy to see by one-variable calculus that $F_s : [(\alpha+1)s, +\infty[\to [0, +\infty[$ is a strictly increasing bijection with strictly positive derivative. Furthermore, if ε_0 is small enough then

$$G_h : [|h|/\varepsilon_0, +\infty[\to G_h([|h|/\varepsilon_0, +\infty[) =: I_h \subseteq [0, +\infty[$$

satisfies $G'_h(x) > 0$ for all $x > |h|/\varepsilon_0$. Then (6.13) can be written as a true change of variable $x = G_h^{-1}(F_s(\overline{x}))$, where $\overline{x} \in F_s^{-1}(I_h) \subset [0, +\infty[$ and by (6.14) we have the following change in the integration element

$$(\overline{x}^\alpha - \alpha s \overline{x}^{\alpha-1})d\overline{x} = (x^\alpha - (\alpha-1)h_2^2 x^{\alpha-2} - \alpha h_1 x^{\alpha-1})dx \simeq x^\alpha dx. \tag{6.15}$$

Then, $x^\alpha dx \le C_\alpha \overline{x}^\alpha d\overline{x}$ and the estimate can be finished by the coarea formula and the Hardy inequality as follows:

$$\begin{aligned} J_2 &\le C_\alpha \int_{\mathbb{R}^2} \frac{1}{|h|^{p+1}} \Big(\int_0^{C|h|} \Big(\int_{\overline{x}>0} |Xf(\overline{x},\overline{y},s\overline{x}^\alpha)|^p \overline{x}^\alpha d\overline{x} d\overline{y} \Big)^{1/p} ds \Big)^p dh \\ &\le C_\alpha \int_0^\infty \Big(\int_0^{Cr} \Big(\int_{\overline{x}>0} |Xf(\overline{x},\overline{y},s\overline{x}^\alpha)|^p \overline{x}^\alpha d\overline{x} d\overline{y} \Big)^{1/p} ds \Big)^p \frac{dr}{r^p} \\ &\le C_{\alpha,p} \int_0^\infty \int_{\overline{x}>0} |Xf(\overline{x},\overline{y},r\overline{x}^\alpha)|^p \overline{x}^\alpha d\overline{x} d\overline{y} dr \\ &\le C_{\alpha,p} \int_{\mathbb{R}^3_+} |Xf(x,y,z)|^p dx dy dz. \end{aligned}$$

Estimate of J_3. The curve connecting u_2 and u_3 is $\gamma_3(t) = (x', y'+t, z'+tx'^\alpha)$, where $t \in [0,\tau]$ and τ solution of (6.5). The quantity z' is defined in (6.3). Using (6.7) and (6.6), we can start from the estimate

$$J_3 \le \int_B \frac{1}{d^{p+2} x^{\alpha-1}} \Big(\int_0^{Cd} |Xf(x', y'+t, z'+tx'^\alpha)|\, dt \Big)^p x^\alpha x'^\alpha \, dx dy dx' dy'. \tag{6.16}$$

Observe that $z' \le \sigma x^\alpha \le C x'^\alpha d$. So, the change of variable in time $z' + tx'^\alpha = sx'^\alpha$ gives $dt = ds$ and the integration set in s is contained in $[0, Cd]$. Then we get

$$J_3 \le C_\alpha \int_B \frac{1}{d^{p+2} x^{\alpha-1}} \Big(\int_0^{Cd} |Xf(x', y'+\widehat{t}, sx'^\alpha))|\, ds \Big)^p x^\alpha x'^\alpha \, dx dy dx' dy',$$

where $\widehat{t} = s - z'/x'^\alpha$.

Next we change variables from (x,y) to $h = (h_1, h_2)$ as in (6.11) with Jacobian $dh = \frac{x}{2h_2} dx dy$, and so we obtain $x dx dy = |2h_2| dh \le C|h| dh$. Therefore

$$J_3 \le C_\alpha \int_{\mathbb{R}^2} \frac{1}{|h|^{p+1}} \Big(\int_0^{C|h|} |Xf(x', y'+\widehat{t}, sx'^\alpha))|\, ds \Big)^p x'^\alpha \, dx' dy' \, dh, \tag{6.17}$$

where $\widehat{t} = s - z'/x'^\alpha = s - \frac{1}{x'^\alpha} \Big((\frac{h_2^2}{x'+h_1} + h_1)(x'+h_1)^\alpha - \frac{(x'+h_1)^{\alpha+1} - x'^{\alpha+1})}{\alpha+1} \Big)$ does not depend on y'.

We use the Minkowski inequality to interchange integration in ds and $dx'dy'$:

$$J_3 \le C_\alpha \int_{\mathbb{R}^2} \frac{1}{|h|^{p+1}} \left(\int_0^{C|h|} \Big(\int_{x'>0} |Xf(x', y'+\widehat{t}, sx'^\alpha))|^p x'^\alpha dx' dy' \Big)^{1/p} ds \right)^p dh.$$

The change of variable $\bar{y} = y' + \widehat{t}$ satisfies $d\bar{y} = dy'$, and we finally obtain

$$J_3 \le C_\alpha \int_{\mathbb{R}^2} \frac{1}{|h|^{p+1}} \left(\int_0^{C|h|} \Big(\int_{x'>0} |Xf(x', \bar{y}, sx'^\alpha))|^p x'^\alpha dx' d\bar{y} \Big)^{1/p} ds \right)^p dh.$$

Ultimately, we conclude using the coarea formula and the Hardy inequality.

Estimate of J_4. The curve connecting u_3 and u_4 is

$$\gamma_4(t) = \big(x'+t, y'+\tau+t, z'+\tau x'^\alpha + [(x'+t)^{\alpha+1} - x'^{\alpha+1}]/(\alpha+1) \big), \quad t \in [0,\tau].$$

Using (6.7) and (6.6), we can start from the estimate

$$J_4 \le \int_B \Big(\int_0^{Cd} |Xf(\gamma_4(t))|\, dt \Big)^p \frac{x^\alpha x'^\alpha\, dxdydx'dy'}{d^{p+2} x^{\alpha-1}}. \tag{6.18}$$

With the change of variables (6.11) from variables (x, y) to $h = (h_1, h_2)$, we obtain

$$J_4 \le C_\alpha \int_{\{x' \ge |h|/\varepsilon_0\}} \Big(\int_0^{C|h|} |Xf(\cdots)| dt \Big)^p \frac{x'^\alpha dx'dy'dh}{|h|^{p+1}},$$

where

$$(\cdots) = \Big(x' + t, y' + \widehat{\tau} + t, \widehat{z'} + \widehat{\tau} x'^\alpha + \frac{(x'+t)^{\alpha+1} - x'^{\alpha+1}}{\alpha+1} \Big),$$

and

$$\widehat{z'} = \widehat{z'}(x', h) = h_2^2 (h_1 + x')^{\alpha-1} + h_1 (h_1 + x')^\alpha - \frac{(x'+h_1)^{\alpha+1} - x'^{\alpha+1}}{\alpha+1}. \tag{6.19}$$

Notice that the unique solution $\widehat{\tau} = \widehat{\tau}(x', h)$ of $\tau((x'+\tau)^\alpha - x'^\alpha) = \widehat{z'}$ does not depend on y'.

In the next step, we perform the change of variable in time

$$\begin{aligned} s = \widehat{\varphi}_{x',h}(t) &:= \frac{\widehat{z'}(x', h) + \widehat{\tau}(x', h) x'^\alpha + [(x'+t)^{\alpha+1} - x'^{\alpha+1}]/(\alpha+1)}{(x'+t)^\alpha} \\ &= \frac{x'+t}{\alpha+1} + (x'+t)^{-\alpha} \Big\{ \widehat{z'}(x', h) + \widehat{\tau}(x', h) x'^\alpha - \frac{x'^{\alpha+1}}{\alpha+1} \Big\} \end{aligned} \tag{6.20}$$

By (6.4), (6.5) and the noncharacteristic case, we have $0 < \widehat{z'}(x', h) + \widehat{\tau}(x', h) x'^\alpha \le C d x'^\alpha \le C \varepsilon_0 x'^{\alpha+1}$. Furthermore, an easy computation furnishes $\frac{1}{2(\alpha+1)} \le \varphi'(t) \le \frac{2}{\alpha+1}$ for all $0 \le t \le C|h|$, if ε_0 is small enough. Therefore, $\varphi_{x',h} : [0, C|h|] \to \varphi_{x',h}([0, C|h|]) =: A_{x',h} \subset [0, \widehat{C}|h|]$ is a monotone increasing bijection. Letting $\widehat{t} = \widehat{\varphi}^{-1}_{x',h}(s)$, we obtain

$$J_4 \le C_\alpha \int_{\{x' \ge |h|/\varepsilon_0\}} \frac{1}{|h|^{p+1}} \Big(\int_0^{\widehat{C}|h|} |Xf(x' + \widehat{t}, y' + \widehat{\tau} + \widehat{t}, s(x' + \widehat{t})^\alpha)| ds \Big)^p x'^\alpha dx'dy'dh.$$

We use the Minkowski inequality to interchange integration in ds and $dx'dy'$:

$$J_4 \le C_\alpha \int_{\mathbb{R}^2} \Big(\int_0^{\widehat{C}|h|} \Big[\int_{\{x' \ge |h|/\varepsilon_0\}} |Xf(x'+\widehat{t}, y'+\widehat{\tau}+\widehat{t}, s(x'+\widehat{t})^\alpha)|^p x'^\alpha dx'dy' \Big]^{1/p} ds \Big)^p \frac{dh}{|h|^{p+1}}.$$

The functions $\widehat{\tau}$ and $\widehat{t}$ do not depend on y'. So the change of variable $\bar{y} = y' + \widehat{\tau} + \widehat{t}$ is a translation and $d\bar{y} = dy'$.

Next we look at the transformation $\bar{x} = x' + \widehat{t} = x' + \widehat{\varphi}^{-1}_{x',h}(s)$, where we know that $\widehat{t} \in [0, \widehat{\tau}(x', h)] \subset [0, C|h|]$. Such transformation is equivalent to $\widehat{\varphi}_{x',h}(\overline{x} - x') = s$. Since the explicit form of (6.20) gives

$$\begin{aligned} \widehat{\varphi}_{x',h}(t) = \frac{1}{(x'+t)^\alpha} \Big\{ & h_2^2 (x'+h_1)^{\alpha-1} + h_1 (x'+h_1)^\alpha \\ & + \frac{(x'+t)^{\alpha+1} - (x'+h_1)^{\alpha+1}}{\alpha+1} + \widehat{\tau}(x', h) x'^\alpha \Big\}, \end{aligned}$$

the transformation can be written as

$$F_s(\overline{x}) := \frac{\overline{x}^{\alpha+1}}{\alpha+1} - s\overline{x}^{\alpha} = \frac{(x'+h_1)^{\alpha+1}}{\alpha+1} - h_2^2(x'+h_1)^{\alpha-1} - h_1(x'+h_1)^{\alpha} - \widehat{\tau}(x',h)x'^{\alpha} =: \widehat{G}_h(x'). \tag{6.21}$$

We are in a situation similar to (6.14) in the estimate of J_2, but here the right-hand side is slightly more complicated. As in the previous case, we see by one-variable calculus that $F_s : [(\alpha+1)s, +\infty[\to [0, +\infty[$ is a strictly increasing bijection with strictly positive derivative. Concerning the right-hand side, it suffices to show that

$$\widehat{G}_h : [|h|/\varepsilon_0, +\infty[\to \widehat{G}_h([|h|/\varepsilon_0, +\infty[) =: \widehat{I}_h \subseteq [0, +\infty[$$

satisfies $\frac{d}{dx'}\widehat{G}_h(x') > 0$ for all $x' > |h|/\varepsilon_0$. All terms are similar to those appearing in J_2, but here we need to show the following further inequality:

Lemma 6.2. *We have the estimate*

$$\Big|\frac{\partial}{\partial x'} x'^{\alpha}\widehat{\tau}(x',h)\Big| \le \sigma_0 x'^{\alpha}, \quad \textit{for all } x' > |h|/\varepsilon_0,$$

where the constant σ_0 can be made small by choosing ε_0 small enough.

The proof of the claim is postponed after the end of the estimate of J_4. To conclude the estimate of J_4, as a consequence of Lemma 6.2, we discover that we may write $x' = \widehat{G}_h^{-1}F_s(\overline{x})$ and the change of variable has strictly positive derivative, the variable $\overline{x}$ is nonnegative and differentiating (6.21), we get $x'^{\alpha}dx' \le C\overline{x}^{\alpha}d\overline{x}$.

Ultimately, we obtain the estimate

$$J_4 \le C_\alpha \int_{\mathbb{R}^2} \frac{1}{|h|^{p+1}} \Big(\int_0^{C|h|} \Big(\int_{\overline{x}>0} |Xf(\bar{x},\bar{y},s\bar{x}^{\alpha})|^p \bar{x}^{\alpha} d\bar{x}d\bar{y} \Big)^{1/p} ds \Big)^p dh,$$

and the argument is concluded in the usual way.

Proof of Lemma 6.2. To prove claim 6.2, we first get an explicit form of $\widehat{\tau}$. Starting from

$$\tau((x'+\tau)^{\alpha} - x'^{\alpha}) = \widehat{z}' := h_2^2(x'+h_1)^{\alpha-1} + h_1(x'+h_1)^{\alpha} - \frac{(x'+h_1)^{\alpha+1} - x'^{\alpha+1}}{\alpha+1} \tag{6.22}$$

and letting $v(s) = s((1+s)^{\alpha}-1)$, we see that $\frac{\tau}{x'} = v^{-1}(\frac{z'}{x'^{\alpha+1}})$. Recall that the ratio $\frac{z'}{x'^{\alpha+1}}$ is close to zero if ε_0 is small (see (6.4)). Furthermore, we have $v(s) \simeq s^2$ and $v'(s) \simeq s$ for s close to 0. So (6.2) is equivalent to

$$\Big|\frac{\partial}{\partial x'}\Big(x'^{\alpha+1}v^{-1}\Big(\frac{z'}{x'^{\alpha+1}}\Big)\Big)\Big| \le \sigma_0 x'^{\alpha} \quad \Leftrightarrow \quad \Big|(\alpha+1)\frac{\tau}{x'} + \frac{x'^{\alpha+1}}{v'(\tau/x')}\frac{\partial}{\partial x'}\frac{z'}{x'^{\alpha+1}}\Big| < \sigma_0.$$

The first term is easily controlled. In order to control the second one, observe that

$$\frac{x'}{v'(\tau/x')} \simeq \frac{x'}{\tau/x'} = \tau\frac{x'^2}{\tau^2} \simeq \frac{\tau x'^{\alpha+1}}{z'}, \tag{6.23}$$

by the quadratic behaviour of v calculated on the small argument $\frac{\tau}{x'}$. Then we are left to prove that

$$\Big|\Big(\frac{\tau}{z'}x'^{\alpha+1}\Big)\Big(\frac{\partial_{x'}z'}{x'^{\alpha+1}}-\frac{\alpha+1}{x'^{\alpha+2}}z'\Big)\Big|\le\sigma_0. \tag{6.24}$$

The second term is easily estimated. To conclude, we show that $\frac{\tau}{z'}|\frac{\partial z'}{\partial x'}|\le\sigma_0$. By a direct calculation of $\partial_{x'}z'$, we are reduced to the proof that the inequality

$$\begin{aligned}&\Big|\tau\Big((\alpha-1)h_2^2(x'+h_1)^{\alpha-2}+\alpha h_1(x'+h_1)^{\alpha-1}-((x'+h_1)^\alpha-x'^\alpha)\Big)\Big|\\&\le\sigma_0\Big(h_2^2(x'+h_1)^{\alpha-1}+h_1(x'+h_1)^\alpha-\frac{(x'+h_1)^{\alpha+1}-x'^{\alpha+1}}{\alpha+1}\Big)\end{aligned}$$

holds for some σ_0 as small as we wish for small ε_0. The ratio $\frac{\tau(\alpha-1)h_2^2(x'+h_1)^{\alpha-2}}{h_2^2(x'+h_1)^{\alpha-1}}$ enjoys this property, by the estimate $\tau\le Cd\le C\varepsilon_0x'$. Thus, it suffices to prove the inequality with $h_2=0$. This can be achieved by looking at the following Taylor expansions in h_1/x'

$$\begin{aligned}x'\Big(\alpha h_1(x'+h_1)^{\alpha-1}-((x'+h_1)^\alpha-x'^\alpha)\Big)&=\frac{\alpha(\alpha-1)}{2}x'^{\alpha-1}h_1^2+x'^{\alpha-1}h_1^2o(1)\\h_1(x'+h_1)^\alpha-\frac{(x'+h_1)^{\alpha+1}-x'^{\alpha+1}}{\alpha+1}&=\frac{\alpha}{2}x'^{\alpha-1}h_1^2+x'^{\alpha-1}h_1^2o(1),\end{aligned}$$

where $o(1)\to0$ as $h_1/x'\to0$. The proof of Lemma 6.2 is concluded. □

Estimate of J_5. The (backward) curve connecting u_4 and u_5 is

$$\gamma_5(t)=\Big(x'+\tau,y'+\tau+t,\frac{(x'+\tau)^{\alpha+1}-x'^{\alpha+1}}{\alpha+1}+(x'+\tau)^\alpha t\Big),\quad t\in[0,\tau],$$

and we have

$$J_5=\int_B\Big(\int_0^\tau|Xf(\gamma_5(t))|dt\Big)^p\frac{x^\alpha x'^\alpha dxdydx'dy'}{d^{p+2}x^{\alpha-1}}.$$

We change variable $t\mapsto s$ letting $(x'+\tau)^\alpha s=(x'+\tau)^\alpha t+\frac{(x'+\tau)^{\alpha+1}-x'^{\alpha+1}}{\alpha+1}$. Using $\tau\le Cd$, we get $0\le s\le Cd$ and we have

$$J_5\le\int_B\Big(\int_0^{Cd}\Big|Xf\Big(x'+\tau,y'+\tau+\widehat{t},(x'+\tau)^\alpha s\Big)\Big|ds\Big)^p\frac{x^{\alpha+1}dxdydx'dy'}{d^{p+2}},$$

where $\widehat{t}=\widehat{t}(s,x,x',y'-y)=s-\frac{1}{\alpha+1}\frac{(x'+\tau)^{\alpha+1}-x'^{\alpha+1}}{(x'+\tau)^\alpha}$.

Next we pass from variables x,y to variables $h_1=x-x'$ and $h_2=\sqrt{(y'-y)x}$. As in the previous cases, the Jacobian satisfies the estimate $xdxdy\le C|h|dh$. The unique solution $\widehat{\tau}=\widehat{\tau}(x',h)$ of (6.22) does not depend on y'. Then, the function $\widehat{t}=s-\frac{1}{\alpha+1}\frac{(x'+\widehat{\tau})^{\alpha+1}-x'^{\alpha+1}}{(x'+\widehat{\tau})^\alpha}$ defined above, depends on x',h_1,h_2 but not on y'. Thus, after the Minkowski inequality and the change of variable $\overline{y}=y'+\widehat{\tau}+\widehat{t}$, we obtain

$$J_5\le C_\alpha\int_{\mathbb{R}^2}\Big(\int_0^{C|h|}\Big[\int_{x'>|h|/\varepsilon_0}|Xf(x'+\widehat{\tau},\overline{y},(x'+\widehat{\tau})^\alpha s)|^px'^\alpha dx'd\overline{y}\Big]^{1/p}ds\Big)^p\frac{dh}{|h|^{p+1}}.$$

Finally, we exploit the transformation $\overline{x} = x' + \widehat{\tau}(x', h)$. With a slight modification of the argument used in the estimate of J_4, see especially (6.22), (6.23) and (6.24), we see that $|\partial_{x'}\widehat{\tau}(x', h)| < \frac{1}{2}$, if ε_0 is small and $x' > |h|/\varepsilon_0$. Therefore we have a correct change of variable and moreover $x'^{\alpha}dx' \simeq \overline{x}^{\alpha}d\overline{x}$. The argument is then concluded as in the estimate of J_4.

Estimate of J_6. We have to estimate the integral

$$J_6 = \int_B \Big(\int_0^{\tau} \Big| Xf\Big(x'+t, y'+t, \frac{(x'+t)^{\alpha+1} - x'^{\alpha+1}}{\alpha+1}\Big)\Big| dt\Big)^p \frac{x^{\alpha}x'^{\alpha}dxdydx'dy'}{d^{p+2}x^{\alpha-1}}.$$

We use $\tau \le Cd$ and we change variables from (x, y) to $h = (h_1, h_2)$ letting $h_1 = x - x'$ and $h_2 = \sqrt{(y'-y)x'}$. Then $xdxdy \le C|h|dh$, with $|h| \simeq d$ and we get

$$J_6 \le C_\alpha \int_{\mathbb{R}^2}\int_{E_h} \Big(\int_0^{C|h|} \Big| Xf\Big(x'+t, y'+t, \frac{(x'+t)^{\alpha+1} - x'^{\alpha+1}}{\alpha+1}\Big)\Big| dt\Big)^p x'^{\alpha}dx'dy' \frac{dh}{|h|^{p+1}},$$

where $E_h := \{(x', y') \in \mathbb{R}^2 : x' > |h|/\varepsilon_0\}$. Next we perform the change of variable

$$t \mapsto s = \frac{1}{(x'+t)^{\alpha}} \frac{(x'+t)^{\alpha+1} - x'^{\alpha+1}}{\alpha+1} =: \varphi_{x'}(t).$$

An explicit calculation gives $\frac{d}{dt}\varphi_{x'}(t) \in \big[\frac{1}{\alpha+1}, 1\big]$, for all $t > 0$. Therefore $\varphi_{x'}(t) \simeq t$, on $t \in [0, C|h|]$ and $ds \simeq dt$. Denoting $\widehat{t} = \varphi_{x'}^{-1}(s)$, we get

$$J_6 \le C_\alpha \int_{\mathbb{R}^2}\int_{E_h} \Big(\int_0^{C|h|} |Xf(x'+\widehat{t}, y'+\widehat{t}, (x'+\widehat{t})^{\alpha}s)| ds\Big)^p x'^{\alpha}dx'dy' \frac{dh}{|h|^{p+1}}.$$

An application of the Minkowski inequality and the change of variable $y' \mapsto \overline{y} = y' + \overline{t}$, where $\widehat{t}$ does not depend on y, lead us to

$$J_6 \le C_\alpha \int_{\mathbb{R}^2} \frac{dh}{|h|^{p+1}} \Big(\int_0^{C|h|} \Big[\int_{E_h} |Xf(x'+\widehat{t}, \overline{y}, (x'+\widehat{t})^{\alpha}s)|^p x'^{\alpha}dx'd\overline{y}\Big]^{1/p} ds\Big)^p.$$

Finally, we analyze the change of variable $\overline{x} = x' + \varphi_{x'}^{-1}(s)$. This is equivalent to $\varphi_{x'}(\overline{x} - x') = s$ and using the definition of φ we get $x' = (\overline{x}^{\alpha+1} - (\alpha+1)s\overline{x}^{\alpha})^{1/(\alpha+1)}$. Note that $\overline{x} \ge x'$. Since $s \le Cd \le C\varepsilon_0 x' \le C\varepsilon_0\overline{x}$, if ε_0 is small enough, we get $x'^{\alpha}dx' \simeq x^{\alpha}dx$ and ultimately

$$J_6 \le C_\alpha \int_{\mathbb{R}^2} \Big(\int_0^{C|h|} \Big[\int_{\mathbb{R}^2} |Xf(\overline{x}, \overline{y}, |\overline{x}|^{\alpha}s)|^p |\overline{x}|^{\alpha} d\overline{x}d\overline{y}\Big]^{1/p} ds\Big)^p \frac{dh}{|h|^{p+1}}.$$

The estimate can be concluded as in the previous cases.

Remark 6.3. So far, we assumed that $y < y'$ and $x > x' > 0$. If $y < y'$ and $0 < x < x'$ we add to the points $u = (x, y, 0)$ and $v = u' = (x', y', 0)$ a third point $u'' = (2x - x', 2y' - y, 0)$. Then we have $d(u, u') \simeq d(u, u'') \simeq d(u', u'')$ and $\mu(B(u, d(u, u'))) \simeq \mu(B(u, d(u, u''))) \simeq \mu(B(u', d(u', u''))))$. Thus, the estimates in this case can be obtained as explained in Remark 5.2.

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