Heir-equations for partial differential equations: a 25-year review

A7. Heir-equations for partial differential equations: a 25-year review

M.C. Nucci

Dipartimento di Matematica e Informatica
Università degli Studi di Perugia, 06123 Perugia, Italy

Abstract

Heir-equations were found by iterating the nonclassical symmetry method. Apart from inheriting the same Lie symmetry algebra of the original partial differential equation, and thus yielding more (and different) symmetry solutions than expected, the heir-equations are connected to conditional Lie-Bäcklund symmetries, and generalized conditional symmetries; moreover they solve the inverse problem, namely a special solution corresponds to the nonclassical symmetry. A 25-year review of work is presented, and open problems are brought forward.

1 Introduction

The most famous and established method for finding exact solutions of differential equations is the classical symmetry method, also called group analysis, which originated in 1881 from the pioneering work of Sophus Lie [54]. Many textbooks have been dedicated to this subject and its generalizations, e.g., [4], [13], [75], [71], [14], [78], [83], [47], [35], [52], [49], [21], [11], [5].

The nonclassical symmetry method was introduced fifty years ago in a seminal paper by Bluman and Cole [12] to obtain new exact solutions of the linear heat equation, i.e. solutions not deducible from the classical symmetry method. The nonclassical symmetry method consists of adding the invariant surface condition to the given equation, and then applying the classical symmetry method on the system consisting of the given differential equation and the invariant surface condition. The main difficulty of this approach is that the determining equations are no longer linear. On the other hand, the nonclassical symmetry method may give more solutions than the classical symmetry method.

After twenty years and few occasional papers, e.g. [74], [15], in the early 1990s there was a sudden spur of interest and several papers began to appear, e.g. [53], [30], [63], [70], [77], [56], [39], [27], [29], [6], [69], [7], [37], [32]. Since then the nonclassical symmetry method has been applied to various equations and systems in hundreds of published papers, e.g., [41], [57], [25], [42], [26], [82], [22], [18], [20], [24] [76], [8], [23], [48], [85], [16], the latest being [86], [9], [17].

One should be aware that some authors call nonclassical symmetries Q-conditional symmetries[1] of the second type, e.g. [34] and [22], while others call them reduction operators, e.g. [76].

[1] In [38] this name was introduced for the first time.

The nonclassical symmetry method can be viewed as a particular instance of the more general differential constraint method that, as stated by Kruglikov [51], *dates back at least to the time of Lagrange... and was introduced into practice by Yanenko* [91]. The method was set forth in detail in Yanenko's monograph [81] that was not published until after his death [31]. A more recent account and generalization of Yanenko's work can be found in [59].

Less than thirty years ago in [40] and [50], solutions were found, which apparently did not seem to follow either the classical or nonclassical symmetries method. Twenty-five years ago, we showed [65] how these solutions could be obtained by iterating the nonclassical symmetries method. A special case of the nonclassical symmetries method generates a new nonlinear equation (the so-called G-equation [64]), which inherits the prolonged symmetry algebra of the original equation. Another special case of the nonclassical symmetries method is then applied to this heir-equation to generate another heir-equation, and so on. Invariant solutions of these heir-equations are just the solutions derived in [40] and [50].

The heir-equations can also yield nonclassical symmetries (as well as classical symmetries) as shown in [68]. The difficulty in applying the method of nonclassical symmetries consists in solving nonlinear determining equations in contrast to the linear determining equations in the case of classical symmetries. The concept of the Gröbner basis has been used [29] for this purpose.

In [68] it was shown that one can find the nonclassical symmetries of any evolution equation of any order by using a suitable heir-equation and searching for a given particular solution among all its solutions, thus avoiding any complicated calculations.

Fokas and Liu [36] and Zhdanov [92] independently introduced the method of generalised conditional symmetries, i.e., conditional Lie-Bäcklund symmetries. In [66] it was shown that the heir-equations can retrieve all the conditional Lie-Bäcklund symmetries found by Zhdanov.

In [43] Goard has shown that Nucci's method of constructing heir-equations by iterating the nonclassical symmetry method is equivalent to the generalized conditional symmetries method.

In [10] Bîlă and Niesen presented another method that reduces the partial differential equation to an ordinary differential equation by using the invariant surface condition and then applying the Lie classical symmetry method in order to find nonclassical symmetries of the original partial differential equation. Recently, Goard in [44] has shown that Bîlă and Niesen's method, and its extension by Bruzón and Gandarias in [20], are equivalent to Nucci's method [68].

The use of a symbolic manipulator became imperative, because the heir-equations can be quite long: one more independent variable is added at each iteration. We employed our own interactive REDUCE programs [67] to calculate both the classical and the nonclassical symmetries, while we use MAPLE in order to generate the heir-equations.

In the next sections, after recalling what heir-equations are and how to construct them, we show some illustrative examples and applications that have been drawn from our publications of the last 25 years, and include some open problems.

2 Constructing the heir-equations

Let us consider an evolution equation in two independent variables and one dependent variable:

$$u_t = H(t, x, u, u_x, u_{xx}, u_{xxx}, \ldots). \tag{2.1}$$

The invariant surface condition is given by:

$$V_1(t, x, u)u_t + V_2(t, x, u)u_x = G(t, x, u). \tag{2.2}$$

Let us take the case with $V_1 = 0$ and $V_2 = 1$, so that (2.2) becomes:

$$u_x = G(t, x, u). \tag{2.3}$$

Applying the nonclassical symmetry method leads to an equation for G. We call this the G-equation [64], which has the following invariant surface condition:

$$\xi_1(t, x, u, G)G_t + \xi_2(t, x, u, G)G_x + \xi_3(t, x, u, G)G_u = \eta(t, x, u, G). \tag{2.4}$$

Let us consider the case $\xi_1 = 0$, $\xi_2 = 1$, and $\xi_3 = G$, so that (2.4) becomes:

$$G_x + GG_u = \eta(t, x, u, G). \tag{2.5}$$

Applying the nonclassical symmetry method leads to an equation for η called the η-equation. Clearly:

$$G_x + GG_u \equiv u_{xx} \equiv \eta. \tag{2.6}$$

We could keep iterating to obtain the Ω-equation, which corresponds to:

$$\eta_x + G\eta_u + \eta\eta_G \equiv u_{xxx} \equiv \Omega(t, x, u, G, \eta), \tag{2.7}$$

the ρ-equation, which corresponds to:

$$\Omega_x + G\Omega_u + \eta\Omega_G + \Omega\Omega_\eta \equiv u_{xxxx} \equiv \rho(t, x, u, G, \eta, \Omega), \tag{2.8}$$

and so on. Each of these equations inherits the symmetry algebra of the original equation, with the correct prolongation: first prolongation for the G-equation, second prolongation for the η-equation, and so on. Therefore, these equations were named heir-equations in [65]. This implies that even in the case of few Lie point symmetries, many more Lie symmetry reductions can be performed by using the invariant symmetry solution of any of the possible heir-equations, as was shown in [65], [3] and [58].

Also, it should be noticed that the $u\underbrace{xx\ldots}_{r}$–equation of (2.1) is just one of many possible r-extended equations as defined by Guthrie in [45].

We point out that the above described iteration method is strongly connected to the definition of partial symmetries given by Vorobev in [87]. To exemplify, we consider the heat equation:

$$u_t = u_{xx}. \tag{2.9}$$

Its G-equation is:

$$2GG_{xu} + G^2 G_{uu} - G_t + G_{xx} = 0. \tag{2.10}$$

Its η-equation is:

$$2\eta\eta_{xG} + 2G\eta\eta_{uG} + \eta^2\eta_{GG} + 2G\eta_{xu} + \eta_{xx} - \eta_t + G^2\eta_{uu} = 0. \tag{2.11}$$

The G-equation corresponds to the zeroth-order differential constraint as given by Vorobev [88] on p. 76, formula (3.5), while the η-equation gives the partial symmetry of the heat equation as in [87] on p. 324, formula (14), and in [88] on p. 83, formula (4.10).

Now, let us consider a hyperbolic equation in two independent variables and one dependent variable:

$$u_{tt} = u_{xx} + h(t, x, u, u_x, u_t), \tag{2.12}$$

and take $V_1 = 1$, and $V_2 = 1$, so that (2.2) becomes[2]:

$$u_t + u_x = G(t, x, u). \tag{2.13}$$

Applying the nonclassical symmetry method leads to an equation for G. We call this equation the G-equation. If we take $\xi_1 = 1$, $\xi_2 = 1$, and $\xi_3 = G$, then (2.4) becomes[3]:

$$G_t + G_x + GG_u = \eta(t, x, u, G). \tag{2.14}$$

Applying the nonclassical symmetry method leads to an equation for η. We call this equation the η-equation. Clearly:

$$G_t + G_x + GG_u \equiv u_{tt} + 2u_{tx} + u_{xx} \equiv \eta. \tag{2.15}$$

We could keep iterating to obtain the Ω-equation, which corresponds to:

$$\eta_t + \eta_x + G\eta_u + \eta\eta_G \equiv u_{ttt} + 3u_{ttx} + 3u_{txx} + u_{xxx} \equiv \Omega(t, x, u, G, \eta), \tag{2.16}$$

and so on.

[2] There exists another case with $V_1 = 1$, and $V_2 = -1$, which leads to $u_t - u_x = G(t, x, u)$.

[3] There exists another case with $\xi_1 = 1$, $\xi_2 = -1$, and $\xi_3 = G$, which leads to $G_t - G_x + GG_u = \eta(t, x, u, G)$.

Let us consider an elliptic equation in two independent variables and one dependent variable:

$$u_{tt} + u_{xx} = h(t, x, u, u_x, u_t), \tag{2.17}$$

and take $V_1 = 1$, and $V_2 = i$, so that (2.2) becomes[4]:

$$u_t + iu_x = G(t, x, u). \tag{2.18}$$

Then the η-equation will be given through:

$$G_t + iG_x + GG_u \equiv u_{tt} + 2iu_{tx} - u_{xx} \equiv \eta(t, x, u, G), \tag{2.19}$$

the Ω-equation will be given through:

$$\eta_t + i\eta_x + G\eta_u + \eta\eta_G \equiv u_{ttt} + 3iu_{ttx} - 3u_{txx} - iu_{xxx} \equiv \Omega(t, x, u, G, \eta), \tag{2.20}$$

and so on.

3 Symmetry solutions of heir-equations

In [65] we have shown that solutions obtained in [40] are just invariant solutions of the $u_{xx} \equiv \eta$–equation.

We seek t-independent invariant solutions, which have x as the similarity variable of the heir-equations. In this way, we obtain ordinary differential equations of order two. Their general solution depends on arbitrary functions of t. Substituting into the original equation yields ordinary differential equations to be satisfied by these t-dependent functions.

We recall Galaktionov's equation:

$$u_t = u_{xx} + u_x^2 + u^2. \tag{3.1}$$

Its G-equation is:

$$2GG_{xu} + G^2G_{uu} + G^2G_u - u^2G_u - G_t + G_{xx} + 2GG_x + 2uG = 0. \tag{3.2}$$

Its η-equation is:

$$\begin{aligned} &2\eta\eta_{xG} + 2G\eta\eta_{uG} + \eta^2\eta_{GG} - 2uG\eta_G + 2G\eta_{xu} + \eta_{xx} \\ &+2G\eta_x - \eta_t + G^2\eta_{uu} + G^2\eta_u - u^2\eta_u + 2\eta^2 + 2u\eta + 2G^2 = 0. \end{aligned} \tag{3.3}$$

The symmetry algebra of (3.1) is spanned by the two vector fields $X_1 = \partial_t$, and $X_2 = \partial_x$. Therefore, t-independent invariant solutions of (3.3) are given in the form $\eta = \eta(x, u, G)$. A particular case is $\eta_u = 0$, which implies $\eta = L(x, G)$. Substituting this expression for η into (3.3) leads to $L = f(x)G$ with[5]:

$$f(x) = \frac{-c_1 \sin x + c_2 \cos x}{c_2 \sin x + c_1 \cos x}. \tag{3.4}$$

[4]There exists another case with $V_1 = 1$, and $V_2 = -i$, which leads to $u_t - iu_x = G(t, x, u)$.
[5]c_n $(n = 1, 2, 3)$ are arbitrary constants.

If we let $c_1 = 0$, then:

$$\eta = \cot(x)G, \tag{3.5}$$

which is just the differential constraint for (3.1) given by Olver in [72], i.e.:

$$u_{xx} = \cot(x)u_x. \tag{3.6}$$

Integrating (3.6) with respect to x gives rise to[6]:

$$u = w_1(t)\cos(x) + w_2(t). \tag{3.7}$$

Finally, the substitution of (3.7) into (3.1) leads to:

$$\dot{w}_1 = w_1^2 + w_2^2, \qquad \dot{w}_2 = 2w_1w_2 - w_2 \tag{3.8}$$

This is the solution derived by Galaktionov for (3.1).

4 Zhdanov's conditional Lie-Bäcklund symmetries and heir-equations

We recall Zhdanov's conditional Lie-Bäcklund symmetries [92] and their relationship with heir-equations as determined in [66].

In [92], Zhdanov introduced the concept of conditional Lie-Bäcklund symmetry, i.e., given an evolution-type equation

$$u_t = H(t, x, u, u_x, u_{xx}, u_{xxx}, \ldots) \tag{4.1}$$

and some smooth Lie-Bäcklund vector field (LBVF)

$$Q = S\partial_u + (D_tS)\partial_{u_t} + (D_xS)\partial_{u_x} + \ldots \tag{4.2}$$

with $S = S(t, x, u, u_t, u_x, \ldots)$, equation (4.1) is said to be conditionally invariant under LBVF (4.2) if the condition

$$Q(u_t - H)|_{M\cap L_x} = 0 \tag{4.3}$$

holds. Here M is a set of all differential consequences of the equation (4.1), and L_x is the set of all x-differential consequences of the equation $S = 0$. Zhdanov claimed that this definition can be applied to construct new exact solutions of (4.1), which cannot be obtained by either Lie point or Lie-Bäcklund symmetries.

However, $S = 0$ is just a particular invariant solution of a suitable heir-equation. Of course, we assume that $S = 0$ can be written in explicit form with respect to the highest derivative of u.

[6] w_n $(n = 1, 2)$ are arbitrary functions of t.

We present here one example from [92], which we have also discussed in [66]. Zhdanov introduced the following nonlinear heat conductivity equation with a logarithmic-type nonlinearity

$$u_t = u_{xx} + (\alpha + \beta \log(u) - \gamma^2 \log(u)^2)u, \tag{4.4}$$

and obtained new solutions by showing that (4.4) is conditionally invariant with respect to LBVF (4.2) with

$$S = u_{xx} - \gamma u_x - u_x^2/u. \tag{4.5}$$

It can easily be shown that equation

$$S \equiv u_{xx} - \gamma u_x - u_x^2/u = 0 \tag{4.6}$$

admits an eight-dimensional Lie point symmetry algebra and therefore is linearizable.[7] In fact, the change of dependent variable $u = \exp(v)$ transforms (4.6) into $v_{xx} - \gamma v_x = 0$. Therefore, the following general solution of (4.6) can be obtained [92]

$$u(t,x) = \exp\left(\phi_1(t) + \phi_2(t)\exp(\gamma x)\right),$$

which, substituted into (4.4), gives rise to the following system of two ordinary differential equations

$$\dot{\phi}_1 = \alpha + \beta\phi_1 - \gamma^2\phi_1^2, \qquad \dot{\phi}_2 = (\beta + \gamma^2 - 2\gamma^2\phi_1)\phi_2,$$

and its general solution can easily be derived [92].

Now, let us apply the heir-equation method to equation (4.4). Its G-equation is

$$\begin{aligned} &2G_{xu}G + G_{uu}G^2 + G_u\log(u)^2\gamma^2 u - G_u\log(u)\beta u - G_u\alpha u - G_t \\ &+G_{xx} - \log(u)^2 G\gamma^2 + \log(u)\beta G - 2\log(u)G\gamma^2 + \alpha G + \beta G = 0. \end{aligned} \tag{4.7}$$

Its η-equation is

$$\begin{aligned} &2\eta_{uG}\eta G u + \eta_{GG}\eta^2 u + \eta_G\log(u)^2\gamma^2 G u - \eta_G\log(u)\beta G u - \eta_G\alpha G u \\ &\quad +2\eta_G\log(u)\gamma^2 G u - \eta_G\beta G u + \eta_{uu}G^2 u + \eta_u\log(u)^2\gamma^2 u^2 \\ &\quad -\eta_u\log(u)\beta u^2 - \eta_u\alpha u^2 - \log(u)^2\gamma^2\eta u + \log(u)\beta\eta u \\ &-2\log(u)\gamma^2\eta u - 2\log(u)\gamma^2 G^2 + \alpha\eta u + \beta\eta u + \beta G^2 - 2\gamma^2 G^2 = 0. \end{aligned} \tag{4.8}$$

The Lie point symmetry algebra of (4.4) is spanned by the two vector fields $X_1 = \partial_t$, and $X_2 = \partial_x$. Therefore, (x,t)-independent invariant solutions of (4.8) are given in the form $\eta = \eta(u, G)$. A particular case is $\eta = r_1(u)G^2 + r_2(u)G + r_3(u)$, i.e., a polynomial of second degree in G. Substituting into (4.8) and assuming $r_3 = 0$ gives rise to

$$\eta = \frac{G^2}{u} \pm \gamma G. \tag{4.9}$$

Finally, substituting $\eta = u_{xx}$, and $G = u_x$ into (4.9) yields (4.6).

All Zhdanov's examples in [92] were similarly framed within the heir-equation method in [66].

[7] Zhdanov integrated equation (4.6) without any mention of this property.

5 Nonclassical symmetries as special solutions of heir-equations

We recall the method that allows one to find nonclassical symmetries of an evolution equation by using a suitable heir-equation [68].

For the sake of simplicity, let us assume that the highest order x-derivative in the equation is two, i.e.:

$$u_t = H(t, x, u, u_x, u_{xx}). \tag{5.1}$$

First, we use (5.1) to replace u_t in (2.2), with the condition $V_1 = 1$, i.e.:

$$H(t, x, u, u_x, u_{xx}) + V_2(t, x, u)u_x = F(t, x, u). \tag{5.2}$$

Then we generate the η-equation with $\eta = \eta(x, t, u, G)$, and replace $u_x = G$, $u_{xx} = \eta$ in (5.2), i.e.:

$$H(t, x, u, G, \eta) = F(t, x, u) - V_2(t, x, u)G. \tag{5.3}$$

By the implicit function theorem, we can isolate η in (5.3), e.g.:

$$\eta = [h_1(t, x, u, G) + F(t, x, u) - V_2(t, x, u)G]\, h_2(t, x, u, G), \tag{5.4}$$

where $h_i(t, x, u, G)(i = 1, 2)$ are known functions. Thus, we have obtained a particular solution of η which must yield an identity if substituted in the η-equation. The only unknown functions are $V_2 = V_2(t, x, u)$ and $F = F(t, x, u)$. We remind the reader that there are two kinds of nonclassical symmetries, namely those with $V_1 \neq 0$ in (2.2) or those with $V_1 = 0$ in (2.2) [29]. In the first case, we can assume without loss of generality that $V_1 = 1$, while in the second case we can assume $V_2 = 1$, which generates the G-equation. If there does exist a nonclassical symmetry[8], our method will recover it. Otherwise, only the classical symmetries will be found. If we are only interested in finding nonclassical symmetries, impose F and V_2 to be functions only of the dependent variable u. Moreover, any such solution should be singular, i.e. should not form a group.

If we are dealing with a third order equation, then we need to construct the heir-equation of order three, i.e. the Ω-equation. Then, a similar procedure will yield a particular solution of the Ω-equation given by a formula of the form:

$$\Omega = [h_1(t, x, u, G, \eta) + F(t, x, u) - V_2(t, x, u)G]\, h_2(t, x, u, G, \eta) \tag{5.5}$$

where $h_i(t, x, u, G, \eta)(i = 1, 2)$ are known functions.

In the case of a fourth order equation, we need to construct the heir-equation of order four, i.e. the ρ-equation. Then, a similar procedure will yield a particular solution of the ρ-equation given by a formula of the form:

$$\rho = [h_1(t, x, u, G, \eta, \Omega) + F(t, x, u) - V_2(t, x, u)G]\, h_2(t, x, u, G, \eta, \Omega) \tag{5.6}$$

[8] Of course, we mean one such that $V_1 \neq 0$, i.e. $V_1 = 1$.

where $h_i(t,x,u,G,\eta,\Omega)(i=1,2)$ are known functions.

And so on.

We would like to underline how easy this method is in comparison with the nonclassical symmetry method itself since one has just to check if a particular solution is admitted by the same-order heir-equation instead of solving nonlinear determining equations. The only difficulty consists of deriving the heir-equations, which become longer and longer. However, they can be automatically determined by using any computer algebra system.

We now present some examples to show how the method works.

In [68], the following family of second order evolution equations:

$$u_t = u_{xx} + R(u, u_x), \tag{5.7}$$

with $R(u,u_x)$ a known function of u and u_x, was considered. Several well-known equations that possess nonclassical symmetries belong to (5.7). In particular, Burgers' equation [4], Fisher's equation [28], real Newell-Whitehead's equation [62], Fitzhugh-Nagumo's equation [70], and Huxley's equation [28], [7].
The G-equation of (5.7) is:

$$R_G(GG_u + G_x) + GR_u + 2G_{xu}G + G_{uu}G^2 - G_uR - G_t + G_{xx} = 0. \tag{5.8}$$

The η-equation of (5.7) is:

$$\begin{aligned} 2R_{uG}\eta G + R_{GG}\eta^2 + R_G\eta_x + GR_G\eta_u + R_{uu}G^2 - GR_u\eta_G + R_u\eta \\ +2\eta_{xG}\eta + 2\eta_{uG}\eta G + \eta_{GG}\eta^2 - \eta_t + 2\eta_{xu}G + \eta_{xx} + \eta_{uu}G^2 - R\eta_u = 0. \end{aligned} \tag{5.9}$$

The particular solution (5.4) that we are looking for is:

$$\eta = -R(u,G) + F(t,x,u) - V_2(t,x,u)G \tag{5.10}$$

which, substituted in (5.9), yields an overdetermined system in the unknowns functions F and V_2, whereby $R(u,G)$ is given explicitly. Otherwise, after solving a first-order linear partial differential equation in $R(u,G)$, we obtain that equation (5.7) may possess a nonclassical symmetry (2.2) with $V_1 = 1, V_2 = v(u), F = f(u)$ if $R(u,u_x)$ has the following form

$$R(u,u_x) = \frac{u_x}{f^2}\left(\left(-\frac{\mathrm{d}f}{\mathrm{d}u}fu_x + \frac{\mathrm{d}v}{\mathrm{d}u}\right)fu_x^2 + \Psi(\xi)u_x^2 + 2f^2v - 3fu_xv^2 + u_x^2v^3\right) \tag{5.11}$$

with f, v arbitrary functions of u, and Ψ arbitrary function of

$$\xi = \frac{f(u)}{u_x} - v(u). \tag{5.12}$$

This means that infinitely many cases can be found. For example, equation (5.7) with $R(u,u_x)$ given by

$$R(u,u_x) = (2u_x + u^4)\frac{u_x}{u} \tag{5.13}$$

i.e.

$$u_t = u_{xx} + (2u_x + u^4)\frac{u_x}{u}, \tag{5.14}$$

admits a nonclassical symmetry[9] with $v = u^3/2$ and $f = -u^7/12$. It is interesting to note that the corresponding reduction leads to the solution of the following ordinary differential equation in $u(x)$:

$$u_{xx} = -2\frac{u_x^2}{u} - \frac{3}{2}u^3 u_x - \frac{u^7}{12}, \tag{5.15}$$

which is linearizable. In fact, it admits a Lie symmetry algebra of dimension eight [55], and consequently the two-dimensional abelian intransitive subalgebra generated by the following operators:

$$\frac{1}{6}u^3\left(6\partial_x - u^4\partial_u\right), \quad \frac{1}{12}(2 - xu^3)\left(6\partial_x - u^4\partial_u\right)$$

which yields the transformation:

$$\tilde{x} = \frac{2u^3}{2 - xu^3}, \quad \tilde{u} = \frac{x(4 - xu^3)}{2(2 - xu^3)}$$

that takes equation (5.15) into

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tilde{x}} = 0.$$

Thus, the general solution of (5.15) is

$$\frac{x^2u^3}{2(2 - xu^3)} + x - \frac{2s_1u^3}{2 - xu^3} = s_2,$$

with s_1, s_2 arbitrary functions of t. Maple 16 solves this third degree polynomial in u,

$$u = \left(\frac{4(x - s_2)}{4s_1 - 2s_2x + x^2}\right)^{1/3},$$

which, substituted into (5.14), yields the following solution:

$$u = \left(\frac{4(x - a_2)}{4a_1 + 2t - 2a_2x + x^2}\right)^{1/3}.$$

[9] We remark that equation (5.14) admits a five-dimensional Lie symmetry algebra, generated by the following operators:

$$\Gamma_1 = t^2\partial_t + tx\partial_x - \frac{x + tu^3}{3u^2}\partial_u,\ \Gamma_2 = t\partial_t + 2x\partial_x + \partial_u,\ \Gamma_3 = \partial_t,\ \Gamma_4 = t\partial_x - \frac{1}{3u^2}\partial_u,\ \Gamma_5 = \partial_x.$$

6 Final remarks

As stated in [68], we have determined an algorithm which is easier to implement than the usual method to find nonclassical symmetries admitted by an evolution equation in two independent variables. Moreover, one can retrieve both classical and nonclassical symmetries with the same method.

While one can apply the heir-equations and their properties to a system of evolution equations [3], [46], [19], it is still an open problem to determine suitable heir-equations for more than two independent variables.

Moreover, the heir-equation method raises many other intriguing questions [68]:

- Could an a priori knowledge of the existence of nonclassical symmetries apart from classical symmetries be achieved by looking at the properties of the right-order heir-equation? We have shown that our method leads to both classical and nonclassical symmetries. Nonclassical symmetries could exist if we impose F and V_2 to be functions only of the dependent variable u in either (5.4), or (5.5), or (5.6), etc. Of course, any such solution of F and V_2 does not yield a nonclassical symmetry, unless it is isolated, i.e. does not form a group.

- What is integrability? The existence of infinitely many higher order symmetries is one of the criteria [60], [73]. In [66], we have shown that invariant solutions of the heir-equations yield Zhdanov's conditional Lie-Bäcklund symmetries [92]. Higher order symmetries may be interpreted as special solutions of heir-equations (up to which order? see [80], [73]). Another criterion for integrability consists of looking for Bäcklund transformations [2], [79]. In [64], we have found that a nonclassical symmetry of the G-equation for the modified Korteweg-deVries equation gives the known Bäcklund transformation between the modified Korteweg-deVries and Korteweg-deVries equations [61]. Another integrability test is the Painlevé property [89] which when satisfied leads to Lax pairs (hence, inverse scattering transform) [2], Bäcklund transformations, and Hirota bilinear formalism [84]. In [33], the singularity manifold of the modified Korteweg-deVries equation was found to be connected to an equation which is exactly the G-equation for the modified Korteweg-deVries, and the same was done for five other equations. Could heir-equations be the common link among all the integrability methods?

- In order to reduce a partial differential equation to ordinary differential equations, one of the first steps is to find the admitted Lie point symmetry algebra. In most instances, it is very small, and therefore not many reductions can be obtained. However, if heir-equations are considered, then many more ordinary differential equations can be derived using the same Lie algebra [65], [3], [58]. Of course, the classification of all subalgebras [90] becomes imperative [58]. In the case of known integrable equations, it would be interesting to investigate which ordinary differential equations result from using the admitted Lie point symmetry algebra and the corresponding heir-equations. Do all these ordinary differential equations possess the Painlevé property (see the Painlevé conjecture as stated in [1])?

- Researchers often find solutions of partial differential equations which apparently do not follow from any symmetry reduction. Are the heir-equations the ultimate method which keeps Lie symmetries at center stage?

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