DE GRUYTER Adv. Calc. Var. 2024; 17(4): 1519–1541

Research Article

Stefano Biagi, Francesco Esposito*, Luigi Montoro and Eugenio Vecchi

Symmetry and monotonicity of singular solutions to p-Laplacian systems involving a first order term

https://doi.org/10.1515/acv-2023-0043
Received April 15, 2023; revised August 23, 2023

Abstract: We consider positive singular solutions (i.e. with a non-removable singularity) of a system of PDEs driven by p-Laplacian operators and with the additional presence of a nonlinear first order term. By a careful use of a rather new version of the moving plane method, we prove the symmetry of the solutions. The result is already new in the scalar case.

Keywords: Singular solutions, p-Laplacian systems, moving plane method

MSC 2020: 35B06, 35J75, 35J62, 35B51

Communicated by: Enrico Valdinoci

1 Introduction

The study of symmetry properties of solutions of PDEs is a long–standing topic which dates back to the foundational works of Alexandrov [1], Serrin [24], Gidas, Ni and Nirenberg [18], and Berestycki and Nirenberg [2]. The common point among such papers is the use of the celebrated *moving plane method* in the case of semilinear elliptic equations. This powerful technique relies heavily on the validity of (weak and strong) maximum or comparison principles. The relevance of the ideas which are at the core of the aforementioned papers is clearly witnessed by the huge number of contributions which are now available in the literature, whose aim has been to extend the results mentioned above to several different cases in the local framework: quasilinear equations (see, e.g., [8–11]) and cooperative elliptic systems (see, e.g., [5, 7, 12–14, 28]).

A slightly different line of research concerns the study of symmetry properties of *singular* solutions. In this case, we refer, e.g., to [6, 26] for the case of point-singularity. More recently, Sciunzi [23] developed a new method which allows to study symmetry properties of solutions which are singular on sets of *small capacity.*

The aim of this paper is to focus on systems of PDEs driven by the p-Laplacian with the additional presence of a nonlinear gradient term. To be more precise, we will consider solutions

$$u_i \in C^{1,\alpha}(\overline{\Omega} \setminus \Gamma), \quad i = 1, \dots, m,$$

***Corresponding author: Francesco Esposito,** Dipartimento di Matematica e Informatica, Università della Calabria, Ponte Pietro Bucci 31B, 87036 Arcavacata di Rende, Cosenza, Italy, e-mail: francesco.esposito@unical.it. https://orcid.org/0000-0002-3056-3627
Stefano Biagi, Dipartimento di Matematica, Politecnico di Milano, Via Bonardi 9, 20133 Milano, Italy, e-mail: stefano.biagi@polimi.it
Luigi Montoro, Dipartimento di Matematica e Informatica, Università della Calabria, Ponte Pietro Bucci 31B, 87036 Arcavacata di Rende, Cosenza, Italy, e-mail: montoro@mat.unical.it
Eugenio Vecchi, Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy, e-mail: eugenio.vecchi2@unibo.it

to the following system of PDEs:

$$\begin{cases} -\Delta_{p_i} u_i + a_i(u_i)|\nabla u_i|^{q_i} = f_i(u_1, \dots, u_m) & \text{in } \Omega \setminus \Gamma, \\ u_i > 0 & \text{in } \Omega \setminus \Gamma, \\ u_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathcal{S}$}$$

Here, $\Omega \subseteq \mathbb{R}^N$ is a smooth and bounded domain and, for suitable smooth functions,

$$-\Delta_p v := \operatorname{div}(|\nabla v|^{p-2} \nabla v)$$

denotes the p-Laplacian. Generally, solutions to equations involving the p-Laplace operator are of class $C^{1,\alpha}$. This assumption is natural according to classical regularity results [15, 20, 27]. We point out that, in this paper, we deal with singular solutions that are $C^{1,\alpha}$ far from the critical set Γ. We consider solutions with a non-removable singularity: we mean that solutions possibly do not admit a smooth extension in Ω, i.e. it is not possible to find $\tilde{u}_i \in W_0^{1,p}(\Omega)$ such that $u_i \equiv \tilde{u}_i$ in $\Omega \setminus \Gamma$. Indeed, without any a priori assumption, the gradient of the solutions possibly blows up near the critical set, and hence each equations of ($\mathcal{S}$) may exhibit both a degenerate and a singular nature at the same time.

Before stating our main result, we first introduce the main *structural assumptions* we require on problem ($\mathcal{S}$); these assumptions will be tacitly understood in the sequel.

Assumptions. Throughout what follows, we suppose the following:

(h_Ω) Ω is a bounded domain in $\mathbb{R}^N$ (with $N \geq 2$), which is *convex* in the x_1-direction and symmetric with respect to the hyperplane

$$\Pi_0 := \{x \in \mathbb{R}^N : x_1 = 0\}.$$

(h_Γ) $\Gamma \subseteq \Omega \cap \Pi_0$ is a compact set satisfying $\operatorname{Cap}_p(\Gamma) = 0$, where

$$p := \max_{1 \leq k \leq m} p_k.$$

($h_{p,q}$) For every $i = 1, \dots, m$, we have

$$\begin{cases} \dfrac{2N+2}{N+2} < p_i \leq N \quad \text{and} \quad \max\{p_i - 1, 1\} \leq q_i < p_i, \\ \text{or} \\ 2 \leq p_i \leq N \quad \text{and} \quad q_i = p_i. \end{cases} \tag{1.1}$$

(h_a) The functions $a_i(\cdot)$ are *locally Lipschitz-continuous* on $\mathcal{I} := [0, +\infty)$, that is, for every $M > 0$ there exists a constant $L = L_M > 0$ such that

$$|a_i(t) - a_i(s)| \leq L|t - s| \quad \text{for every } t, s \in [0, M].$$

(h_f) The functions $f_i(\cdot)$ are *of class* C^1 on $\mathcal{I}^m$. Moreover, we assume that
(a) $f_i(t_1, \dots, t_m) > 0$ for every $t_1, \dots, t_m > 0$;
(b) $\partial_{t_k} f_i(\cdot) \geq 0$ for every $1 \leq i, k \leq m$ with $i \neq k$.

Remark 1.1. We note that assumption (h_f) (a) is kind of natural, indeed for $i = m = 1$ and $p > 2$ there are counterexamples to symmetry results in the literature (see, e.g., [25, Section 6] and [4]), while assumption (h_f) (b) is the usual cooperativity condition, a natural hypothesis in the study of qualitative properties of solutions already in the case of systems driven by the standard Laplacian; see [28].

Now, we properly define what we mean by a *weak solution* to problem ($\mathcal{S}$).

Definition 1.2. We say that a vector-valued function

$$\mathbf{u} = (u_1, \dots, u_m) \in C^{1,\alpha}(\overline{\Omega} \setminus \Gamma; \mathbb{R}^m)$$

is a *weak solution* to (S) if it satisfies the properties listed below:
(i) For every $i = 1, \dots, m$ and every $\varphi \in C_c^1(\Omega \setminus \Gamma)$, one has

$$\int_\Omega |\nabla u_i|^{p_i-2}\langle \nabla u_i, \nabla\varphi\rangle\,dx + \int_\Omega a_i(u_i)|\nabla u_i|^{q_i}\varphi\,dx = \int_\Omega f_i(u_1, \dots, u_m)\varphi\,dx. \tag{1.2}$$

(ii) $u_i > 0$ pointwise in $\Omega \setminus \Gamma$ (for every $i = 1, \dots, m$).
(iii) $u_i = 0$ on $\partial\Omega$ (for every $i = 1, \dots, m$).

Our main result is the following theorem.

Theorem 1.3. *Let assumptions* (h_Ω)–(h_f) *be in force, and let* $\mathbf{u} \in C^{1,\alpha}(\overline{\Omega} \setminus \Gamma; \mathbb{R}^m)$ *be a weak solution to problem* (S). *Then the following facts hold:*
(i) $\mathbf{u}$ *is symmetric with respect to the hyperplane* Π_0, *namely*

$$\mathbf{u}(x_1, x_2, \dots, x_N) = \mathbf{u}(-x_1, x_2, \dots, x_N) \quad \textit{in } \Omega.$$

(ii) $\mathbf{u}$ *is non-decreasing in the* x_1*-direction in the set* $\Omega_0 = \{x_1 < 0\}$, *and moreover*

$$\partial_{x_1} u_i > 0 \quad \textit{in } \Omega_0, \tag{1.3}$$

for every $i \in \{1, \dots, m\}$.

We remark that, although the technique that we will develop to prove Theorem 1.3 works for any $p_i > N$, the result is stated in the range (1.1) since there are no sets (different from the empty-set) of zero p_i-capacity when $p_i > N$.

We now want to give a few comments about Theorem 1.3. Firstly, we notice that Theorem 1.3 extends the results in [16] to the case of singular solutions. The main novelty of our result concerns the singular case $p_i < 2$ (for some $i = 1, \dots, m$). In order to explain this, let us consider the case of a scalar equation with $a(u) \equiv 0$: our problem then boils down to the case considered in [17] (which is therefore our benchmark case), that is,

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \setminus \Gamma, \\ u > 0 & \text{in } \Omega \setminus \Gamma, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In [17], Esposito, Montoro and Sciunzi, already in the scalar case, require some a priori assumptions on the nonlinearity f involved in the problem, while here we are able to remove the a priori growth assumption made in [17] on the nonlinearities f_i. This aspect is strictly related with a technical issue that has to be faced when using the *integral version* of the moving plane method. In particular, one needs to apply a weighted Sobolev inequality due to Trudinger [29], whose validity depends on proper summability conditions of the weight, which happens to be of the form $|\nabla u|^{p-2}$. In [17], this condition is proved to be satisfied after a nice study of the behavior of the gradient of the solution near the singular set Γ, which is based on a subtle growth estimate proved in [22], thanks to a priori assumption on the nonlinearity f involved in the problem. Since we are lacking an analogous estimate for our solutions (even in the simpler case $a_i \equiv 0$ for every $i = 1, \dots, m$), we are not even entitled to profit of the weighted Sobolev inequality of Trudinger. Nevertheless, we can avoid such a priori assumptions by exploiting some a priori regularity results that we prove in this paper; see Lemma 2.3 and Lemma 2.4 below. We stress that this is a considerable simplification of the proof in [17] and it is crucially based on the new Lemma 2.4.

Now, by scrutinizing the proof of Theorem 1.3, one can easily recognize that a key ingredient for our argument is the fact that, under assumption ($h_{p,q}$), we have

$$p_i^* = \frac{Np_i}{N - p_i} \geq 2 \quad \text{for all } i = 1, \dots, m.$$

However, since the above inequality is equivalent to require that $p_i \geq 2N/(N+2)$, it is natural to wonder why we *require* p_i to satisfy the worse lower bound

$$p_i > \frac{2N+2}{N+2} =: \beta_N.$$

The main reason behind this choice is that, when $p \leq \beta_N$, we *do not have* at our disposal a strong comparison principle for the operator $-\Delta_p u + a(u)|\nabla u|^q$ which allows us to do not take care of the *critical set* $\mathcal{Z} = \{\nabla u = 0\}$, which is, in general, a crucial point when one deals with the p-Laplace operator.

In view of this fact, in order to establish Theorem 1.3 when

$$\frac{2N}{N+2} \leq p_i \leq \frac{2N+2}{N+2},$$

we would need to recover a technical lemma analogous to [17, Lemma 3.2], which also requires suitable estimates for the second-order derivatives of the solution of $(\mathbb{S})$. Even if these estimates are available in our setting (see [16, Theorem 2.1]), we prefer to avoid this technicality here: indeed, we believe that the main novelties of our work are both to consider systems involving general first order terms and the possibility of considering the case $p_i < 2$ without the need of prescribing precise (a priori) growth estimates for the solution of $(\mathbb{S})$. To the best of our knowledge, our result is new also in the case of a single scalar equation.

This paper is organized as follows: we prove some technical results in Section 2 that we will exploit in Section 3 to prove Theorem 1.3.

2 Notation and preliminary results

In this section, we collect all relevant notation which shall be used throughout the paper. Moreover, we review some results *already known* in the literature which shall be fundamental for the proof of our Theorem 1.3. We will adopt the symbol $|A|$ to denote the N-dimensional Lebesgue measure of a measurable set $A \subseteq \mathbb{R}^N$.

2.1 Capacity

Due to the important role played by assumption (h_Γ) in our arguments, we briefly recall a few notions and results concerning the *Sobolev capacity.*

Let $K \subseteq \mathbb{R}^N$ be a *compact set,* and let $\mathcal{O} \subseteq \mathbb{R}^N$ be an open set such that $K \subseteq \mathcal{O}$. Given any $1 < r \leq N$, the r-capacity of the *condenser* $\mathcal{C} := (K, \mathcal{O})$ is defined by

$$\operatorname{Cap}_r(K, \mathcal{O}) := \inf \Big\{ \int_{\mathbb{R}^N} |\nabla \phi|^r \, dx : \phi \in C_c^\infty(\mathcal{O}) \text{ and } \phi \geq 1 \text{ on } K \Big\}.$$

We then say that K has *vanishing r-capacity,* and we write $\operatorname{Cap}_r(K) = 0$, if

$$\operatorname{Cap}_r(K, \mathcal{O}) = 0 \quad \text{for every open set } \mathcal{O} \supset K.$$

As is reasonable to expect, compact sets with vanishing r-capacity have to be *very small;* the next theorem shows that this is actually true in the sense of Hausdorff measure.

Theorem 2.1. *Let $K \subseteq \mathbb{R}^N$ be a compact set. Then the following assertions hold:*
(i) *If $\operatorname{Cap}_r(K) = 0$, then $\mathcal{H}^s(K) = 0$ for every $s > N - r$.*
(ii) *If $r < N$ and $\mathcal{H}^{N-r}(K) < \infty$, then $\operatorname{Cap}_r(K) = 0$.*

For a proof of Theorem 2.1, we refer the reader to [19, Section 2.24].

Corollary 2.2. *Let $1 < r_1 < r_2 \leq N$ and let $K \subseteq \mathbb{R}^N$ be compact. Then*

$$\operatorname{Cap}_{r_2}(K) = 0 \implies \operatorname{Cap}_{r_1}(K) = 0.$$

Proof. Since, by assumption, $\operatorname{Cap}_{r_2}(K) = 0$, by Theorem 2.1 (i) we have $\mathcal{H}^s(K) = 0$ for every $s > N - r_2$. In particular, since $r_1 < r_2$, we derive that

$$r_1 < N \quad \text{and} \quad \mathcal{H}^{N-r_1}(K) = 0.$$

Using this fact and Theorem 2.1 (ii), we then conclude that $\operatorname{Cap}_{r_1}(K) = 0$. □

On account of Corollary 2.2, if $\Gamma \subseteq \Omega \cap \Pi_0$ is as in Theorem 1.3, we have

$$\mathrm{Cap}_{p_i}(\Gamma) = 0 \quad \text{for every } i = 1, \dots, m. \tag{2.1}$$

Let now $K \subseteq \mathbb{R}^N$ be a fixed compact set *with vanishing r-capacity,* and let $\mathcal{O} \subseteq \mathbb{R}^N$ be an open set containing K. We aim to show that it is possible to construct a family of functions in $\mathbb{R}^N$, say $\{\psi_\varepsilon\}_\varepsilon$, which satisfies the following properties:

(i) $\psi_\varepsilon \in \mathrm{Lip}(\mathbb{R}^N)$ and $0 \le \psi_\varepsilon \le 1$ pointwise in $\mathbb{R}^N$.

(ii) There exists an open neighborhood $\mathcal{V}_\varepsilon \subseteq \mathcal{O}$ of K such that

$$\psi_\varepsilon \equiv 0 \text{ on } \mathcal{V}_\varepsilon.$$

(iii) $\psi_\varepsilon(x) \to 1$ as $\varepsilon \to 0^+$ for a.e. $x \in \mathbb{R}^N$.

(iv) There exists a constant $C_0 > 0$, only depending on r, such that

$$\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^r \, dx \le C_0 \varepsilon.$$

In fact, let $\varepsilon_0 = \varepsilon_0(K, \mathcal{O}) > 0$ be such that

$$\mathcal{B}_\varepsilon := \{x \in \mathbb{R}^N : \mathrm{dist}(x, K) < \varepsilon\} \Subset \mathcal{O} \quad \text{for every } 0 < \varepsilon < \varepsilon_0.$$

Since $\mathrm{Cap}_r(K) = 0$ and $\mathcal{B}_\varepsilon \subseteq \mathbb{R}^N$ is an open neighborhood of K, for every $\varepsilon \in (0, \varepsilon_0)$ there exists a smooth function $\phi_\varepsilon \in C_c^\infty(\mathcal{B}_\varepsilon)$ such that

$$\phi_\varepsilon \ge 1 \text{ on } K, \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla \phi_\varepsilon|^r \, dx < \varepsilon. \tag{2.2}$$

We then consider the following Lipschitz functions:

(a) $T(s) := \max\{0; \min\{s; 1\}\}$ (for $s \in \mathbb{R}$).

(b) $g(t) := \max\{0; 1 - 2t\}$ (for $t \ge 0$).

Furthermore, we define, for every $\varepsilon \in (0, \varepsilon_0)$,

$$\psi_\varepsilon : \mathbb{R}^N \to \mathbb{R}, \quad \psi_\varepsilon(x) := g(T(\varphi_\varepsilon(x))). \tag{2.3}$$

Clearly, $\psi_\varepsilon \in \mathrm{Lip}(\mathbb{R}^N)$ and $0 \le \psi_\varepsilon \le 1$ in $\mathbb{R}^N$. Moreover, by (2.2) we have

$$\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^r \, dx \le 2^r \int_{\mathbb{R}^N} |\nabla \phi_\varepsilon|^r \, dx \le 2^r \varepsilon.$$

Hence, the family $\{\psi_\varepsilon\}_\varepsilon$ satisfies properties (i) and (iv) (with $C_0 = 2^r$). As for the validity of property (ii), we observe that, by the explicit definitions of T and g, we have

$$\psi_\varepsilon \equiv 0 \quad \text{on } \mathcal{V}_\varepsilon := \left\{\phi_\varepsilon > \frac{1}{2}\right\}.$$

As a consequence, since $\mathcal{V}_\varepsilon$ is an open neighborhood of K and since $\mathcal{V}_\varepsilon \subseteq \mathcal{B}_\varepsilon \subseteq \mathcal{O}$ (remember that $\phi_\varepsilon \ge 1$ on K, and $\mathrm{supp}(\phi_\varepsilon) \subseteq \mathcal{B}_\varepsilon$), we immediately conclude that also property (ii) is satisfied. Finally, the validity of property (iii) follows from the fact that

$$\psi_\varepsilon \equiv g(0) = 1 \quad \text{on } \mathbb{R}^N \setminus \mathcal{B}_\varepsilon. \tag{2.4}$$

Indeed, since the family $\{\mathcal{B}_\varepsilon\}_\varepsilon$ shrinks to K as $\varepsilon \to 0^+$, by (2.4) we get

$$\psi_\varepsilon(x) \to 1 \quad \text{as } \varepsilon \to 0^+, \text{ for every } x \in \mathbb{R}^N \setminus K.$$

From this, since $|K| = \mathcal{H}^N(K) = 0$ (remember that $\mathrm{Cap}_p(K) = 0$ and see Theorem 2.1), we immediately conclude that the family $\{\psi_\varepsilon\}_\varepsilon$ satisfies also property (iii), as claimed.

Throughout what follows, we will repeatedly use the family $\{\psi_\varepsilon\}_\varepsilon$ with different choices of K and $\mathcal{O}$; hence, to simplify the notation, we shall refer to this family as a *cut-off family for the compact set K, related with the open set* $\mathcal{O}$.

2.2 Notations for the moving plane method

Let $\mathbf{u} \in C^{1,\alpha}(\overline{\Omega} \setminus \Gamma; \mathbb{R}^m)$ be a weak solution to problem $(\mathcal{S})$. For every fixed $\lambda \in \mathbb{R}$, we denote by R_λ the reflection trough the hyperplane

$$\Pi_\lambda := \{x \in \mathbb{R}^N : x_1 = \lambda\},$$

that is,

$$R_\lambda(x) = x_\lambda := (2\lambda - x_1, x_2, \dots, x_N). \tag{2.5}$$

Accordingly, we introduce the vector-valued function

$$\mathbf{u}_\lambda(x) = (u_{1,\lambda}(x), \dots, u_{m,\lambda}(x)) := \mathbf{u}(x_\lambda) \quad \text{for } x \in R_\lambda(\overline{\Omega} \setminus \Gamma). \tag{2.6}$$

We point out that, since $\mathbf{u}$ solves $(\mathcal{S})$, it is easy to see that
(i) $\mathbf{u}_\lambda \in C^{1,\alpha}(R_\lambda(\overline{\Omega} \setminus \Gamma); \mathbb{R}^m)$.
(ii) For every $i = 1, \dots, m$ and every $\varphi \in C_c^1(R_\lambda(\Omega \setminus \Gamma))$, one has

$$\int_{R_\lambda(\Omega)} |\nabla u_{i,\lambda}|^{p_i - 2}\langle \nabla u_{i,\lambda}, \nabla\varphi\rangle \, dx + \int_{R_\lambda(\Omega)} a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{q_i}\varphi \, dx = \int_{R_\lambda(\Omega)} f_i(\mathbf{u}_\lambda)\varphi \, dx. \tag{2.7}$$

(iii) $u_{i,\lambda} > 0$ pointwise in $R_\lambda(\Omega \setminus \Gamma)$ (for every $i = 1, \dots, m$).
(iv) $u_{i,\lambda} \equiv 0$ on $R_\lambda(\partial\Omega)$ (for every $i = 1, \dots, m$).
To proceed further, we let

$$\varrho = \varrho_\Omega := \inf_{x \in \Omega} x_1 \tag{2.8}$$

and we observe that, since Ω is bounded and symmetric with respect to Π_0, we certainly have $-\infty < \varrho < 0$. As a consequence, for every $\lambda \in (\varrho, 0)$ we can set

$$\Omega_\lambda := \{x \in \Omega : x_1 < \lambda\}. \tag{2.9}$$

We explicitly point out that, since Ω is convex in the x_1-direction, we have

$$\Omega_\lambda \subseteq R_\lambda(\Omega) \cap \Omega. \tag{2.10}$$

Finally, for every $\lambda \in (\varrho, 0)$ we define the function

$$\mathbf{w}_\lambda(x) = (w_{1,\lambda}(x), \dots, w_{m,\lambda}(x)) := (\mathbf{u} - \mathbf{u}_\lambda)(x) \quad \text{for } x \in (\overline{\Omega} \setminus \Gamma) \cap R_\lambda(\overline{\Omega} \setminus \Gamma).$$

On account of (2.10), $\mathbf{w}_\lambda$ is surely well-posed on $\overline{\Omega}_\lambda \setminus R_\lambda(\Gamma)$.

2.3 Preliminary results

After these preliminaries, we devote the remaining part of this section to collect some auxiliary results which shall be useful for the proof of Theorem 1.3. In what follows, we tacitly inherit all notation introduced so far.

To begin with, we recall some identities between vectors in $\mathbb{R}^N$ which are useful in dealing with quasilinear operators: *for every $s > 1$ there exist constants $C_1, \dots, C_4 > 0$, only depending on s, such that, for every $\eta, \eta' \in \mathbb{R}^N$, one has*

$$\begin{cases} \langle |\eta|^{s-2}\eta - |\eta'|^{s-2}\eta', \eta - \eta'\rangle \geq C_1(|\eta| + |\eta'|)^{s-2}|\eta - \eta'|^2, \\ \big||\eta|^{s-2}\eta - |\eta'|^{s-2}\eta'\big| \leq C_2(|\eta| + |\eta'|)^{s-2}|\eta - \eta'|, \\ \langle |\eta|^{s-2}\eta - |\eta'|^{s-2}\eta', \eta - \eta'\rangle \geq C_3|\eta - \eta'|^s & \text{if } s \geq 2, \\ \big||\eta|^{s-2}\eta - |\eta'|^{s-2}\eta'\big| \leq C_4|\eta - \eta'|^{s-1} & \text{if } 1 < s < 2. \end{cases} \tag{2.11}$$

We refer, e.g., to [8] for a proof of (2.11).

We then establish the following fundamental lemma.

Lemma 2.3. *Let assumptions* (h_Ω)–(h_f) *be in force. Let* $i \in \{1, \dots, m\}$ *be fixed, and let* $\lambda \in (\varrho, 0)$ *be such that*

$$R_\lambda(\Gamma) \cap \overline{\Omega}_\lambda \neq \emptyset.$$

Then there exists a constant $\mathbf{c} = \mathbf{c}_i > 0$ *such that*

$$\int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \, dx \leq \mathbf{c}_i. \tag{2.12}$$

Proof. We first notice that, since $\lambda < 0$, by assumption (h_Γ), we have $\overline{\Omega}_\lambda \cap \Gamma = \emptyset$. Therefore, the function $\mathbf{u} = (u_1, \dots, u_m)$ is of class $C^{1,\alpha}$ on $\overline{\Omega}_\lambda$, and we can set

$$M = M_{\mathbf{u}} := \max_{1 \leq j \leq m} \big(\|u_j\|_{L^\infty(\overline{\Omega}_\lambda)} + \|\nabla u_j\|_{L^\infty(\overline{\Omega}_\lambda)} \big) < +\infty.$$

Moreover, since u_i and $u_{i,\lambda}$ are non-negative, we have

$$0 \leq w_{i,\lambda}^+ = (u_i - u_{i,\lambda})^+ \leq u_i \leq M \quad \text{pointwise in } \overline{\Omega}_\lambda \setminus R_\lambda(\Gamma). \tag{2.13}$$

Now, to prove (2.12) we distinguish two cases:
(i) $\max\{1, p_i - 1\} \leq q_i < p_i$.
(ii) $q_i = p_i$.

Case (i). First of all, we observe that, since R_λ is a bijective linear map and since Γ has vanishing p_i-capacity (see (2.1)), the compact set $\Gamma_\lambda := R_\lambda(\Gamma)$ satisfies

$$\mathrm{Cap}_{p_i}(\Gamma_\lambda) = 0.$$

As a consequence, if $\mathcal{O}_\lambda \subseteq \mathbb{R}^N$ is a fixed open neighborhood of Γ_λ, we can choose a *cut-off family* $\{\psi_\varepsilon\}_{\varepsilon < \varepsilon_0}$ for Γ_λ, related with the open set $\mathcal{O}_\lambda$. This means, precisely, that the following assertions hold:
(i) $\psi_\varepsilon \in \mathrm{Lip}(\mathbb{R}^N)$ and $0 \leq \psi_\varepsilon \leq 1$ pointwise in $\mathbb{R}^N$.
(ii) There exists an open neighborhood $\mathcal{V}_\varepsilon^\lambda \subseteq \mathcal{O}_\lambda$ of Γ_λ such that

$$\psi_\varepsilon \equiv 0 \quad \text{on } \mathcal{V}_\varepsilon^\lambda.$$

(iii) $\psi_\varepsilon(x) \to 1$ as $\varepsilon \to 0^+$ for a.e. $x \in \mathbb{R}^N$.
(iv) There exists a constant $C_0 > 0$, independent of ε, such that

$$\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^{p_i} \, dx \leq C_0 \varepsilon.$$

We now define, for every $\varepsilon \in (0, \varepsilon_0)$, the map

$$\varphi_{i,\varepsilon}(x) := \begin{cases} w_{i,\lambda}^+(x) \psi_\varepsilon^{p_i}(x) = (u_i - u_{i,\lambda})^+(x) \psi_\varepsilon^{p_i}(x) & \text{if } x \in \Omega_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

We then claim that the following assertions hold:
(i) $\varphi_{i,\varepsilon} \in \mathrm{Lip}(\mathbb{R}^N)$.
(ii) $\mathrm{supp}(\varphi_{i,\varepsilon}) \subseteq \Omega_\lambda$ and $\varphi_{i,\varepsilon} \equiv 0$ near Γ_λ.
In fact, since

$$u_i \in C^{1,\alpha}(\overline{\Omega}_\lambda) \quad \text{and} \quad u_{i,\lambda} \in C^{1,\alpha}(\overline{\Omega}_\lambda \setminus \Gamma_\lambda),$$

we readily see that

$$w_{i,\lambda}^+ \in \mathrm{Lip}(\overline{\Omega}_\lambda \setminus V) \quad \text{for every open set } V \supseteq \Gamma_\lambda.$$

From this, since $\psi_\varepsilon \in \mathrm{Lip}(\mathbb{R}^N)$ and $\psi_\varepsilon \equiv 0$ on $\mathcal{V}_\varepsilon^\lambda \supseteq \Gamma_\lambda$, we get $\varphi_{i,\varepsilon} \in \mathrm{Lip}(\overline{\Omega}_\lambda)$. On the other hand, since $\varphi_{i,\varepsilon} \equiv 0$ on $\partial\Omega_\lambda$, we easily conclude that $\varphi_{i,\varepsilon} \in \mathrm{Lip}(\mathbb{R}^N)$, as claimed. As for assertion (ii), it follows from the definition of $\varphi_{i,\varepsilon}$, jointly with the fact that

$$\psi_\varepsilon \equiv 0 \quad \text{on } \mathcal{V}_\varepsilon^\lambda \supseteq \Gamma_\lambda.$$

On account of properties (i) and (ii) of $\varphi_{i,\varepsilon}$, a standard density argument allows us to use $\varphi_{i,\varepsilon}$ as a test function *both in* (1.2) *and* (2.7), obtaining

$$\begin{aligned}\int_{\Omega_\lambda} \langle |\nabla u_i|^{p_i-2}\nabla u_i - |\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda}, \nabla\varphi_{i,\varepsilon}\rangle\,dx + \int_{\Omega_\lambda} (a_i(u_i)|\nabla u_i|^{q_i} - a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{q_i})\varphi_{i,\varepsilon}\,dx \\ = \int_{\Omega_\lambda} (f_i(\mathbf{u}) - f_i(\mathbf{u}_\lambda))\varphi_{i,\varepsilon}\,dx.\end{aligned} \tag{2.14}$$

By unraveling the very definition of $\varphi_{i,\varepsilon}$, we then obtain

$$\begin{aligned}&\int_{\Omega_\lambda} \langle |\nabla u_i|^{p_i-2}\nabla u_i - |\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda}, \nabla w_{i,\lambda}^+\rangle \psi_\varepsilon^{p_i}\,dx \\ &\quad + p_i \int_{\Omega_\lambda} \langle |\nabla u_i|^{p_i-2}\nabla u_i - |\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda}, \nabla\psi_\varepsilon\rangle w_{i,\lambda}^+ \psi_\varepsilon^{p_i-1}\,dx \\ &\quad + \int_{\Omega_\lambda} (a_i(u_i)|\nabla u_i|^{q_i} - a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{q_i}) w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\,dx \\ &= \int_{\Omega_\lambda} (f_i(\mathbf{u}) - f_i(\mathbf{u}_\lambda)) w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\,dx.\end{aligned} \tag{2.15}$$

We now observe that the integral on the right-hand side of (2.15) is actually performed on the set

$$A_{i,\lambda} := \{x \in \Omega_\lambda : u_i \geq u_{i,\lambda}\} \setminus \Gamma_\lambda.$$

Moreover, we have

$$0 \leq u_{i,\lambda}(x) \leq u_i(x) \leq M \quad \text{for every } x \in A_{i,\lambda}. \tag{2.16}$$

The right-hand side of (2.15) can be arranged as follows:

$$\begin{aligned}&\int_{\Omega_\lambda} (f_i(\mathbf{u}) - f_i(\mathbf{u}_\lambda)) w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\,dx \\ &= \int_{\Omega_\lambda} [f_i(u_1,\dots,u_m) - f_i(u_{1,\lambda},\dots,u_{m,\lambda})] w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\,dx \\ &= \int_{\Omega_\lambda} [f_i(u_1,\dots,u_m) - f_i(u_{1,\lambda},u_2,\dots,u_m) \\ &\qquad + f_i(u_{1,\lambda},u_2,\dots,u_m) - f_i(u_{1,\lambda},u_{2,\lambda},\dots,u_{m,\lambda})] w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\,dx \\ &= \int_{\Omega_\lambda} [f_i(u_1,\dots,u_m) - f_i(u_{1,\lambda},u_2,\dots,u_m) \\ &\qquad + f_i(u_{1,\lambda},u_2,\dots,u_m) - f_i(u_{1,\lambda},u_{2,\lambda},\dots,u_m) \\ &\qquad + \cdots + f_i(u_{1,\lambda},u_{2,\lambda},\dots,u_i,\dots,u_m) - f_i(u_{1,\lambda},u_{2,\lambda}\dots,u_{i,\lambda},\dots,u_m) \\ &\qquad + \cdots + f_i(u_{1,\lambda},u_{2,\lambda},u_{3,\lambda}\dots,u_m) - f_i(u_{1,\lambda},u_{2,\lambda},u_{3,\lambda}\dots,u_{m,\lambda})] w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\,dx.\end{aligned} \tag{2.17}$$

By (2.17), we have

$$\begin{aligned}&\int_{\Omega_\lambda} (f_i(\mathbf{u}) - f_i(\mathbf{u}_\lambda)) w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\,dx \\ &= \int_{\Omega_\lambda} [f_i(u_1,\dots,u_m) - f_i(u_{1,\lambda},\dots,u_{m,\lambda})] w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\,dx \\ &= \int_{\Omega_\lambda} \frac{f_i(u_1,\dots,u_m) - f_i(u_{1,\lambda},u_2,\dots,u_m)}{u_1 - u_{1,\lambda}} (u_1 - u_{1,\lambda}) w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\,dx \\ &\qquad + \int_{\Omega_\lambda} \frac{f_i(u_{1,\lambda},u_2,\dots,u_m) - f_i(u_{1,\lambda},u_{2,\lambda},\dots,u_m)}{u_2 - u_{2,\lambda}} (u_2 - u_{2,\lambda}) w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\,dx\end{aligned}$$

$$
\begin{aligned}
&+\dots+\int\limits_{\Omega_\lambda}\frac{f_i(u_{1,\lambda},u_2,\dots,u_i,\dots,u_m)-f_i(u_{1,\lambda},u_{2,\lambda},\dots,u_{i,\lambda},\dots,u_m)}{u_i-u_{i,\lambda}}(u_i-u_{i,\lambda})w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx\\
&+\dots+\int\limits_{\Omega_\lambda}\frac{f_i(u_{1,\lambda},u_{2,\lambda},\dots,u_m)-f_i(u_{1,\lambda},u_{2,\lambda},\dots,u_{m,\lambda})}{u_m-u_{m,\lambda}}(u_m-u_{m,\lambda})w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx\\
\le&\int\limits_{\Omega_\lambda}\frac{f_i(u_1,\dots,u_m)-f_i(u_{1,\lambda},u_2,\dots,u_m)}{u_1-u_{1,\lambda}}(u_1-u_{1,\lambda})^+w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx\\
&+\int\limits_{\Omega_\lambda}\frac{f_i(u_{1,\lambda},u_2,\dots,u_m)-f_i(u_{1,\lambda},u_{2,\lambda},\dots,u_m)}{u_2-u_{2,\lambda}}(u_2-u_{2,\lambda})^+w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx\\
&+\dots+\int\limits_{\Omega_\lambda}\frac{f_i(u_{1,\lambda},u_2,\dots,u_i,\dots,u_m)-f_i(u_{1,\lambda},u_{2,\lambda},\dots,u_{i,\lambda},\dots,u_m)}{u_i-u_{i,\lambda}}(u_i-u_{i,\lambda})^+w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx\\
&+\dots+\int\limits_{\Omega_\lambda}\frac{f_i(u_{1,\lambda},u_{2,\lambda},\dots,u_m)-f_i(u_{1,\lambda},u_{2,\lambda},\dots,u_{m,\lambda})}{u_m-u_{m,\lambda}}(u_m-u_{m,\lambda})^+w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx\\
\le&\sum_{j=1}^m L_j\int\limits_{\Omega_\lambda}w_{i,\lambda}^+w_{j,\lambda}^+\psi_\varepsilon^{p_i}\,dx,
\end{aligned}
\tag{2.18}
$$

where in the last inequality we define

$$u_j-u_{j,\lambda}:=w_{j,\lambda}=w_{j,\lambda}^+-w_{j,\lambda}^-$$

and we used the cooperativity assumption of each f_j for $j\neq i$, together with the fact that f_j is of class C^1 (see assumption (h_f)). Here, $L_j>0$ is the Lipschitz constant of f_j on $[0,M]\times\dots\times[0,M]$. Hence, resuming the computations above, we have

$$\int\limits_{\Omega_\lambda}(f_i(\mathbf{u})-f_i(\mathbf{u}_\lambda))w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx\le\sum_{j=1}^m L_j\int\limits_{\Omega_\lambda}w_{j,\lambda}^+w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx. \tag{2.19}$$

On the other hand, by using the estimates in (2.11), we get

$$\int\limits_{\Omega_\lambda}\langle|\nabla u_i|^{p_i-2}\nabla u_i-|\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda},\nabla w_{i,\lambda}^+\rangle\psi_\varepsilon^{p_i}\,dx\ge C_1\int\limits_{\Omega_\lambda}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx. \tag{2.20}$$

Gathering together (2.19) and (2.20), from (2.15) we then obtain

$$
\begin{aligned}
C_1&\int\limits_{\Omega_\lambda}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx\\
&\le\int\limits_{\Omega_\lambda}\langle|\nabla u_i|^{p_i-2}\nabla u_i-|\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda},\nabla w_{i,\lambda}^+\rangle\psi_\varepsilon^{p_i}\,dx\\
&\le p_i\int\limits_{\Omega_\lambda}\big||\nabla u_i|^{p_i-2}\nabla u_i-|\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda}\big||\nabla\psi_\varepsilon|w_{i,\lambda}^+\psi_\varepsilon^{p_i-1}\,dx\\
&\qquad+\int\limits_{\Omega_\lambda}\big|a_i(u_i)|\nabla u_i|^{q_i}-a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{q_i}\big|w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx+\sum_{j=1}^m L_j\int\limits_{\Omega_\lambda}w_{j,\lambda}^+w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx\\
&\le p_i\int\limits_{\Omega_\lambda}\big||\nabla u_i|^{p_i-2}\nabla u_i-|\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda}\big||\nabla\psi_\varepsilon|w_{i,\lambda}^+\psi_\varepsilon^{p_i-1}\,dx\\
&\qquad+\int\limits_{\Omega_\lambda}|a_i(u_i)-a_i(u_{i,\lambda})|\,|\nabla u_{i,\lambda}|^{q_i}w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx\\
&\qquad+\int\limits_{\Omega_\lambda}|a_i(u_i)|\big||\nabla u_i|^{q_i}-|\nabla u_{i,\lambda}|^{q_i}\big|w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx+\sum_{j=1}^m L_j\int\limits_{\Omega_\lambda}w_{j,\lambda}^+w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx.
\end{aligned}
\tag{2.21}
$$

To proceed further, we now turn to provide ad-hoc estimates for the integrals on the right-hand side of (2.21). To this end, we first introduce the notation

$$F_i := \sum_{j=1}^{m} L_j \int_{\Omega_\lambda} w_{j,\lambda}^+ w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\, dx.$$

Moreover, we split the set Ω_λ as $\Omega_\lambda = \Omega_\lambda^{(1)} \cup \Omega_\lambda^{(2)}$, where

$$\begin{aligned}\Omega_\lambda^{(1)} &= \{x \in \Omega_\lambda \setminus \Gamma_\lambda : |\nabla u_{i,\lambda}(x)| < 2|\nabla u_i|\},\\ \Omega_\lambda^{(2)} &= \{x \in \Omega_\lambda \setminus \Gamma_\lambda : |\nabla u_{i,\lambda}(x)| \geq 2|\nabla u_i|\}.\end{aligned}$$

Then the following assertions ensue:

- By the definition of $\Omega_\lambda^{(1)}$, one has
$$|\nabla u_i| + |\nabla u_{i,\lambda}| < 3|\nabla u_i|. \tag{2.22}$$
- By the definition of the set $\Omega_\lambda^{(2)}$ and standard triangular inequalities, one has
$$\frac{1}{2}|\nabla u_{i,\lambda}| \leq |\nabla u_{i,\lambda}| - |\nabla u_i| \leq |\nabla w_{i,\lambda}| \leq |\nabla u_{i,\lambda}| + |\nabla u_i| \leq \frac{3}{2}|\nabla u_{i,\lambda}|. \tag{2.23}$$

Accordingly, we define

$$\begin{aligned}
P_{i,1} &:= \int_{\Omega_\lambda^{(1)}} \big||\nabla u_i|^{p_i-2}\nabla u_i - |\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda}\big| |\nabla \psi_\varepsilon| w_{i,\lambda}^+ \psi_\varepsilon^{p_i-1}\, dx\\
P_{i,2} &:= \int_{\Omega_\lambda^{(2)}} \big||\nabla u_i|^{p_i-2}\nabla u_i - |\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda}\big| |\nabla \psi_\varepsilon| w_{i,\lambda}^+ \psi_\varepsilon^{p_i-1}\, dx\\
I_{i,1} &:= \int_{\Omega_\lambda^{(1)}} |a_i(u_i) - a_i(u_{i,\lambda})| |\nabla u_{i,\lambda}|^{q_i} w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\, dx\\
I_{i,2} &:= \int_{\Omega_\lambda^{(2)}} |a_i(u_i) - a_i(u_{i,\lambda})| |\nabla u_{i,\lambda}|^{q_i} w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\, dx\\
J_{i,1} &:= \int_{\Omega_\lambda^{(1)}} |a_i(u_i)| \big||\nabla u_i|^{q_i} - |\nabla u_{i,\lambda}|^{q_i}\big| w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\, dx\\
J_{i,2} &:= \int_{\Omega_\lambda^{(2)}} |a_i(u_i)| \big||\nabla u_i|^{q_i} - |\nabla u_{i,\lambda}|^{q_i}\big| w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\, dx.
\end{aligned} \tag{2.24}$$

We then turn to estimate all integrals above. In what follows, we denote by the same C any positive constant which is independent of ε (but possibly depending on i).

Estimate of $P_{i,1}$. If $1 < p_i < 2$, from (2.11), (2.22) and by Hölder's inequality, we obtain

$$\begin{aligned}
P_{i,1} &\leq C_4 \int_{\Omega_\lambda^{(1)}} |\nabla w_{i,\lambda}^+|^{p_i-1} |\nabla \psi_\varepsilon| \psi_\varepsilon^{p_i-1} w_{i,\lambda}^+\, dx\\
&\leq C_4 \Big(\int_{\Omega_\lambda^{(1)}} |\nabla w_{i,\lambda}^+|^{p_i} \psi_\varepsilon^{p_i}\, dx\Big)^{\frac{p_i-1}{p_i}} \Big(\int_{\Omega_\lambda^{(1)}} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^{p_i}\, dx\Big)^{\frac{1}{p_i}}\\
&\leq C_4 \Big(\int_{\Omega_\lambda^{(1)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i} \psi_\varepsilon^{p_i}\, dx\Big)^{\frac{p_i-1}{p_i}} \Big(\int_{\Omega_\lambda^{(1)}} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^{p_i}\, dx\Big)^{\frac{1}{p_i}}
\end{aligned}$$

$$\le C_4\Big(3^{p_i}\int_{\Omega_\lambda^{(1)}}|\nabla u_i|^{p_i}\psi_\varepsilon^{p_i}\,dx\Big)^{\frac{p_i-1}{p_i}}\Big(\int_{\Omega_\lambda^{(1)}}|\nabla\psi_\varepsilon|^{p_i}(w_{i,\lambda}^+)^{p_i}\,dx\Big)^{\frac{1}{p_i}}$$
$$\le C\Big(\int_{\Omega_\lambda}|\nabla u_i|^{p_i}\,dx\Big)^{\frac{p_i-1}{p_i}}\Big(\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^{p_i}\,dx\Big)^{\frac{1}{p_i}}. \tag{2.25}$$

If $p_i \ge 2$, from (2.11) and the weighted Young's inequality, we have

$$P_{i,1} \le C_2\int_{\Omega_\lambda^{(1)}}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|\,|\nabla\psi_\varepsilon|\psi_\varepsilon^{p_i-1}w_{i,\lambda}^+\,dx$$
$$\le C\delta\int_{\Omega_\lambda^{(1)}}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx+\frac{C}{\delta}\int_{\Omega_\lambda^{(1)}}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}|\nabla\psi_\varepsilon|^2\psi_\varepsilon^{p_i-2}(w_{i,\lambda}^+)^2\,dx.$$

Using (2.22) and Hölder's inequality, we obtain

$$P_{i,1} \le C\delta\int_{\Omega_\lambda^{(1)}}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx+\frac{C}{\delta}\int_{\Omega_\lambda^{(1)}}|\nabla u_i|^{p_i-2}|\nabla\psi_\varepsilon|^2\psi_\varepsilon^{p_i-2}(w_{i,\lambda}^+)^2\,dx$$
$$\le C\delta\int_{\Omega_\lambda^{(1)}}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx+\frac{C}{\delta}\Big(\int_{\Omega_\lambda^{(1)}}|\nabla u_i|^{p_i}\psi_\varepsilon^{p_i}\,dx\Big)^{\frac{p_i-2}{p_i}}\Big(\int_{\Omega_\lambda^{(1)}}|\nabla\psi_\varepsilon|^{p_i}(w_{i,\lambda}^+)^{p_i}\,dx\Big)^{\frac{2}{p_i}} \tag{2.26}$$
$$\le C\delta\int_{\Omega_\lambda}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx+\frac{C}{\delta}\Big(\int_{\Omega_\lambda}|\nabla u_i|^{p_i}\,dx\Big)^{\frac{p_i-2}{p_i}}\Big(\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^{p_i}\,dx\Big)^{\frac{2}{p_i}}.$$

Estimate of $P_{i,2}$. If $1 < p_i < 2$, using the weighted Young's inequality and (2.23), we get

$$P_{i,2} \le C_4\int_{\Omega_\lambda^{(2)}}|\nabla w_{i,\lambda}^+|^{p_i-1}|\nabla\psi_\varepsilon|\psi_\varepsilon^{p_i-1}w_{i,\lambda}^+\,dx$$
$$\le C\delta\int_{\Omega_\lambda^{(2)}}|\nabla w_{i,\lambda}^+|^{p_i}\psi_\varepsilon^{p_i}\,dx+\frac{C}{\delta}\int_{\Omega_\lambda^{(2)}}|\nabla\psi_\varepsilon|^{p_i}(w_{i,\lambda}^+)^{p_i}\,dx$$
$$\le C\delta\int_{\Omega_\lambda^{(2)}}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^2\psi_\varepsilon^{p_i}\,dx+\frac{C}{\delta}\int_{\Omega_\lambda^{(2)}}|\nabla\psi_\varepsilon|^{p_i}(w_{i,\lambda}^+)^{p_i}\,dx$$
$$\le C\delta\int_{\Omega_\lambda^{(2)}}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}|\nabla u_{i,\lambda}|^2\psi_\varepsilon^{p_i}\,dx+\frac{C}{\delta}\int_{\Omega_\lambda^{(2)}}|\nabla\psi_\varepsilon|^{p_i}(w_{i,\lambda}^+)^{p_i}\,dx \tag{2.27}$$
$$\le C\delta\int_{\Omega_\lambda^{(2)}}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx+\frac{C}{\delta}\int_{\Omega_\lambda^{(2)}}|\nabla\psi_\varepsilon|^{p_i}(w_{i,\lambda}^+)^{p_i}\,dx$$
$$\le C\delta\int_{\Omega_\lambda}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx+\frac{C}{\delta}\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^{p_i}\,dx.$$

If $p_i \ge 2$, by the weighted Young's inequality and (2.11), we deduce that

$$P_{i,2} \le C_i\int_{\Omega_\lambda^{(2)}}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+||\nabla\psi_\varepsilon|\psi_\varepsilon^{p_i-1}w_{i,\lambda}^+\,dx$$
$$\le C\delta\int_{\Omega_\lambda^{(2)}}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{\frac{p_i(p_i-2)}{p_i-1}}|\nabla w_{i,\lambda}^+|^{\frac{p_i}{p_i-1}}\psi_\varepsilon^{p_i}\,dx+\frac{C}{\delta}\int_{\Omega_\lambda^{(2)}}|\nabla\psi_\varepsilon|^{p_i}(w_{i,\lambda}^+)^{p_i}\,dx$$
$$= C\delta\int_{\Omega_\lambda^{(2)}}(|\nabla u_i|+|\nabla u_{i,\lambda}|)^{\frac{p_i(p_i-2)}{p_i-1}}|\nabla w_{i,\lambda}^+|^2|\nabla w_{i,\lambda}^+|^{\frac{p_i}{p_i-1}-2}\psi_\varepsilon^{p_i}\,dx+\frac{C}{\delta}\int_{\Omega_\lambda^{(2)}}|\nabla\psi_\varepsilon|^{p_i}(w_{i,\lambda}^+)^{p_i}\,dx.$$

Using (2.23) and noticing that

$$\frac{p_i}{(p_i-1)} - 2 \le 0,$$

we obtain the following estimate:

$$\begin{aligned} P_{i,2} &\le C\delta \int_{\Omega_\lambda^{(2)}} |\nabla u_{i,\lambda}|^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx + \frac{C}{\delta} \int_{\Omega_\lambda^{(2)}} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^{p_i}\, dx \\ &\le C\delta \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx + \frac{C}{\delta} \int_{\Omega_\lambda^{(2)}} |\nabla \psi_\varepsilon|^{p_i}\, dx \\ &\le C\delta \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx + \frac{C}{\delta} \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^{p_i}\, dx. \end{aligned} \tag{2.28}$$

In the second line of (2.28), we exploited the fact that, since $p_i \ge 2$,

$$|\nabla u_{i,\lambda}|^{p_i-2} \le (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}.$$

Collecting (2.25)–(2.28), we deduce that for $p_i \ge 2$ it holds

$$\begin{aligned} P_{i,1} + P_{i,2} \le C\delta \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx + \frac{C}{\delta} \Big(\int_{\Omega_\lambda} |\nabla u_i|^{p_i} \psi_\varepsilon^{p_i}\, dx \Big)^{\frac{p_i-2}{p_i}} \Big(\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^{p_i} |w_{i,\lambda}^+|^{p_i}\, dx \Big)^{\frac{2}{p_i}} \\ + \frac{C}{\delta} \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^{p_i}\, dx \quad \text{for every } \delta > 0, \end{aligned}$$

where $C > 0$ is a suitable constant depending on p_i, λ, Ω, and M.

In the same way, if $1 < p_i < 2$, we deduce that

$$\begin{aligned} P_{i,1} + P_{i,2} \le C\delta \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx + C \Big(\int_{\Omega_\lambda} |\nabla u_i|^{p_i}\, dx \Big)^{\frac{p_i-1}{p_i}} \Big(\int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^{p_i}\, dx \Big)^{\frac{1}{p_i}} \\ + \frac{C}{\delta} \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^{p_i}\, dx \quad \text{for every } \delta > 0, \end{aligned}$$

where $C > 0$ is a suitable constant depending on p_i, λ, Ω, and M.

In both cases, taking into account (2.13) and the properties of ψ_ε, we derive the estimate

$$P_{i,1} + P_{i,2} \le C\delta \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx + C_\delta \varepsilon^{\frac{1}{p_i}}, \tag{2.29}$$

holding true for every choice of $\delta > 0$.

Estimate of $I_{i,1}$. Since the integral $I_{i,1}$ is actually performed on the set

$$A_{i,\lambda} = \{u_i \ge u_{i,\lambda}\} \setminus \Gamma_\lambda,$$

from (2.16), assumption (h_a) and the definition of $\Omega_\lambda^{(1)}$, we immediately obtain

$$\begin{aligned} I_{i,1} &\le L \int_{\Omega_\lambda^{(1)}} |\nabla u_{i,\lambda}|^{q_i} |w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx \\ &\le 2^{q_i} L \int_{\Omega_\lambda^{(1)}} |\nabla u_i|^{q_i} |w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx \quad (\text{since } u_i \in C^{1,\alpha}(\overline{\Omega}_\lambda) \text{ and } 0 \le \psi_\varepsilon \le 1) \\ &\le C \int_{\Omega_\lambda} |w_{i,\lambda}^+|^2\, dx. \end{aligned} \tag{2.30}$$

Estimate of $I_{i,2}$. First of all, since also the integral $I_{i,2}$ is actually performed on $A_{i,\lambda}$, we can use again estimate (2.16) and assumption (h_a), obtaining

$$I_{i,2} \le L \int_{\Omega_\lambda^{(2)}} |\nabla u_{i,\lambda}|^{q_i} |w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx \le L \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{q_i} |w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx =: (\star).$$

From this, since $q_i < p_i$, using (2.23) and by Young's inequality, we obtain

$$\begin{aligned}
(\star) &= L \int_{\Omega_\lambda^{(2)}} [(|\nabla u_i| + |\nabla u_{i,\lambda}|)^{q_i} \psi_\varepsilon^{q_i}] \cdot [|w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i - q_i}]\, dx \\
&\le C\delta \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i} \psi_\varepsilon^{p_i}\, dx + \frac{C}{\delta} \int_{\Omega_\lambda^{(2)}} |w_{i,\lambda}^+|^{\frac{2p_i}{p_i - q_i}} \psi_\varepsilon^{p_i}\, dx \quad (\text{since } 0 \le \psi_\varepsilon \le 1 \text{ and } p_i/(p_i - q_i) > 1) \\
&\le C\delta \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i - 2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx + \frac{C}{\delta} \int_{\Omega_\lambda} |w_{i,\lambda}^+|^2\, dx \quad \text{for every } \delta > 0.
\end{aligned} \tag{2.31}$$

Estimate of $J_{i,1}$. We first observe that, since $0 \le u_i \le M$ pointwise in $\overline{\Omega}_\lambda$, by exploiting assumption (h_a) and the mean value theorem, we obtain the following estimate:

$$\begin{aligned}
J_{i,1} &\le C \int_{\Omega_\lambda^{(1)}} \big||\nabla u_i|^{q_i} - |\nabla u_{i,\lambda}|^{q_i}\big| w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\, dx \\
&\le C \int_{\Omega_\lambda^{(1)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{q_i - 1} |\nabla w_{i,\lambda}^+| w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\, dx \\
&\le C \int_{\Omega_\lambda^{(1)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{\frac{2q_i - p_i}{2}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{\frac{p_i - 2}{2}} |\nabla w_{i,\lambda}^+| w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\, dx \\
&\le C\delta \int_{\Omega_\lambda^{(1)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{2q_i - p_i} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i - 2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx + \frac{C}{\delta} \int_{\Omega_\lambda} |w_{i,\lambda}^+|^2\, dx.
\end{aligned}$$

Finally, using (2.22), since $q_i \ge \max\{p_i - 1, 1\}$ and therefore $q_i \ge p_i/2$, we conclude that

$$J_{i,1} \le C\delta \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i - 2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx + \frac{C}{\delta} \int_{\Omega_\lambda} |w_{i,\lambda}^+|^2\, dx, \tag{2.32}$$

and this estimate holds for every choice of $\delta > 0$.

Estimate of $J_{i,2}$. Using once again the fact that $0 \le u_i \le M$ in $\overline{\Omega}_\lambda$, and taking into account assumption (h_a), we get

$$J_{i,2} \le C \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{q_i} w_{i,\lambda}^+ \psi_\varepsilon^{p_i}\, dx =: (\bullet).$$

By Young's inequality and estimate (2.23), we obtain

$$\begin{aligned}
(\bullet) &= C \int_{\Omega_\lambda^{(2)}} [(|\nabla u_i| + |\nabla u_{i,\lambda}|)^{q_i} \psi_\varepsilon^{q_i}] \cdot [w_{i,\lambda}^+ \psi_\varepsilon^{p_i - q_i}]\, dx \\
&\le C\delta \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i} \psi_\varepsilon^{p_i}\, dx + \frac{C}{\delta} \int_{\Omega_\lambda^{(2)}} |w_{i,\lambda}^+|^{\frac{p_i}{p_i - q_i}} \psi_\varepsilon^{p_i}\, dx \\
&\le C\delta \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i - 2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx + \frac{C}{\delta} \int_{\Omega_\lambda} |w_{i,\lambda}^+|^2\, dx \\
&\le C\delta \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i - 2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\, dx + \frac{C}{\delta} \int_{\Omega_\lambda} |w_{i,\lambda}^+|^2\, dx,
\end{aligned} \tag{2.33}$$

where we used the facts that $p_i/(p_i - q_i) \geq 2$ (because $q_i \geq p_i/2$) and $0 \leq u_i \leq M$ in $\overline{\Omega}_\lambda$; this estimate holds for every $\delta > 0$.

Estimate of F_i. Since $0 \leq \psi_\varepsilon \leq 1$, by Young's inequality, we immediately get

$$F_i \leq \sum_{j=1}^{m} L_j\Big(\int_{\Omega_\lambda} |w_{i,\lambda}^+|^2\,dx + \int_{\Omega_\lambda} |w_{j,\lambda}^+|^2\,dx\Big) \leq C\sum_{j=1}^{m}\int_{\Omega_\lambda} |w_{j,\lambda}^+|^2\,dx. \tag{2.34}$$

Thanks to all estimates above, we can finally complete the proof of (2.12): in fact, by combining (2.29)–(2.34), from (2.21) we infer that

$$\begin{aligned} &C_1\int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx \\ &\quad\leq C\delta\int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx + C_\delta\varepsilon^{\frac{1}{p_i}} + \frac{C}{\delta}\int_{\Omega_\lambda} |w_{i,\lambda}^+|^2\,dx + C\sum_{j=1}^{m}\int_{\Omega_\lambda} |w_{j,\lambda}^+|^2\,dx, \end{aligned} \tag{2.35}$$

and this estimate holds for every $\delta > 0$. As a consequence, if we choose δ sufficiently small and if we let $\varepsilon \to 0^+$ with the aid of Fatou's lemma, we obtain

$$\int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\,dx \leq C\sum_{j=1}^{m}\int_{\Omega_\lambda} |w_{j,\lambda}^+|^2\,dx, \tag{2.36}$$

where $C > 0$ is a suitable constant depending on $p_i, q_i, a_i, f_i, \lambda, \Omega, M$. This, together with the estimate in (2.13), immediately implies the desired (2.12).

Case (ii). Let us define the function

$$A_i(t) = \int_0^t a_i(s)\,ds.$$

Using the family of *cut-off* functions $\{\psi_\varepsilon\}_{\varepsilon<\varepsilon_0}$ defined in the previous case, we now define in a very similar way, for every $\varepsilon \in (0, \varepsilon_0)$, the maps

$$\varphi_{i,\varepsilon}^{(1)}(x) := \begin{cases} e^{-A_i(u_i(x))}w_{i,\lambda}^+(x)\psi_\varepsilon^{p_i}(x) & \text{if } x \in \Omega_\lambda, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\varphi_{i,\varepsilon}^{(2)}(x) := \begin{cases} e^{-A_i(u_{i,\lambda}(x))}w_{i,\lambda}^+(x)\psi_\varepsilon^{p_i}(x) & \text{if } x \in \Omega_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

It is possible to prove the following assertions (see, e.g., [16]):
(i) $\varphi_{i,\varepsilon}^{(1)}, \varphi_{i,\varepsilon}^{(2)} \in \mathrm{Lip}(\mathbb{R}^N)$.
(ii) $\mathrm{supp}(\varphi_{i,\varepsilon}^{(1)}) \subseteq \Omega_\lambda$, $\mathrm{supp}(\varphi_{i,\varepsilon}^{(2)}) \subseteq \Omega_\lambda$ and $\varphi_{i,\varepsilon}^{(1)} \equiv \varphi_{i,\varepsilon}^{(2)} \equiv 0$ near Γ_λ.
Hence, taking into account properties (i) and (ii) of $\varphi_{i,\varepsilon}^{(1)}$ and $\varphi_{i,\varepsilon}^{(2)}$, a standard density argument allows us to use $\varphi_{i,\varepsilon}^{(1)}$ and $\varphi_{i,\varepsilon}^{(2)}$ as test functions respectively in (1.2) and (2.7). We then subtract the latter from the former, getting

$$\begin{aligned} &\int_{\Omega_\lambda} \langle|\nabla u_i|^{p_i-2}\nabla u_i, \nabla\varphi_{i,\varepsilon}^{(1)}\rangle\,dx - \int_{\Omega_\lambda} \langle|\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda}, \nabla\varphi_{i,\varepsilon}^{(2)}\rangle\,dx \\ &\qquad + \int_{\Omega_\lambda} a_i(u_i)|\nabla u_i|^{p_i}\varphi_{i,\varepsilon}^{(1)}\,dx - \int_{\Omega_\lambda} a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{p_i}\varphi_{i,\varepsilon}^{(2)}\,dx \\ &= \int_{\Omega_\lambda} f_i(\mathbf{u})\varphi_{i,\varepsilon}^{(1)}\,dx - \int_{\Omega_\lambda} f_i(\mathbf{u}_\lambda)\varphi_{i,\varepsilon}^{(2)}\,dx. \end{aligned}$$

Now, we use the explicit expressions of both $\varphi_{i,\varepsilon}^{(1)}$ and $\varphi_{i,\varepsilon}^{(2)}$ to get

$$\begin{aligned}
-\int_{\Omega_\lambda} a_i(u_i)e^{-A_i(u_i)}|\nabla u_i|^{p_i} w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx + \int_{\Omega_\lambda} e^{-A_i(u_i)}|\nabla u_i|^{p_i-2}\langle\nabla u_i,\nabla w_{i,\lambda}^+\rangle\psi_\varepsilon^{p_i}\,dx \\
+ p_i\int_{\Omega_\lambda} e^{-A_i(u_i)}|\nabla u_i|^{p_i-2}\langle\nabla u_i,\nabla\psi_\varepsilon\rangle w_{i,\lambda}^+\psi_\varepsilon^{p_i-1}\,dx \\
+ \int_{\Omega_\lambda} a_i(u_{i,\lambda})e^{-A_i(u_{i,\lambda})}|\nabla u_{i,\lambda}|^{p_i} w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx - \int_{\Omega_\lambda} e^{-A_i(u_{i,\lambda})}|\nabla u_{i,\lambda}|^{p_i-2}\langle\nabla u_{i,\lambda},\nabla w_{i,\lambda}^+\rangle\psi_\varepsilon^{p_i}\,dx \\
- p_i\int_{\Omega_\lambda} e^{-A_i(u_{i,\lambda})}|\nabla u_{i,\lambda}|^{p_i-2}\langle\nabla u_{i,\lambda},\nabla\psi_\varepsilon\rangle w_{i,\lambda}^+\psi_\varepsilon^{p_i-1}\,dx \\
+ \int_{\Omega_\lambda} a_i(u_i)|\nabla u_i|^{p_i}e^{-A_i(u_i)} w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx - \int_{\Omega_\lambda} a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{p_i}e^{-A_i(u_{i,\lambda})} w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx \\
= \int_{\Omega_\lambda} f_i(\mathbf{u})e^{-A_i(u_i)} w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx - \int_{\Omega_\lambda} f_i(\mathbf{u}_\lambda)e^{-A_i(u_{i,\lambda})} w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx.
\end{aligned}$$

After a simplification, we add on both sides the term

$$\int_{\Omega_\lambda} e^{-A_i(u_{i,\lambda})}|\nabla u_i|^{p_i-2}\langle\nabla u_i,\nabla w_{i,\lambda}^+\rangle\psi_\varepsilon^{p_i}\,dx,$$

and, on the left-hand side, we add and subtract the term

$$p_i\int_{\Omega_\lambda} e^{-A_i(u_{i,\lambda})}|\nabla u_i|^{p_i-2}\langle\nabla u_i,\nabla\psi_\varepsilon\rangle w_{i,\lambda}^+\psi_\varepsilon^{p_i-1}\,dx.$$

Rearranging the terms, we find

$$\begin{aligned}
\int_{\Omega_\lambda} e^{-A_i(u_{i,\lambda})}\langle|\nabla u_i|^{p_i-2}\nabla u_i - |\nabla u_{i,\lambda}|^{p_i-2}\nabla u_i,\nabla w_{i,\lambda}^+\rangle\psi_\varepsilon^{p_i}\,dx \\
= \int_{\Omega_\lambda} (e^{-A_i(u_{i,\lambda})} - e^{-A_i(u_i)})|\nabla u_i|^{p_i-2}\langle\nabla u_i,\nabla w_{i,\lambda}^+\rangle\psi_\varepsilon^{p_i}\,dx \\
+ p_i\int_{\Omega_\lambda} (e^{-A_i(u_{i,\lambda})} - e^{-A_i(u_i)})|\nabla u_i|^{p_i-2}\langle\nabla u_i,\nabla\psi_\varepsilon\rangle w_{i,\lambda}^+\psi_\varepsilon^{p_i-1}\,dx \\
+ p_i\int_{\Omega_\lambda} e^{-A_i(u_{i,\lambda})}\langle|\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda} - |\nabla u_i|^{p_i-2}\nabla u_i,\nabla\psi_\varepsilon\rangle w_{i,\lambda}^+\psi_\varepsilon^{p_i-1}\,dx \\
+ \int_{\Omega_\lambda} (e^{-A_i(u_i)}f_i(\mathbf{u}) - e^{-A_i(u_{i,\lambda})}f_i(\mathbf{u}_\lambda))w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx.
\end{aligned} \tag{2.37}$$

Arguing similarly to case (i), see (2.20), the left-hand side can be estimated from below by

$$C\int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx. \tag{2.38}$$

Indeed, in the set $\Omega_\lambda \cap \operatorname{supp}(w_{i,\lambda}^+)$, one has

$$e^{-A_i(u_{i,\lambda})} \geq C > 0.$$

We now focus on the right-hand side, which we firstly bound from above passing to the absolute values, getting

$$\begin{aligned}\int_{\Omega_\lambda} &|e^{-A_i(u_{i,\lambda})} - e^{-A_i(u_i)}||\nabla u_i|^{p_i-1}|\nabla w_{i,\lambda}^+|\psi_\varepsilon^{p_i}\,dx\\ &+ p_i\int_{\Omega_\lambda}|e^{-A_i(u_{i,\lambda})} - e^{-A_i(u_i)}||\nabla u_i|^{p_i-1}|\nabla\psi_\varepsilon|w_{i,\lambda}^+\psi_\varepsilon^{p_i-1}\,dx\\ &+ p_i\int_{\Omega_\lambda} e^{-A_i(u_{i,\lambda})}\big||\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda} - |\nabla u_i|^{p_i-2}\nabla u_i\big||\nabla\psi_\varepsilon|w_{i,\lambda}^+\psi_\varepsilon^{p_i-1}\,dx\\ &+ \int_{\Omega_\lambda}(e^{-A_i(u_i)}f_i(\mathbf{u}) - e^{-A_i(u_{i,\lambda})}f_i(\mathbf{u}_\lambda))w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx\\ =:&\ \mathcal{J}_{i,1} + \mathcal{J}_{i,2} + \mathcal{J}_{i,3} + \mathcal{J}_{i,4}.\end{aligned}$$

For the reader's convenience, we recall that, in this case, we are assuming $2 \le p_i \le N$. We have the following estimate.

Estimate of $\mathcal{J}_{i,1}$. Using the Lipschitzianity of $t \mapsto e^{-A_i(t)}$, we get that there exists a positive constant $C > 0$ such that

$$\mathcal{J}_{i,1} \le C\int_{\Omega_\lambda}|\nabla u_i|^{p_i-1}|\nabla w_{i,\lambda}^+|\psi_\varepsilon^{\frac{p_i}{2}}\,w_{i,\lambda}^+\psi_\varepsilon^{\frac{p_i}{2}}\,dx.$$

Now, by exploiting the weighted Young's inequality, we obtain that for every $\delta > 0$ it holds

$$\begin{aligned}\mathcal{J}_{i,1} &\le C\delta\int_{\Omega_\lambda}|\nabla u_i|^{2p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx + \frac{C}{\delta}\int_{\Omega_\lambda}(w_{i,\lambda}^+)^2\psi_\varepsilon^{p_i}\,dx\\ &\le C\delta\int_{\Omega_\lambda}(|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx + \frac{C}{\delta}\int_{\Omega_\lambda}(w_{i,\lambda}^+)^2\psi_\varepsilon^{p_i}\,dx,\end{aligned} \tag{2.39}$$

where in the last step $C > 0$ depends also on $\|\nabla u\|_{L^\infty(\Omega_\lambda)}$.

Estimate of $\mathcal{J}_{i,2}$. Using once again the Lipschitzianity of $t \mapsto e^{-A_i(t)}$ together with the boundedness of ∇u in Ω_λ, and then exploiting Hölder's inequality with exponents $(p_i, p_i/(p_i-1))$, we get

$$\mathcal{J}_{i,2} \le C\Big(\int_{\Omega_\lambda}|\nabla\psi_\varepsilon|^{p_i}(w_{i,\lambda}^+)^{p_i}\,dx\Big)^{\frac{1}{p_i}}\Big(\int_{\Omega_\lambda}\psi_\varepsilon^{p_i}\,dx\Big)^{\frac{p_i}{p_i-1}}. \tag{2.40}$$

Estimate of $\mathcal{J}_{i,3}$. We notice first that $e^{-A_i(u_{i,\lambda})} \le 1$. Arguing similarly to the computations that led to (2.25) and subsequently to (2.29), we deduce

$$\mathcal{J}_{i,3} \le C\delta\int_{\Omega_\lambda}(|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx + C_\delta\varepsilon^{\frac{1}{p_i}}\quad\text{for every } \delta > 0.$$

Estimate of $\mathcal{J}_{i,4}$. We first consider the function

$$g_i(t) := e^{-A_i(t_i)}f_i(t),\quad t \in \mathcal{I}^m,$$

which is still a C^1 function satisfying the cooperativity condition $\partial_{t_k}g_i \ge 0$ on $\mathcal{I}^m$ for every $k \ne i$. Hence, we can repeat the computations made in (2.17)–(2.19) to get

$$\mathcal{J}_{i,4} \le \sum_{j=1}^m L_{g_j}\int_{\Omega_\lambda}w_{j,\lambda}^+w_{i,\lambda}^+\psi_\varepsilon^{p_i}\,dx \le C\sum_{j=1}^m\int_{\Omega_\lambda}|w_{j,\lambda}^+|^2\,dx. \tag{2.41}$$

Putting everything together, we can conclude as in the previous case (i). Hence, we obtain the desired (2.12) for every $i = 1, \dots, m$. □

We conclude this section by proving the following useful lemma.

Lemma 2.4. *Let assumptions* (h_Ω)–(h_f) *be in force. Let* $i \in \{1,\dots,m\}$ *be fixed, let* $1 < p_i < 2$ *and let* $\lambda \in (\varrho, 0)$. *Then there exists a constant* $\mathbf{c} = \mathbf{c}_i > 0$ *such that*

$$\int_{\Omega_\lambda} |\nabla w_{i,\lambda}^+|^{p_i}\,dx \le \mathbf{c}\Big(\int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\,dx\Big)^{\frac{p_i}{2}}. \tag{2.42}$$

Proof. Let us define the set $\Omega_\lambda^+ := \Omega_\lambda \cap \operatorname{supp}(w_{i,\lambda}^+)$ and let ψ_ε be the cut-off function defined in (2.3). Using Hölder's inequality with conjugate exponents $(2/(2-p_i), 2/p_i)$, we obtain

$$\begin{aligned}\int_{\Omega_\lambda^+} |\nabla w_{i,\lambda}^+|^{p_i}\psi_\varepsilon^{p_i}\,dx &= \int_{\Omega_\lambda^+} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{\frac{p_i(2-p_i)}{2}}(|\nabla u_i| + |\nabla u_{i,\lambda}|)^{\frac{p_i(p_i-2)}{2}}|\nabla w_{i,\lambda}^+|^{p_i}\psi_\varepsilon^{p_i}\,dx\\ &\le \Big(\int_{\Omega_\lambda^+} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i}\psi_\varepsilon^{p_i}\,dx\Big)^{\frac{2-p_i}{p_i}}\Big(\int_{\Omega_\lambda^+} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx\Big)^{\frac{p_i}{2}}.\end{aligned} \tag{2.43}$$

Now we are going to give an estimate to the first term on the right-hand side of (2.43), i.e. the term

$$\mathcal{I} := \int_{\Omega_\lambda^+} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i}\psi_\varepsilon^{p_i}\,dx.$$

In the same spirit of the previous lemma, we split the set Ω_λ as $\Omega_\lambda = \Omega_\lambda^{(1)} \cup \Omega_\lambda^{(2)}$, where

$$\begin{aligned}\Omega_\lambda^{(1)} &= \{x \in \Omega_\lambda^+ \setminus \Gamma_\lambda : |\nabla u_{i,\lambda}(x)| < 2|\nabla u_i|\},\\ \Omega_\lambda^{(2)} &= \{x \in \Omega_\lambda^+ \setminus \Gamma_\lambda : |\nabla u_{i,\lambda}(x)| \ge 2|\nabla u_i|\}.\end{aligned}$$

Hence,

$$\mathcal{I} = \int_{\Omega_\lambda^{(1)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i}\psi_\varepsilon^{p_i}\,dx + \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i}\psi_\varepsilon^{p_i}\,dx =: \mathcal{I}_1 + \mathcal{I}_2.$$

Estimate of $\mathcal{I}_1$. Since $\psi_\varepsilon^{p_i} \le 1$, using (2.22), we immediately get

$$\mathcal{I}_1 \le C\int_{\Omega_\lambda^{(1)}} |\nabla u_i|^{p_i}\,dx, \tag{2.44}$$

Estimate of $\mathcal{I}_2$. Using (2.23), we obtain

$$\begin{aligned}\mathcal{I}_2 &= \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}(|\nabla u_i| + |\nabla u_{i,\lambda}|)^2\psi_\varepsilon^{p_i}\,dx\\ &\le C\int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla u_{i,\lambda}|^2\psi_\varepsilon^{p_i}\,dx\\ &\le C\int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx.\end{aligned} \tag{2.45}$$

Collecting (2.44) and (2.45) in order to estimate $\mathcal{I}$, we have

$$\mathcal{I} \le C\Big(\int_{\Omega_\lambda^{(1)}} |\nabla u_i|^{p_i}\,dx + \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\psi_\varepsilon^{p_i}\,dx\Big).$$

Now, since each $\nabla u_i \in L^\infty(\Omega_\lambda)$ for every $\lambda < 0$ and the second term is bounded by Lemma 2.3, we deduce that

$$\mathcal{I} \le C,$$

where C is a positive constant that does not depends on ε. Since $\psi_\varepsilon^{p_i} \le 1$, inequality (2.43) becomes

$$\int_{\Omega_\lambda^+} |\nabla w_{i,\lambda}^+|^{p_i} \psi_\varepsilon^{p_i}\, dx \le C\Big(\int_{\Omega_\lambda^+} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2\, dx\Big)^{\frac{p_i}{2}}, \tag{2.46}$$

Finally, by Fatou's lemma, from (2.46) we get (2.42). □

3 Proof of Theorem 1.3

Thanks to all results established so far, we are ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3. Our approach relies on a suitable adaptation of the integral version of the moving plane method. We consider the set

$$\Lambda := \{\eta \in (\varrho, 0) : u_i \le u_{i,\lambda} \text{ on } \Omega_\lambda \setminus R_\lambda(\Gamma) \text{ for all } \lambda \in (\varrho, \eta] \text{ and for all } i = 1, \dots, m\},$$

and we claim that the following facts hold:
(a) $\Lambda \neq \emptyset$.
(b) Setting $\lambda_0 := \sup(\Lambda)$, one has $\lambda_0 = 0$.

Proof of (a). First of all, we observe that, since Γ is compact and contained in $\Omega \cap \Pi_0$, it is possible to find a small $\tau_0 > 0$ such that $\varrho + \tau_0 < 0$ and

$$R_\lambda(\Gamma) \cap \overline{\Omega}_\lambda = \emptyset \quad \text{for every } \lambda \in I_0 := (\varrho, \varrho + \tau_0].$$

In particular, for every $\lambda \in I_0$ we have $\mathbf{u}, \mathbf{u}_\lambda \in C^{1,\alpha}(\overline{\Omega}_\lambda)$. On the other hand, since both u_i and $u_{i,\lambda}$ are non-negative (for every $i = 1, \dots, m$), it is immediate to recognize that

$$u_i \le u_{i,\lambda} \quad \text{on } \partial\Omega_\lambda, \text{ for every } i = 1, \dots, m.$$

Now, it is possible to start the moving plane procedure using [16, Proposition 2.5]. For the reader's convenience we state such a result adapted to our context. The proof is exactly the same, and therefore we skip it.

Proposition 3.1. *Let assumptions* (h_a) *and* (h_f) *be in force, and suppose that*

$$p_i > 1 \quad \textit{and} \quad q_i \ge \max\{p_i - 1, 1\} \qquad \textit{for every } i = 1, \dots, m.$$

In addition, suppose that $\mathbf{u}, \mathbf{u}_\lambda \in C^{1,\alpha}(\overline{\Omega}_{\varrho+\tau_0})$. *Then there exists a number* $\delta > 0$, *depending on* m, p_i, q_i, a_i, f_i, $\|u_i\|_{L^\infty(\Omega_{\varrho+\tau_0})}$, $\|\nabla u_i\|_{L^\infty(\Omega_{\varrho+\tau_0})}$, *and* $\|\nabla \tilde{u}_i\|_{L^\infty(\Omega_{\varrho+\tau_0})}$, *with the following property: if* $\Omega_\lambda \subseteq \Omega_{\varrho+\tau_0}$ *is such that* $|\Omega_\lambda| \le \delta$, *and if*

$$u_i \le u_{i,\lambda} \quad \textit{on } \partial\Omega_\lambda, \textit{ for every } i = 1, \dots, m,$$

then $u_i \le u_{i,\lambda}$ *in* Ω_λ *for every* $i = 1, \dots, m$.

Therefore, by possibly shrinking τ_0 in such a way that $|\Omega_\lambda| \le \delta$ for every $\lambda \in I_0$, we derive that $u_i \le u_{i,\lambda}$ in Ω_λ for every $i = 1, \dots, m$. Hence, $\eta := \varrho + \tau_0 \in \Lambda$, and thus

$$\Lambda \neq \emptyset.$$

Proof of (b). On account of (a), λ_0 is well-defined and $\lambda_0 \le 0$. Arguing by contradiction, we then suppose that $\lambda_0 < 0$, and we prove that there exists some $\tau_0 > 0$ such that

$$u_i \le u_{i,\lambda} \text{ on } \Omega_\lambda \setminus R_\lambda(\Gamma) \quad \text{for all } i = 1, \dots, m \text{ and } \lambda \in (\lambda_0, \lambda_0 + \tau_0]. \tag{3.1}$$

Since (3.1) is clearly in contrast with the very definition of λ_0, we can conclude that

$$\lambda_0 = 0.$$

In order to establish the needed (3.1), we proceed as follows: first of all, since both $\mathbf{u}$ and $\mathbf{u}_\lambda$ are continuous on $\Omega_\lambda \setminus R_\lambda(\Gamma)$, we observe that

$$u_i \leq u_{i,\lambda_0} \text{ on } \Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma) \quad \text{for every } i = 1, \dots, m. \tag{3.2}$$

We then claim that, as a consequence of (3.2), one actually has

$$u_i < u_{i,\lambda_0} \text{ on } \Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma) \quad \text{for every } i = 1, \dots, m. \tag{3.3}$$

To prove (3.3), we arbitrarily fix $i \in \{1, \dots, m\}$.

We first point out that, since $\Omega_{\lambda_0} = \Omega \cap \{x_1 < \lambda_0\}$ is connected and the compact set $\Gamma_{\lambda_0} := R_{\lambda_0}(\Gamma)$ has vanishing p_i-capacity (as the same is true of Γ, see assumption (h_Γ)), it is not difficult to check that $\Omega_{\lambda_0} \setminus \Gamma_{\lambda_0}$ is *connected* (see, e.g., [3, Lemma 2.4]). Thus, owing to assumption (h_f), we deduce that holds the following distributional inequality holds:

$$-\Delta_{p_i} u_i + a_i(u_i)|\nabla u_i|^{q_i} + \Lambda_i u_i \leq -\Delta_{p_i} u_{i,\lambda_0} + a_i(u_{i,\lambda_0})|\nabla u_{i,\lambda_0}|^{q_i} + \Lambda_i u_{i,\lambda_0} \quad \text{in } \Omega_{\lambda_0} \setminus \Gamma_{\lambda_0},$$

where Λ_i is a positive constant. As a consequence, since $u_i \leq u_{i,\lambda_0}$ in $\Omega_{\lambda_0} \setminus \Gamma_{\lambda_0}$, we can apply the strong comparison principle (see, e.g., [21, Theorem 1.2]), ensuring that

$$\text{either } u_i \equiv u_{i,\lambda_0} \text{ or } u_i < u_{i,\lambda_0} \quad \text{in } \Omega_{\lambda_0} \setminus \Gamma_{\lambda_0}. \tag{3.4}$$

On the other hand, since $\mathbf{u}$ solves ($\mathcal{S}$), we have $u_i - u_{i,\lambda_0} = -u_{i,\lambda_0} < 0$ on $\partial\Omega_{\lambda_0} \cap \partial\Omega$. This immediately gives (3.3), since the alternative $u_i \equiv u_{i,\lambda_0}$ in $\Omega_{\lambda_0} \setminus \Gamma_{\lambda_0}$ obviously cannot be achieved because of the Dirichlet boundary condition and the fact that $u_i > 0$ in $\Omega \setminus \Gamma$.

Now we have fully established (3.3), and we can continue with the proof of (3.1). To begin with, we arbitrarily fix a compact set $\mathcal{K} \subseteq \Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$ and we observe that, since $R_{\lambda_0}(\Gamma)$ is compact, we can find $\tau_0 = \tau_0(\mathcal{K}) > 0$ so small that

$$\mathcal{K} \subseteq \Omega_\lambda \setminus R_\lambda(\Gamma) \quad \text{for every } \lambda_0 \leq \lambda \leq \lambda_0 + \tau_0.$$

Moreover, since $\mathbf{u}$ is continuous on $\mathcal{K}$, a simple uniform-continuity argument based on (3.3) shows that, by possibly shrinking τ_0, we also have (for all $i = 1, \dots, m$)

$$u_i < u_{i,\lambda} \text{ on } \mathcal{K} \quad \text{for every } \lambda_0 \leq \lambda \leq \lambda_0 + \tau_0. \tag{3.5}$$

We now turn to prove that, for every $\lambda_0 \leq \lambda \leq \lambda_0 + \tau_0$ and every $i = 1, \dots, m$, one can find a constant $C_i > 0$, depending on p_i, q_i, a_i, f_i, λ, Ω, and M, with

$$M = M_{\mathbf{u}} := \max_{1 \leq j \leq m} \big(\|u_j\|_{L^\infty(\overline{\Omega}_{\lambda_0+\tau_0})} + \|\nabla u_j\|_{L^\infty(\overline{\Omega}_{\lambda_0+\tau_0})} \big) < +\infty,$$

such that

$$\int\limits_{\Omega_\lambda \setminus \mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \, dx \leq C \sum_{j=1}^m \int\limits_{\Omega_\lambda \setminus \mathcal{K}} |w_{j,\lambda}^+|^2 \, dx. \tag{3.6}$$

Taking (3.6) for granted for a moment, let us show how this integral estimate can be used in order to prove (3.1). First of all, by taking the sum in (3.6) for $i = 1, \dots, m$, we get

$$\sum_{i=1}^m \int\limits_{\Omega_\lambda \setminus \mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \, dx \leq C' \sum_{i=1}^m \int\limits_{\Omega_\lambda \setminus \mathcal{K}} |w_{i,\lambda}^+|^2 \, dx. \tag{3.7}$$

Now, we have to distinguish the singular case from the degenerate one. To this end, let us suppose (up to a rearrangement of the sum on both sides of inequality (3.7)) that

$$\frac{(2N+2)}{(N+2)} < p_i \leq 2 \quad \text{for every } i = 1, \dots, m',$$

for some $1 \leq m' \leq m$. In this case, we have that $p_i^* > 2$. Applying the Hölder inequality with conjugate exponents $((p_i^* - 2)/p_i^*, p_i^*/2)$ and the Sobolev inequality to the first m' terms of the right-hand side of (3.7), we deduce that

$$\begin{aligned}
&\sum_{i=1}^{m'} \int_{\Omega_\lambda \setminus \mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2\, dx + \sum_{i=m'+1}^{m} \int_{\Omega_\lambda \setminus \mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2\, dx \\
&\quad \leq C \sum_{i=1}^{m'} \int_{\Omega_\lambda \setminus \mathcal{K}} |w_{i,\lambda}^+|^2\, dx + C \sum_{i=m'+1}^{m} \int_{\Omega_\lambda \setminus \mathcal{K}} |w_{i,\lambda}^+|^2\, dx \\
&\quad \leq C \sum_{i=1}^{m'} |\Omega_\lambda \setminus \mathcal{K}|^{\frac{p_i^*-2}{p_i^*}} \Big(\int_{\Omega_\lambda \setminus \mathcal{K}} |w_{i,\lambda}^+|^{p_i^*}\, dx \Big)^{\frac{2}{p_i^*}} + C \sum_{i=m'+1}^{m} \int_{\Omega_\lambda \setminus \mathcal{K}} |w_{i,\lambda}^+|^2\, dx \\
&\quad \leq C \sum_{i=1}^{m'} |\Omega_\lambda \setminus \mathcal{K}|^{\frac{p_i^*-2}{p_i^*}} \Big(\int_{\Omega_\lambda \setminus \mathcal{K}} |\nabla w_{i,\lambda}^+|^{p_i}\, dx \Big)^{\frac{2}{p_i}} + C \sum_{i=m'+1}^{m} \int_{\Omega_\lambda \setminus \mathcal{K}} |w_{i,\lambda}^+|^2\, dx.
\end{aligned} \tag{3.8}$$

In the first m' terms of the right-hand side of (3.8) we apply Lemma 2.4, while in the last terms (from $m' + 1$ to m) of the same inequality, since $p_i \geq 2$, we do apply the weighted Sobolev inequality [21, Theorem 2.3]. Hence, we obtain

$$\begin{aligned}
&\sum_{i=1}^{m'} \int_{\Omega_\lambda \setminus \mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2\, dx + \sum_{i=m'+1}^{m} \int_{\Omega_\lambda \setminus \mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2\, dx \\
&\quad \leq C \sum_{i=1}^{m'} |\Omega_\lambda \setminus \mathcal{K}|^{\frac{p_i^*-2}{p_i^*}} \int_{\Omega_\lambda \setminus \mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2\, dx \\
&\qquad + C \sum_{i=m'+1}^{m} C_P(\Omega_\lambda \setminus \mathcal{K}) \int_{\Omega_\lambda \setminus \mathcal{K}} |\nabla u_i|^{p_i-2} |\nabla w_{i,\lambda}^+|^2\, dx \\
&\quad \leq C \sum_{i=1}^{m'} |\Omega_\lambda \setminus \mathcal{K}|^{\frac{p_i^*-2}{p_i^*}} \int_{\Omega_\lambda \setminus \mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2\, dx \\
&\qquad + C \sum_{i=m'+1}^{m} C_P(\Omega_\lambda \setminus \mathcal{K}) \int_{\Omega_\lambda \setminus \mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2\, dx,
\end{aligned} \tag{3.9}$$

where $C_P(\Omega_\lambda \setminus \mathcal{K})$ is the Poincaré constant that tends to zero, when the Lebesgue measure $|\Omega_\lambda \setminus \mathcal{K}|$ tends to zero. From (3.9), up to redefining constants, we have

$$\sum_{i=1}^{m} \int_{\Omega_\lambda \setminus \mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2\, dx \leq C(\Omega_\lambda \setminus \mathcal{K}) \sum_{i=1}^{m} \int_{\Omega_\lambda \setminus \mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2\, dx, \tag{3.10}$$

where $C(\Omega_\lambda \setminus \mathcal{K})$ tends to zero when the Lebesgue measure $|\Omega_\lambda \setminus \mathcal{K}|$ tends to zero. Now, we choose the compact set $\mathcal{K}$ sufficiently large such that

$$C(\Omega_\lambda \setminus \mathcal{K}) < 1.$$

From this fact, we immediately deduce that $u_i \leq u_{i,\lambda}$ for each $\lambda_0 < \lambda \leq \lambda_0 + \tau$, and this gives a contradiction with the definition of λ_0. Therefore $\lambda_0 = 0$, and (i) is proved. Since the moving plane procedure can be performed in the same way but in the opposite direction, this proves the desired symmetry result. To prove (ii), we observe that the monotonicity of the solution is in fact implicit in the moving plane method, and in particular we get that $\partial_{x_1} u_i \geq 0$ in Ω_0. To get (1.3) it is sufficient to apply the strong maximum principle [16, Theorem 2.3].

Hence, we are left to prove (3.6). To this end, for every *fixed* $\lambda \in (\lambda_0, \lambda_0 + \tau_0]$ we choose an open neighborhood $\mathcal{O}_\lambda \subseteq \Omega_\lambda \setminus \mathcal{K}$ of $\Gamma_\lambda = R_\lambda(\Gamma)$, and a *cut-off family* $\{\psi_\varepsilon\}_{\varepsilon<\varepsilon_0}$ for Γ_λ related with $\mathcal{O}_\lambda$. This means, precisely, that the following assertions hold:

(i) $\psi_\varepsilon \in \operatorname{Lip}(\mathbb{R}^N)$ and $0 \leq \psi_\varepsilon \leq 1$ pointwise in $\mathbb{R}^N$.

(ii) There exists an open neighborhood $\mathcal{V}_\varepsilon^\lambda \subseteq \mathcal{O}_\lambda$ of Γ_λ such that

$$\psi_\varepsilon \equiv 0 \quad \text{on } \mathcal{V}_\varepsilon^\lambda.$$

(iii) $\psi_\varepsilon(x) \to 1$ as $\varepsilon \to 0^+$ for a.e. $x \in \mathbb{R}^N$.
(iv) There exists a constant $C_0 > 0$, independent of ε, such that

$$\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^{p_i}\,dx \le C_0\varepsilon.$$

We also fix $i \in \{1,\dots,m\}$, and we distinguish two cases:
(i) $q_i < p_i$.
(ii) $q_i = p_i$.

Case (i). In this case, for every $\varepsilon \in (0,\varepsilon_0)$ we consider the map

$$\varphi_{i,\varepsilon}(x) := \begin{cases} w_{i,\lambda}^+(x)\psi_\varepsilon^{p_i}(x) = (u_i - u_{i,\lambda})^+(x)\psi_\varepsilon^{p_i}(x) & \text{if } x \in \Omega_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

As already recognized in Lemma 2.3, $\varphi_{i,\varepsilon}$ satisfies the following properties:
(i) $\varphi_{i,\varepsilon} \in \operatorname{Lip}(\mathbb{R}^N)$.
(ii) $\operatorname{supp}(\varphi_{i,\varepsilon}) \subseteq \Omega_\lambda$ and $\varphi_{i,\varepsilon} \equiv 0$ near Γ_λ.
Moreover, since we know that $u_i < u_{i,\lambda}$ on $\mathcal{K}$, we also have

$$w_{i,\lambda}^+ = (u_i - u_{i,\lambda})^+ \equiv 0 \quad \text{on } \mathcal{K}. \tag{3.11}$$

As a consequence, a standard density argument allows us to use $\varphi_{i,\varepsilon}$ as a test function *both in* (1.2) *and* (2.7). From (3.11), we obtain

$$\begin{aligned}\int_{\Omega_\lambda\setminus\mathcal{K}} \langle |\nabla u_i|^{p_i-2}\nabla u_i - |\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda}, \nabla\varphi_{i,\varepsilon}\rangle\,dx + \int_{\Omega_\lambda\setminus\mathcal{K}} (a_i(u_i)|\nabla u_i|^{q_i} - a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{q_i})\varphi_{i,\varepsilon}\,dx \\ = \int_{\Omega_\lambda\setminus\mathcal{K}} (f_i(\mathbf{u}) - f_i(\mathbf{u}_\lambda))\varphi_{i,\varepsilon}\,dx.\end{aligned}$$

Starting from this identity, and proceeding *exactly* as in case (i) of the proof of Lemma 2.3, we obtain the following estimate, which is the analogue of (2.36):

$$\int_{\Omega_\lambda\setminus\mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\,dx \le C\sum_{j=1}^m \int_{\Omega_\lambda\setminus\mathcal{K}} |w_{j,\lambda}^+|^2\,dx. \tag{3.12}$$

Here, $C > 0$ is a suitable constant depending on $p_i, q_i, a_i, f_i, \lambda, \Omega, M$, with

$$M = M_{\mathbf{u}} := \max_{1\le j\le m}\big(\|u_j\|_{L^\infty(\overline{\Omega}_{\rho+\tau_0})} + \|\nabla u_j\|_{L^\infty(\overline{\Omega}_{\rho+\tau_0})}\big) < +\infty.$$

But, in this case, (3.12) coincides with (3.6).

Case (ii). In this case, for every $\varepsilon \in (0,\varepsilon_0)$ we consider the maps

$$\varphi_{i,\varepsilon}^{(1)}(x) := \begin{cases} e^{-A_i(u_i(x))}w_{i,\lambda}^+(x)\psi_\varepsilon^{p_i}(x) & \text{if } x \in \Omega_\lambda, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_{i,\varepsilon}^{(2)}(x) := \begin{cases} e^{-A_i(u_{i,\lambda}(x))}w_{i,\lambda}^+(x)\psi_\varepsilon^{p_i}(x) & \text{if } x \in \Omega_\lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$A_i(t) := \int_0^t a_i(s)\,ds.$$

Also in the present case, we have already recognized in the proof of Lemma 2.3 that $\varphi_{i,\varepsilon}^{(1)}, \varphi_{i,\varepsilon}^{(2)}$ satisfy the following properties:

(i) $\varphi_{i,\varepsilon}^{(1)}, \varphi_{i,\varepsilon}^{(2)} \in \operatorname{Lip}(\mathbb{R}^N)$.

(ii) $\operatorname{supp}(\varphi_{i,\varepsilon}^{(1)}) \subseteq \Omega_\lambda$, $\operatorname{supp}(\varphi_{i,\varepsilon}^{(2)}) \subseteq \Omega_\lambda$ and $\varphi_{i,\varepsilon}^{(1)} \equiv \varphi_{i,\varepsilon}^{(2)} \equiv 0$ near Γ_λ.

As a consequence, a standard density argument shows that it is possible to use $\varphi_{i,\varepsilon}^{(1)}, \varphi_{i,\varepsilon}^{(2)}$ as test functions in (1.2) and (2.7), respectively. By subtracting the resulting identities, and by taking into account (3.11), we then obtain

$$\begin{aligned}\int_{\Omega_\lambda\setminus\mathcal{K}} \langle|\nabla u_i|^{p_i-2}\nabla u_i, \nabla\varphi_{i,\varepsilon}^{(1)}\rangle\,dx - \int_{\Omega_\lambda\setminus\mathcal{K}} \langle|\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda}, \nabla\varphi_{i,\varepsilon}^{(2)}\rangle\,dx \\ + \int_{\Omega_\lambda\setminus\mathcal{K}} a_i(u_i)|\nabla u_i|^{p_i}\varphi_{i,\varepsilon}^{(1)}\,dx - \int_{\Omega_\lambda\setminus\mathcal{K}} a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{p_i}\varphi_{i,\varepsilon}^{(2)}\,dx \\ = \int_{\Omega_\lambda\setminus\mathcal{K}} f_i(\mathbf{u})\varphi_{i,\varepsilon}^{(1)}\,dx - \int_{\Omega_\lambda\setminus\mathcal{K}} f_i(\mathbf{u}_\lambda)\varphi_{i,\varepsilon}^{(2)}\,dx.\end{aligned}$$

Starting from this identity, and proceeding *exactly* as in case (ii) of the proof of Lemma 2.3, we obtain the following estimate, which is again the analogue of (2.36):

$$\int_{\Omega_\lambda\setminus\mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2\,dx \le C\sum_{j=1}^{m}\int_{\Omega_\lambda\setminus\mathcal{K}} |w_{j,\lambda}^+|^2\,dx. \tag{3.13}$$

Here $C > 0$ is a suitable constant depending on $p_i, q_i, a_i, f_i, \lambda, \Omega, M$, with

$$M = M_{\mathbf{u}} := \max_{1\le j\le m}\big(\|u_j\|_{L^\infty(\overline{\Omega}_{\rho+\tau_0})} + \|\nabla u_j\|_{L^\infty(\overline{\Omega}_{\rho+\tau_0})}\big) < +\infty.$$

But (3.13) coincides with (3.6) in this case. □

Acknowledgment: We would like to thank the anonymous referee for the careful reading of our manuscript.

Funding: The authors are members of INdAM. F. Esposito and L. Montoro are partially supported by PRIN project 2017JPCAPN (Italy): Qualitative and quantitative aspects of nonlinear PDEs. L. Montoro is partially supported by Agencia Estatal de Investigación (Spain): project PDI2019-110712GB-100.

References

[1] A. D. Alexandrov, A characteristic property of spheres, *Ann. Mat. Pura Appl. (4)* **58** (1962), 303–315.

[2] H. Berestycki and L. Nirenberg, On the method of moving planes and the sliding method, *Bol. Soc. Brasil. Mat. (N. S.)* **22** (1991), no. 1, 1–37.

[3] S. Biagi, E. Valdinoci and E. Vecchi, A symmetry result for cooperative elliptic systems with singularities, *Publ. Mat.* **64** (2020), no. 2, 621–652.

[4] F. Brock, Continuous rearrangement and symmetry of solutions of elliptic problems, *Proc. Indian Acad. Sci. Math. Sci.* **110** (2000), no. 2, 157–204.

[5] J. Busca and B. Sirakov, Symmetry results for semilinear elliptic systems in the whole space, *J. Differential Equations* **163** (2000), no. 1, 41–56.

[6] L. Caffarelli, Y. Y. Li and L. Nirenberg, Some remarks on singular solutions of nonlinear elliptic equations. II. Symmetry and monotonicity via moving planes, in: *Advances in Geometric Analysis*, Adv. Lect. Math. (ALM) 21, International Press, Somerville (2012), 97–105.

[7] P. Clément, R. Manásevich and E. Mitidieri, Positive solutions for a quasilinear system via blow up, *Comm. Partial Differential Equations* **18** (1993), no. 12, 2071–2106.

[8] L. Damascelli, Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **15** (1998), no. 4, 493–516.

[9] L. Damascelli and F. Pacella, Monotonicity and symmetry of solutions of p-Laplace equations, 1<; p<; 2, via the moving plane method, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **26** (1998), no. 4, 689–707.

[10] L. Damascelli and B. Sciunzi, Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations, *J. Differential Equations* **206** (2004), no. 2, 483–515.
[11] L. Damascelli and B. Sciunzi, Harnack inequalities, maximum and comparison principles, and regularity of positive solutions of m-Laplace equations, *Calc. Var. Partial Differential Equations* **25** (2006), no. 2, 139–159.
[12] E. N. Dancer, Moving plane methods for systems on half spaces, *Math. Ann.* **342** (2008), 245–254.
[13] D. G. de Figueiredo, Monotonicity and symmetry of solutions of elliptic systems in general domains, *NoDEA Nonlinear Differential Equations Appl.* **1** (1994), no. 2, 119–123.
[14] D. G. De Figueiredo and J. Yang, Decay, symmetry and existence of solutions of semilinear elliptic systems, *Nonlinear Anal.* **33** (1998), no. 3, 211–234.
[15] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.* **7** (1983), no. 8, 827–850.
[16] F. Esposito, S. Merchán and L. Montoro, Symmetry results for p-Laplacian systems involving a first-order term, *Ann. Mat. Pura Appl. (4)* **200** (2021), no. 5, 2313–2331.
[17] F. Esposito, L. Montoro and B. Sciunzi, Monotonicity and symmetry of singular solutions to quasilinear problems, *J. Math. Pures Appl. (9)* **126** (2019), 214–231.
[18] B. Gidas, W. M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), no. 3, 209–243.
[19] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Math. Monogr., Oxford University, New York, 1993.
[20] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12** (1988), no. 11, 1203–1219.
[21] L. Montoro, Harnack inequalities and qualitative properties for some quasilinear elliptic equations, *NoDEA Nonlinear Differential Equations Appl.* **26** (2019), no. 6, Paper No. 45.
[22] P. Poláčik, P. Quittner and P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems, *Duke Math. J.* **139** (2007), no. 3, 555–579.
[23] B. Sciunzi, On the moving plane method for singular solutions to semilinear elliptic equations, *J. Math. Pures Appl. (9)* **108** (2017), no. 1, 111–123.
[24] J. Serrin, A symmetry problem in potential theory, *Arch. Ration. Mech. Anal.* **43** (1971), 304–318.
[25] J. Serrin and H. Zou, Symmetry of ground states of quasilinear elliptic equations, *Arch. Ration. Mech. Anal.* **148** (1999), no. 4, 265–290.
[26] S. Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, *Adv. Differential Equations* **1** (1996), no. 2, 241–264.
[27] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations* **51** (1984), no. 1, 126–150.
[28] W. C. Troy, Symmetry properties in systems of semilinear elliptic equations, *J. Differential Equations* **42** (1981), no. 3, 400–413.
[29] N. S. Trudinger, Linear elliptic operators with measurable coefficients, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (3)* **27** (1973), 265–308.