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A Modal Logic for Temporal and Jurisdictional Classifier Models

Cecilia Di Florio[1,3], Huimin Dong[2], and Antonino Rotolo[1]

[1]University of Bologna, [2]TU WIEN, [3]University of Luxembourg

Abstract. Logic-based models can be used to build verification tools for machine learning classifiers employed in the legal field. ML classifiers predict the outcomes of new cases based on previous ones, thereby performing a form of case-based reasoning (CBR). In this paper, we introduce a modal logic of classifiers designed to formally capture legal CBR. We incorporate principles for resolving conflicts between precedents, by introducing into the logic the temporal dimension of cases and the hierarchy of courts within the legal system.

Keywords: Modal logic, Legal Case-based Reasoning, Classifier Models

1 Introduction

The use of machine learning (ML) classifiers to predict legal outcomes has been widely discussed in both academic literature and policy circles (see, e.g., [12,20,5,3]). In particular, judges have voiced specific concerns regarding the integration of ML into judicial decision-making, especially given that there is no guarantee that ML outcomes are normatively correct, accurate, or robust. To address these concerns, symbolic methods can be employed to formally verify the outcomes produced by judicial ML classifiers.

Especially within common law systems, one critical constraint that an ML classifier should satisfy is that of *precedential constraint*, consistent with the doctrine of *stare decisis*, which holds that prior judicial decisions constitute binding case law guiding future rulings. To develop verification methods that account for this constraint, we begin with an intuitive observation: ML classifiers derive the outcome of new cases based on a body of past cases, thereby performing a form of case-based reasoning (CBR). This observation is significant because the legal CBR literature already includes models of precedential constraint. Two well-known such models are the reason and result models proposed by Horty [14]. In fact, Liu and others [17] establish a correspondence between Horty's models and classifier models (CMs) developed within the binary input classifier logic (BCL) framework [16]. A key assumption in Horty's models is the *consistency* of the initial case base: no prior case violated the precedential constraint [14]. As a consequence, the models preclude the possibility of retrieving conflicting precedents.

Yet, the consistency requirement may not hold in real-world legal systems [9]. Conflicts of precedent do occur in legal practice, which is precisely why certain principles have been developed to resolve them. In common law systems, one of the most important is a *temporal-hierarchical principle*: when precedents conflict, the more recent decision issued by the higher court should prevail [19]. The importance of incorporating both temporal and hierarchical elements into models of precedential constraint, to better reflect actual legal reasoning, has already been emphasised in literature of AI and Law [8,25,11]. In particular, an extension of the semantics of CMs is proposed in [11].

In this work, we follow the idea that legal case-based reasoners are nothing but binary classifiers [14,17]. We propose a modal logic for handling conflict of precedents. To this end, we extend BCL with both temporal and hierarchical operators. The resulting framework is theoretical, offering semantic and proof-foundations that may serve as a basis for future applications such as verification algorithms.

The paper is structured as follows. Section 2 introduces the notion of conflicting precedents. Section 3 extends BCL with the temporal and hierarchical elements. Building on these elements, Section 4 provides a formal definition of both precedents and potentially binding precedents. A potentially binding precedent is a case that may constrain future decisions, unless it falls under some exceptions. Finally, Section 5 formalises a temporal and hierarchical principle for resolving conflicts among precedents.

2 Legal CBR and Conflict of Precedents

Following the literature on legal CBR, our focus is on civil cases within common law systems, where judgments are rendered in favor of one of two opposing parties — the plaintiff or the defendant. Suppose a ML classifier predicts the outcome of a case as favoring either the plaintiff or the defendant. The central question we want to address is: *Is this outcome consistent with the precedential constraint?* To answer this question, we need to develop a suitable model.

Two well-known models of precedent constraint — the *reason* model and the *result* model — were introduced by Horty [14]. Both represent cases using factors, which are legally relevant factual patterns supporting one party over the other. However within these models we cannot handle *conflict of precedents*. To provide some intuition on the matter, in this section we focus on the result model. The result model captures *a fortiori reasoning*: a case is forced to have a given outcome if there exists a precedent with the same outcome such that all differences between the two cases make the new case at least as strong in favor of that outcome as the precedent. Roughly, a new case should have the same outcome as a precedent if it includes all the factors supporting the precedent's outcome (or more), and no additional factors opposing it. Within the result model, a *case base* — that is, a set of decided cases — is *consistent* if no case violates this *a fortiori* constraint. In line with the CATO [2] framework for CBR, we concentrate on cases pertaining trade secret misappropriation and draw on a selected set of its factors:

- Pro-plaintiff factors: $\mathsf{measure}$: "The plaintiff had taken security measures to keep the secret", $\mathsf{deceived}$: "The defendant had obtained the secret by deceiving the plaintiff";
- Pro-defendant factors: $\mathsf{reverse}$: "The product is reverse-engineerable", $\mathsf{disclosed}$: "The plaintiff had voluntarily disclosed the secret to outsiders".

Now consider two cases, as summarized below:

- Case 1: $\{\mathsf{reverse}, \mathsf{disclosed}, \mathsf{measure}\}$ — decided for the plaintiff;
- Case 2: $\{\mathsf{reverse}, \mathsf{measure}, \mathsf{deceived}\}$ — decided for the defendant.

The given case base violates the *a fortiori* constraint: Case 2 contains fewer pro-defendant factors and more pro-plaintiff factors than Case 1, yet it is decided for the defendant, whereas Case 1 is decided for the plaintiff. The case base inconsistent. Suppose a ML classifier have to decide a new case:

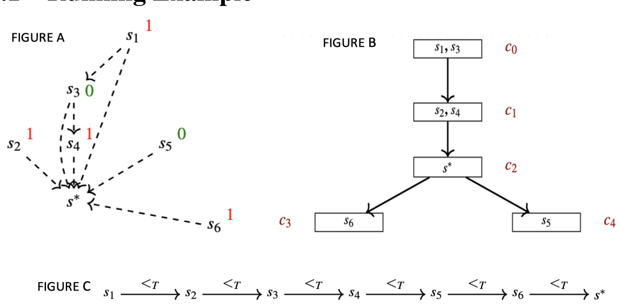
– Case 3: $\{\mathsf{reverse}, \mathsf{measure}, \mathsf{deceived}, \mathsf{disclosed}\}$ — outcome not known.

Under the *a fortiori* constraint, Case 3 should be decided for the plaintiff, based on Case 1, and for the defendant, based on Case 2. We have *a conflict of precedents*. How should Case 3 be decided? Based solely on the available factual elements, the answer is uncertain. In this paper, we show that by taking into account other elements beside facts, conflicts can be effectively managed. In particular, we will model a principle for resolving conflicts based on the temporal dimension of the cases — when the cases were assessed — and the hierarchical relationship between the courts that decided them.

In the example, we identified binding cases solely on the basis of factual comparison. That is, we considered as binding those cases that are applicable to the case at hand purely due to factual relevance. This relevance is grounded in *a fortiori* reasoning. Other criteria for determining factual relevance are adopted in different approaches. In the strict model [8], a precedent is considered applicable (and therefore binding) whenever its underlying reason applies to the new case. In many arguments-based approaches, the similarities and differences between cases can be strategically emphasized or downplayed [22]. In all these approaches, factual relevance is regarded as sufficient for determining when a precedent is binding. In this paper, we do not commit to any specific model of legal CBR. Our primary interest lies in addressing the problem of conflicting precedents. Henceforth, when we refer to a case as relevant to the case under consideration, we mean that it is factually applicable to the case at hand. Relevant cases, depending on specific criteria that we will introduce in the next section and formally define in Sections 4 and 5, may qualify as binding precedents.

3 Language and Temporal Jurisdictional Classifiers Models

3.1 Running Example



In the example from the previous section, we identified a conflict of precedents for Case 3: both Case 1 and Case 2 were relevant — based on the *a fortiori* constraint — to Case 3, but they were decided in opposite directions. From this point onward, we will consider a more complex and abstract example than the one presented above, in order to better illustrate the scenarios in legal practice that we aim to model in this paper. Let s^* represent a new case to be decided based on previous cases $s_1, \ldots, s_6$. We denote a pro-defendant decision as 0 and a pro-plaintiff decision as 1. The relevance relations among the cases, as well as their decisions, are shown in Figure A. All cases s_i are all relevant to s^*, yet some have been decided as 0 and others as 1. For example, both s_3 and s_1 are relevant for s^*, but s_3 is decided for the defendant, while s_1 is decided for the plaintiff — we can think of s^* as analogous to Case 3, with s_1 corresponding to Case 1 and s_3 to Case 2. Again, this seems to be a conflict among precedents. However, we overlooked three foundational elements of legal decision-making.

Jurisdiction. The precedent constraint is shaped by the legal system where cases are decided. A court is *bound* only by decisions from courts with binding authority over it. Courts *hierarchical and binding relationships* vary depending on the jurisdiction.

> We draw on the civil court system of England and Wales. At the top is the UK Supreme Court (c_0), which, is not formally bound by its own decisions. [1] Next is the Court of Appeal (c_1), also self-bound, followed by the High Court (c_2), which we likewise assume binds itself. At the bottom are the County Courts (which are about 170 [24]), which do not issue binding decisions; we model only two of them (c_3 and c_4). Vertical *stare decisis* applies: higher courts bind lower ones, but not the reverse.

Returning to our example, assume the cases are decided or to be decided by one of the four courts above. Specifically, there exists a hierarchical relationship between the cases, as illustrated in Figure B. Consider cases s_5 and s_6, which are relevant to our case s^*. However, they were decided by courts c_3 and c_4, which do not hold binding authority over court c_2 deciding s^*. Therefore, s_6 *and* s_5 *are not binding for* s^*. We can eliminate s_5 and s_6 from binding precedents. But s_1, s_2, s_3, and s_4 may still be binding.

Time. Obviously, a case is constrained only by earlier decisions. Assume the timeline for our cases is as in Figure C. Then, case s^* is decided after s_1, s_2, s_3, and s_4, which could therefore be binding for s^*. Time is also useful to model *exceptions*.

Exceptions. The precedential constraint is subject to exceptions: there may be situations in which earlier cases lose their binding authority over subsequent ones. One example is *overruling*. Overruling occurs when a court rules against a relevant precedent [19,23]. Not all courts can overrule. In line with common law systems, we assume a higher court can always overrule a lower court's decisions, and that a court may overrule itself when it is not self-bound [19].

For example, let us consider case s_3. This case was decided by the same court — the Supreme Court c_0 — *after* the relevant precedent s_1. The decision in s_3 differs from that in s_1. Since we assumed that c_0 is not bound by its own prior decisions, it has the authority to overrule them. Accordingly, s_1 was overruled by s_3 and lost its binding authority over s^*. Therefore, we can eliminate s_1 which is another source of conflict.

Now, it seems that s_4, s_3, s_2 are left. What would be the decision for s^*? In this paper, we will formalise the notions introduced so far and attempt to answer this question.

We conclude by mentioning that each case will be assigned a *name* [2]. This is consistent with legal practice, where every case has an identifiable name — such as "A vs B" or a unique case identifier. Case naming will enables a hybrid approach in the logic.

[1] Actually c_0 treat its own decision as binding when it sit in a partial panel [18].
[2] As in [13], we use term *names* rather than usual hybrid logic term *nominals*.

3.2 Temporal and Jurisdictional Classifier Models

In this section we instantiate the hierarchical and temporal dimensions in the framework of Classifier Models [16]. First, we define the notion of jurisdiction, which characterises the structure of a specific legal system. This includes the identification of courts, the hierarchical relation among them and which courts issue binding decision.

Definition 1 (Jurisdiction). *A triple* $(\texttt{Courts}, \mathscr{H}, \mathscr{B})$ *is called a* jurisdiction*, denoted as Jur, when:* $\mathscr{H}, \mathscr{B} \subseteq \texttt{Courts} \times \texttt{Courts}$*;* $\mathscr{H}$ *is transitive, irreflexive* [3].

For $c_i, c_j \in \texttt{Courts}$, $c_i \mathscr{H} c_j$ is read as "court c_i is hierarchically higher than c_j"; $c_i \mathscr{B} c_j$ is read as "c_i has binding authority on c_j."

Example 1. The jurisdiction in Figure B of running example can be modeled as $Jur_{ex} = (\texttt{Courts}_{ex}, \mathscr{H}_{ex}, \mathscr{B}_{ex})$, where: the set of courts is $\texttt{Courts}_{ex} = \{c_0, c_1, c_2, c_3, c_4\}$ and the hierarchy relation $\mathscr{H}_{ex} \subseteq \texttt{Courts}_{ex} \times \texttt{Courts}_{ex}$ is $\mathscr{H}_{ex} = \{(c_i, c_j) \mid i < j, 0 \leq i \leq 2, 1 \leq j \leq 4\}$. The binding relation is $\mathscr{B}_{ex} = \mathscr{H}_{ex} \cup \{(c_1, c_1), (c_2, c_2)\}$.

We introduce now the temporal and jurisdictional classifier models. We start by considering a set of possible input values of the classifiers, Atm_0. In Atm_0 there are: input variables describing the facts of the case, belonging to the countable set `Facts`; the courts that can make the decision, belonging to the finite set `Courts`; the name of each case belonging to the countable set `Names`. Pertaining the `Names` we assume that there is a finite subset of names $\texttt{Names}_d \subseteq \texttt{Names}$ for decided cases. The sets `Facts`, `Courts`, and `Names` are non empty and pairwise disjoint to form Atm_0. In other words, $Atm_0 = \texttt{Facts} \cup \texttt{Names} \cup \texttt{Courts}$. The classifier outputs have values in $Val = \{1, 0, ?\}$ where elements stand for *plaintiff wins*, *defendant wins* and *absent decision* respectively. For $o \in \{0, 1\}$, the "opposite" $\overline{o}$ is noted for the value $1 - o$.

Definition 2. *A temporal jurisdictional classifier model (TJCM) is a tuple* $C = (S, f, Jur, \leq_T, \mathscr{R})$ *which satisfies these conditions:*
- $S \subseteq 2^{Atm_0}$ *is a non-empty set s.t.* $\forall s \in S \exists! c \in \texttt{Courts} : c \in s$*;*
- $f : S \longrightarrow Val$ *is called a decision (or classification) function;*
- $\mid S_d \mid$ *is finite, with* $S_d = \{s \mid f(s) \neq ?\}$ *be the set of decided states;*
- *Jur is a jurisdiction;*
- $\leq_T$ *is a total preorder on S* [4]*;*
- $\mathscr{R} \subseteq S \times S$ *is a relation of relevant cases;*
- $\forall s \exists n \in \texttt{Names} : n \in s$*;*
- $\forall s \in S \, \forall n \in \texttt{Names} \cap s : s \in S_d \Leftrightarrow n \in \texttt{Names}_d$*;*
- $\forall s, s' \in S : (s \neq s' \Rightarrow s \cap s' \cap \texttt{Names} = \emptyset)$.

The class of temporal and jurisdictional classifier models is **TJCM** .

S is the set of states of the model, where each state contains a unique court. In this sense, a state $s \in S$ represents a case presented to a specific court $c \in \texttt{Courts}$. The classification function maps each state to a value in the set $\{0, 1, ?\}$, representing the decision associated with the case. For each $s \in S$, we either have $f(s) = o$, with

[3] $\mathscr{H}$ is irreflexive, when: for all $c \in \texttt{Courts}$, it is **not** the case that $c \mathscr{H} c$.

[4] $\leq_T$ is a transitive and reflexive relation, with $\forall s, s' \in S$, either $s \leq_T s'$ or $s' \leq_T s$.

$o \in \{0,1\}$ — indicating that s represents an assessed case — or $f(s) = ?$, meaning s is an unassessed or new case. The set S_d of decided cases is finite. The structure *Jur*, as defined in Definition 1, specifies the hierarchical and binding relations among the courts. The relation $\leq_T$ is a temporal preorder, where $s \leq_T s'$ is interpreted as "s' was not assessed before s". We say that s is simultaneous to s', denoted $s =_T s'$, if both $s \leq_T s'$ and $s' \leq_T s$. From $\leq_T$, we derive the strict temporal order $<_T$: $s <_T s'$ if and only if $s \leq_T s'$ and $s' \not\leq_T s$. Thus, $s <_T s'$ means "s was decided strictly before s'". We write $\leq_T[s] = \{s' \mid s \leq_T s'\}$ for the set of states that are assessed simultaneous or strictly later than s. By using a temporal *preorder*, we allow two cases to be considered simultaneous. While it could be argued that, for any two cases, it is always possible to determine which was decided first, the temporal dimension is introduced here primarily to determine which precedents may be considered binding. We claim that, given a case s, a relevant case s' decided on the same day or in the same week cannot be considered binding: the court assessing s would not have had the opportunity to take s' into account (e.g., because the reasons of s' had not yet been published). Therefore, depending on the chosen temporal granularity, cases decided on the same day or in the same week may be considered simultaneous. The binary relation $\mathscr{R}$ captures the notion of relevance: $s'\mathscr{R}s$ is read as "s' is relevant for s". We write $\mathscr{R}(s) = \{s' \mid s'\mathscr{R}s\}$ for the set of states relevant for s. The relevance relation is crucial in defining the notion of a precedent — roughly, a precedent for a case is a relevant, previously assessed case. No specific definition of relevance is imposed in the model, nor are any particular properties (such as symmetry or transitivity) required for the relevance relation. This flexibility allows the model to accommodate various notions of relevance. Each state is associated with a name, and no two states share the same name. [5] All and only the decided state have names in the finite subset of names $Names_d$. We will say that state s is n-named if $n \in s$.

For notational convenience, we extend the relationships between courts to states: $s \prec s'$ iff $\mathscr{H}(c', c)$ where $c \in s \cap \texttt{Courts}$, $c' \in s' \cap \texttt{Courts}$; $s \cong s'$ iff $c \in s \cap s' \cap \texttt{Courts}$.

We have not specified the relationship between hierarchical and binding relations, allowing our framework to accommodate different legal systems. However, in this work, we focus on *common law systems*. Within such systems, typically, 1) *vertical stare decisis* applies, i.e. decisions of higher courts bind lower courts: if $c_i\mathscr{H}c_j$ then $c_i\mathscr{B}c_j$; 2) lower courts have no binding power on higher courts and two non-hierarchically-related courts don't issue binding decisions one for the other: if not $c_i\mathscr{H}c_j$ then not $c_i\mathscr{B}c_j$; 3) *horizontal stare decisis* may apply, i.e. some c_i may be self-bound, namely it may be $c_i\mathscr{B}c_i$. Ultimately from 1), 2) and 3) it follows

$$\mathscr{H} \subseteq \mathscr{B} \subseteq \mathscr{H} \cup \mathscr{I} \tag{SD}$$

where $\mathscr{I} = \{(c,c) \mid c \in \mathsf{Courts}\}$ is the identity relation to identify the self-bound courts.

Henceforth, we impose a condition on the temporal dimension of states of classifier models: the new and undecided cases are strictly after already assessed cases.

$$\forall s, s' \in S : (f(s) \neq ? \,\&\, f(s') = ?) \Rightarrow s <_T s' \tag{O}$$

From now on we focus on $\overline{\textbf{TJCM}}$, the subclass of TJCMs satisfying (O), and (SD).

[5] As it happens in hybrid logic [6], a state can have multiple names. This is not problematic: legal cases may have a common name like "A vs B" alongside a unique identifier.

Example 2. We model the running example through TJCMs. Fix Atm_0, assuming the finite set of decided names is $\texttt{Names}_d = \{n_1, ..., n_6\}$. Let $S_{ex} = \{s_1, ..., s_6, s^*\} \subseteq 2^{Atm_0}$. Pertaining names, assume $n_i \in s_i$ and $s^* \cap \texttt{Names} = \{n^*\}$. $C_{ex} = (S_{ex}, f_{ex}, Jur_{ex}, \leq_T^{ex}, \mathscr{R}^{ex})$ is the classifier model, where $f_{ex}(s^*) = ?$, $f_{ex}(s_1) = f_{ex}(s_2) = f_{ex}(s_4) = f_{ex}(s_6) = 1$, $f_{ex}(s_3) = f_{ex}(s_5) = 0$; Jur_{ex} is defined in Ex. 1; $\mathscr{R}^{ex}(s^*) = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, $\mathscr{R}^{ex}(s_3) = \{s_1\}$, $\mathscr{R}^{ex}(s_4) = \{s_3\}$; $s_1 <_T s_2 <_T \ldots <_T s_6 <_T s^*$. C_{ex} satisfies (O), (SD).

3.3 Temporal and Jurisdictional Classifers Logic

Now we introduce the temporal and jurisdictional classifiers logic $TJCL$. First, we will introduce the language to express the legal decision process in which the temporal and jurisdictional constraints are considered. For any $o \in Val$, we call $\mathtt{t}(o)$ a decision atom, to be read as "the actual decision (or output) takes value o". $Dec = \{\mathtt{t}(o) : o \in Val\}$ is the set of decision atoms. $Atm = Atm_0 \cup Dec$.

Definition 3 (Language). *The modal language $\mathscr{L}(Atm)$ is hence defined as:*

$$\varphi ::= p \mid \mathtt{t}(o) \mid \mathtt{H}(c,c) \mid \mathtt{B}(c,c) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid [\leq_T]\varphi \mid \mathtt{R}^{\forall}\varphi$$

where $p \in Atm_0$, $\mathtt{t}(o) \in Dec$, and $c \in \texttt{Courts}$.

Let $n \in \texttt{Names}$. We define the formula $@_n\varphi$ as $\Box(n \to \varphi)$. The formula has a hybrid logic flavour but is expressed in our language without explicit hybrid operators.

$\mathtt{H}(c_i, c_j)$ is read as "The court c_i is hierarchically higher than the court c_j". $\mathtt{B}(c_i, c_j)$ is read as "The court c_i issues binding decisions for the court c_j". $\Box\varphi$ is read as "φ is universally true". $[\leq_T]\varphi$ is read as "φ is the case and always will be the case". Finally, $\mathtt{R}^{\forall}\varphi$, is read "$\varphi$ is true at all states that are relevant for the current one". We define the dual operators as usual: $\Diamond\varphi =_{def} \neg\Box\neg\varphi$, $\mathtt{R}^{\exists}\varphi =_{def} \neg\mathtt{R}^{\forall}\neg\varphi$, $\langle\leq_T\rangle\varphi =_{def} \neg[\leq_T]\neg\varphi$. Formulas of $\mathscr{L}(Atm)$ are interpreted with respect to classifier models as follows.

Definition 4 (Satisfaction relation). *Let $C = (S, f, Jur, \leq_T, \mathscr{R})$ be a temporal jurisdictional classifier model. Let $s \in S$. Then:*

$(C,s) \models p \iff p \in s$;
$(C,s) \models \mathtt{t}(c) \iff f(s) = c$;
$(C,s) \models \mathtt{H}(c_i, c_j) \iff c_i \mathscr{H} c_j$;
$(C,s) \models \mathtt{B}(c_i, c_j) \iff c_i \mathscr{B} c_j$;
$(C,s) \models \neg\varphi \iff (C,s) \not\models \varphi$;
$(C,s) \models \varphi \wedge \psi \iff (C,s) \models \varphi$ *and* $(C,s) \models \psi$;
$(C,s) \models \Box\varphi \iff \forall s' \in S : (C,s') \models \varphi$;
$(C,s) \models [\leq_T]\varphi \iff \forall s' \in \leq_T[s] : (C,s') \models \varphi$;
$(C,s) \models \mathtt{R}^{\forall}\varphi \iff \forall s' \in \mathscr{R}(s) : (C,s') \models \varphi$.

A formula φ of $\mathscr{L}(Atm)$ is satisfiable relative to the class **TJCM** if there exists a pointed classifier model (C,s) with $C \in$ **TJCM** such that $(C,s) \models \varphi$. Validity is defined as usual.

By using our unary temporal modality $[\leq_T]$ and name formulas, we can define the binary modalities of future and past — "strictly later" and "strictly earlier". The use of names allows us to capture irreflexivity in the temporal dimension. Specifically, we define $\hat{\mathtt{F}}_n\varphi =_{def} n \wedge \langle\leq_T\rangle(\neg\langle\leq_T\rangle n \wedge \varphi)$ and $\hat{\mathtt{P}}_{n,m}\varphi =_{def} n \wedge \Diamond(\varphi \wedge \hat{\mathtt{F}}_m n)$. The operators $\hat{\mathtt{F}}_n$ and $\hat{\mathtt{P}}_{n,m}$ act as existential future and past operators — indicating the existence of a strictly later or strictly earlier point — as shown in the Proposition below.

Proposition 1. *I) $(C,s) \models \hat{\mathtt{F}}_n\varphi$ iff $(C,s) \models n$ and $\exists s' \in S$ s.t. $s <_T s'$ and $(C,s') \models \varphi$.*
II) $(C,s) \models \hat{\mathtt{P}}_{n,m}\varphi$ iff $(C,s) \models n$ and $\exists s' \in S$ s.t. $(C,s') \models m \wedge \varphi$ and $s' <_T s$.

Further, we can define two new modalities of hierarchy, which are useful to define binding constraints in Section 4.2: $\mathsf{Lower}\varphi =_{def} \bigvee_{c,c' \in \mathsf{Courts}}(c \wedge \mathtt{H}(c,c') \wedge \diamond(c' \wedge \varphi))$; $\mathsf{Higher}\varphi =_{def} \bigvee_{c,c' \in \mathsf{Courts}}(c \wedge \mathtt{H}(c',c) \wedge \diamond(c' \wedge \varphi))$; $\mathsf{SameCourt}\varphi =_{def} \bigvee_{c \in \mathsf{Courts}}(c \wedge \diamond(c \wedge \varphi))$. $\mathsf{Lower}\varphi$ reads "a lower hierachical state is a φ-state". $\mathsf{Higher}\varphi$ reads "a higher hierachical state is a φ-state". $\mathsf{SameCourt}\varphi$ reads "a state with same court is a φ-state".

Proposition 2. *I) $(C,s) \models \mathsf{Lower}\varphi$ iff there is $s' \in S$ s.t. $s' \prec s$ and $(C,s') \models \varphi$; II) $(C,s) \models \mathsf{Higher}\varphi$ iff there is $s' \in S$ s.t. $s \prec s'$ and $(C,s') \models \varphi$; III) $(C,s) \models \mathsf{SameCourt}\varphi$ iff there is $s' \in S$ s.t. $s \cong s'$ and $(C,s') \models \varphi$.*

3.4 Axiomatics and Completeness

Definition 5 (Axiomatics of $TJCL$). *We define $TJCL$ to be the extension of propositional logic with the following axioms and rules:*

$\Box\varphi \wedge \Box(\varphi \to \psi) \to \Box\psi$ **(K$_\Box$)**

$\Box\varphi \to \varphi$ **(T$_\Box$)**

$\Box\varphi \to \Box\Box\varphi$ **(4$_\Box$)**

$\varphi \to \Box\diamond\varphi$ **(B$_\Box$)**

$\bigvee_{o \in Val} \mathtt{t}(o)$ **(AtLeastValue)**

$\mathtt{t}(o) \to \neg\mathtt{t}(o')$ if $o \neq o'$ **(AtMostValue)**

$\big([\leq_T]\varphi \wedge [\leq_T](\varphi \to \psi)\big) \to [\leq_T]\psi$ **(K$_{[\leq_T]}$)**

$[\leq_T]\varphi \to \varphi$ **(T$_{[\leq_T]}$)**

$[\leq_T]\varphi \to [\leq_T][\leq_T]\varphi$ **(4$_{[\leq_T]}$)**

$@_n\big([\leq_T]\varphi \to @_m\varphi\big) \vee @_m\big([\leq_T]\varphi \to @_n\varphi\big)$ **(Total$_{[\leq_T]}$)**

$\Box\varphi \to [\leq_T]\varphi$ **(MIX$_{\Box,[\leq_T]}$)**

$\bigvee_{c_i \in \mathtt{Courts}} c_i$ **(AtLeastCourt)**

$c_i \to \neg c_j$ if $c_i \neq c_j, c_i, c_j \in \mathtt{Courts}$ **(AtMostCourt)**

$\neg\mathtt{H}(c_i,c_i)$ **(IrrHierarchy)**

$\mathtt{H}(c_i,c_j) \wedge \mathtt{H}(c_j,c_k) \to \mathtt{H}(c_i,c_k)$ **(TrHierarchy)**

$\mathtt{H}(c_i,c_j) \to \Box\mathtt{H}(c_i,c_j)$ **(GlobHier)**

$\mathtt{B}(c_i,c_j) \to \Box\mathtt{B}(c_i,c_j)$ **(GlobBind)**

$\big(\mathtt{R}^\forall\varphi \wedge \mathtt{R}^\forall(\varphi \to \psi)\big) \to \mathtt{R}^\forall\psi$ **(K$_{\mathtt{R}^\forall}$)**

$\Box\varphi \to \mathtt{R}^\forall\varphi$ **(MIX$_{\Box,\mathtt{R}^\forall}$)**

$\dfrac{n \to \varphi}{\varphi}$ where n does not occur in φ **(NAME)**

$\diamond\big(n \wedge \varphi\big) \to @_n\varphi$ **(nam1)**

$\neg\mathtt{t}(?) \leftrightarrow \bigvee_{n \in \mathtt{Names}_d} n$ **(nam2)**

$\dfrac{\varphi \to \psi, \quad \varphi}{\psi}$ **(MP)**

$\dfrac{\varphi}{\Box\varphi}$ **(Nec$_\Box$)**

$\Box$ is an S5 operator. $[\leq_T]$ is a S4 operator. **Total$_{[\leq_T]}$** reflects totality of temporal relation. $\mathtt{R}^\forall$ is a normal operator. **AtLeastValue**, **AtMostValue** syntactically represent the decision function [16]. **TrHierarchy**, **IrrHierarchy** capture the hierarchical structure. **GlobHier** and **GlobBind** express the universality of the hierarchical and binding relations. **AtLeastCourt**, **AtMostCourt** ensure the existence and uniqueness of courts. **nam1**, adapted from [13], guarantees that each name uniquely identifies a single state; **nam2** ensures that all and only decided states are associated to names in $\mathtt{Names}_d$, and so the set of decided states is finite; **NAME**, from [6], guarantees the construction of canonical models with named states.

We will focus on $TJCL^+$ the extension of $TJCL$ given by axioms:

$\bigwedge_{c_i \neq c_j}\big((\mathtt{H}(c_i,c_j) \to \mathtt{B}(c_i,c_j)) \wedge (\neg\mathtt{H}(c_i,c_j) \to \neg\mathtt{B}(c_i,c_j)\big)$ **(SL)** $\quad \Box(\mathtt{t}(?) \to [\leq_T]\mathtt{t}(?))$ **(OL)**

Axioms **SL** and **OL** corresponds respectively to SD and O.

Proposition 3. *TJCL is sound and complete with respect to* **TJCM**. *$TJCL^+$ is sound and complete with respect to* $\overline{\mathbf{TJCM}}$.

4 Precedents, Binding Precedents and Exceptions

A precedent for a case under consideration is a relevant case that was previously decided. However, not all precedents are *potentially binding*. Potentially binding precedents are those established by a court that holds binding authority over the court deciding the case at hand. For example, a precedent set by the UK Supreme Court is potentially binding on the High Court — but the reverse does not hold. We use the term *potentially binding precedent* because such precedents are subject to exceptions: they may lose their binding force in certain circumstances. Indeed, a case may be *overruled* by a court with the power to do so; or it may have been decided in disregard of a binding precedent without the authority to do so, and thus be considered *per incuriam* [15]. Accurately modeling the notion of exceptions will require particular care.

4.1 Potentially Binding precedents

A precedent for a case at hand is a relevant case that was decided earlier. Specifically, s' is a supporting precedent — or simply a precedent — for s if s' is relevant for s and was decided before s. For $s, s', s'' \in S$, we denote their courts c, c', c'', (e.g. $c \in s \cap \texttt{Courts}$).

Definition 6 (Precedent). *Let $s, s' \in S$ and $o \in \{0,1\}$. s' is a (supporting) precedent for s in the direction of o, noted $\Pi(s', s, o)$, iff $f(s') = o$, $s' \in \mathscr{R}(s)$, and $s' <_T s$.*

Supporting precedents are the relevant past cases. In this context, we use a hybrid style to identify which states qualify as precedents, represented by using the intersection of both relevance and temporal modalities, as illustrated below.

Definition 7. *The notion that for the n-named state a supporting state is m-named and a φ-state, denoted as $\mathsf{Supporting}_{n,m}\varphi$, is defined as $\hat{\mathtt{P}}_{n,m}\varphi \wedge \mathtt{R}^{\exists}(\varphi \wedge m)$.*

Proposition 4. *Let $o \in \{0,1\}$. So, $(C, s) \models \mathsf{Supporting}_{n,m}(\mathtt{t}(o) \wedge \varphi)$ iff $(C, s) \models n$ and exists $s' \in S$, s.t. $\Pi(s', s, o)$ and $(C, s') \models m \wedge \varphi$.*

Example 3. $(C_{ex}, s^*) \models \mathsf{Supporting}_{n*,n_1}\mathtt{t}(1)$: s_1 named n_1 is decided as 1 and is relevant and before s^*, named n^*. Similarly, $(C_{ex}, s^*) \models \mathsf{Supporting}_{n*,n_2}\mathtt{t}(1) \wedge \mathsf{Supporting}_{n*,n_3}\mathtt{t}(0) \wedge \mathsf{Supporting}_{n*,n_4}\mathtt{t}(1) \wedge \mathsf{Supporting}_{n*,n_5}\mathtt{t}(0) \wedge \mathsf{Supporting}_{n*,n_6}\mathtt{t}(1)$.

Not all supporting precedents are potentially binding, i.e. not all of them may force the decision in current case. The potentially binding precedents for a state s, decided by court c, are those precedents s' issued by a court c' that holds binding authority over c.

Definition 8 (Pot. bind.). *Let $s, s' \in S$ and $o \in \{0,1\}$. s' is potentially a binding precedent for s for a decision as o, denoted as $\beta(s', s, o)$, iff $\Pi(s', s, o)$ and $c' \mathscr{B} c$. We simply write $\beta(s', s)$ iff there is $o \in \{0,1\}$ s.t. $\beta(s', s, o)$. We define $\beta_s = \{s' \mid \beta(s', s)\}$.*

Definition 9. *For the n-named state a potentially binding state is a m-named φ-state, noted as* $\mathsf{PBinding}_{n,m}\varphi$*, is defined as* $\bigvee_{c_i,c_j \in \texttt{Courts}} \Big(\texttt{B}(c_i,c_j) \wedge (\mathsf{Supporting}_{n,m}(c_i \wedge \varphi)\Big)$.

Proposition 5. *Let $s \in S$, $o \in \{0,1\}$. $(C,s) \models \mathsf{PBinding}_{m,n}(\mathtt{t}(o) \wedge \varphi)$ iff $(C,s) \models n$ and exists $s' \in S$, s.t $\beta(s',s,o)$ and $(C,s') \models m \wedge \varphi$.*

Example 4. $(C_{ex},s^*) \models \mathsf{PBinding}_{n*,n_1}\mathtt{t}(1)$ means: s_1 is supporting for s^* and decided by court c_0, which is binding for court c_2 in s^*. Same can be verified for s_2, s_4 (with 1) and s_3 (with 0). However, $(C_{ex},s^*) \not\models \mathsf{PBinding}_{n*,n_5}\mathtt{t}(0)$. Although s_5 is supporting for s^*, s_5 is decided by c_4 which is not binding for s^* (same can be verified for s_6).

4.2 Exceptions: Incuriam and Overruling

Figure D $\quad s^* \xleftarrow{\text{p-binding}} s_1 \xleftarrow{\text{overrules}} s_2 \xrightarrow{\text{per incuriam}} s_3$

Figure E $\quad s^* \xleftarrow{\text{p-binding}} s_1 \xrightarrow{\text{per incuriam}} s_2 \xleftarrow{\text{overrules}} s_3$

Potentially binding precedents are subject to two exceptions: they may be overruled or decided *per incuriam*. Intuitively, a case s is overruled when a later, relevant case is decided differently by a court that holds overruling power with respect to the court in s. Conversely, a case s is decided *per incuriam* if it goes against a binding precedent, without having authority to do so. To formally define these exceptions we begin by considering these aspects:

- *Per incuriam* and overruled cases lose their bindingness on subsequent cases, i.e. we can remove them from binding precedents.
- The model does not specify which cases are *per incuriam* or overruled. Rather, as shown later in this section, this information is computed through a recursive process involving the following interactions between *per incuriam* and overruling.
- Consider Figure D. Suppose we have a case s^* to be decided, for which s_1 is a potentially binding precedent. Now assume that s_2 — for which s_1 was relevant — is decided later and differently by a court with overruling authority. Intuitively, this suggests that s_1 has been overruled and should no longer be binding for s^*. However, suppose we then discover that s_2 itself contradicts a binding precedent s_3, without having the authority to do so — that is, s_2 was decided *per incuriam*. In that case, s_2 was not binding on s_1, and we can reinstate s_1 as a binding precedent for s^*. Hence, a case loses its overruling power when it is *per incuriam.*
- Consider Figure E. Suppose we have a case s^* to be decided, and a potentially binding precedent s_1. Now assume that s_1 went against s_2 without the authority to do so. Intuitively, this would suggest that s_1 *per incuriam* and should be excluded from the binding precedents for s^*. However, suppose we later discover that s_2 had already been overruled by s_3 before s_1 was decided. In that case, s_2 was no longer binding at the time of s_1's decision, and s_1 can be "reinstated" as a valid binding precedent for s^*. A case is not *per incuriam* if it goes against an overruled case.

To capture such interactions, we proceed as follows. We first identify potential overruling cases — those that can overrule others unless deemed *per incuriam*. We then formally define *per incuriam*. Finally, we define overruled cases.

Not all courts have the authority to overrule prior decisions. Within common law systems typically higher courts can overrule lower courts decision. Viceversa, lower court cannot overrule a higher court. More attention requires self-overruling: intuitively, if a court can overrule its own previous decisions, then it is not bound by its own previous decisions. So we assume that a court c' may overrule a decision made by court c when deciding s', if: (1) c' is hierarchically higher to c; or (2) $c' = c$ and not-$c\mathscr{B}c$.

Definition 10 (Overruling Power). *Court c' has the power to overrule (a decision by) court c, denoted $O(c',c)$, iff $\mathscr{H}(c',c)$ or ($c' = c$ and not-$c\mathscr{B}c$).*

Definition 11. *That a decision made by c_i can be overruled in current state, denoted as $\mathsf{PwOver}(c_i)$, is defined as $\bigvee_{c_j \neq c_i} \big(c_j \wedge \mathtt{H}(c_j,c_i)\big) \vee \big(c_i \wedge \neg\mathtt{B}(c_i,c_i)\big)\big)$.*

Proposition 6. *Let $c \in s$. So, $(C,s) \models \mathsf{PwOver}(c_i)$ iff $O(c,c_i)$.*

We now define potential overruling cases. Intuitively, s' may overrule a precedent s, when: 1) s' is decided by c' in the opposite direction wrt s and 2) c' has overruling power over c when deciding s' (i.e. $O(c',c)$).

Definition 12 (Pot. Over.). *The case s' potentially overrules s, denoted as $O(s',s)$, iff $\Pi(s,s',o)$, $f(s') = \overline{o}$, and $O(c',c)$. We write $\omega_s = \{s' \mid O(s',s)\}$ for the set of potentially overruling states wrt. s. Given a case $\tilde{s} \in S$, we write $Overrule_T(s,\tilde{s}) = \{s' \mid O(s',s),\ s' <_T \tilde{s}\}$ to represent the set of states potentially overruling s and that were assessed before $\tilde{s}$ was decided.*

Remark 1. It can be verified that: If a case s is potentially bounded by case s', then s cannot potentially overule s'. Namely, $s' \in \beta_s \Rightarrow s \notin \omega_{s'}$.

Definition 13. *The notion that for the n-named state a potentially overruling state is m-named and a φ-state, denoted as $\mathsf{POverruling}_{n,m}\varphi$, is defined as $\bigvee_{o\in\{0,1\},c} \Big(n \wedge c \wedge \diamond\big(\varphi \wedge \mathsf{t}(\overline{o}) \wedge \mathsf{PwOver}(c) \wedge \mathsf{Supporting}_{m,n}\mathsf{t}(o)\big)\Big)$.*

Proposition 7. $(C,s) \models \mathsf{POverruling}_{n,m}\varphi$ *iff $(C,s) \models n$ and exists $s' \in S$ s.t. $O(s',s)$ and $(C,s') \models m \wedge \varphi$.*

Example 5. $(C_{ex},s_3) \models \mathsf{PwOver}(c_0) \wedge \mathsf{t}(0)$ means: $c_0 \in s_3$, c_0 is not self-bound, and $(C_{ex},s_3) \models \mathsf{t}(0)$. It can be verified that $(C_{ex},s_3) \models \mathsf{Supporting}_{n_3,n_1}(\mathsf{t}(1))$. Hence $(C_{ex},s_1) \models \mathsf{POverruling}_{n_1,n_3}\mathsf{t}(0)$ — s_3 potentially overruled s_1.

Intuitively, a *per incuriam* case is a case that went *against* a binding precedent. To model this, we define the notions of going against/according a binding precedent.

Definition 14. *The case s went against s', denoted as $Against(s,s')$, iff $f(s) = o \in \{0,1\}$ and $\beta(s',s,\overline{o})$. The case s was decided according s', denoted as $According(s,s')$, iff $f(s) = o \in \{0,1\}$ and $\beta(s',s,o)$.*

Definition 15. *The notion that the n-named state went against a m-named and potentially binding φ-state, denoted as* $\mathsf{Against}_{n,m}\varphi$*, is defined as* $\bigvee_{o\in\{0,1\}}\Big(\mathsf{t}(o)\wedge\mathsf{PBinding}_{n,m}(\varphi\wedge\mathsf{t}(\overline{o}))\Big)$*. The n-named state is decided according to a m-named and potentially binding φ-state, denoted as* $\mathsf{According}_{n,m}\varphi$*, is defined* $\bigvee_{o\in\{0,1\}}\Big(\mathsf{t}(o)\wedge\mathsf{PBinding}_{n,m}(\varphi\wedge\mathsf{t}(o))\Big)$.

Proposition 8. *I)* $(C,s)\models\mathsf{Against}_{n,m}\varphi$ *iff* $(C,s)\models n$ *and* $\exists s'$ *s.t.* $Against(s,s')$ *and* $(C,s')\models m\wedge\varphi$*; II)* $(C,s)\models\mathsf{According}_{n,m}\varphi$ *iff* $(C,s)\models n$ *and* $\exists s'$ *s.t.* $According(s,s')$ *and* $(C,s')\models m\wedge\varphi$.

Consider a state s that went against a binding precedent s', i.e. $Against(s,s')$. By Remark 1, s had no overruling power wrt s'. This suggests that s was decided *per incuriam*. However, to properly establish this, four interrelated aspects must be considered.

1) Not all courts can disregard a previous decision *per incuriam* [15]. We assume that a lower court cannot disregard a *per incuriam* precedent by a higher court, to which it remains bound. [6] But, a court may disregard its own *per incuriam* precedents. [7]
2) As mentioned, we infer from the model which cases are decided *per incuriam*, such information is not given. Returning to our question: if $Against(s,s')$, does it mean s was decided *per incuriam*? It depends. If s' was decided by a higher court, then s is necessarily *per incuriam*, since it lacks the authority to disregard s'. If s and s' were decided by the same court, then the court — when deciding s — may have found that s' went against a binding $\bar{s}$. But to confirm that s' was *per incuriam* — so not binding for s — we have to check whether $\bar{s}$ was *per incuriam*, etc. Thus, determining if s is *per incuriam* requires tracing its full chain of binding precedents.
3) Using the same reasoning done for Figure E, we state that s is not *per incuriam* if s' was overruled by another case $\tilde{s}$. The reason is that, if s' was overruled, then it no longer held binding authority over s. However, we must also require that $\tilde{s}$ was decided prior to s — otherwise, s' had not yet been overruled at the time s was assessed and thus s' still bound s. Moreover, $\tilde{s}$ must itself not be *per incuriam*; otherwise, it would not represent a legitimate overruling.
4) For a given case s, there may exist conflicting binding precedents: s' and s'' within β_s, such that $Against(s,s')$ and $According(s,s'')$ hold. In this scenario, any decision made in s would have contradicted one potentially binding precedent. We argue that s can be considered *per incuriam* if s'' fails to satisfy the previously established criteria — namely, if s'' is itself *per incuriam* and by a court not higher than s, or if it was overruled (prior to s) by a non-*per incuriam* precedent. If s'' fits neither category, there remains another case in which s may still be regarded as *per incuriam*: following a principle for resolving conflicts of precedents (to be detailed later), s should have followed the later precedent from the higher court. Therefore, we examine the temporal and hierarchical relationship between s' and s''. s qualifies as *per incuriam* if: s'' was decided by a lower court than s', or by the same court but strictly before.

[6] In Cassell v Broome (UK), the Court of Appeal was held bound by *per incuriam* House of Lords decision [19]. Canada follows same principle [21].
[7] Clearly stated for UK Court of App. in *Young v. Bristol Aeroplane Co.*.

As observed, determining whether a case was decided *per incuriam* is a chain-like evaluation process of binding precedents. Here, a graph $G_s = (V_s, E_s)$ constructed recursively is given to compute whether a case s was decided *per incuriam.* At step 0, the graph ($G_0 = (V_0, E_0)$) has only one node s ($V_0 = \{s\}$) and no edge ($E_0 = \emptyset$). At step 1, we add to the nodes the s', the potentially binding precedents or the potentially overruling states for s ($V_1 = \{s\} \cup \{s' \mid s' \in \beta_s \text{or } s' \in \omega_s\}$). Edges joining s to each s' are added ($E_1 = \{(s,s') \mid s' \in \beta_s \text{ or } s' \in \omega_s\}$). We repeat the procedure recursively for each node.

Definition 16. *(s-Graph) A structure $G_s = (V_s, E_s) = \bigcup_{n \geq 0} G_n$ is called a s-graph, when $f(s) \neq ?$ and every G_n satisfies:*
- *$G_0 = (V_0, E_0)$, where $V_0 = \{s\}$ and $E_0 = \emptyset$;*
- *$G_{n+1} = (V_{n+1}, E_{n+1})$, where*
  - *$V_{n+1} = V_n \cup \{s' \mid s' \in \beta_{s''} \text{ or } s' \in \omega_{s''},\ \text{with } s'' \in V_n\}$ and*
  - *$E_{n+1} = E_n \cup \{(s'', s') \mid s' \in \beta_{s''} \text{ or } s' \in \omega_{s''},\ \text{with } s'' \in V_n\}$.*

G_s is a finite directed acyclic graph. We thus define the *height* of a state s, denoted $\text{height}(s)$, as the number of edges in the longest path from s to a sink node in G_s. [8]

By recursively exploring graph G_s, we can compute whether s is *per incuriam* (i.e. $Incuriam(s)$). We claim that s is *per incuriam* if:
- there is s' adjacent node to s (i.e. $(s,s') \in E_s$), such that: a) s is against s' (i.e. $Against(s,s')$); b) if s' was decided *per incuriam* then it is hiearchically superior to s (i.e. $Incuriam(s') \Rightarrow s \prec s'$); and c) every $\tilde{s}$ potentially overruling s' (before s) is itself *per incuriam* (i.e. $\forall \tilde{s} \in Overrule_T(s',s) : Incuriam(\tilde{s})$).
- for all s'', adjacent node to s (i.e. $(s,s'') \in E_s$) s.t. $According(s,s'')$ we have: a) s'' was decided *per incuriam* by a court that is not higher than the court of s (i.e. $Incuriam(s'')$ and $s \not\prec s''$)); or b) there is $\tilde{s}$ potentially overruling s' (before s) and not *per incuriam* (i.e. $\exists \tilde{s} \in Overrule_T(s'',s)$ s.t. not $Incuriam(\tilde{s})$); or c) s'' was decided by a lower court of s' or s'' was decided by the same court of s' but strictly before (i.e. ($s'' <_T s'$ and $s' \cong s''$) or $s'' \prec s'$).

Definition 17. *(Incuriam) Let $G_s = (V_s, E_s)$ be a s-graph. The state s was decided* per incuriam*, denoted as $Incuriam(s)$, iff*
- *$\exists s' \in S$ s.t. $(s,s') \in E_s$, $Against(s,s')$, ($Incuriam(s') \Rightarrow s \prec s'$), and $\forall \tilde{s} \in Overrule_T(s',s)$: $Incuriam(\tilde{s})$.*
- *$\forall s'' \in S$ if $(s,s'') \in E_s$ and $According(s,s'')$ then either [$Incuriam(s'')$ and $s \not\prec s''$], or [$\exists \tilde{s} \in Overrule_T(s'',s)$: not $Incuriam(\tilde{s})$], or [($s'' <_T s'$ and $s' \cong s''$) or $s'' \prec s'$].*

We now use our modal language to represent the notion of *per incuriam*. As previously shown, *per incuriam* involves temporal and jurisdictional hierarchies, as well as potential overruling. Given a finite set $\texttt{Names}_d$ of decided names, we define the following useful formulas to express *per incuriam*: $\hat{\texttt{F}}\varphi =_{def} \bigvee_{n \in \texttt{Names}_d} \left(n \wedge \hat{\texttt{F}}_n \varphi\right)$; $\mathsf{Against}\varphi =_{def} \bigvee_{n,m \in \texttt{Names}_d} \left(n \wedge \mathsf{Against}_{n,m} \varphi\right)$; $\mathsf{POverruling}\varphi =_{def} \bigvee_{n,m \in \texttt{Names}_d} \left(n \wedge \mathsf{POverruling}_{n,m} \varphi\right)$; $\mathsf{According}^{\forall}\varphi =_{def} \bigvee_{n \in \texttt{Names}_d} \left(n \wedge \bigwedge_{m \in \texttt{Names}_d} \neg \mathsf{According}_{n,m} \neg \varphi\right)$; $\mathsf{POverruling}^{\forall}\varphi =_{def} \bigvee_{n \in \texttt{Names}_d} \left(n \wedge \bigwedge_{m \in \texttt{Names}_d} \neg \mathsf{POverruling}_{n,m} \neg \varphi\right)$. They will be used to define the expressions that characterize the recursive nature of computing whether a case was decided *per incuriam*.

[8] A sink node is a node with no outgoing edges.

Definition 18. *We define formula ι^n, recursively on $n \in \mathbb{N}$:*

$$\iota^1 =_{def} \bigvee_{m \in \mathtt{Names}_d} \Big(\mathsf{Against}(m) \wedge \mathsf{According}^{\forall}\big((\hat{\mathtt{F}}m \wedge \mathsf{SameCourt}(m)) \vee \mathsf{Higher}(m)\big)\Big),$$

$$\iota^{n+1} =_{def} \bigvee_{n,m \in \mathtt{Names}_d} \Big(n \wedge \mathsf{Against}\big(m \wedge (\neg\iota^n \vee \mathsf{Lower}(n)) \wedge \mathsf{POverruling}^{\forall}(\hat{\mathtt{F}}n \to \iota^n)\big) \wedge$$
$$\mathsf{According}^{\forall}\big((\iota^n \wedge \neg\mathsf{Lower}(m)) \vee \mathsf{POverruling}(\hat{\mathtt{F}}n \wedge \neg\iota^n) \vee ((\hat{\mathtt{F}}m \wedge \mathsf{SameCourt}(m)) \vee \mathsf{Higher}(m))\big)\Big).$$

Let $h = height(s)$. It can be proven that if a state satisfies a ι^k for some $k \geq h = height(s)$, then it satisfies ι^j for all $j \geq h = height(s)$:

$$\text{If } \exists k \geq h : (C,s) \models \iota^k \text{ then } \forall j \geq h : (C,s) \models \iota^j. \qquad (\star)$$

We can finally define the notion of *per incuriam* in the language.

Definition 19. $\mathsf{Incuriam} =_{def} \bigvee_{k \leq |\mathtt{Names}_d|} \bigwedge_{k \leq j \leq |\mathtt{Names}_d|} \iota^j$.

We aim to show that semantic and syntactic definitions of *per incuriam* are equivalent, i.e. $Incuriam(s)$ iff exists $k \leq |\mathtt{Names}_d|$ s.t. s satisfies ι^j, for all j with $|\mathtt{Names}_d| \geq j \geq k$. We claim k is actually *at most* the height s ($h = height(s)$)[9]. The formal proof is by induction; here, we provide hints focusing on $(C,s) \models \mathsf{Incuriam} \Rightarrow Incuriam(s)$.

- Assume $(C,s) \models \mathsf{Incuriam}$. Then $(C,s) \models \iota^h$, exactly at $h = height(s)$. Indeed, by definition of $\mathsf{Incuriam}$, there is $k \leq |\mathtt{Names}_d|$ s.t. $(C,s) \models \iota^j$, with $|\mathtt{Names}_d| \geq j \geq k$. If $k \leq h$, then trivially $(C,s) \models \iota^h$. If $k > h = height(s)$, $(\star)$ guarantees that $(C,s) \models \iota^h$.
- So, if we want to prove $(C,s) \models \mathsf{Incuriam} \Rightarrow Incuriam(s)$ we just need to prove that $(C,s) \models \iota^h \Rightarrow Incuriam(s)$, for $h = height(s)$.
- We give an intuition that previous point holds for $h = 1$. Suppose $(C,s) \models \iota^1$, from this we can verify 1) there exists a state s' s.t. $Against(s,s')$ and 2) for all s'' s.t. $According(s,s'')$: ($s \prec s'$ or ($s' \cong s''$ and $s' <_T s''$)). Is this enough to say $Incuriam(s)$? Yes — provided that $Incuriam(s')$ and s' has not been overruled. This is the case. Indeed, since $height(s) = 1$ then $height(s') = 0$ — from $Against(s',s)$, we know $s' \in \beta_s$, so s' is a successor of s in the graph. Since $height(s') = 0$, s' has no successors in the graph. This means s' admits neither binding precedents nor potentially overruling cases. Hence, not $Incuriam(s')$ and s' is not overruled.

Proposition 9. *Let $s \in S$.* $(C,s) \models \mathsf{Incuriam} \Leftrightarrow Incuriam(s)$.

We have modeled *per incuriam*. We now turn to another exception: overruled states — those for which there exists a potentially overruling state that is not *per incuriam*.

Definition 20. *$Overruled(s)$ iff there is $s' \in \omega_s$ s.t. not $Incuriam(s')$.*
$\mathsf{Overruled} =_{def} \mathsf{POverruling}(\neg\mathsf{Incuriam})$.

Proposition 10. $(C,s) \models \mathsf{Overruled}$ *iff* $Overruled(s)$.

Finally, we arrive at the notion of binding precedents *without exception*: potentially binding precedents that are neither *per incuriam* (by same court) nor overruled.

[9] $h \leq |\mathtt{Names}_d|$: states of G_s are in S_d and we can prove that $|S_d| \leq |\mathtt{Names}_d|$.

Definition 21 (Binding). *s' is a binding precedent (without exception) for s iff $s' \in \beta_s$ and not (Incuriam(s) and $s \cong s'$) and not Overruled(s'). $\overline{\beta}_s$ is the set of binding precedents without exception for s.*

Definition 22. *The notion that for the n-named state a binding state without exception is m-named and a φ-state, denoted $\mathsf{Binding}_n\varphi$, is defined as $\bigvee_{m\in\mathtt{Names}_d}\mathsf{PBinding}_{n,m}\big(\varphi\wedge\neg\mathsf{Overruled}\wedge\neg(\mathsf{Incuriam}\wedge\mathsf{SameCourt}(n))\big)$.*

As shown in the proposition below, our modal language can be used to represent the concept of binding precedents without exceptions.

Proposition 11. *$(C,s)\models\mathsf{Binding}_n$ iff $(C,s)\models n$ and there is $s'\in\overline{\beta}_s$ s.t. $(C,s')\models\varphi$.*

Example 6. By Ex. 4, s_1, s_2, s_3, s_4 are potentially binding for s^*. s_4 is *per incuriam*, i.e. $(C_{ex},s_4)\models\mathsf{Incuriam}$. Indeed, $(C_{ex},s_4)\models\iota^j$ for all $j\geq height(s_4)$, where $height(s_4)=1$ (the s_4-graph contains only s_4 and its binding s_3 — that in turn has no binding precedent nor potentially overruling case). By ($\star$), we need to verify that $(C_{ex},s_4)\models\iota^1$. This holds since s_4 goes against s_3 and s_4 has no according state. It can also be verified that s_1 is overruled by s_3. Also, s_2 and s_3 are not *per incuriam* or overruled. So, the binding precedents without exception for s^* are s_2 and s_3, decided resp. as 1 and 0: $(C_{ex},s^*)\models\mathsf{Binding}_{n*}(\mathsf{t}(1))\wedge\mathsf{Binding}_{n*}(\mathsf{t}(0))$. We need a principle to solve the conflict.

5 Resolving Precedents Conflict for New Cases

When binding precedents conflict, the most recent decision from the highest court should be followed — we call this the *Temporal Hierarchical Principle*. It reflects the idea that political, economic, or social changes may influence a court's stance [19]. U.S. courts, for example, adopt this principle [19,8]. Here, we formalise a decision process for a new case s^* (i.e. $f(s^*)=?$) based on this principle. First we define the best temporal hierarchical binding precedents — those more recent and from higher courts.

Definition 23. *The set of best temporal hierarchical binding precedents for $s\in S$ is $Best_{TH}(\overline{\beta}_s)=\{s'\in\overline{\beta_s}\mid\forall s''\in\overline{\beta}_s\colon s'\not\prec s''$ and not $(s'\cong s''$ and $s'<_T s'')\}$.*

Definition 24. *That for the n-named state, there is a best temporal hierarchial binding state without exception in which φ holds, denoted as $\mathsf{BestBinding}_n\varphi$, is defined as*

$$\bigvee_{m\in\mathtt{Names}_d}\Big(\mathsf{Binding}_n(\,m\wedge\varphi)\wedge\neg\mathsf{Binding}_n\big((\mathsf{Lower}(m))\vee(\mathsf{SameCourt}(m)\wedge\hat{\mathsf{P}}m)\big)\Big).$$

The best binding precedents thus can be expressed by our modal formulas, as shown above, capturing those that satisfy *Temporal Hierarchical Principle*.

Proposition 12. *$(C,s)\models\mathsf{BestBinding}_n\varphi$ iff $(C,s)\models n$ and $\exists s'\in Best_{TH}(\overline{\beta}_s)$ s.t. $(C,s')\models\varphi$.*

We define a decision making process based on the *Temporal Hierarchical Principle*.

Definition 25 (Temp. Hier. Principle). *Let s^* s.t. $f(s^*)=?$. The decision making function $f^*:S\to 2^{\{0,1,?\}}$, based on best temporal hierarchical binding precedents is*

$$f^*(s)=\begin{cases}\{f(s)\} & \text{if } s\neq s^*,\\ \{f(s')\mid s'\in Best_{TH}(\overline{\beta}_{s^*})\} & \text{otherwise.}\end{cases}$$

The decision process f^* assigns to s^* the decisions among the best temporal hierarchical binding precedents in $Best_{TH}(\overline{\beta}_{s^*})$. Note that there may be two such precedents with conflicting outcomes — two binding precedents decided differently (as 0 and as 1), by higher courts and considered to apply "simultaneously". In such a case, the decision process may yield both outcomes, $f^*(s^*) = \{0,1\}$. Conversely, if all the best binding precedents agree on a single outcome $o \in \{0,1\}$, we say that the decision is *forced* to o, namely $f^*(s^*) = \{o\}$. Both scenarios can be expressed using our language. The formula $\mathsf{BestBinding}_{n*}(\mathsf{t}(o)) \wedge \mathsf{BestBinding}_{n*}(\mathsf{t}(\overline{o}))$ holds for s^* in the first scenario. While, in the second scenario $\mathsf{BestBinding}_{n*}(\mathsf{t}(o)) \wedge \neg\mathsf{BestBinding}_{n*}(\mathsf{t}(\overline{o}))$ holds for s^*.

Definition 26. *The notion that for the n-named state the decision is forced to be* $o \in \{0,1\}$*, denoted* $\mathsf{Cl}_n(o)$*, is defined as* $\mathsf{BestBinding}_n(\mathsf{t}(o)) \wedge \neg\mathsf{BestBinding}_n(\mathsf{t}(\overline{o}))$.

Proposition 13. $(C,s^*) \models \mathsf{Cl}_n(o)$ *iff* $(C,s^*) \models n$ *and* $f^*(s^*) = \{o\}$.

Example 7. $(C_{ex},s^*) \models \mathsf{BestBinding}_{n*}(\mathsf{t}(0)) \wedge \neg\mathsf{BestBinding}_{n*}(\mathsf{t}(1))$: s_2, s_3 are binding without exception for s^*. s_3 is decided (as 0), by a higher court than s_2. So, $(C_{ex},s^*) \models \mathsf{Cl}_{n*}(0)$. s^* is forced to be 0. We have solved the conflict in Example 6.

6 Conclusion and Related Works

We proposed an extension of BCL logic for classifiers [16], by introducing new modalities that capture the relevance relationships between cases, their temporal ordering, and the hierarchical structure of the legal system under consideration. We modeled the notion of precedents, including binding precedents and those that admit exceptions — such as precedents decided *per incuriam* or overruled. Finally, we formalised within the language a temporal and hierarchical principle for resolving conflicts of precedents.

The relevance relation merits further attention. We imposed no properties on the relevance relation. As shown in [11], the relevance relation derived from Horty result model is neither transitive nor reflexive nor symmetric. Also, leaving relevance unconstrained allows for cases that are both relevant to a new case but not to each other — a feature important for modeling decisions involving multiple concerns (see [7,4,10]).

Also, the notion of binding is "binary": a court issues binding decisions or not. But, sometimes, as suggested in [1], a binding relation between two agents of an institution may be context-dependent, as it happens for the UK supreme court (see footnote 1). In the future, we plan to consider a more sophisticated binding relation.

Other than [17], no modal logics for legal CBR are known to us. Nonetheless, two prior works explored the role of time and hierarchy in legal CBR, from a semantic perspective. [8] combines two models of constraint, based on the assumption that vertical constraints are stronger than horizontal ones. In [8], if a case in the starting case base violated the vertical constraint, then a decision for any new case cannot be forced. Instead, for us a case decided ignoring a vertical precedent is *per incuriam* and can simply be discarded. Also, differently than [8], we don't assume that horizontal constraints always apply; this depends on the binding relation associated with the model.

[25] extended argumentation frameworks to analyze judicial reasoning. The framework determines which claims are justified based also on the court level, procedure type, and applicable precedents. Future work could explore the link between our modal logic and the argumentation-based approach in [25].

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