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Robustness of Solutions to the Capacitated Facility Location Problem with Uncertain Demand

Roberto Baldacci · Marco Caserta · Emiliano Traversi · Roberto Wolfler Calvo

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Abstract We investigate properties of robust solutions of the Capacitated Facility Location Problem with uncertain demand with different uncertainty sets. We show that the monotonic behavior of the price of robustness is not guaranteed, and that one cannot discriminate among alternative robust solutions by simply relying on the trade-off price-vs-robustness. Furthermore, we report a computational study on benchmark instances from the literature and on instances derived from a real-world application, which demonstrates the validity of our findings.

Keywords integer programming · robust optimization · network/graphs: applications

1 Introduction

The Capacitated Facility Location Problem (CFLP) (Rosenwein 1994; Daskin 1995) considered in this paper can be described as follows. We are given a

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set M of potential facility locations and a set N of customers. Associated with each $i \in M$ is a maximum capacity of the facility s_i, and each user $j \in N$ is associated with a demand a_j. Two type of costs arise: (i) the decision to establish a facility at i incurs a fixed-charge (setup) cost d_i and (ii) for $i \in M$ and $j \in N$, a unitary transportation cost c_{ij} for serving the demand of customer j from facility i must be paid. The problem consists of minimizing the sum of the setup costs of opened facilities and of the transportation costs, while satisfying demand requirements and capacity constraints. The CFLP can be mathematically formulated as follows:

$$
\begin{aligned}
\text{(CFLP)} \quad \min \quad & \sum_{i \in M} d_i y_i + \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i \in M} x_{ij} = 1, && \forall j \in N, && (1) \\
& \sum_{j \in N} a_j x_{ij} \leq s_i y_i, && \forall i \in M, && (2) \\
& x \in X, && && (3) \\
& y_i \in \{0, 1\}, && \forall i \in M, && (4)
\end{aligned}
$$

where $X \subseteq \mathbb{R}_+^{m \times n}$ or $X = \{0,1\}^{m \times n}$. Problem $CFLP$ deals with the following two-level decision variables: (i) *strategic* variables $\mathbf{y}$, used to model the long-term decisions, and (ii) *operational* variables $\mathbf{x}$, which model the short-term, operational part of the problem. In the literature, there are two main variants of the CFLP: the single source CFLP (SS-CFLP), where a customer must be completely assigned to a single facility, and the multiple source CFLP (MS-CFLP) where a customer could be assigned to several facilities (Laporte et al. 2015). Several applications in transportation, logistics, telecommunications, and production planning, such as capacitated network design problems (Gendron et al. 1999), can be formulated using models similar to model $CFLP$.

We assume that the demand coefficients a_j are *uncertain*, and that the model of data uncertainty is based upon the following *scenario space*.

Definition 1 Let each coefficient $a_j \geq 0$ be an (independent), symmetric, and bounded random variable $\tilde{a}_j$ taking values in the interval $\bar{a}_j \pm \varepsilon \bar{a}_j$, i.e., $\bar{a}_j(1-\varepsilon) \leq \tilde{a}_j \leq \bar{a}_j(1+\varepsilon)$ with $\varepsilon > 0$ and $\bar{a}_j$ corresponding to the nominal value of the uncertain parameter. We define the *scenario space* $U(\varepsilon)$ as follows:

$$
U(\varepsilon) = \{\tilde{a}_j \in \mathbb{R}_+ : \bar{a}_j(1-\varepsilon) \leq \tilde{a}_j \leq \bar{a}_j(1+\varepsilon), \forall j \in N\}.
$$

Therefore, the scenario space is the uncertainty set within which the decision maker hypothesizes every realization of the uncertain parameter will occur. In addition, we assume that the uncertainty can affect the objective function of problem $CFLP$, as the cost vector $\mathbf{c}$ can be computed as a function of the uncertainty coefficients a_j.

One way of dealing with the uncertainty of the parameters of the problem is provided by Robust Optimization (RO) (Bertsimas and Sim 2004). RO ensures that a robust solution $(\mathbf{y}, \mathbf{x})$ be feasible with respect to every realization of the robust parameters in the pre-specified interval defined by ε. However, while in certain applications it might be of paramount importance to guarantee feasibility of the robust solution 100% of the times (e.g., in medical applications, or some critical engineering applications), in most realms of application of robust optimization, e.g., in management, it is not necessary to produce a solution that hedges against every possible realization of the uncertain parameters in the scenario space, since full immunization comes at a high cost in terms of objective function value. Therefore, while recognizing the nature of the data in a support or uncertainty set U which defines the scenario space, to avoid being overly conservative and, therefore, to mitigate the adverse effect of full immunization on the objective function value, one might want to optimize over a smaller support $\tilde{U} \subseteq U$. The rationale is that, if $\tilde{U}$ is properly crafted, one might obtain important benefits in terms of costs while keeping the risk of incurring infeasible scenarios very low.

Such strategy is especially appealing to the business community, where the fact that customers demands might vary does not typically have as drastic repercussions as in, e.g., medical or engineering applications. Therefore, in line with the reasoning put forth by (Bertsimas and Sim 2004), given a set of uncertain parameters taking values in U, the goal of a practitioner using a robust model could be to define a support $\tilde{U} \subseteq U$ and to find a robust solution such that (i) if nature selects a realization of the uncertain parameters from $\tilde{U}$, the solution is deterministically feasible, and (ii) if nature selects a realization of the parameters in $U \setminus \tilde{U}$, the probability of incurring an infeasible scenario is still very low.

Our distinct contributions in this paper are as follows: (i) It is the first time that a limitation/drawback associated with the price of robustness is identified for the CFLP: the monotonicy of the trade-off price-vs-robustness cannot be guaranteed *a priori*. Furthermore, we show that there always exists a situation in which a nominal solution can be more robust, and cheaper, than a robust solution, (ii) We conduct an extensive computational study on both real-world and benchmark instances, to ascertain the significance of such phenomenon. Empirical evidence suggests that such phenomenon cannot be neglected by the decision maker, since it has a significant impact in terms of costs and reliability of the implemented solution.

The remainder of the paper is organized as follows. The next section motivate our study by means of an introductory example. Section 2 reviews contributions related to the optimization problems addressed in this paper. The main results of this paper are given in Section 3, where the properties of nominal and robust solutions are investigated. Section 4 reports the computational studies. Finally, in Section 5, managerial insights are discussed, together with future research directions.

1.1 Motivation and introductory example

The work in this paper is also motivated by a real-world application from a major dairy company whose core business is the production and distribution of perishable products (fresh milk, cheese, and butter, to name a few). Our study addresses a strategic problem of defining the partition of the customers to a set of depots under uncertain customer demands. We consider the following robust formulation of SS-CFLP:

$$
\begin{aligned}
\text{(R-SS-CFLP)} \min & \sum_{i \in M} d_i y_i + \max_{\mathbf{a} \in U(\varepsilon)} \sum_{i \in M} \sum_{j \in N} (a_j \bar{c}_{ij}) x_{ij} \\
\text{s.t.} \quad & \sum_{j \in N} a_j x_{ij} \leq s_i y_i, \quad \forall i \in M, \mathbf{a} \in U(\varepsilon), \qquad (5) \\
& (1), (4), x_{ij} \in \{0,1\}, \forall i \in M, j \in N,
\end{aligned}
$$

where $\mathbf{a} \in \mathbb{R}^n_+$, and $U(\varepsilon) \subseteq \mathbb{R}^n_+$ denotes the support (box uncertainty set) at hand. We then consider the following experimental setup.

(i) *Optimization over the support* $U(\varepsilon)$. We solve the robust counterpart of formulation SS-CFLP with box support $U(\varepsilon)$, with $\varepsilon \in \{0.1, 0.2, \ldots, 0.9\}$, and we obtain the corresponding robust solution $(\mathbf{y}, \mathbf{x})^{BOX}_{\varepsilon}$ of cost $(z)^{BOX}_{\varepsilon}$. Clearly, we have $(z)^{BOX}_{\varepsilon''} \geq (z)^{BOX}_{\varepsilon'}$ if $\varepsilon'' > \varepsilon'$;

(ii) *Evaluation over the scenario space* $U(\hat{\varepsilon})$. We define a family of nested scenarios spaces $U(\hat{\varepsilon})$, with $\hat{\varepsilon} \in \{0.1, 0.2, \ldots, 0.9\}$, and for each $\hat{\varepsilon}$ we generate 1,000,000 realizations of coefficients a_j and we evaluate, in terms of violation of constraints (5), the solutions $(\mathbf{y}, \mathbf{x})^{BOX}_{\varepsilon}$.

Figure 1 presents the results of the aforementioned experiment on a SS-CFLP real-world instance (see instance A of Section 4). On the horizontal axis, we indicate the size of the support used for the optimization step (ε), and on the vertical axis we report the number of infeasible scenarios evaluated by using the robust solution $(\mathbf{y}, \mathbf{x})^{BOX}_{\varepsilon}$ over the nested scenarios spaces ($\hat{\varepsilon}$).

In the figure, consider the pair of values $\varepsilon = 0.45$ and $\varepsilon = 0.5$, for the line given by $\hat{\varepsilon} = 0.85$. We observe that the robust solution associated to the smaller support, i.e., $(\mathbf{y}, \mathbf{x})^{BOX}_{0.45}$, is more robust than the solution associated to the larger uncertainty set, i.e., $(\mathbf{y}, \mathbf{x})^{BOX}_{0.5}$. A similar behaviour can also be observed for other values of ε and lines given by $\hat{\varepsilon}$. This empirical observation poses a problem when it comes to comparing and evaluating two robust solutions using the "price of robustness" (Bertsimas and Sim 2004) as a proxy: due to the fact that the monotonic behavior of the curve is not guaranteed, one cannot discriminate among alternative robust solutions by simply relying on the trade-off price-vs-robustness. As evinced from Figure 1, a more expensive solution, obtained over a larger support, could be less robust than a cheaper solution, obtained over a smaller support.

In the sequel, we will show that, for the robust formulation of problem $CFLP$, it is always possible to find an instance of the problem and a realization of the uncertain parameters such that the nominal, i.e., non-robust, solution

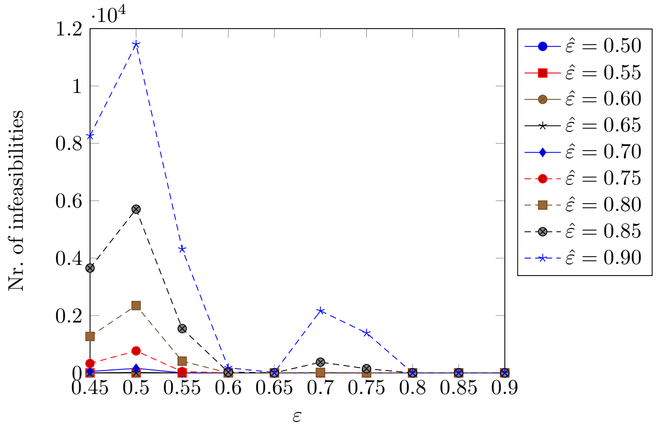

Fig. 1 R-SS-CFLP: example about the real-world instance `A`.

is feasible while the robust solution is infeasible for that specific realization of the uncertain parameters. In addition, we will show that it is possible to prove that a robust solution obtained using a larger support could be less robust, i.e., more likely to incur violation of constraints (2), than a robust solution obtained using a smaller support of the same type.

2 Literature review

Robust optimization (RO) has received a considerable increase of interest over the last decade. There exists a vast literature on this topic, and for a detailed introduction to robust optimization, we refer the reader to (Ben-Tal et al. 2009) and (Bertsimas et al. 2011), whereas for an overview of developments in robust optimization since 2007 the reader is referred to (Gabrel et al. 2014) and (Lu and Shen 2020), where the latter provides a review with a special focus on the application of robust optimization to operations management. More recent works on robust optimization have developed solution frameworks to produce less conservative solutions such as two-stage RO and more general multi-stage RO, also known as robust adjustable or adaptable optimization (Ben-Tal et al. 2004).

When alternative robust solutions are compared, two prominent criteria have been proposed in the literature, the Pareto Robust Optimality criterion of (Iancu and Trichakis 2014) and the price of robustness of (Bertsimas and Sim 2004). With respect to the former, the idea is to exploit the degeneracy of many RO models and select a Pareto-optimal solution, i.e., a solution that is optimal with respect to the worst-case scenario, and is not dominated by any other solution for all the possible scenarios of the uncertainty set. For example, if the uncertainty of the problem is in the objective function, a robust solution $\mathbf{x}^*$ is Pareto-optimal if the cost associated to the worst case realization of the uncertain parameter is minimal and, in addition, no other solution is cheaper that $\mathbf{x}^*$ for any realization of the uncertain parameter in the uncertainty set. Therefore, the Pareto-optimality criterion focuses on the selection of alternative robust solution within the same uncertainty set. On the other hand, the price of robustness is employed to discriminate among alternative robust solutions obtained using different uncertainty sets. Thus, it considers the trade-off between the size of $\tilde{U}$, and therefore the value of the objective function, and the probability of violation of the uncertain constraints (2) has been employed to select and discriminate among alternative robust solutions. More specifically, as mentioned, one would expect a monotonically non-decreasing behavior of the price of robustness curve with respect to the size of the support $\tilde{U}$. In other words, if we assume that the size of $\tilde{U}$ is proportional to the value of ε, then the trade-off captured by the price of robustness states that a larger values of ε should lead to optimal solutions associated to an increase in the objective function value and a decrease in the probability of violation of constraints (2). Pareto-optimal solutions were also investigated by Chassein and Goerigk (2016). The authors considered linear programming problems with uncertain cost functions and presented a column generation approach that requires no direct solution of the computationally expensive worst-case problem.

It is worth pointing out that our study goes beyond the search of a Pareto Robust Optimal (PRO) solution, as done in (Iancu and Trichakis 2014) and Chassein and Goerigk (2016). A PRO solution is defined as a minimum cost solution which provides the maximum slack of the robust constraint (e.g., unused capacity in a facility location problem) for every realization of the uncertain parameter in the uncertainty set. Our approach departs from that of (Iancu and Trichakis 2014) since (i) their analysis is based on the fact that the uncertainty sets used in the optimization and evaluation phases are the same. We want to explore whether the price-of-robustness framework holds when the uncertainty set used in the evaluation phase is decoupled from the one used in the evaluation phase; (ii) our analysis is not limited to PRO solutions, but we extend the analysis by comparing solutions in terms of expected values of constraints violation, and (iii) we compare alternative robust solutions obtained over different uncertainty sets extending the reasoning to other measures to capture the robustness of a solution, which go beyond the slack measure proposed in (Iancu and Trichakis 2014).

Facility location problems find application in a number of realms, e.g., supply chain management, distribution system design and telecommunication network design, among others. (Melo et al. 2009) and (Laporte et al. 2015) present a comprehensive literature review of facility location models in the context of supply chain management.

Uncertain versions of the facility location problem consider different types of uncertainties (for instance, regarding demand, costs, and facility reliability), and mainly focus on the uncertainty of the demands. Different methods have been proposed to deal with uncertainty, among them RO. For a comprehensive overview of robust facility location approaches we direct the interested reader to (Snyder 2006) and (Correia and da Gama 2015) and references therein. A detailed review of facility location with uncertain parameters and their solution methods can also be found in (Laporte et al. 2015).

While the literature on the deterministic CFLP is vast, the robust version of the CFLP has not received much attention so far. (Opher et al. 2010) studied a variant of the multi-period CFLP in which the maximum capacity of each facility must be determined, and showed that the robust approach offers significant improvements when compared with the nominal solution. In a similar fashion, (Gülpinar et al. 2013) presented different robust models for the CFLP, with both known and ambiguous demand probability distribution function. Both papers ascertain the superiority of the robust solution over the deterministic one on an extensive set of benchmark instances. (An et al. 2014) investigated two-stage RO models for the reliable p-median facility location problem by considering two practical features, i.e., facility capacity and demand change due to site disruption. The authors designed an exact algorithm based on column-and-constraint generation and Bender decomposition methods, and solved to optimality instances with up to 49 sites. (Zeng and Zhao 2013) described a column-and-constraint generation algorithm to solve two-stage RO problems as an alternative to Benders-style cutting plane methods. The proposed solution framework was used to solve a two-stage robust location-transportation problem. Recently, (Du et al. 2020) have considered two-stage facility location problems in an uncertain and dynamic environment, aiming at building a network that serves demand in both general and disruptive situations. The paper compared a two-stage robust model and a two-stage stochastic model for the reliable p-center problem, and the experiments reported showed that the solutions produced by the two-stage robust model are not overly conservative.

3 Non-monotonicity of the price of robustness

In this section, we investigate properties of the nominal and robust solutions. We first show that, given an optimal robust solution over a box support $U(\varepsilon)$ (i.e., optimization scenario), it is possible to construct a scenario from an uncertainty set of size $U(\hat{\varepsilon}) \supset U(\varepsilon)$ (i.e., evaluation scenario) for which the nominal solution remains feasible while the robust solution becomes infeasible.

As defined in the introduction, we assume a pre-specified symmetric scenario space model of data uncertainty for the uncertain parameters of the problem. However, for optimization purposes, and to reduce the conservativism of a robust solution, we aim at finding an optimal robust solution on a support which is different from the scenario space hypothesized for the real data. More specifically, at the cost of incurring in potentially infeasible realizations of the uncertain parameters, we restrict the optimization process to all the realizations of the uncertain parameters a_j within a less conservative support or uncertainty set.

Definition 2 A polyhedral support for the uncertain parameters a_j is defined as:

$$Q(\varepsilon) = \{\mathbf{a} \geq 0 : W\mathbf{a}^\top \leq \mathbf{h}\},$$

where $W \in \mathbb{R}^{r\times n}$, and $\mathbf{h} = \mathbf{h}(\varepsilon) \in \mathbb{R}^r$ is increasing on the value of ε, i.e., the parameter which controls the size of the support and, therefore, the degree of immunization of the robust solution.

The robust counterpart of $CFLP$ can be derived by reformulating the semi-infinite constraints (5) as shown by the following theorem.

Theorem 1 (Ben-Tal and Nemirovski (2007)) *The robust counterpart of problem $CFLP$ can be formulated as a the following MILP:*

$$\begin{aligned}
\textit{(RP)} \quad \min \quad & \sum_{i\in M} d_i y_i + \sum_{i\in M}\sum_{j\in N} c_{ij}x_{ij} & \\
s.t. \quad & (1), (3), (4), & \\
& \sum_{t=1}^{r} h_t \psi_{it} \leq s_i y_i, & \forall i \in M, \\
& \sum_{t=1}^{r} w_{tj} \psi_{it} \geq x_{ij}, & \forall j \in N, i \in M, \\
& \psi_i \in \mathbb{R}^r_+, & \forall i \in M,
\end{aligned}$$

where $w_{tj} \in W$ and $h_t \in \mathbf{h}$, with W and $\mathbf{h}$ specified as in Definition 2.

We observe that the robust counterpart of $CFLP$ is still a MILP, i.e., it retains the same complexity of the original problem (albeit with a larger number of variables and constraints.)

Let $(\mathbf{y}^*, \mathbf{x}^*)$ be an optimal solution to the nominal problem $CFLP$ with nominal values $\bar{a}_{ij}$ and cost z^*. Moreover, let $(\mathbf{y}, \mathbf{x})^{BOX}_{\varepsilon}$ be an optimal solution to formulation RP over the box support $U(\varepsilon)$, $\varepsilon > 0$, of cost $(z)^{BOX}_{\varepsilon} > z^*$. The following theorem holds.

Theorem 2 *If solution $(\boldsymbol{y}, \boldsymbol{x})^{BOX}_{\varepsilon}$ is evaluated over a scenario space $U(\hat{\varepsilon})$ with $\hat{\varepsilon} > \varepsilon > 0$, there might exist a sample $[\hat{a}_j] \in U(\hat{\varepsilon})$ such that the nominal solution $(\boldsymbol{y}^*, \boldsymbol{x}^*)$ is feasible whereas solution $(\boldsymbol{y}, \boldsymbol{x})^{BOX}_{\varepsilon}$ is infeasible.*

Proof. The proof is by an example. Consider an instance of $CFLP$ where $m = 2$ (i.e., $M = \{1, 2\}$) and $n = 3$ (i.e., $N = \{1, 2, 3\}$) with $X \subseteq \mathbb{R}^{2\times 3}_+$, defined by the following parameters:

- $\bar{a}_1 = \bar{a}_2 = \bar{a}_3 = \bar{a}$; $d_1 = d_2$; $b_{ij} = 1, \forall i \in M, j \in N$, and $e_j = 1$, $\forall j \in N$;
- The cost matrix $[c_{ij}]$ is defined as follows: $\mathbf{c} = \begin{bmatrix} c_{11} & 0 & B \\ c_{21} & B & 0 \end{bmatrix}$ where $c_{11} < c_{21}$ and B is a large constant;
- $s_1 \geq 2\bar{a}$, $s_2 \geq 2\bar{a}$, $s_1 \geq \bar{a}(2 + \varepsilon)$, $s_1 < 2\bar{a}(1 + \varepsilon)$, $s_2 \geq \bar{a}(1 + \varepsilon)$.

Then, the nominal solution $(\mathbf{y}^*, \mathbf{x}^*)$ is $\mathbf{y}^* = \begin{bmatrix}1\\1\end{bmatrix}$, $\quad \mathbf{x}^* = \begin{bmatrix}1\ 1\ 0\\0\ 0\ 1\end{bmatrix}$ while the robust solution $(\mathbf{y}, \mathbf{x})^{BOX}_{\varepsilon}$ depends on ε and is $\mathbf{y} = \begin{bmatrix}1\\1\end{bmatrix}$, $\quad \mathbf{x} = \begin{bmatrix}\frac{s_1-\bar{a}(1+\varepsilon)}{\bar{a}(1+\varepsilon)} & 1\ 0\\ 1-\alpha & 0\ 1\end{bmatrix}$ where the term $x_{11} = \alpha = \frac{s_1-\bar{a}(1+\varepsilon)}{\bar{a}(1+\varepsilon)} < 1$. Clearly, $(z)^{BOX}_{\varepsilon} > z^*$.

Consider $\hat{\varepsilon}$ such that $\hat{\varepsilon} > \varepsilon$, then there exists a scenario $[\hat{a}_j] \in U(\hat{\varepsilon})$ which makes the robust solution $(\mathbf{y}, \mathbf{x})^{BOX}_{\varepsilon}$ infeasible while the nominal solution $(\mathbf{y}^*, \mathbf{x}^*)$ remains feasible. Consider $[\hat{a}_{ij}] = \left[\bar{a}(1+\delta)\ \bar{a}(1-\hat{\varepsilon})\ \right]$, with $0 \le \delta \le \hat{\varepsilon}$. Below, we aim to find a value of δ such that constraints (2) are satisfied by the nominal solution $\mathbf{x}^*$ for the scenario $[\hat{a}_j]$, that is the constraints with $i = 1, 2$ are satisfied by solution $\mathbf{x}^*$:

$$\begin{cases} x^*_{11}\bar{a}(1+\delta) + x^*_{12}\bar{a}(1-\hat{\varepsilon}) \le s_1 & (a)\\ x^*_{23}\bar{a}(1+\hat{\varepsilon}) \le s_2, & (b)\end{cases} \tag{6}$$

whereas the constraint (2) with $i = 2$ is violated by solution $\mathbf{x}$, i.e.:

$$x_{21}\bar{a}(1+\delta) + x_{23}\bar{a}(1+\hat{\varepsilon}) > s_2,$$

or, equivalently,

$$\begin{cases} \bar{a}(1+\delta) + \bar{a}(1-\hat{\varepsilon}) \le s_1,\\ \bar{a}(1+\hat{\varepsilon}) \le s_2,\\ (1 - \frac{s_1-\bar{a}(1+\varepsilon)}{\bar{a}(1+\varepsilon)})\bar{a}(1+\delta) + \bar{a}(1+\hat{\varepsilon}) > s_2,\end{cases}$$

which implies $\hat{\varepsilon} \le \frac{s_2-\bar{a}}{\bar{a}}$, and $\max\{0, \delta_L\} < \delta \le \min\{\hat{\varepsilon}, \delta_U\}$ with $\delta_U = \frac{s_1-\bar{a}(1-\hat{\varepsilon})}{\bar{a}} - 1$ and $\delta_L = \frac{s_2-\bar{a}(1+\hat{\varepsilon})}{2\bar{a}(1+\varepsilon)-s_1}(1+\varepsilon) - 1$.

Note that if constraint (a) (case $i = 1$) of inequalities (6) is satisfied, then also constraint $x_{11}\bar{a}(1+\delta) + x_{12}\bar{a}(1-\hat{\varepsilon}) \le s_1$ is satisfied by any $\mathbf{x}$ solution since $x_{12} = x^*_{12}$ and $x_{13} = 0$ and $x_{11} < 1$.

Since $s_1 \ge 2\bar{a}$, we have $s_1 - \bar{a}(1-\hat{\varepsilon}) > \bar{a}$, hence $\delta_U > 0$. Sufficient conditions for the existence of a set of values of δ such that the statement of the theorem is verified are:

$$\begin{cases} s_1 - \bar{a}(1-\hat{\varepsilon}) \ge [s_2 - \bar{a}(1+\hat{\varepsilon})]\,(1+\varepsilon)\\ \bar{a} \le 2\bar{a}(1+\varepsilon) - s_1\end{cases},$$

which lead to $\varepsilon \ge \frac{s_1-\bar{a}}{2\bar{a}}$, $\hat{\varepsilon} > \varepsilon$, $\hat{\varepsilon} \le \frac{s_2-\bar{a}}{\bar{a}}$, and $\hat{\varepsilon} \ge \frac{s_2(1+\varepsilon)-\bar{a}\varepsilon-s_1}{\bar{a}(2+\varepsilon)}$.

The choice of values of ε and $\hat{\varepsilon}$ in the halfspaces derived above guarantees that $\delta_L \le \delta_U$. Thus, a suitable value of δ exists. $\Box$

As previously stated, consider a nominal optimal solution $(\mathbf{y}^*, \mathbf{x}^*)$ and an optimal solution $(\mathbf{y}', \mathbf{x}')^{BOX}_{\varepsilon'}$ to RP over the box support $U(\varepsilon')$ with cost $(z')^{BOX}_{\varepsilon'} > z^*$. In the following, we show that by optimizing RP on a larger support $U(\varepsilon'')$ (with $\varepsilon'' > \varepsilon'$) does not necessarily provide a solution which is more robust than the solution associated with $U(\varepsilon')$, if the solutions are evaluated over a scenario space $U(\hat{\varepsilon})$, with $U(\hat{\varepsilon}) \supset U(\varepsilon'') \supset U(\varepsilon')$. Indeed, the following theorem holds.

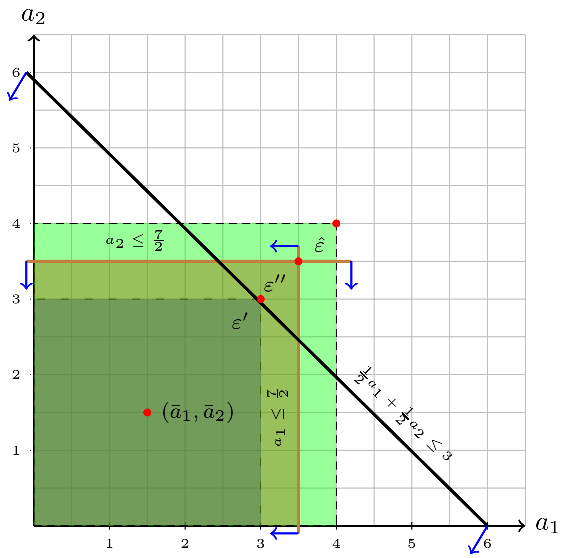

Fig. 2 Representation of Theorem 3.

Theorem 3 *Let $(\boldsymbol{y}'', \boldsymbol{x}'')^{BOX}_{\varepsilon''}$ be an optimal robust solution of RP over the box support $U(\varepsilon'')$, with $\varepsilon'' > \varepsilon'$. If solution $(\boldsymbol{y}'', \boldsymbol{x}'')^{BOX}_{\varepsilon''}$ is evaluated over a scenario space $U(\hat{\varepsilon})$ with $U(\hat{\varepsilon}) \supset U(\varepsilon'') \supset U(\varepsilon')$, and the uncertain parameters a_{ij} are uniformly distributed in $U(\hat{\varepsilon})$, then the violation probability of solution $(\boldsymbol{y}'', \boldsymbol{x}'')^{BOX}_{\varepsilon''}$ can be strictly greater than the violation probability of solution $(\boldsymbol{y}', \boldsymbol{x}')^{BOX}_{\varepsilon'}$ and the ratio between solution costs $(z')^{BOX}_{\varepsilon'}$ and $(z'')^{BOX}_{\varepsilon''}$ can be a very small number.*

Proof. The proof is by an example. Consider an instance of $CFLP$ where $m = 3$ (i.e., $M = \{1, 2, 3\}$) and $n = 2$ (i.e., $N = \{1, 2\}$) with $X \subseteq \mathbb{R}_+^{2\times 3}$, defined by the following parameters:

- The nominal values are $\bar{a}_1 = \frac{3}{2}$ and $\bar{a}_2 = \frac{3}{2}$; $b_{ij} = 1, \forall i \in M, j \in N$, and $e_j = 1$, $\forall j \in M$; $d_1 < d_2 < d_3$; $s_1 = s_2 = s_3 = 3$.
- The cost matrix $[c_{ij}]$ is defined as follows: $\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ B & c_{32} \end{bmatrix}$ where $c_{11} = c_{12}$, $c_{21} = c_{22}$, $c_{11} < c_{21}$, $c_{12} < c_{22}$, $c_{32} < c_{22}$, $c_{12} < c_{32} + d_3$ and B is a large constant;

The nominal solution $(\mathbf{y}^*, \mathbf{x}^*)$ is $\mathbf{y}^* = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathbf{x}^* = \begin{bmatrix}1 & 1\\0 & 0\\0 & 0\end{bmatrix}$. Consider $\varepsilon' = 1$; then the robust optimal solution $(\mathbf{y}', \mathbf{x}')^{BOX}_{\varepsilon'}$ is $\mathbf{y}' = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{x}' = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2}\\0 & 0\end{bmatrix}$. Clearly, $(z')^{BOX}_{\varepsilon'} > z^*$.

Consider now a value $\varepsilon'' > \varepsilon'$, with $\varepsilon'' = \frac{4}{3}$; we have $U(\varepsilon'') \supset U(\varepsilon')$. For the support $U(\varepsilon'')$, a corresponding optimal solution $(\mathbf{y}'', \mathbf{x}'')^{BOX}_{\varepsilon''}$ is $\mathbf{y}'' = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \mathbf{x}'' = \begin{bmatrix}\frac{6}{7} & 0\\\frac{1}{7} & \frac{1}{7}\\0 & \frac{6}{7}\end{bmatrix}$. Let $(z'')^{BOX}_{\varepsilon''}$ be the cost of solution $(\mathbf{y}'', \mathbf{x}'')^{BOX}_{\varepsilon''}$. We have $(z'')^{BOX}_{\varepsilon''} > (z')^{BOX}_{\varepsilon'} > z^*$. Note that, if $d_3 \gg d_2$, the ratio between solution costs $(z')^{BOX}_{\varepsilon'}$ and $(z'')^{BOX}_{\varepsilon''}$ can be a very small number.

For solution $(\mathbf{y}', \mathbf{x}')^{BOX}_{\varepsilon'}$ constraints (2) for $i = 1, 2$ correspond to: $\frac{1}{2}a_1 + \frac{1}{2}a_2 \leq 3$, whereas the constraint for $i = 3$ is redundant. For solution $(\mathbf{y}'', \mathbf{x}'')^{BOX}_{\varepsilon''}$ constraints (2) are as follows:

$$i = 1 : \frac{6}{7}a_1 \leq 3; i = 2 : \frac{1}{7}a_1 + \frac{1}{7}a_2 \leq 3; i = 3 : \frac{6}{7}a_2 \leq 3,$$

where the constraint for $i = 2$ is dominated by the constraints for $i = 1, 3$. Figure 2 illustrates the different regions corresponding to the above inequalities. Consider a value $\hat{\varepsilon} > \varepsilon''$, such that $U(\hat{\varepsilon}) \supset U(\varepsilon'') \supset U(\varepsilon')$, with $\hat{\varepsilon} = \frac{5}{3}$. Assuming that the uncertain parameters a_{ij} are uniformly distributed in the box $U(\hat{\varepsilon})$, the probability that solution $(\mathbf{y}', \mathbf{x}')^{BOX}_{\varepsilon'}$ is not feasible can be computed as:

$$P_{infeas}(\mathbf{y}', \mathbf{x}') = \frac{Vol(U(\hat{\varepsilon}) \setminus \{\mathbf{a} \in U(\hat{\varepsilon}) : \frac{1}{2}a_1 + \frac{1}{2}a_2 \leq 3\})}{Vol(U(\hat{\varepsilon}))} = \frac{8}{64} = 0.125,$$

where $Vol(S)$ denotes the volume of region S, whereas the probability that solution $(\mathbf{y}'', \mathbf{x}'')^{BOX}_{\varepsilon''}$ is not feasible can be computed as:

$$P_{infeas}(\mathbf{y}'', \mathbf{x}'') = \frac{Vol(U(\hat{\varepsilon}) \setminus \{\mathbf{a} \in U(\hat{\varepsilon}) : a_1 \leq \frac{7}{2}, a_2 \leq \frac{7}{2}\}}{Vol(U(\hat{\varepsilon}))} = \frac{15}{64} \simeq 0.23.$$

We therefore have $P_{infeas}(\mathbf{y}', \mathbf{x}') < P_{infeas}(\mathbf{y}'', \mathbf{x}'')$, thus the probability of violation of a supposedly more robust solution can be higher than the probability of the less robust solution. $\square$

The counter-intuitive behavior showed in this section leads to less expensive solutions obtained by considering a specified uncertainty set that are more robust than more expensive solutions obtained by considering a larger uncertainty set. In the following section we investigate how often this behaviour happens in practice.

4 Computational study

5 Conclusions

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