Time-varying dependence and currency tail risk during the Covid-19 pandemic

# The impact of the time-varying dependence structure on the tail risk of a portfolio of foreign currencies during the Covid-19 pandemic

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April 4, 2023

**Abstract**

**Purpose** - The purpose of this paper is to introduce a generalization of the time varying correlation elliptical copula models and to analyze its impact on the tail risk of a portfolio of foreign currencies during the Covid-19 pandemic.
**Design/methodology/approach** - We consider a multivariate time series model where marginal dynamics are driven by an ARMA-GJR-GARCH model and the dependence structure among the residuals is given by an elliptical copula function. The correlation coefficient follows an autoregressive equation where the auregressive coefficient is a function of the past values of the correlation. The model is applied to a portfolio of a couple of exchange rates, specifically U.S. dollar - Japanese yen and U.S. dollar - Euro and compared with two alternative specifications of the correlation coefficient: constant and with auroregressive dynamics.
**Findings** - The use of the new model results in a more conservative evaluation of the tail risk of the portfolio measured by the Value-at-Risk and the Expected Shortfall suggesting a more prudencial capital allocation policy.
**Originality** - The main contribution of the paper consists in the introduction of a time-varying correlation model where the past values of the correlation coefficient impact on the autoregressive structure.

JEL classification: G11, G15, C22

**Keywords**: copula functions, time-varying dependence structure, tail risk, exchange rates.

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# 1 Introduction

The recent financial crisis due to the Covid-19 pandemic has highlighted the importance of the systemic risk in the dynamics of financial markets. The conditional dependence among financial instruments can drastically change in periods of financial or economic distress and the recent pandemic has had a profound impact on the markets. In this paper, we propose to investigate if the nature of the correlation structure between exchange rates affects the tail risk of a portfolio of foreign currencies. The Covid-19 pandemic can be considered one of the most major landmark event from an economic, social and political point of view of the 21 century (together with the great depression started in 2008) which involved a great number of countries around the world in less than one year. Due to the novelty of the topic, the empirical impact on the global economy in general and on exchange rates in particular of the Covid-19 pandemic has not yet been developed with accuracy.

Being one of our aim to model the impact of the pandemic on the dependence structure between exchange rates, we believe that a particularly effective tool is the copula function. The main advantage to build multivariate models using copula functions is that the contribution to the global risk may be partitioned into components only relating to the marginal distributions and components only relating to the copula. In our approach, we consider non-gaussian conditional marginal distributions to reflect the stylized facts about daily exchange rates returns, such as serial dependence in the conditional mean and strongly persistence in the conditional variance, which will likely be accentuated in the presence of violent and systemic shocks such as the Covid-19 pandemic.

In this paper we generalize the approach of Patton (2006a and 2006b) by assuming a conditional dependence structure modeled by a fixed copula with a parameter $\delta_t$ whose dynamics is modeled by a nonlinear autoregressive process and to measure its effects on the risk of a portfolio of currencies. The parameter evolution is given by a state-dependent autoregressive type models as introduced and discussed in Cherubini, Gobbi and Mulinacci (2016) and in Gobbi and Mulinacci (2021). This approach assumes that in the evolution equation of the parameter $\delta_t$, the coefficient of the lagged value $\delta_{t-1}$ is not constant as in the Patton specification but has a specified functional form which depends on $\delta_{t-1}$, e.g., $\psi(\delta_{t-1})$. Another extension of Patton models, applied to exchange rates time series, is introduced in Ahdika et al. (2021) where a wide class of copula functions is considered with different ARMA process dynamics of the time varying coefficient $\delta_t$.

The model is tested on two time series of exchange rates, e.g., U.S. dollar - Japanese yen and U.S. dollar - Euro in the Covid pandemic period. The dependence structure may change due to the financial distress and this effect can be more or less strong in relation to the financial reputation of countries. Our methodology consists in estimating the ARMA-GJR-

GARCH models for the exchange rates univariate returns to obtain the conditional marginal distributions of the residuals whose dependence structure is given by a copula function with time-varying parameter. The new model with depenence parameter driven by a state-dependent autoregressive coefficien is introduced as an alternative model to the existing copula approaches with constant parameter and with time varying autoregressive dynamics.

The above models are applied to the analysis of the tail risk of a portfolio of currencies. In order to compare their performance we consider the Value-at-Risk and the Expected Shortfall being tail risk measures with a monetary value that are used to define capital allocation requirements. More precisely, we analyze the 10-days Value-at-Risk and Expected Shortfall of an equally-weighted portfolio expressed in dollar built on U.S. dollar - Japanese yen and U.S. dollar - Euro exchange rates. The comparison will be conducted through a Monte Carlo simulation experiment which allows to obtain the distribution of the portfolio returns. The results suggest that the the new method corresponds to a more prudential capital allocation strategy.

The plan of the paper is the following. In Section 2 we discuss the literature review. Section 3 specifies the models for the marginal distributions and the models for the copulas describing the time-varying parameter. Section 4 is devoted to an empirical application to exchange rates. More precisely, in subsection 4.2 we present and discuss the estimation results, while in subsection 4.3 a portfolio of Euro and Yen currencies is constructed and a Monte Carlo simulation to compute its Value-at-Risk and Expected Shortfall is performed. Section 5 concludes.

# 2 Literature review

Some studies on the impact of the pandemic on exchange rates and on related economic and financial quantities are present in literature. Among others, Aslam et al. (2020) analyze the efficiency of foreign exchange markets during the initial periods of the Covid-19 outbreak, Li et al. (2021) used the Covid-19 pandemic impact on the USA and China exchange rates and Singh et al. (2021) explore the time-varying pattern caused by the pandemic between exchange rates and other variables like stock market returns, temperature and number of Covid-19 cases of G7 countries. Iyke and Ho (2020) examined the impact of exchange rate exposure on different sectors of the South African stock market before and during the Covid-19 pandemic and Villarreal-Samaniego (2021) used the autoregressive distributed lag (ARDL) procedure to investigate the connection between oil prices and exchange rates during the Covid-19 pandemic in a sample of five emerging economies. Kinateder et al. (2021) apply the bivariate Dynamic Conditional Correlation Generalized Autoregressive Conditional

Heteroskedasticity model to analize the effect of the Global Financial Crisis and COVID pandemic on major financial assets while Ahdika et al. (2021) investigate the dynamic dependency among exchange rates of five Asian countries during the pandemic. In a recent paper Aliu et al. (2023) address the problem of modeling the joint dynamics of European exchange rates considering the shock scenario due to the Ukraina war.

From a univariate point of view, there is a number of papers which address the problem of analysing the exchange rates dynamics using different models (see, among others, Krager and Kugler, 1993, Clements and Smith, 1997, Chappel, Padmore, Mistry and Ellis, 1996, and Peel and Speight, 1994, who evaluate forecasts from self-exciting threshold autoregressive (SETAR) models for exchange rates). On the other hand, the estimate of the conditional joint distribution of daily exchange rates can be found in Patton (2006a, 2006b) whose analysis focuses on the U.S. dollar - Japanese yen and the U.S. dollar - Euro exchange rates during 1990s and early 2000s.

The problem of exchange rate determination and its predictability is a very controversial issue in the international economics literature. In particular, the specification of non-linear models for exchange rates has been largely motivated in many papers. Among them, we mention those which concentrate on the most commonly applied non-linear models, e.g., the GARCH (generalized autoregressive conditional heteroscedastic), including its generalizations, and the SETAR. Such models have proved successful in describing the dynamic behaviour of many economic and financial variables; moreover, they offer the advantage of being readily interpretable in economic terms (see, among others, Krager and Kugler, 1993, Peel and Speight, 1994; Chappell et al., 1996). The generalized family of GARCH models allow one to specify the process governing both the mean and the variance of the series, while the SETAR models represent a stochastic process generated by different regimes depending on fixed conditions. According to Doman and Doman (2014), Nurrahmat et al. (2017), He and Hamori (2019), in this paper we will focus on GARCH or GJR-GARCH with a Student's t or skewed Student's t distribution for residuals that are suitable for liquid assets like exchange rates (see Tsay, 2010).

To investigate the dependence structure of exchange rates, we make use of a theorem due to Sklar (1959), which shows that an $d$-dimensional joint distribution function may be decomposed into its $d$ margins distributions, and a copula, which completely describes the dependence between the $d$ variables. For a general discussion on copulas the reader can consult Nelsen (2006) and Joe (2015). The construction of joint distributions using copula functions has many applications in finance: see among others, Cherubini and Luciano (2002), Rosenberg (2003) and Patton (2004, 2006a and 2006b). For a detailed review of dynamic copula-based models in finance, the reader can consult Cherubini et al. (2012) and the vast literature therein.

A number of papers have shown that the dependence structure between financial asset returns, and therefore the copula parameters, cannot be considered constant, but, rather, time-varying (see, among others, Engle, 2002, Patton, 2006a and 2006b, and Manner and Reznikova, 2012, and references therein). Furthermore, as these studies indicate, the correlation between asset returns tends to be more pronounced during downward phases of the stock market than upward phases: this is a feature that can be captured either considering time-varying dependence or considering asymmetric copula functions as in Junker et al. (2006). In general, disregarding these facts could be misleading from the point of view of risk assessment and could lead to incorrect inferences. In literature many papers have studied specific models with time varying dependence structure also based on regime switching: among others, see Tse and Tsui (2002), Patton (2006a and 2006b), Lu et al. (2014), Ahdika et al. (2021) for copula models with time varying parameters, Creal et al. (2013) for the generalized autoregressive score models and Doman and Doman (2014) and Jondeau and Rockinger (2006) for time variation based on regimes. The correct modelling of the dependence structure among assets is crucial for the analysis of the risk of a portfolio and many papers study its impact in the evaluation of Value-at-Risk and Expected shorfall measures (see, among the others, Lu et al., 2014, Nurrahmat et al., 2017, and He and Hamori, 2019). The contribution of this paper is in line with this stream of literature since we introduce a new more general dynamics for the time varying correlation coefficient and we analyze its effect in a portfolio Vale-at-Risk and Expected Shortfall values.

# 3 The model: ARMA-GARCH with time-varying copulas

In this section we introduce the models for the marginal distributions and the models for the dependency structure. For the former we will consider autoregressive moving average–generalized autoregressive conditional heteroskedasticity (ARMA-GARCH) models that established themselves in the early 80s, drawing inspiration from the works of Engle (1982) and Bollerslev (1986) and their ARMA-GJR-GARCH extension of Glosten et al. (1993). On the other hand, the dependence structure will be modeled by a copula function or more precisely by a conditional copula in the spirit of Patton (2006a and 2006b).

## 3.1 ARMA-GARCH marginal models

The models employed for the marginal distributions are presented below. We will denote the log-returns of two time series of prices as the variables $X_{1,t}$ and $X_{2,t}$. We assume that the marginal distributions are completely characterized by an ARMA$(p,q)$-GJR-GARCH(1,1) specification. The error term is distributed as a Student's t random variable with $\nu$ degrees of freedom. The functional form is the following

$$\begin{cases} X_{j,t} = c_j + \sum_{i=1}^{p} \phi_{j,i} X_{j,t-i} + \sum_{i=1}^{q} \theta_{j,i} \eta_{j,t-i} + \eta_{j,t}, \\ h_{j,t}^2 = \omega_{j,0} + \omega_{j,1} \eta_{j,t-1}^2 + \gamma_j \mathbf{1}_{j,t-1} \eta_{j,t-1}^2 + \omega_{j,2} h_{j,t-1}^2, \\ \frac{\eta_{j,t}}{h_{j,t}} \Big| \mathcal{F}_{j,t-1} \sim \text{ Skewed Student's t with } \nu_j \text{ d.f.} \end{cases} . \tag{3.1}$$

with $j = 1,2$ and $\mathbf{1}_{j,t-1} = \begin{cases} 1, & \text{if } \eta_{j,t-1} < 0 \\ 0, & \text{if } \eta_{j,t-1} \geq 0 \end{cases}$. The structure of an ARMA$(p,q)$-GJR-GARCH(1,1) combines two needs: that of having a conditional mean component and that of modeling the heteroskedasticity of the time series of returns. In the equation above the conditional mean is an ARMA$(p,q)$ process where the AR part involves the lagged values of the variable and the MA component involves a linear combination of the innovation and its lagged values. The conditional variance $h_{j,t}^2$, $j = 1,2$, is a GJR-GARCH(1,1) process which models the heteroskadasticity and the asymmetric volatility clustering with an equation involving both the lagged value of the variance and the lagged value of the squared innovation.

## 3.2 State-dependent copula model

The theoretical framework of the multivariate model is based on Sklar's theorem. Let $H$ be the joint cumulative distribution function of a random pair of continuous random variables $(X,Y)$ and let $F$ and $G$ be the marginal distributions functions of $X$ and $Y$, respectively. Sklar's theorem (Sklar, 1959) states that there exists a unique 2-dimensional copula function $C$ on $[0,1]^2$ such that for all $(x,y) \in \mathbb{R}^2$ we have $H(x,y) = C(F(x), G(y))$. The functional form of the copula $C$ and its parameters determine the shape and the strength of the dependence structure between $X$ and $Y$.

Our analysis focuses on multivariate time series and both marginal distributions and copula functions must be considered in their conditioned versions: this extension was introduced and studied in Patton (2006a, 2006b). Therefore, assuming that the time series represent the log-returns of some financial instrument at time $t$, e.g., $(X_{1,t})_t$ and $(X_{2,t})_t$, adapted to the filtration $(\mathcal{F}_t)_t$ with $\mathcal{F}_{t-1}$-conditional marginal distributions $F_{1,t}$ and $F_{2,t}$ respectively, the $\mathcal{F}_{t-1}$-conditional joint distribution $H_t$ of the pair $(X_{1,t}, X_{2,t})_t$ can be expressed in terms of an $\mathcal{F}_{t-1}$-conditional copula $C_t$

$$H_t(x_1,x_2) = \mathbb{P}\left(X_{1,t} \leq x_1, X_{2,t} \leq x_2\right) = C_t(F_{1,t}(x_1), F_{2,t}(x_2)), \quad (x_1,x_2) \in \mathbb{R}^2.$$

The copula function $C_t$ is characterized by a functional form and a set of parameters: in this paper we focus on elliptical copulas, in particular we consider Gaussian and t-copula functions. The novelty of the approach consists in assuming a dynamics in which the correlation coefficient is a

nonlinear function of the lagged values extending the models presented in Patton (2006a and 2006b) in the spirit of the state-dependent autoregressive models discussed in Gobbi and Mulinacci (2021).
Patton (2006a, 2006b) suggests to model time variation in the conditional copula parameter in order to capture time-varying volatility using a specification similar to the GARCH model in the spirit of Hansen (1994). Explicitly, the time-varying correlation coefficient $\rho_t$ has the following functional forms:

$$\rho_t = \Lambda\left(\beta_0 + \beta_1\rho_{t-1} + \beta_2\frac{1}{m}\sum_{i=1}^{m}\Phi^{-1}\left(u_{1,t-i}\right)\Phi^{-1}\left(u_{1,t-i}\right)\right)$$

for the Gaussian copula ($\Phi^{-1}$ is the inverse of the standard normal distribution) and

$$\rho_t = \Lambda\left(\beta_0 + \beta_1\rho_{t-1} + \beta_2\frac{1}{m}\sum_{i=1}^{m}T_\nu^{-1}(u_{1,t})T_\nu^{-1}(u_{2,t})\right)$$

for the Student's $t$ copula with $\nu$ degrees of freedom being $T_\nu^{-1}$ the inverse cumulative distribution function of a standard Student's $t$ distribution with $\nu$ degrees of freedom. In both cases, $u_{1,t}$ and $u_{2,t}$ are the conditional distribution functions values of the two marginal lagged variables ($m$ is the lag order) while $\Lambda(x) = (1+e^{-x})/(1+e^{-x})$ is a modified logistic function that is used to ensure that the correlation coefficient is in $(-1,1)$ for all $t$ and $(\beta_0, \beta_1, \beta_2)$ is the vector of parameters.

In this paper we consider a slighly modified version of the time-varying correlation coefficient

$$\rho_t = \Lambda\left(\beta_0 + \beta_1\rho_{t-1} + \beta_2\frac{1}{m}\sum_{i=1}^{m}\bar{\eta}_{1,t-i}\bar{\eta}_{2,t-i}\right), \tag{3.2}$$

where $\bar{\eta}_{j,t}$, $j = 1,2$, are the standardized lagged innovations. This approach, allows to disentangle the nature of the copula from the distribution of the lagged variables on which it depends.
In order to evaluate the impact of a powerful and systemic shock such as the Covid-19 pandemic, a possible drawback of the Patton model may be the fact that the autoregressive coefficient $\beta_1$ is constant. To overcome this limit we try to generalize the Patton approach in the spirit of the state-dependent autoregressive models considered by Gobbi and Mulinacci (2021) where the autoregressive coefficient in equation (3.2) is a specified function of $\rho_{t-1}$, say, $\psi(\rho_{t-1})$. Therefore, the state-dependent time-varying correlation coefficient has the form

$$\rho_t = \Lambda\left(\alpha_0 + \psi(\rho_{t-1}; \alpha_1, ..., \alpha_{k-1})\rho_{t-1} + \alpha_k\frac{1}{m}\sum_{i=1}^{m}\bar{\eta}_{1,t-i}\bar{\eta}_{2,t-i}\right), \tag{3.3}$$

where the function $\psi(\rho_{t-1}; \alpha_1, ..., \alpha_{k-1})$ is characterized by a vector of parameters $(\alpha_1, ..., \alpha_{k-1})$, that modify its properties. The idea behind this specification of the correlation coefficient arises from the consideration that the dynamics of $\rho_t$ also influences the "weight" that is attributed to the autoregressive component rather than being constant over time. In particular, in this article we propose an exponential form of $\psi$ characterized by two parameters; more precisely

$$\psi(x; \alpha_1, \alpha_2) = \alpha_1 e^{\alpha_2 |x|}, x \in [-1, 1] \tag{3.4}$$

As a function of $\rho_{t-1}$, $\psi$ has some properties which characterize the persistence. As can be seen from the formula, it can assume positive and negative values depending on the sign of the first parameter $\alpha_1$. Not only that: if $\alpha_1 > 0$, $\psi$ is positive and, if $\alpha_2 > 0$ it is an increasing function of $|\rho_{t-1}|$ and it assumes its minimum value $(\alpha_1 > 0)$ when $\rho_{t-1} = 0$ while, if $\alpha_2 < 0$, it is a decreasing function of $|\rho_{t-1}|$ and it assumes its maximum value $(\alpha_1 > 0)$ when $\rho_{t-1} = 0$. The opposite behaviour is performed for $\alpha_1 < 0$ when $\psi$ assumes only negative values. This implies that we can model different relationships between $\rho_{t-1}$ and the persistence.

# 4 Empirical application to exchange rates: estimation and portfolio tail risk

## 4.1 The data

The data set analysed in this work is composed by a couple of daily exchange rate returns: U.S. dollar - Japanese yen and U.S. dollar - Euro from 28/03/2018 to 06/10/2021. The period of observations covers the Covid-19 pandemic whose beginning can be considered December 2019. In fact, already in that month the news coming from Wuhan had alarmed the financial markets. The total number of observations is 920. The data (in percentage form) are plotted in Figure 2 in the Appendix. The decision to use the log-returns of exchange rates arises from the consideration that these variables are very reactive in the face of sudden economic and financial shocks. Market operators react immediately to bad news and in this specific case of the Covid-19 pandemic they responded violently anticipating the disastrous effects on the real economy during 2020. The descriptive statistics relating to log-returns are reported in Table 2 in the Appendix.

## 4.2 Estimation results

The estimation was made using the rugarch-package of R and the selection of the models of the marginal distributions was carried out with the criterion of the minimum AIC (Akaike, 1973). This allowed to identify the orders of the conditional mean model in the ARMA($p$,$q$)-GJR-GARCH(1,1) specification. For both time series of exchange rates log-returns the best fitted models are ARMA(5,5)-GARCH(1,1) with Student's t distributed innovations: more precisely, the leverage parameter $\gamma_j$ of the GJR-GARCH model is not significantly different from zero for both exchange rates returns. The estimation results are shown in Table 3 in the Appendix that reports the estimates of parameters of the marginal distribution models, the value of the log-likelihood, the AIC value and the $p$-values relating to the Ljung-Box test (Ljung and Box, 1978) and the McLeod-Li (McLeod and Li, 1983) test on residuals and squared residuals respectively.

Some insights are needed to investigate the main difference in the behavior of the marginal distributions. The drift term (measured by the estimate of the parameter $c_j$) in the conditional mean equation is close to zero and positive for both time series, indicating a slight appreciation of the dollar against the European and Japanese currencies: we can ask ourselves whether the dollar has better resisted the arrival of the pandemic. The AR coefficients are similar except for the coefficient relating to the fourth lag $\phi_{j,4}$ which is close to zero for the U.S. dollar - Japanese yen exchange rate (0.070984) and strongly negative for the U.S. dollar - Euro exchange rate (-0.723295). Exactly the opposite change of sign occurs for the fourth MA coefficient, e.g., $\theta_{j,4}$. As for the variance equation, the ARCH component, measured by $\omega_{j,1}$, is definitely higher in the U.S. dollar - Japanese yen exchange rate (0.108841) whereas the GARCH component, measured by $\omega_{j,2}$, behaves in a mirror being higher for the U.S. dollar - Euro (0.949227 against 0.756531). The estimates of the degrees of freedom parameter are coherent with the hypothesis of heavy tails in the marginal distributions.

We now focus on the method discussed in subsection 3.2 in order to study the conditional joint distribution of daily marginal log-returns of the pair of exchange rates estimated above. In particular, we are interested in investigating the impact of the pandemic on the nature of the dependence structure between exchange rates. The estimation technique is the quasi-maximum likelihood introduced and discussed in White (1994).
Once the marginal models have been estimated, a preliminary analysis of the standardized residuals highlights the absence of any tail dependence. However we have tested the Gumbel copula, the Clayton copula, the $t$-copula and the gaussian copula with constant parameters: based on AIC criterium the best fitted one is the latter confirming the observed nature of the data. For this reason we consider the gaussian copula with constant and time-varying correlation coefficient. In practice, we compare three joint distributions obtained with the gaussian copula which differ in the functional specification of the dependence coefficient: the constant copula parameter model, the Patton specification in (3.2) and the state-dependent model in (3.3) with autoregressive functional coefficient of exponential type as in (3.4). As for specifications (3.2) and (3.3) we have selected $m = 5$ since it coincides with the lags considered in the ARMA part of the marginal dynamics: this choice is in line with Patton (2006a

and 2006b). However experiments with different values of $m$ show that the estimates do not change significantly. In comparing the estimates obtained and their goodness-of-fit, the structural break that occurred with the Covid-19 pandemic plays a decisive role both in the selection of the most suitable model and in the interpretation of the results.

A premise is necessary before any comparative analysis of the estimates. The number of observations is not very large and therefore the comparison between the models is not easy. We will use the AIC value for comparing the goodness-of-fit of the different models, but we stress the fact that an higher number of observations would be needed to better distinguish them. We can certainly consider the following insights. Firstly, it should be noted that, as shown in Tables 4-6 in the Appendix, the AIC of the constant parameter copula model is the highest among the three considered models, indicating that the temporal dependence of the correlation coefficient is a necessary assumption that the empirical data support.

The AIC value corresponding to the Patton model (Table 5) and the state dependent autoregressive coefficient (Table 6) cases are very close each other, being that corresponding to the Patton model slighly lower indicating a better goodness-of-fit. Nonetheless, analyzing Table 6, the estimates of $\alpha_2$ is significantly different from zero indicating a state dependent autoregressive component in the correlation coefficient (see (3.4)). Given these facts, we stress again that a larger number of observations would be needed to obtain a clearer distinction. In the sequel we will analyze the different impact of the different selection between the Patton and the state dependent model.

Figure 3 in the Appendix shows the dynamics of the correlation coefficient according to the Patton model and the state dependent one. The vertical dotted line identifies the beginning of the pandemic that we place on 01/01/2020. We see a slightly increasing trend after the onset of the pandemic with a peak in the weeks immediately following (in line with the findings in Ahdika et al., 2021, and Kinateder et al. 2021). On the contrary, the dynamics appear to be mean stationary in the pre-Covid period. In the post-Covid period the trends are apparently similar but the higher variability induced by the state-dependent exponential model is highlighted in Figure 4 in the Appendix where the plot of the dynamics of the differences in the correlation coefficient $\rho_t^{EXP} - \rho_t^{P}$ implied by the two models are shown. We observe that the correlation dynamics induced by the state-dependent exponential model tends to be higher than that induced by Patton model after the arrival of the pandemic with a very marked peak in the immediacy of the lockdown measures. Figure 5 in the Appendix compares the constant autoregressive coefficient of the Patton's model and the dynamic of the state dependent one with exponential shape: it is evident that the constant one typically overestimates the state-dependent exponential autoregressive coefficient, while it is lower in turbolent situations (post-Covid period of time).

## 4.3 Portfolio of exchange rates: the impact of the correlation structure on tail dependence

In this section, we consider an equi-weighted portfolio of Euros and Yen and we will analyze the 10 days Value-at-Risk and Expected Shortfall comparing the standard autoregressive model, the Patton model and the time-varying exponential autoregressive coefficient model. Through a Monte Carlo simulation experiment we will prove that, although the Patton and state-dependent models produce a very similar contribution from a statistical point of view, the difference in impact in the calculation of the two above risk indicators will be significant.

Let $(E_t)_t$ and $(Y_t)_t$ denote the exchange rates from USD dollar to Euro and to Japanese Yen, respectively. We consider a portfolio whose value at time $t$ is

$$V_t = 0.5C\frac{E_0}{E_t} + 0.5C\frac{Y_0}{Y_t}, \tag{4.5}$$

where $V_0 = C$ is the initial capital in USD dollars. We present the results of a Monte Carlo study based on the simulation of portfolio returns. The aim of the experiment is to capture the effects on the VaR and the expected shortfall of the three correlation models under considerations. The data generating process (DGP) is designed to reflect the estimates obtained in subection 4.2. In fact, we consider three different DGPs characterized by the same marginal distributions given by equations (3.1) but different correlation structures. More precisely, the DGPs differ in the amount of dependence between the two exchange rates induced by the three specifications of the correlation dynamics. Notice that this is the only source of risk in which the model differ. We simulate a 10-day trajectory of exchange rates, e.g., $\tilde{X}_{j,T+10}$, $j = 1, 2$. The value of the equi-weighted portfolio after 10 days, e.g. $\tilde{V}_{T+10}$, is obtained using equation (4.5). The number of simulated trajectories is $M = 50000$. The empirical distribution of returns of $\tilde{V}_{T+10}$ is needed to compute the Value-at-Risk and Expected Shortfall with level 0.01 and 0.05 of the returns. The simulation algorithm relating to the Patton and the state-dependent exponential models is the following. Consider the step $T + n$, with $n = 1, ..., 10$.

1. We use the vector $(\bar{\eta}_{j,T+n-5}, ..., \bar{\eta}_{j,T+n-1})$, $j = 1, 2$, to compute $\tilde{\rho}_{T+n}$ using equations (3.2) or (3.3). We simulate $(\tilde{u}_{T+n}, \tilde{v}_{T+n}) \sim C(\cdot, \cdot; \tilde{\rho}_{T+n}$.
2. Simulate the standardized residuals

$$\begin{cases} \tilde{\varepsilon}_{1,T+n} = T_{\hat{\nu}_1}^{-1}\left(\tilde{u}_{T+n}\right)\sqrt{\frac{\hat{\nu}_1-2}{\hat{\nu}_1}} \\ \tilde{\varepsilon}_{2,T+n} = T_{\hat{\nu}_2}^{-1}\left(\tilde{v}_{T+n}\right)\sqrt{\frac{\hat{\nu}_2-2}{\hat{\nu}_2}}, \end{cases}. \tag{4.6}$$

3. Simulate the residuals

$$\begin{cases} \tilde{\eta}_{1,T+n} = \tilde{\varepsilon}_{1,T+n}\tilde{h}_{1,T+n} \\ \tilde{\eta}_{2,T+n} = \tilde{\varepsilon}_{2,T+n}\tilde{h}_{2,T+n}, \end{cases} . \tag{4.7}$$

using

$$\begin{cases} \tilde{h}^2_{1,T+n} = \hat{\omega}_{1,0} + \hat{\omega}_{1,1}\tilde{\eta}^2_{1,T+n-1} + \hat{\omega}_{1,2}\tilde{h}^2_{1,T+n-1} \\ \tilde{h}^2_{2,T+n} = \hat{\omega}_{2,0} + \hat{\omega}_{2,1}\tilde{\eta}^2_{2,T+n-1} + \hat{\omega}_{2,2}\tilde{h}^2_{2,T+n-1}, \end{cases} . \tag{4.8}$$

4. The returns are

$$\begin{cases} \tilde{X}_{1,T+n} = \hat{c}_1 + \sum_{i=1}^5 \hat{\phi}_{1,i}\tilde{X}_{1,T+n-i} + \sum_{i=1}^5 \hat{\theta}_{1,i}\tilde{\eta}_{1,T+n-i} + \tilde{\eta}_{1,T+n}, \\ \tilde{X}_{2,T+n} = \hat{c}_2 + \sum_{i=1}^5 \hat{\phi}_{2,i}\tilde{X}_{2,T+n-i} + \sum_{i=1}^5 \hat{\theta}_{2,i}\tilde{\eta}_{2,T+n-i} + \tilde{\eta}_{2,T+n}, \end{cases} . \tag{4.9}$$

5. Simulate the exchange rate levels

$$\begin{cases} \tilde{E}_{1,T+n} =, \tilde{E}_{1,T+n-1}e^{\tilde{X}_{1,T+n}} \\ \tilde{E}_{2,T+n} =, \tilde{E}_{2,T+n-1}e^{\tilde{X}_{2,T+n}}, \end{cases} . \tag{4.10}$$

6. The portfolio value is $\tilde{V}_{T+n} = 0.5\tilde{E}_{1,T+n} + 0.5\tilde{E}_{2,T+n}$
7. Repeat steps 2-6 $M$ times.

The simulation algorithm described above generates a simulated distribution of the portfolio value after 10 days, i.e., $\tilde{V}_{T+10}$, which can be used to obtain the VaR and the expected shortfall of the portfolio. The results of the Monte Carlo study are displayed in Table 1. For both risk measures, the constant autoregressive model returns the lowest value and the state-dependent exponential the highest one, while the Patton model is always in between the two. Furthermore, they show that the model with constant correlation coefficient and the model with Patton dynamics produce very similar risk indicators, while the model with the state-dependent exponential dynamics differs markedly both in terms of VaR and expected shortfall. In fact, though the Patton and exponential modesl are substantially equivalent as goodness-of-fit, the choice of the state-dependent exponential model corresponds to a more conservative approach. Figure 1 depicts the empirical densities of the worst 5% of the simulated portfolio returns generated by the Patton and exponential models. Figure 1 shows the empirical density obtained from Monte Carlo simulations for the portfolio returns below the 5th percentile, comparing the Patton model with the state-dependent exponential one. Observe how the exponential model generates much more negative returns than the Patton model, thus highlighting a greater risk in the portfolio of currencies.

Given the monetary meaning of the VaR and ES risk measures as regards capital allocation policies, the above analysis suggests that that the choice of the new proposed model for the time-varying dynamics of the correlation coefficient is the most prudential among those analyzed and considered as benchmarks in this paper.

| | Constant | Patton | state-dependent exponential |
|---|---|---|---|
| Minimum | -5.8117 | -5.3786 | -6.7012 |
| Maximum | 4.6651 | 4.0606 | 6.1182 |
| SD | 0.7790 | 0.7867 | 0.8281 |
| VaR(5%) | 2.0465 | 2.0475 | 2.1064 |
| Expected shortfall (5%) | 2.4403 | 2.4463 | 2.5358 |
| VaR(1%) | 2.6954 | 2.6863 | 2.7989 |
| Expected shortfall (1%) | 3.0554 | 3.0741 | 3.2280 |

Table 1: Monte Carlo simulation results on the portfolio of exchange rates.

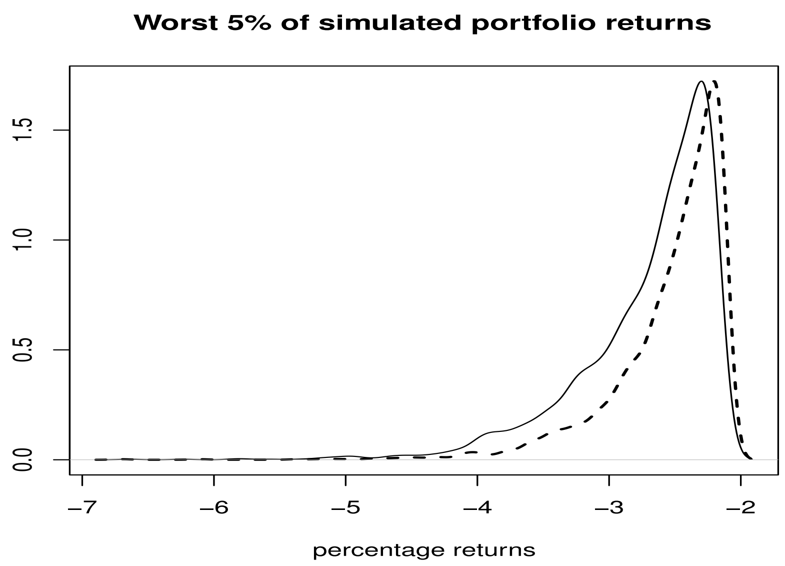


Figure 1: Empirical density of the worst 5% portfolio returns. Solid line: Patton model; dashed line: state-dependent exponential model.

# 5 Conclusion

In this paper we have introduced a new dynamics for the dependence coefficient of an elliptical copula and we have tested three different types of dependent structures between time series of exchange rates, e.g., U.S. dollar - Japanese yen and U.S. dollar - Euro during the Covid-19 pandemic. In particular, we estimated three correlation models based on copula functions with three different parameter specifications: constant over time, autoregressive with constant coefficient (from Patton, 2006) and a new autoregressive model with state-dependent coefficient of exponential type. The study of goodness-of-fit showed that the first model underperforms compared to the remaining two, indicating that the time-varying dynamics is needed for the exchange rates, while the Patton model and the state-dependent model offer a very similar fit. However, this does not prevent the two models from offering a significantly different performance in the simulation experiment we conducted. In fact, we have applied these three models to measure the riskiness of a portfolio of currencies, showing that the choice of a state dependent autoregressive coefficient corresponds to a more conservative approach since it returns the highest values of both risk measures analyzed, e.g., the Vaue-at-Risk and the Expected Shortfall. The sample period analysed is characterized by the Covid-19 pandemic and its financial impact is strongly noticeable on the behaviour of the time-varying dependence parameter, pointing out the importance of a temporally adaptive dynamics especially in the period of economic and financial turbulence.

The implications of the obtained results are related to the problem of capital allocation: the study suggests that the model with state-dependent autoregressive coefficient corresponds to a more prudential strategy in the turbolent Covid pandemic period, compared with the standard alternative models with constant or with autoregressive dynamics correlation coefficient.

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# 6 Appendix: tables and figures

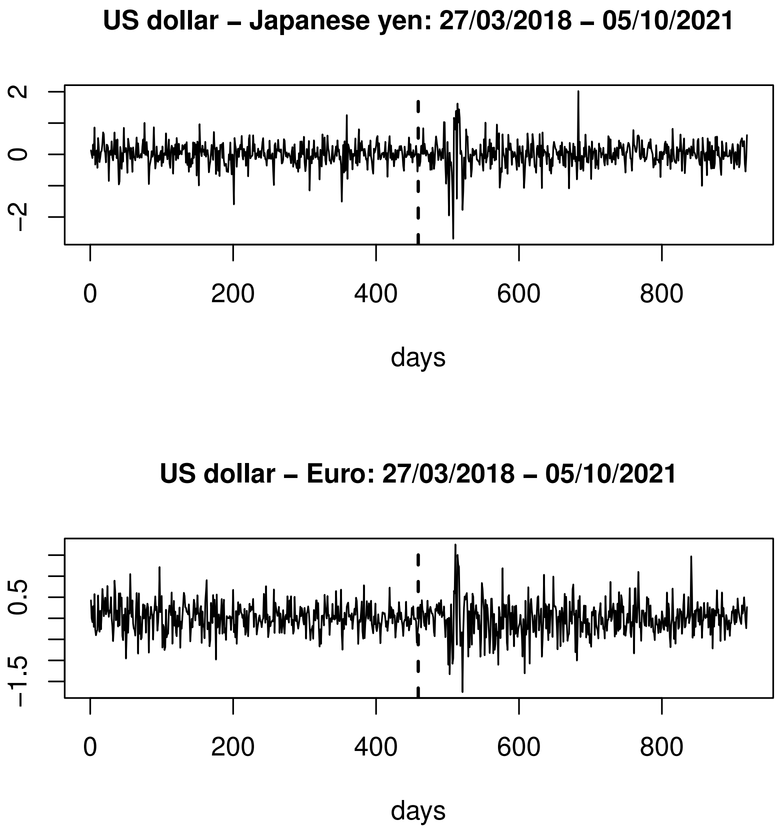
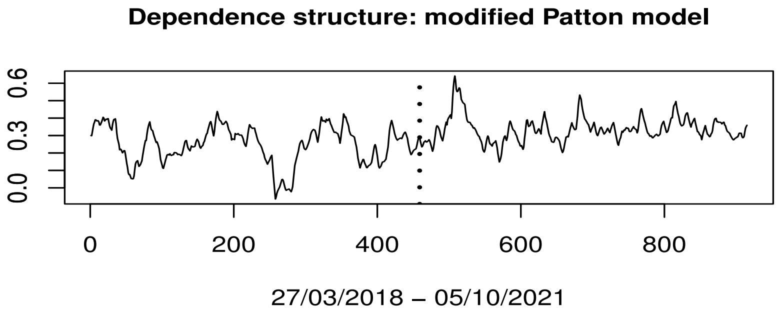



Figure 2: Log-returns.



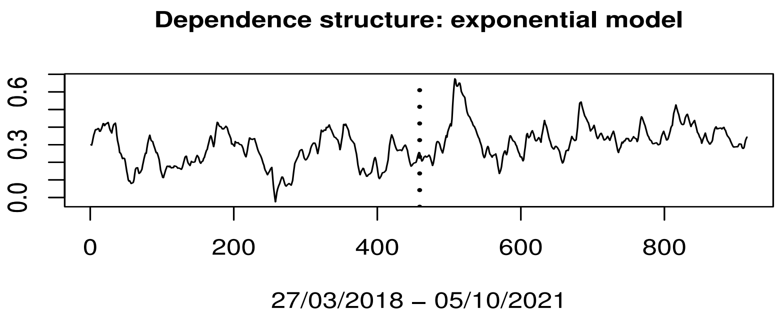




Figure 3: Dependence structure: dynamics of the correlation coefficient according to the modified Patton model (bottom) and the state-dependent exponential model (down). The vertical dashed line identifies the beginning of the Covid pandemic phase in most countries of the world.

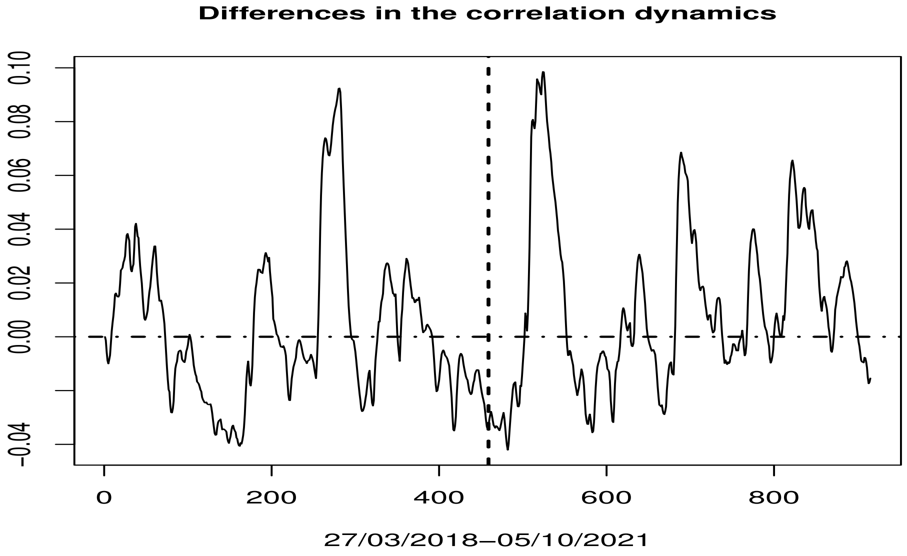



Figure 4: Absolute differences of the correlation coefficient dynamics generated by the two alternative models: $\rho_t^{EXP} - \rho_t^{P}$. The vertical dashed line identifies the beginning of the Covid pandemic phase in most countries of the world.

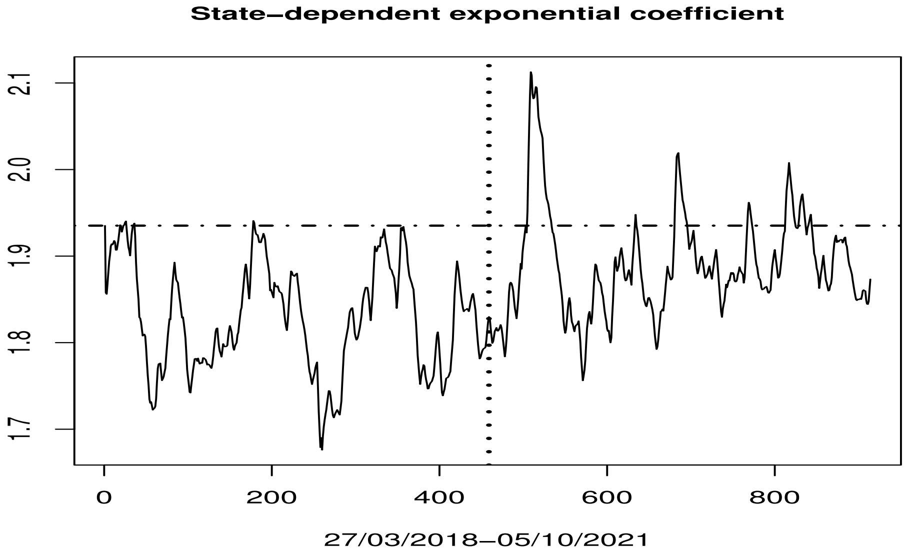



Figure 5: Relationships between the exponential autoregressive coefficient and the constant autoregressive coefficient. The vertical dashed line identifies the beginning of the Covid pandemic phase in most countries of the world. The horizontal line represents the value of the autoregressive coefficient in the modified Patton dynamics.

| | U.S. dollar - Japanese yen | U.S. dollar - Euro |
|---|---|---|
| N. of obs. | 920 | 920 |
| Mean | 0.005323 | 0.06892 |
| Maximum | 2.021916 | 1.753970 |
| Minimum | -2.693895 | -1.752801 |
| Std. dev. | 0.395612 | 0.375652 |
| Skewness | -0.469995 | 0.024288 |
| Kurtosis | 5.044356 | 1.85223 |

Table 2: Descriptive statistics relating to log-returns in percentage form.

| | U.S. dollar - Japanese yen | U.S. dollar - Euro |
|---|---|---|
| | **ARMA(5,5)-GARCH(1,1)** | **ARMA(5,5)-GARCH(1,1)** |
| $c$ | 0.014225 (0.008727) | 0.010236 (0.010521) |
| $\phi_{j,1}$ | -0.215870 (0.024025) | -0.369444 (0.207732) |
| $\phi_{j,2}$ | 0.617358 (0.023607) | 1.212758 (0.067431) |
| $\phi_{j,3}$ | 0.663843 (0.021679) | 0.866613 (0.269389) |
| $\phi_{j,4}$ | 0.070984 (0.024867) | -0.723295 (0.067497) |
| $\phi_{j,5}$ | -0.728333 (0.021705) | -0.616614 (0.201820) |
| $\theta_{j,1}$ | 0.241345 (0.000578) | 0.454922 (0.184798) |
| $\theta_{j,2}$ | -0.611589 (0.000804) | -0.984135 (0.248401) |
| $\theta_{j,3}$ | -0.744960 (0.000102) | -0.407830 (0.117236) |
| $\theta_{j,4}$ | -0.152233 (0.002636) | 0.653537 (0.075092) |
| $\theta_{j,5}$ | 0.742006 (0.000105) | 0.702825 (0.174705) |
| $\omega_{j,0}$ | 0.018884 (0.005984 | 0.001342 (0.000886) |
| $\omega_{j,1}$ | 0.108841 (0.034312) | 0.040429 (0.017437) |
| $\omega_{j,2}$ | 0.756531 (0.058537) | 0.949227 (0.019528) |
| $\nu_j$ | 5.662673 (1.045171) | 7.539101 (2.045167) |
| Log-lik | -339.6638 | -327.7351 |
| AIC | 0.77185 | 0.74589 |
| LB (Lag=1) | 0.4434 | 0.4724 |
| LB (Lag=29) | 1.000 | 1.000 |
| LB (Lag=49) | 0.9835 | 1.000 |
| ML (Lag=1) | 0.3826 | 0.3432 |
| ML (Lag=29) | 0.7842 | 0.2177 |
| ML (Lag=49) | 0.2885 | 0.1737 |

Table 3: Results for the marginal distributions. Estimated parameters and relative standard errors. We report the value of the loglikelihood at maximum, the AIC value and $p$-values relating to the Ljung-Box test (LB) and McLeod-Li test (ML) on residuals.

| Constant parameter | |
|---|---|
| $\rho$ | 0.304810** |
| | (0.028621) |
| $\ell_C$ | 44.83 |
| AIC | -0.095803 |

Table 4: Quasi-maximum likelihood estimates of the model with constant parameter. The asterisks denote that the parameter is significantly different from zero at 10% level (*) or at 5% level (**). We also report the value of the copula likelihood at optimum and the AIC value.

| Patton's model | |
|---|---|
| $\beta_0$ | 0.018081 |
| $\beta_1$ | 1.935189** |
| $\beta_2$ | 0.085531** |
| $\ell_C$ | 51.623670 |
| AIC | -0.106398 |

Table 5: Quasi-maximum likelihood estimates of the copula parameters. The table considers the **gaussian copula** and the time-varying parameters as in Patton (2006a). The asterisks denote that the parameter is significantly different from zero at 10% level (*) or at 5% level (**). We also report the value of the copula likelihood at optimum and the AIC value.

| state-dependent exponential model | |
|---|---|
| $\alpha_0$ | 0.040819** |
| $\alpha_1$ | 1.675904** |
| $\alpha_2$ | 0.343329* |
| $\alpha_3$ | 0.074377** |
| $\ell_C$ | 52.408260 |
| AIC | -0.105926 |

Table 6: Quasi-maximum likelihood estimates of the copula parameters. The table considers the **gaussian copula** and the time-varying parameters proposed in this papers. The asterisk denotes that the parameter is significantly different from zero at 10% level (*) or at 5% level (**). We also report the value of the copula likelihood at optimum and the AIC value.