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Foundations of Context-Aware Preference Propagation*

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Preferences are a fundamental ingredient in a variety of fields, ranging from economics to computer science, for deciding the best choices among possible alternatives. Contexts provide another important aspect to be considered in the selection of the best choices since, very often, preferences are affected by context. In particular, the problem of *preference propagation* from more generic to more specific contexts naturally arises. Such a problem has only been addressed in a very limited way and always resorting to practical, ad hoc approaches. In order to fill this gap, in this paper we analyze preference propagation in a principled way, and adopt an abstract context model without making any specific assumptions on how preferences are stated. Our framework only requires that the contexts form a partially ordered set and that preferences define a strict partial order on the objects of interest. We first formalize the basic properties that any propagation process should satisfy. We then introduce an algebraic model for preference propagation that relies on two abstract operators for combining preferences, and, under mild assumptions, we prove that the only possible interpretations for such operators are the well-known Pareto and Prioritized composition. We then study several propagation methods based on such operators, and precisely characterize them in terms of the stated properties. We finally identify a method meeting all the requirements, on the basis of which we provide an efficient algorithm for preference propagation.

CCS Concepts: • **Information systems** → *Data management systems*; *Database design and models*; Information systems applications.

Additional Key Words and Phrases: Preferences, Context Awareness, Partial Order

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1 INTRODUCTION

Choices are part of our everyday life, and any decision we make reflects our preferences among the alternatives we have. For this reason, preferences and their influence on choices have been studied for a long time in a variety of scientific fields, including psychology, sociology, economics, artificial intelligence, and data management. Depending on the field, specific aspects of the problem have been investigated in more detail. For instance, economists have studied how preferences affect the degree of utility that a consumer obtains from a good [6]; the AI community has investigated the

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influence of preferences on various automated tasks, such as recommendations, planning, and non-monotonic reasoning [40]; research on database systems has mainly focused on preference-based algorithms for efficiently retrieving the most relevant data from large repositories [46].

In all of the above fields, it is widely recognized that, in real-world cases, preferences highly depend on the *context*. For instance, it is well known that the preferences of customers are strongly influenced by a variety of aspects regarding the context in which they lie, as shown for instance in the following example, which will be used throughout the paper.

Example 1. *Assume that we are in Italy and we need to order food at a restaurant: normally, then, we prefer pasta to beef. In Naples, though, we enjoy the world-famous pizza more than pasta. Furthermore, if it is summer and it is very hot, we just opt for something fresh, such as a tomato salad.* □

As it is apparent in this simple example and common in most context models [9, 10], we can make two basic assumptions when dealing with contexts: (i) they can be considered as *states* in which the subject of interest (e.g., the user) is operating (such as "Italy in summer") and (ii) in most cases, contexts can be compared on the basis of a *generalization hierarchy*, which allows us to say that, for instance, "Naples" is more specific than "Italy".

In this framework, a natural behavior of preferences is that they may *propagate* along the hierarchy, from the more generic to the more specific contexts, meaning that, for instance, a preference defined for Italy is expected to also hold in any Italian city. There are however a number of issues to be considered when preferences propagate through contexts, as the following example shows.

Example 2. *Let us consider the contextual preferences illustrated in Example 1 and assume that it is summer, we are in Naples, and we need to decide which is the best choice of food. All of the preferences above should be taken into account since they refer to contexts that are more generic than the current one. However, it is evident that the preferences defined in "Naples" and "Italy in summer" should take precedence over those in the more generic context "Italy". Moreover, the preference in "Naples" should not take precedence over the preference in "Italy in summer", and vice versa, since, in general, the preference in one context does not apply to the other context. It turns out that, in the current context, pizza and salad are both the best alternatives among the mentioned foods since, on the basis of all the preferences stated in the various contexts, no other food is preferable to them, and, thus, both should be recommended.* □

These phenomena have been studied in the past, but always resorting to ad hoc and pragmatic approaches [3, 29, 35, 47, 48, 51], thus failing to uncover the fundamental and challenging aspects of the problem. In this paper, we aim to fill this gap by providing a formal basis to the problem of *context-aware preference propagation*. To the best of our knowledge, this is the first proposal for a principled and general approach to this issue, apart from a previous attempt of ours in which we started studying the basic properties of the propagation process and introduced a preliminary mechanism for preference propagation [18].

In our approach, we consider a very general framework in which the only basic requirements are the following:

- the contexts of interest belong to a *poset*, that is, a set C with a (strict) partial order relation $<_C$ on its elements: $c_1 <_C c_2$ means that the context c_1 is *more specific* than the context c_2 (and that c_2 is *more generic* than c_1);
- preferences define a strict partial order $<_O$ on the domain of objects of interest O, where $o_1 <_O o_2$ means that the object o_2 *is preferable to* the object o_1;

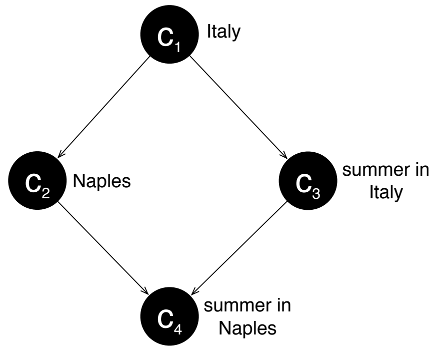


Fig. 1. The context poset for Example 1.

- each stated preference is *contextualized*, i.e., associated with one or more contexts.[1]

In this setting, we start by formalizing the basic properties of the propagation process, which are also implicitly at the basis of earlier approaches to the problem and correspond to the observations made for Example 2:

(1) preferences may only propagate to a context c if they are defined in contexts that are more generic than c (*coherence*),
(2) preferences stated for two unordered (i.e., neither more generic nor more specific) contexts do not take precedence over each other (*fairness*), and
(3) preferences stated for a more specific context take precedence over those stated for a more generic context (*specificity*).

Building on these properties, we focus on three main problems:

- (CFS) To determine whether there exist well-behaved (i.e., Coherent, Fair and Specific) methods for the propagation of preferences.
- (OrdRel) To determine the preference holding between any two objects in a context, according to a given propagation method, i.e., to efficiently compute their order relation.
- (Best) To establish the best objects in a context c according to the preferences propagated to c.

In order to tackle these problems, we follow a principled approach that axiomatizes the required properties of propagation. To this end, we introduce an algebraic framework that allows us to define an operational semantics for preference propagation. The framework is based on two abstract binary operators that can be used to express, in a procedural way, preference propagation by means of *Preference Composition* (PC) expressions. The two operators are:

- $+$, which combines preferences defined in two unordered contexts, and
- $\triangleright$, which combines preferences in two contexts, where one is more specific than the other.

Example 3. *The poset of contexts described in Example 2 is represented in Figure 1.*[2] *An example of PC-expression for propagating preferences to the context "summer in Naples" is the following:*

$$c_4 \triangleright ((c_2 + c_3) \triangleright c_1)$$

[1]This also includes the case of non-contextualized preferences, which can be viewed as associated with a context more generic than all other contexts in the poset.
[2]We represent a context poset C with its *Hasse diagram*, in which nodes are circles, the edges represent the partial order (transitively reduced), and c_1 is drawn lower than c_2 if $c_1 <_C c_2$.

where, e.g., c_1 *denotes the preferences in "Italy" (in our case, pasta is preferable to beef). In this expression, first the preferences in "Naples"* (c_2) *and those in "summer in Italy"* (c_3) *are combined with* $+$*, since the two contexts are unordered. The result is then combined with the preferences in "Italy"* (c_1) *using* $\triangleright$*, since this context is more generic than both "Naples" and "summer in Italy". Finally, the result is combined with the preferences for "summer in Naples"* (c_4) *using* $\triangleright$*, since this is the most specific context.* □

We then study possible interpretations of $+$ and $\triangleright$ complying with the stated axioms for the abstract operators. Our first important result is that, under mild assumptions, Pareto ($\oplus$) and Prioritized ($\rhd\!\!\!\!\bigcirc$) composition [16, 24] are the *only* possible interpretations of $+$ and $\triangleright$, respectively, that satisfy the axioms. This implies that *any* propagation method built on different interpretations would fail to satisfy the fairness and specificity requirements.

These results allow us to concretely address Problem CFS in terms of the identification of coherent, fair and specific propagation methods expressed as PC-expressions based on $\oplus$ and $\rhd\!\!\!\!\bigcirc$.

After proving that two "natural" forms of PC-expressions are indeed unable to enforce specificity, we then discover a method, called $\mathcal{OC}$, that is both fair and specific. Moreover, we show that $\mathcal{OC}$ propagates all and only the preferences that can be propagated while satisfying both fairness and specificity; as such, $\mathcal{OC}$ can be considered the ultimate propagation method.

Then, given any two objects in the domain, we study how to establish if one is preferable to the other according to the propagated preferences (Problem OrdRel). To this end, we present an algorithm for propagating preferences according to the $\mathcal{OC}$ method, and characterize its asymptotic complexity. Consistently with the adopted approach, we remain parametric with respect to the complexity of comparing contexts and comparing objects, since both heavily depend on the specific context model and on the formalism used for expressing preferences over objects, respectively. Remarkably, the algorithm's complexity is independent of the underlying domain size (i.e., number of objects). Finally, building on the above results, we show how to determine the best objects according to the propagated preferences (Problem Best).

In sum, our main contributions are:

- the identification and formalization of the desirable properties of preference propagation in a poset of contexts;
- the definition of an algebra for preference propagation based on two abstract operators, $+$ and $\triangleright$, reflecting such desirable properties;
- the proof that, under mild assumptions, Pareto and Prioritized composition are the only possible interpretations of $+$ and $\triangleright$, respectively;
- the formal analysis of several propagation methods, one of which (called $\mathcal{OC}$) proves to satisfy all the required properties;
- the proof that no natural propagation method other than $\mathcal{OC}$ satisfies such properties;
- a provably correct algorithm for preference propagation according to $\mathcal{OC}$ that works for any context model and preference language, for which we characterize the asymptotic complexity.

To our knowledge, these are the first results that provide a theoretical foundation to the propagation of preferences arising in context-aware scenarios.

The rest of the paper is organized as follows. In Section 2, we introduce the basic notions concerning contexts and preferences. In Section 3 we introduce the fundamental properties of the propagation process (coherence, fairness, and specificity) and precisely state the problems studied in this paper. In Section 4 we propose an algebraic model for combining preferences based on two abstract operators. Section 5 is devoted to analyzing possible interpretations of such abstract operators, and shows that, for the relevant class of the so-called IIO (*independent of irrelevant objects*) operators, Pareto and Prioritized composition are the only choice satisfying the propagation properties. In Section 6, we analyze different propagation methods and, in Section 7, we introduce an algorithm implementing the $\mathcal{OC}$ propagation method and characterize its complexity. In Section 8 we compare our work with the related literature and finally, in Section 9, we draw some conclusions. In the interest of readability, all the proofs of our results are reported in the Appendix.

2 PRELIMINARIES

Symbol	Full name	Meaning and notes
$\leqslant_V$	Non-strict partial order on V	Reflexive, antisymmetric, transitive subset of $V \times V$
$<_V$	Strict partial order on V	Asymmetric, transitive subset of $V \times V$
$\sim_V$	Unordered relation on V	$v_1 \sim_V v_2$ if neither $v_1 \leqslant_V v_2$ nor $v_2 \leqslant_V v_1$
$V\lfloor v \rfloor$	Successor poset of v	$V' = \{v' \in V \mid v \leqslant_V v'\}$, where $v_1 \leqslant_{V'} v_2$ iff $v_1 \leqslant_V v_2$
$v_1 \lessdot_V v_2$	v_2 covers v_1	$v_1 <_V v_2$ and $\nexists v \in V \mid v_1 <_V v <_V v_2$
$\mathrm{cov}^V(v)$	Cover of v	Set of elements in V that cover v
$\langle v_1, \ldots, v_n \rangle$	Chain	Sequence of elements in V such that $v_1 <_V \ldots <_V v_n$
$c_1 <_C c_2$	c_1 more specific than c_2	$c_1 \leqslant_C c_2$ and $c_1 \neq c_2$
$o_1 \prec o_2$	o_2 is preferable to o_1	$o_1 <_O o_2$.
$o_1 \approx o_2$	o_1 and o_2 are indifferent	See Definition 3.
$o_1 \parallel o_2$	o_1 and o_2 are incomparable	$o_1 \sim_O o_2$, but $o_1 \not\approx o_2$.

Table 1. Table of symbols.

2.1 Partial orders

For what follows some basic notions on partial orders and posets are needed. A list of relevant symbols used throughout the paper is shown in Table 1. A *(non-strict) partial order* $\leqslant_V$ on a domain V is a subset of $V \times V$, whose elements are denoted by $v_1 \leqslant_V v_2$, that is *i)* reflexive ($v \leqslant_V v$ for all $v \in V$), *ii)* antisymmetric (if $v_1 \leqslant_V v_2$ and $v_2 \leqslant_V v_1$ then $v_1 = v_2$), and *iii)* transitive (if $v_1 \leqslant_V v_2$ and $v_2 \leqslant_V v_3$ then $v_1 \leqslant_V v_3$) [27]. A *strict* partial order on V, denoted by $<_V$, is an asymmetric (we never have both $v_1 <_V v_2$ and $v_2 <_V v_1$) and transitive subset of $V \times V$; equivalently, a strict partial order $<_V$ can be obtained from a partial order $\leqslant_V$ by removing all relationships of the form $v \leqslant_V v$, i.e., $v_1 <_V v_2$ if and only if $v_1 \leqslant_V v_2$ and $v_1 \neq v_2$. A partially ordered set, or *poset*, is a set V on which a partial order $\leqslant_V$ is defined. Two elements v_1 and v_2 of a poset V are *ordered* if either $v_1 \leqslant_V v_2$ or $v_2 \leqslant_V v_1$, otherwise they are *unordered*, denoted $v_1 \sim_V v_2$. Given an element v of a poset V, we will denote by $V\lfloor v \rfloor$ the poset $V' = \{v' \in V \mid v \leqslant_V v'\}$, called *successor poset* of v, where $v_1 \leqslant_{V'} v_2$ if and only if $v_1 \leqslant_V v_2$.

A *chain* is a subset of a poset V such that any two elements are ordered. For convenience, a chain of elements $v_1, \ldots, v_n$ such that $v_1 <_V \ldots <_V v_n$ is denoted by the sequence $\langle v_1, \ldots, v_n \rangle$. An *antichain* is a subset of V whose elements are pairwise unordered. A chain (resp., antichain) is *maximal* if it is not included into another chain (resp., antichain). An element v is *maximal* (resp., *minimal*) in V if there is no element v' such that $v <_V v'$ (resp., $v' <_V v$). The *width* $w(V)$ of a poset V is the cardinality of the largest maximal antichain of V.

If $v_1 <_V v_2$ and there is no other element $v \in V$ such that $v_1 <_V v <_V v_2$, then we say that v_2 *covers* v_1 (v_1 *is covered by* v_2), denoted $v_1 \lessdot_V v_2$. The *cover* of an element v in a poset V, denoted $\mathrm{cov}^V(v)$, is the set of elements in V that cover v, i.e., $\mathrm{cov}^V(v) = \{v' \in V \mid v \lessdot_V v'\}$.

2.2 Contexts and preferences

Our aim is to study contextual preferences independently of the specific formalisms used to represent contexts and specify preferences. We only focus on a fundamental characteristic of context models: the ability to represent contexts at different levels of detail [10]. We will therefore rely on the general notion of context that follows.

Definition 1 (Context). *A* context c *is an element of a poset* C*, called* context poset*. If* $c_1 <_C c_2$ *we say that* c_1 *is* more specific *than* c_2 *and that* c_2 *is* more generic *than* c_1*.*

Example 4. *A simple example of context poset, which refers to the scenario discussed in Example 2, is shown in Figure 1. In this example, since "summer in Naples" is more specific than "summer in Italy", we have that* $c_4 <_C c_3$*.* □

Later, in Section 7.2.1, we shall present a specific context model of practical relevance that conforms to Definition 1.

In this paper, we consider the well-known *binary relation model* for expressing preferences over a domain of objects O [16, 24, 38].

Definition 2 (Preference relation). *A* preference relation *over objects of a domain* O *is a strict partial order* $<_O$ *on* O*. Given a pair of objects* o_1 *and* o_2 *in* O*, if* $o_1 <_O o_2$ *then* o_2 is preferable to o_1*, also written* $o_1 \prec_O o_2$*.*

A refinement of the unordered relation $\sim_O$ associated with a preference relation $\prec_O$ allows some unordered objects to be considered as *indifferent*, which, as we will see, is a key property for the composition of preference relations.

Definition 3 (Indifferent and incomparable objects). *Given a preference relation* $\prec_O$*, an indifference relation* $\approx_O$ *is a subset of the unordered relation* $\sim_O$ *such that*
i) $\approx_O$ *is reflexive, symmetric, and transitive (thus an equivalence relation);*
ii) if $o_1 \approx o_2$ *then for all* o *in* O *such that* $o_1 \prec o$ $(o \prec o_1)$*, it is* $o_2 \prec o$ $(o \prec o_2)$*.*
If $o_1 \sim_O o_2$*, but* $o_1 \not\approx_O o_2$*, we say that* o_1 *and* o_2 *are* incomparable*, denoted* $o_1 \|_O o_2$*.*

Notice that, since $\|_O = \sim_O - \approx_O$, in order to completely characterize O, it suffices to consider the $\prec_O$ and $\approx_O$ relations, collectively referred to as a *preference structure*. The importance of the distinction between $\sim_O$ and $\approx_O$ will become clear in the next section.

In the following, for simplicity, we shall consider O as understood, and will omit it as a subscript from the relation symbols.

Let θ be one of $\prec, \succ, \approx, \|$, where $\succ$ denotes the inverse of $\prec$; we say that the *order relation* between a pair of objects o_1 and o_2 is θ if $o_1 \theta o_2$.

Example 5. *Let us consider the following objects:* `pasta`, `beef`, `salad`*, and* `pizza`*. A possible preference structure over these objects is:* `beef` $\prec$ `pasta`, `beef` $\prec$ `salad`*, and* `pasta` $\approx$ `salad`*. In words,* `pasta` *and* `salad` *are both preferable to* `beef`*, whereas* `pasta` *and* `salad` *are indifferent. It follows that* `pizza` *is incomparable with all other foods, i.e.,* `pizza` $\|$ o*, for* $o \in \{$`pasta`, `beef`, `salad`$\}$*. Figure 2 provides a graphical representation of this preference structure.*[3] □

For a finite domain O, the best objects (i.e., maximal elements) in O according to the preference relation $\prec$ can be selected by the *Best* operator β [49]:[4]

$$\beta_\prec(O) = \{o \in O \mid \nexists o' \in O, o \prec o'\}$$

[3] Similarly to the graphical representation of context posets, we represent a preference structure by a transitively reduced graph in which the nodes are white rounded rectangles (so as to avoid confusion with context posets), the directed arcs represent the preference relation, and additional undirected arcs represent indifference of objects.
[4] This operator is also called *winnow* [16] and *preference selection* [24].

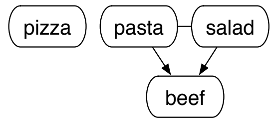


Fig. 2. A graphical representation of the preference structure of Example 5.

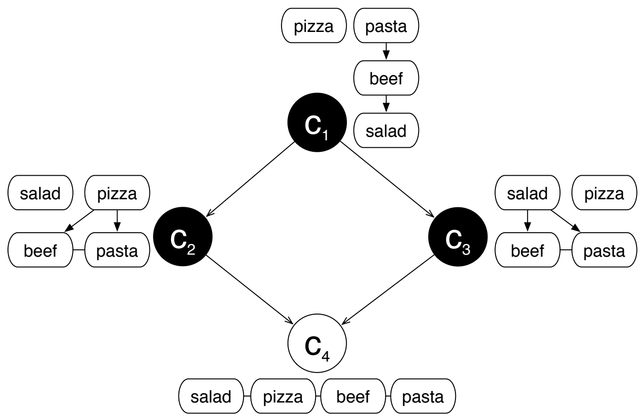


Fig. 3. Preference configuration for context poset C_1 in Figure 1; each ground preference relation is shown next to the corresponding context. Notice that c_4 is inactive (and thus shown as a blank circle).

For instance, according to the preferences of Example 5, the best objects are $\beta_{\prec}(O) = \{\texttt{pasta}, \texttt{salad}, \texttt{pizza}\}$. The restriction of Definition 2 to strict partial orders guarantees that, for any non-empty domain O, $\beta_{\prec}(O)$ is never empty [16]. In case O is infinite, it is customary to apply the $\beta_{\prec}$ operator to a finite subset of O (typically the portion that is stored in a database).

3 PREFERENCE PROPAGATION

Throughout the paper we consider a context poset C and a domain O, and assume that each context $c \in C$ is associated with a preference structure $\langle \prec^c, \approx^c \rangle$ over O, called the *ground preference structure* in c. We say that a context c is *active* whenever $\approx^c \neq O \times O$, i.e., not all objects are indifferent in c; c is *inactive* otherwise. In other words, c is active if there exists a pair of objects that are either incomparable or such that one is preferred to the other. We also call $\langle \prec^C, \approx^C \rangle = \{\langle \prec^c, \approx^c \rangle \mid c \in C\}$ (the set of all ground preference structures for all the contexts in C) a *preference (structure) configuration* over C.

Example 6. *Consider the context poset in Figure 3, which reproduces the context poset of Figure 1. Besides each context, Figure 3 also reports the ground preference structures that include the preferences discussed in Example 2, and others. Notice that in context c_4 all objects are indifferent, and thus c_4 is inactive. For the sake of clarity, we represent active contexts as circles with a black fill, whereas inactive ones are blank. In context c_1 ("Italy"),* `pasta` *is preferable to* `beef` *and* `beef` *to* `salad`*. On the other hand, in context c_2 ("Naples"),* `pizza` *is preferable to both* `pasta` *and* `beef`*, and* `pasta` $\approx^{c_2}$ `beef`*; similarly,* `pasta` $\approx^{c_3}$ `beef` *and* `pasta` $\approx^{c_4}$ `beef`. □

We assume that ground preferences are given or somehow derived from the application, and, for the sake of generality, we make no assumption on the language used to specify them. Specific models for expressing ground preferences will be considered in Section 7, when discussing complexity issues.

Since, as we have seen, preferences *propagate* along the poset C, we call *complete preference relation* in c, denoted by ${}^{\mathcal{P}}{\prec}^{c}$, the result of combining $\prec^c$ with the ground preferences defined in the other contexts in C, according to a *propagation method* $\mathcal{P}$. Such a method also defines how indifference of objects is propagated to c, denoted by ${}^{\mathcal{P}}{\approx}^{c}$ (and thus also the unordered relation, denoted ${}^{\mathcal{P}}{\sim}^{c}$, and the incomparability relation, denoted ${}^{\mathcal{P}}{\|}^{c}$), thereby defining a *complete preference structure* $\langle {}^{\mathcal{P}}{\prec}^{c}, {}^{\mathcal{P}}{\approx}^{c} \rangle$.

In abstract terms, a propagation method $\mathcal{P}$ is a function that associates a context poset C, a preference configuration $\langle \prec^C, \approx^C \rangle$ over C, and a target context $c \in C$, with a complete preference structure $\langle {}^{\mathcal{P}}{\prec}^{c}, {}^{\mathcal{P}}{\approx}^{c} \rangle$. In this paper, we focus on an approach in which such propagation methods are implemented through algebraic expressions, which will be discussed starting from the next section. However, in Section 6.4 we shall also present results that hold for any possible implementation of propagation methods.

Let us now try to capture the basic ideas underlying earlier, practical approaches on preference propagation (described in detail in Section 8). A commonly adopted notion [35, 43, 50] is that, for each context c, all and only the ground preferences in the contexts $c' \in C\lfloor c \rfloor$ (i.e., the contexts more generic than c) are *relevant* for determining ${}^{\mathcal{P}}{\prec}^{c}$. We can capture this requirement with the following notion of *coherent propagation*, which is based on the notion of relevance. In particular, we say that a context c' is relevant for another context c when a change in the ground preferences in c' may affect the propagated preferences in c.

Definition 4 (Coherence). *A context c' in a context poset C is* relevant *for $c \in C$ according to a propagation method $\mathcal{P}$ if there exist two preference configurations $\langle \prec_1^C, \approx_1^C \rangle$ and $\langle \prec_2^C, \approx_2^C \rangle$ differing only in context c' such that $\langle {}^{\mathcal{P}}{\prec_1}^{c}, {}^{\mathcal{P}}{\approx_1}^{c} \rangle \neq \langle {}^{\mathcal{P}}{\prec_2}^{c}, {}^{\mathcal{P}}{\approx_2}^{c} \rangle$.*

A propagation method $\mathcal{P}$ is coherent wrt. *C if, for every context c in C, the relevant contexts for c according to $\mathcal{P}$ are exactly those in $C\lfloor c \rfloor$; $\mathcal{P}$ is* coherent *if it is coherent wrt. every context poset C.*

As discussed in Example 2, two further basic properties should be satisfied by preference propagation. Specifically, given a context c:

(1) for each pair of ordered contexts $c_1 <_C c_2$ in $C\lfloor c \rfloor$, the ground preferences in c_1 should take precedence over those in c_2 in determining ${}^{\mathcal{P}}{\prec}^{c}$; in this case we say that the propagation is *specific*;
(2) for each pair of unordered contexts c_1 and c_2 in $C\lfloor c \rfloor$, the ground preferences in c_1 and c_2 should not take precedence over each other in determining ${}^{\mathcal{P}}{\prec}^{c}$; in this case we say that the propagation is *fair*.

With respect to the property of Point (1), it is natural to give more importance to preferences that hold in contexts "closer" to the one under consideration, as has also been argued in previous works (see, e.g., [35, 48]). As for Point (2), in order to combine preferences expressed in unordered contexts, a simplistic approach would be to take their intersection, i.e., to propagate only the preferences that hold in both contexts (see, e.g., [50]). Our criterion, which is tailored to deal with preferences complying with the binary relation model, is more flexible, just aiming to avoid the propagation of conflicting preferences.

A precise characterization of the above principles can be given as follows.

Definition 5 (Fairness). *A propagation method $\mathcal{P}$ is* fair *for a context c in a context poset C if the following holds for every pair of unordered contexts c_1 and c_2 in $C\lfloor c \rfloor$, every two objects $o_1, o_2 \in O$ and every preference configuration $\langle \prec^C, \approx^C \rangle$:*
if

i) $o_2 \prec^{c_1} o_1$,
ii) $o_1 \prec^{c_2} o_2$,
iii) $o_1 \approx^{c_i} o_2$ *for each* c_i *such that* $c \leqslant_C c_i <_C c_1 \vee c \leqslant_C c_i <_C c_2$,
then o_1 *and* o_2 *are unordered in the complete preferences for* c*, i.e.,* $o_1 \,{}^{\mathcal{P}}\!\!\sim^{c} o_2$.

A propagation method $\mathcal{P}$ *is* fair *if it is fair for every context* c *in every context poset* C.

Basically, Definition 5 asserts that if c_1 and c_2 disagree on how to order o_1 and o_2 while such objects are indifferent in all the more specific contexts, then o_1 and o_2 are not ordered in ${}^{\mathcal{P}}\!\!\prec^{c}$.

A different approach to ensuring fairness would be to settle disagreements by, for instance, a notion of majority, i.e., given n unordered contexts and a pair of objects o_1 and o_2, the preference between o_1 and o_2 that holds in the majority of such contexts would be propagated to c. However, such an approach can lead to the presence of preference cycles, even in the case in which the preference relations in the n contexts are strict partial orders [41].

Definition 6 (Specificity). *A propagation method* $\mathcal{P}$ *is* specific *for a context* c *in a context poset* C *if the following holds for every context* c_1 *in* $C\lfloor c \rfloor$*, every two objects* $o_1, o_2 \in O$ *and every preference configuration* $\langle \prec^C, \approx^C \rangle$*:*
if
i) $o_1 \prec^{c_1} o_2$,
ii) $o_1 \approx^{c_i} o_2$ *for each* $c \leqslant_C c_i <_C c_1$*, and*
iii) it is either $o_1 \prec^{c_2} o_2$ *or* $o_1 \approx^{c_2} o_2$ *for all* $c_2 \in C\lfloor c \rfloor$ *such that: 1)* c_2 *is unordered wrt.* c_1*, and 2)* $o_1 \approx^{c_3} o_2$ $\forall c_3$ *such that* $c \leqslant_C c_3 <_C c_2$,
then $o_1 \,{}^{\mathcal{P}}\!\!\prec^{c} o_2$.

A propagation method $\mathcal{P}$ *is* specific *if it is specific for every context* c *in every context poset* C.

Definition 6 states that, if o_2 is preferable to o_1 in c_1 and such objects are indifferent in all the more specific contexts, then this preference does indeed propagate to context c. In other words, in the propagation, the preferences that hold in any more generic context c' are overridden by those that hold in a more specific context $c_1 <_C c'$. The preference must not propagate, however, if a conflicting ground preference occurs in some other context in C unordered with respect to c_1 (point *iii*), in accordance with the fairness principle.

Example 7. *Consider the context poset in Figure 3. By Definition 5, in a fair propagation method* $\mathcal{P}$*,* `pasta` *and* `pizza` *must be unordered in context* c_4*, since* `pasta` $\prec^{c_2}$ `pizza` *whereas* `pizza` $\prec^{c_3}$ `pasta`*, and similarly for* `beef` *and* `pizza`*, i.e.,* `pasta` ${}^{\mathcal{P}}\!\!\sim^{c_4}$`pizza` *and* `beef` ${}^{\mathcal{P}}\!\!\sim^{c_4}$`pizza`*. By Definition 6, when the propagation method* $\mathcal{P}$ *is specific, then* `pasta` *is preferable to* `beef` *in contexts* c_2*,* c_3*, and* c_4*, i.e.,* `beef` ${}^{\mathcal{P}}\!\!\prec^{c_2}$`pasta`*,* `beef` ${}^{\mathcal{P}}\!\!\prec^{c_3}$`pasta`*, and* `beef` ${}^{\mathcal{P}}\!\!\prec^{c_4}$`pasta`.

□

For preference propagation to occur, point *iii* of Definition 6 requires that in c_2 the two objects o_1 and o_2 be either ordered as in c_1 or indifferent. Note that allowing them to simply be unordered but not indifferent (i.e., $o_1 \parallel^{c_2} o_2$) might lead to a preference relation that is no longer a strict partial order, as illustrated below in Example 8.

Example 8. *Let us consider the objects* o_1*,* o_2*, and* o_3 *and the contexts* c, c_1, c_2, c_3 *such that* $c <_C c_i$*, for* $i = 1, 2, 3$*, with the following ground preferences:*

$$o_2 \prec^{c_1} o_1, \quad o_3 \prec^{c_2} o_2, \quad o_1 \prec^{c_3} o_3,$$

as represented in Figure 4. We have the following unordered pairs of objects in c_1, c_2, c_3*:*

$$o_3 \sim^{c_1} o_2, \quad o_1 \sim^{c_1} o_3, \quad o_1 \sim^{c_2} o_3, \quad o_2 \sim^{c_2} o_1, \quad o_2 \sim^{c_3} o_1, \quad o_3 \sim^{c_3} o_2.$$

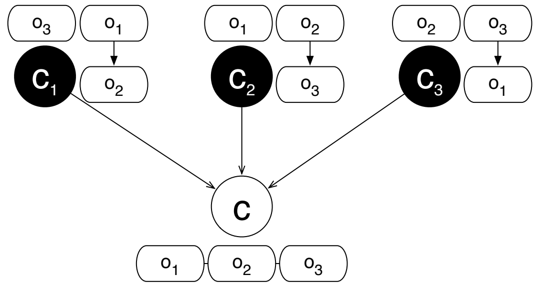


Fig. 4. Contexts and ground preferences for Example 8.

However, they are not *indifferent in such contexts, e.g., $o_3 \not\approx^{c_1} o_2$. Note that if point iii of Definition 6 had allowed the objects to be simply unordered (rather than indifferent), then all ground preferences would have propagated to context c by any specific method $\mathcal{P}$, i.e.: $o_2 \,{}^{\mathcal{P}}\!\!\prec^{c} o_1$, $o_3 \,{}^{\mathcal{P}}\!\!\prec^{c} o_2$ and $o_1 \,{}^{\mathcal{P}}\!\!\prec^{c} o_3$. Observe that, in such a case, $o_3 \,{}^{\mathcal{P}}\!\!\prec^{c} o_1$ would not hold, and thus transitivity would be lost. If we were to add the missing preferences, such as $o_3 \,{}^{\mathcal{P}}\!\!\prec^{c} o_1$, by taking the transitive closure, we would then have a conflict with $o_1 \,{}^{\mathcal{P}}\!\!\prec^{c} o_3$, thus losing asymmetry of strict partial orders.* □

Having established the main desiderata for preference propagation, we can now state the main problems that will be studied in the remainder of the paper.

Problem 1 (CFS). *To determine whether there exist well-behaved (i.e., Coherent, Fair and Specific) methods for the propagation of preferences.*

Problem 2 (OrdRel). *To characterize the complete preference structure $\langle {}^{\mathcal{P}}\!\!\prec^{c}, {}^{\mathcal{P}}\!\!\approx^{c} \rangle$ in a context $c \in C$ given a preference configuration $\langle \prec^{C}, \approx^{C} \rangle$ and a propagation method $\mathcal{P}$, i.e., given any two objects o_1 and o_2, to efficiently compute their order relation in $\langle {}^{\mathcal{P}}\!\!\prec^{c}, {}^{\mathcal{P}}\!\!\approx^{c} \rangle$.*

Problem 3 (Best). *Establish the best objects according to the complete preferences in a context $c \in C$ given a preference configuration $\langle \prec^{C}, \approx^{C} \rangle$.*

4 PREFERENCE COMPOSITION EXPRESSIONS

The properties of specificity and fairness suggest that the complete preference structures can be computed by means of expressions involving two generic binary operators, $+$ and $\triangleright$, that, given two ground preference structures $\langle \prec^{c_1}, \approx^{c_1} \rangle$ and $\langle \prec^{c_2}, \approx^{c_2} \rangle$, return a new preference structure:

- $\langle \prec^{c_1}, \approx^{c_1} \rangle + \langle \prec^{c_2}, \approx^{c_2} \rangle$, which applies when c_1 and c_2 are unordered; with a slight abuse of notation, we denote the resulting structure as $\langle \prec^{c_1} + \prec^{c_2}, \approx^{c_1} + \approx^{c_2} \rangle$;
- $\langle \prec^{c_1}, \approx^{c_1} \rangle \triangleright \langle \prec^{c_2}, \approx^{c_2} \rangle$, which applies when $c_1 <_C c_2$; similarly, we denote the result as $\langle \prec^{c_1} \triangleright \prec^{c_2}, \approx^{c_1} \triangleright \approx^{c_2} \rangle$.

Clearly, $+$ is commutative whereas this is not the case for $\triangleright$. Both operators are associative, since it is reasonable to assume that their composition does not depend on the order in which preferences are considered. Also, they are both idempotent since the combination of the same preferences should not have any effect. Finally, the identity element for both operators is what we call the *full indifference structure* $\emptyset_{\approx} = \langle \emptyset, O \times O \rangle$, i.e., the structure in which all elements are indifferent (thus equivalent). The rationale for this is that contexts in which all elements are considered to be equivalent (i.e., inactive contexts) should not influence the result at all.

Summarizing, these operators are characterized as follows.

Definition 7 (+ operator). *A $+$ operator satisfies the following axioms, for all objects $o_1, o_2 \in O$ and all preference structures $\langle \prec_1, \approx_1 \rangle, \langle \prec_2, \approx_2 \rangle, \langle \prec_3, \approx_3 \rangle$:*

i) $\langle \prec_1, \approx_1 \rangle + \langle \prec_2, \approx_2 \rangle = \langle \prec_2, \approx_2 \rangle + \langle \prec_1, \approx_1 \rangle$ *(commutativity)*
ii) $(\langle \prec_1, \approx_1 \rangle + \langle \prec_2, \approx_2 \rangle) + \langle \prec_3, \approx_3 \rangle = \langle \prec_1, \approx_1 \rangle + (\langle \prec_2, \approx_2 \rangle + \langle \prec_3, \approx_3 \rangle)$ *(associativity)*
iii) $\langle \prec_1, \approx_1 \rangle + \langle \prec_1, \approx_1 \rangle = \langle \prec_1, \approx_1 \rangle$ *(idempotence)*
iv) $\langle \prec_1, \approx_1 \rangle + \emptyset_{\approx} = \langle \prec_1, \approx_1 \rangle$ *(identity element)*
v) $o_1 \prec_1 o_2, o_2 \prec_2 o_1 \Rightarrow \neg(o_1 \prec_1 + \prec_2 o_2) \wedge \neg(o_2 \prec_1 + \prec_2 o_1)$ *(fairness)*

Definition 8 ($\rhd$ operator). *A $\rhd$ operator satisfies the following axioms, for all objects $o_1, o_2 \in O$ and all preference structures $\langle \prec_1, \approx_1 \rangle, \langle \prec_2, \approx_2 \rangle, \langle \prec_3, \approx_3 \rangle$:*
i) $(\langle \prec_1, \approx_1 \rangle \rhd \langle \prec_2, \approx_2 \rangle) \rhd \langle \prec_3, \approx_3 \rangle = \langle \prec_1, \approx_1 \rangle \rhd (\langle \prec_2, \approx_2 \rangle \rhd \langle \prec_3, \approx_3 \rangle)$ *(associativity)*
ii) $\langle \prec_1, \approx_1 \rangle \rhd \langle \prec_1, \approx_1 \rangle = \langle \prec_1, \approx_1 \rangle$ *(idempotence)*
iii) $\langle \prec_1, \approx_1 \rangle \rhd \emptyset_{\approx} = \emptyset_{\approx} \rhd \langle \prec_1, \approx_1 \rangle = \langle \prec_1, \approx_1 \rangle$ *(identity element)*
iv) $o_1 \prec_1 o_2 \Rightarrow o_1 \prec_1 \rhd \prec_2 o_2$ *(specificity)*

Preference structures can then be combined with these two operators to form a so-called *PC-expression*, as specified below.

Definition 9 (PC-expression). *A* preference composition expression*, or* PC-expression*, over a poset C is any expression E of the form:*[5] $E ::= c \mid (E + E) \mid (E \rhd E) \mid \bot$*, where c is a context in C.*

In other words, the base case of a PC-expression is the name of some context c, which denotes the corresponding preference structure $\langle \prec^c, \approx^c \rangle$; additionally, one can compose PC-expressions (thus, ultimately, preference structures) via the $+$ and $\rhd$ operators; finally, one can also denote the full indifference structure $\emptyset_{\approx}$ via the $\bot$ symbol.

In our approach, we express propagation methods in terms of PC-expressions. For this reason, we also write ${}^{E}\!\prec$ and ${}^{E}\!\approx$ instead of ${}^{\mathcal{P}}\!\prec^c$ and ${}^{\mathcal{P}}\!\approx^c$ when referring to the result of applying a specific PC-expression E rather than a general propagation method $\mathcal{P}$ for computing the complete preferences in context c. A standard notion of semantic equivalence between PC-expressions can be given as follows.

Definition 10 (Equivalent PC-expressions). *Given a context poset C, two PC-expressions E and E' over C are* equivalent*, written $E \equiv E'$, iff, for every preference configuration $\langle \prec^C, \approx^C \rangle$, it is $\langle {}^{E}\!\prec, {}^{E}\!\approx \rangle = \langle {}^{E'}\!\prec, {}^{E'}\!\approx \rangle$.*

Clearly, thanks to the axioms of the $+$ and $\rhd$ operators (Definitions 7 and 8, respectively), corresponding equivalence rules are available for PC-expressions. For instance, $E + E \equiv E$ thanks to idempotence of $+$, and $E \rhd \bot \equiv \bot \rhd E \equiv E$ thanks to the identity element axiom of $\rhd$, and so on.

Also observe that the notions of fairness and specificity can be extended to PC-expressions in a straightforward way by considering ${}^{E}\!\prec$ instead of ${}^{\mathcal{P}}\!\prec^c$ in the respective Definitions 5 and 6.

We now introduce some notable PC-expressions that will play a relevant role in the study of propagation methods in the next section. The first of these simply considers the maximal chains in $C\lfloor c \rfloor$, thus reproducing the structure of the poset $C\lfloor c \rfloor$ as a PC-expression.

Definition 11. *Let c be a context in C and let $\mathcal{H}_C(c) = \{H_1, \dots, H_l\}$ be the set of all the maximal chains in $C\lfloor c \rfloor$. For $H = \langle c_1, \dots, c_h \rangle$, let $\rhd(H)$ be a shorthand for the expression: $(c_1 \rhd \dots \rhd c_h)$. Then the expression*

$$Can^C(c) = \rhd(H_1) + \dots + \rhd(H_l) \tag{1}$$

is called the canonical expression *for computing the complete preference structure for a context c in the context poset C.*

[5] Parentheses are omitted when no ambiguity arises in the application of the operators.

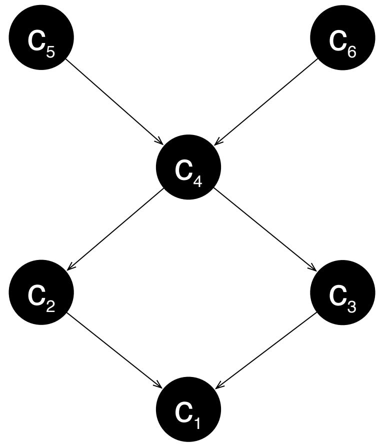


Fig. 5. The context poset C_2.

Example 9. *The maximal chains in the context poset C_1 in Figure 1 are $\langle c_4, c_2, c_1\rangle$ and $\langle c_4, c_3, c_1\rangle$. It follows that the canonical expression for computing the complete preference structure for a context c_4 in the poset C_1 is:*

$$Can^{C_1}(c_4) = (c_4 \rhd c_2 \rhd c_1) + (c_4 \rhd c_3 \rhd c_1)$$

For a more complex example, consider context poset C_2 in Figure 5, in which c_1, the minimal context, is covered by c_2 and c_3 (i.e., $c_1 \lessdot_{C_2} c_2$ and $c_1 \lessdot_{C_2} c_3$), which are both covered by c_4 (i.e., $c_2 \lessdot_{C_2} c_4$ and $c_3 \lessdot_{C_2} c_4$); finally, the two maximal contexts, c_5 and c_6, both cover c_4 (i.e., $c_4 \lessdot_{C_2} c_5$ and $c_4 \lessdot_{C_2} c_6$). The maximal chains in C_2 are $\langle c_1, c_2, c_4, c_5\rangle$, $\langle c_1, c_2, c_4, c_6\rangle$, $\langle c_1, c_3, c_4, c_5\rangle$, and $\langle c_1, c_3, c_4, c_6\rangle$. Therefore, the canonical expression for c_1 is:

$$\begin{aligned} Can^{C_2}(c_1) = \quad & (c_1 \rhd c_2 \rhd c_4 \rhd c_5) + (c_1 \rhd c_2 \rhd c_4 \rhd c_6) \\ & +(c_1 \rhd c_3 \rhd c_4 \rhd c_5) + (c_1 \rhd c_3 \rhd c_4 \rhd c_6) \end{aligned}$$

□

The following Definition 12 provides a recursive way to compute the complete preference structure for a context c.

Definition 12. *Let c be a context in C and let $cov^C(c) = \{c_1, \ldots, c_k\}$ be the cover of c in C. The PC-expression indicated by $Rec^C(c)$ is recursively defined as follows:*

$$\begin{aligned} Rec^C(c) &= c && \text{if } cov^C(c) = \emptyset \\ Rec^C(c) &= c \rhd (Rec^C(c_1) + \ldots + Rec^C(c_k)) && \text{if } cov^C(c) \neq \emptyset \end{aligned} \tag{2}$$

Example 10. *The recursive expression $Rec^{C_1}(c_4)$ for c_4 in poset C_1 in Figure 1 is*

$$\begin{aligned} Rec^{C_1}(c_4) &= c_4 \rhd (Rec^{C_1}(c_2) + Rec^{C_1}(c_3)) \\ &\equiv c_4 \rhd ((c_2 \rhd Rec^{C_1}(c_1)) + (c_3 \rhd Rec^{C_1}(c_1))) \\ &\equiv c_4 \rhd ((c_2 \rhd c_1) + (c_3 \rhd c_1)) \end{aligned}$$

The recursive expression $Rec^{C_2}(c_1)$ for c_1 in poset C_2 in Figure 5 is

$$Rec^{C_2}(c_1) = c_1 \rhd ((c_2 \rhd c_4 \rhd (c_5 + c_6)) + (c_3 \rhd c_4 \rhd (c_5 + c_6)))$$

□

We observe that the PC-expressions $\mathrm{Can}^C(c)$ and $\mathrm{Rec}^C(c)$ are generally different, as is the case, e.g., with $\mathrm{Can}^{C_2}(c_1)$ and $\mathrm{Rec}^{C_2}(c_1)$, shown in Examples 9 and 10, respectively. However, if left-distributivity of $\triangleright$ over $+$ is assumed, as will be done in the next section, then the two expressions become equivalent.

5 INTERPRETING THE PROPAGATION OPERATORS

In this section we investigate on possible interpretations of the operators $+$ and $\triangleright$. As a first step of our analysis, we start with a general result about *idempotent semirings*. We remind that a semiring is an algebraic structure in which there is an associative and commutative additive operator (like $+$) as well as an associative multiplicative operator (like $\triangleright$), which is both left- and right-distributive over addition. Let $(\mathcal{PR}_O, +, \triangleright)$ be an algebraic structure, where $\mathcal{PR}_O$ denotes the set of all preference structures over a domain O. Then, with the additional hypothesis that $\triangleright$ distributes over $+$, $(\mathcal{PR}_O, +, \triangleright)$ would be a semiring in which both operators are idempotent, i.e., an idempotent semiring.

However, the following major result rules out the possibility of using idempotent semirings for providing an interpretation to the propagation operators, since distributivity of $\triangleright$ over $+$ turns out to be incompatible with the axioms of specificity and fairness of the operators.

Theorem 1. *No $(\mathcal{PR}_O, +, \triangleright)$ structure is an idempotent semiring.*

A close inspection of the proof of Theorem 1 (see Appendix) reveals that the cause of incompatibility of distributivity with the axioms of the operators lies only in assuming that $\triangleright$ *right-distributes* over $+$. For this reason, here we consider the case in which $(\mathcal{PR}_O, +, \triangleright)$ is an *idempotent left near-semiring*, that is, an algebraic structure satisfying all requirements of idempotent semirings except for right-distributivity of $\triangleright$. Note that we still assume that $\triangleright$ left-distributes over $+$.

Let us now consider two popular ways to combine preference relations that satisfy all the axioms required for $+$ and $\triangleright$: Pareto and Prioritized composition [16, 24]. As it turns out (Corollary 1 below), such operators form an idempotent left near-semiring.

Definition 13 (Pareto and Prioritized composition). *Let $\langle \prec_1, \approx_1 \rangle$ and $\langle \prec_2, \approx_2 \rangle$ be two preference structures over a domain O. The* Prioritized composition *of $\langle \prec_1, \approx_1 \rangle$ and $\langle \prec_2, \approx_2 \rangle$, written $\langle \prec_1, \approx_1 \rangle \rhd\!\!\!\!\bigcirc \langle \prec_2, \approx_2 \rangle$, is defined as*

$$o_1 \prec_1 \rhd\!\!\!\!\bigcirc \prec_2 o_2 \Leftrightarrow (o_1 \prec_1 o_2) \vee (o_1 \prec_2 o_2 \wedge o_1 \approx_1 o_2)$$
$$o_1 \approx_1 \rhd\!\!\!\!\bigcirc \approx_2 o_2 \Leftrightarrow o_1 \approx_1 o_2 \wedge o_1 \approx_2 o_2.$$

and their Pareto composition, *written $\langle \prec_1, \approx_1 \rangle \oplus \langle \prec_2, \approx_2 \rangle$, is:*

$$o_1 \prec_1 \oplus \prec_2 o_2 \Leftrightarrow (o_1 \prec_1 o_2 \wedge o_1 \prec_2 o_2) \vee (o_1 \prec_1 o_2 \wedge o_1 \approx_2 o_2) \vee (o_1 \approx_1 o_2 \wedge o_1 \prec_2 o_2)$$
$$o_1 \approx_1 \oplus \approx_2 o_2 \Leftrightarrow o_1 \approx_1 o_2 \wedge o_1 \approx_2 o_2.$$

where o_1 and o_2 are any two objects in O.

Intuitively, Prioritized composition gives precedence to preferences in $\prec_1$, while preferences in $\prec_2$ are used only if two objects are indifferent according to $\prec_1$. Conversely, Pareto considers the two preference relations equally important.

Example 11. *Consider the two preference relations*

$$\prec_1 = \{o_1 \prec_1 o_2, o_1 \prec_1 o_3\} \quad \prec_2 = \{o_2 \prec_2 o_1, o_2 \prec_2 o_3\}$$

and let $o_2 \approx_1 o_3$ and $o_1 \approx_2 o_3$. Then, the Prioritized composition $\prec_1 \rhd\!\!\!\!\bigcirc \prec_2$ yields the preferences $o_1 \prec_1 \rhd\!\!\!\!\bigcirc \prec_2 o_2$, $o_1 \prec_1 \rhd\!\!\!\!\bigcirc \prec_2 o_3$, and $o_2 \prec_1 \rhd\!\!\!\!\bigcirc \prec_2 o_3$, whereas the Pareto composition $\prec_1 \oplus \prec_2$ yields $o_1 \prec_1 \oplus \prec_2 o_3$ and $o_2 \prec_1 \oplus \prec_2 o_3$.

For a more concrete example, consider the following objects in O:

- o_1*: a comedy with Adam Sandler.*
- o_2*: a comedy without Adam Sandler.*
- o_3*: a drama (or any other non-comedy genre) with Adam Sandler.*
- o_4*: a drama without Adam Sandler.*

Let $\prec_1$ be the preference relation corresponding to the statement "I prefer comedies to all other movie genres"; similarly, let $\prec_2$ correspond to "I prefer movies with Adam Sandler to all other movies". If we consider preference relation $\prec_1$, then o_1 and o_2 are both preferable to o_3 and o_4; furthermore, $o_1 \approx_1 o_2$ and $o_3 \approx_1 o_4$. With $\prec_2$, we have instead that o_1 and o_3 are both preferable to o_2 and o_4, with $o_1 \approx_2 o_3$ and $o_2 \approx_2 o_4$.

Let $\prec_{\mathtt{Par}} = \prec_1 \oplus \prec_2$; then we have $o_2 \prec_{\mathtt{Par}} o_1$, $o_3 \prec_{\mathtt{Par}} o_1$, $o_4 \prec_{\mathtt{Par}} o_2$, $o_4 \prec_{\mathtt{Par}} o_3$, with o_2 and o_3 that are incomparable ($o_2 \parallel_{\mathtt{Par}} o_3$).

Let now $\prec_{\mathtt{Pri}} = \prec_1 \rhd \prec_2$; then we have $o_4 \prec_{\mathtt{Pri}} o_3$, $o_3 \prec_{\mathtt{Pri}} o_2$, $o_2 \prec_{\mathtt{Pri}} o_1$.

In both cases, the resulting preference structure reflects the intuition that o_1 is always the best alternative, since it satisfies both preferences, while o_4 is the worst one. As for o_2 and o_3, they remain unordered if no priority is assumed between the two preferences (and thus Pareto composition is used), whereas o_2 is preferred to o_3 when $\prec_1$ has priority over $\prec_2$ (and thus Prioritized composition is used). □

Both Prioritized and Pareto composition preserve strict partial orders [25] (i.e., they both yield a strict partial order when applied to two strict partial orders), whereas this is not guaranteed by replacing in their definition $\approx$ with $\sim$ [16]. It is known that $\oplus$ is both commutative and associative and that $\rhd$ is associative [25] (but obviously not commutative). It is also evident that both operators are idempotent and have $\emptyset_{\approx}$ as the identity.

The following lemma shows that $\rhd$ left-distributes (but does not right-distribute) over $\oplus$.

Lemma 1. *Prioritized composition left-distributes over Pareto composition, that is, for all objects $o_1, o_2 \in O$ and all preference structures $\langle \prec_1, \approx_1 \rangle, \langle \prec_2, \approx_2 \rangle, \langle \prec_3, \approx_3 \rangle$, it is:*

$$o_1 \prec_1 \rhd (\prec_2 \oplus \prec_3) o_2 \Leftrightarrow o_1 (\prec_1 \rhd \prec_2) \oplus (\prec_1 \rhd \prec_3) o_2$$

Prioritized composition does not right-distribute over Pareto composition, that is, there exist objects $o_1, o_2 \in O$ and preference structures $\langle \prec_1, \approx_1 \rangle, \langle \prec_2, \approx_2 \rangle, \langle \prec_3, \approx_3 \rangle$ such that:

$$o_1 (\prec_2 \oplus \prec_3) \rhd \prec_1 o_2 \not\Leftrightarrow o_1 (\prec_2 \rhd \prec_1) \oplus (\prec_3 \rhd \prec_1) o_2$$

Due to the known properties of $\oplus$ and $\rhd$, by Lemma 1 we have the following corollary.

Corollary 1. *$(\mathcal{PR}_O, \oplus, \rhd)$ is an idempotent left near-semiring.*

From the above result, the following property follows.

Proposition 1. *When $+$ and $\triangleright$ are interpreted as $\oplus$ and $\rhd$, respectively, $Rec^C(c)$ is equivalent to $Can^C(c)$, for each context c and context poset C.*

Example 12. *Consider the preference configuration for context poset C_1 in Figure 6. Let E_i be the recursive expression $Rec^{C_1}(c_i)$ computed, for $i = 1, 2, 3, 4$, with $\oplus$ and $\rhd$. We have:*

$$\begin{array}{lcl} E_1 & = & c_1 \\ E_2 & = & c_2 \rhd c_1 \\ E_3 & = & c_3 \rhd c_1 \\ E_4 & = & c_4 \rhd ((c_2 \rhd c_1) \oplus (c_3 \rhd c_1)) \quad \equiv \quad c_4 \rhd (E_2 \oplus E_3) \end{array}$$

Preference propagation through E_i in c_i yields the complete preference structures shown in the gray boxes in Figure 6 for each context in C_1.

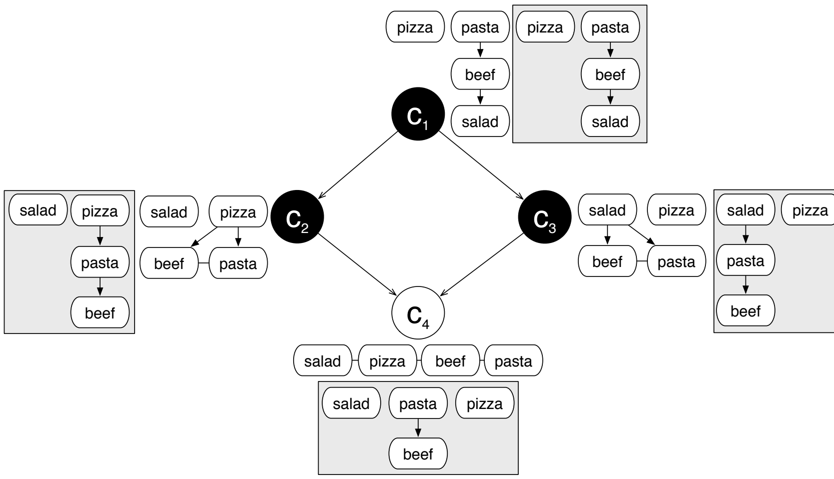


Fig. 6. Preference configuration for context poset C_1 in Figure 1; each ground preference structure is shown next to the corresponding context. Complete preference structures computed with $\mathrm{Can}^{C_1}(c_i)$ are shown in a gray box for each context c_i, $i = 1, 2, 3, 4$.

Clearly, ${}^{E_1}\!\!\prec\; = \prec^{c_1}$*. The complete preference relation in* c_2*, besides all the ground preferences in* $\prec^{c_2}$*, also includes* `beef` ${}^{E_2}\!\!\prec$ `pasta`*, which propagates from* c_1*. Note that we do not propagate preferences concerning* `salad` *to* c_2*, since* `salad` *is incomparable to all other objects in* c_2*. Similarly, the complete preference relation in* c_3 *includes* `beef` ${}^{E_3}\!\!\prec$ `pasta`*, which propagates from* c_1*; here, preferences concerning* `salad` *are not propagated from* c_1 *because they conflict with the ground preferences in* c_3*. Finally, since in* c_4 *all objects are indifferent, we have* `beef` ${}^{E_4}\!\!\prec$ `pasta`*, on which* ${}^{E_2}\!\!\prec$ *and* ${}^{E_3}\!\!\prec$ *agree; yet, we do not propagate to* c_4 *any preference concerning* `salad` *(incomparable to all other objects in* ${}^{E_2}\!\!\prec$*) or* `pizza` *(incomparable to all other objects in* ${}^{E_3}\!\!\prec$*). We remind, as illustrated in Example 8, that preference propagation does not occur when objects are incomparable, as this might cause cycles, thus leading to a preference relation that is no longer a strict partial order.*

Note that, by Proposition 1, E_4 *is equivalent to the expression* $E'_4 = (c_4 ⧁ c_2 ⧁ c_1) \oplus (c_4 ⧁ c_3 ⧁ c_1)$*, i.e., the canonical expression* $Can^C(c_4)$ *computed with Pareto and Prioritized composition.* □

One may wonder whether other interpretations, besides the one based on $\oplus$ and $⧁$, exist for the $+$ and $\rhd$ operators. Our answer is negative for an important class of operators, which we call *independent of irrelevant objects.*

Definition 14 (IIO operator). *An operator* $\diamond$ *for combining preferences is* independent of irrelevant objects *(IIO) if, for any two objects* o *and* o' *in a domain* O*, the order relation between* o *and* o' *according to the combined preference structure* $\langle \prec^1, \approx^1 \rangle \diamond \langle \prec^2, \approx^2 \rangle$ *only depends on the order relation between* o *and* o' *according to* $\langle \prec^1, \approx^1 \rangle$ *and* $\langle \prec^2, \approx^2 \rangle$

Thus, to determine the order relation between any two objects o_1 and o_2, an IIO operator does not need to consider any other objects in the domain O. In this respect, both $\oplus$ and $⧁$ are IIO. In Section 6.4, we provide further insight about non-IIO operators.

The two following theorems show that $\oplus$ and $⧁$ are the only possible IIO interpretations of $+$ and $\rhd$, i.e., there is no other IIO interpretation of $+$ (respectively, $\rhd$) that satisfies all the axioms of Definition 7 (respectively, 8).

Theorem 2. *Operator* $\oplus$ *is the only IIO* $+$ *operator.*

Theorem 3. *Operator* $\circledarrowright$ *is the only IIO* $\triangleright$ *operator.*

Because of the above results, any propagation method built on different interpretations of the $+$ and $\triangleright$ operators would fail to satisfy the fairness and specificity requirements. For this reason, unless otherwise stated, in the remainder of the paper we shall then adopt the semantics of $\oplus$ and $\circledarrowright$ for the operators $+$ and $\triangleright$, respectively, and use them to build expressions for computing the propagation of ground preferences.

6 PROPAGATION METHODS

In this section, we address the problem of building propagation methods via PC-expressions based on $\oplus$ and $\circledarrowright$. While $\oplus$ and $\circledarrowright$ guarantee, respectively, the fairness and specificity properties "locally", the challenge is to guarantee the "global" satisfaction of these properties, when a PC-expression involves an arbitrary number of sets of ground preferences.

A natural way to define a propagation method is to consider the whole structure of the poset. To show the inadequacy of alternative, naive approaches, let us first consider the following propagation methods, $\mathcal{N}_{\texttt{Par}}$ and $\mathcal{N}_{\texttt{Pri}}$.

Definition 15. *Let* c *be a context in a context poset* C *and let* $C\lfloor c \rfloor = \{c_1, \ldots, c_n\}$*. The complete preference structure in* c *under the*

- Naive-Pareto *propagation, denoted* $\mathcal{N}_{\texttt{Par}}$*, is computed as* $Par^C(c) = c_1 \oplus \ldots \oplus c_n$*;*
- Naive-Priori *propagation, denoted* $\mathcal{N}_{\texttt{Pri}}$*, is computed as* $Pri^C(c) = c_{\pi(1)} \circledarrowright \ldots \circledarrowright c_{\pi(n)}$*, where* $\pi(1), \ldots \pi(n)$ *is a permutation of* $1, \ldots, n$ *corresponding to a* linear extension *of the poset* $C\lfloor c \rfloor$*, i.e., if* $c_{\pi(i)} <_C c_{\pi(j)}$ *then* $i < j$*.*

It is straightforward to see that $\mathcal{N}_{\texttt{Par}}$ is fair (since it only uses $\oplus$ and not $\circledarrowright$) and $\mathcal{N}_{\texttt{Pri}}$ is specific (since, if c' is a successor of c, then the PC-expression generated by $\mathcal{N}_{\texttt{Pri}}$ is of the form $\ldots c \circledarrowright (\ldots c' \ldots) \ldots$). However, as shown in Example 13 below, $\mathcal{N}_{\texttt{Par}}$ is not specific and $\mathcal{N}_{\texttt{Pri}}$ is not fair.

Example 13. *Consider the poset in Figure 1. For context* c_4*,* $\mathcal{N}_{\texttt{Par}}$ *yields the PC-expression* $c_1 \oplus c_2 \oplus c_3 \oplus c_4$*. To see why* $\mathcal{N}_{\texttt{Par}}$ *violates specificity, consider a preference configuration such that* $o_1 \prec^{c_3} o_2$*,* $o_2 \prec^{c_1} o_1$*,* $o_1 \approx^{c_2} o_2$*, and* $o_1 \approx^{c_4} o_2$*. In such a case, specificity requires that the preference* $o_1 \prec o_2$ *be propagated to* c_4*, whereas* $\mathcal{N}_{\texttt{Par}}$ *yields* $o_1 \parallel o_2$ *in* c_4*.*

Consider now a preference configuration such that $o_1 \prec^{c_3} o_2$*,* $o_2 \prec^{c_2} o_1$*,* $o_1 \approx^{c_1} o_2$*, and* $o_1 \approx^{c_4} o_2$*. Any linear extension of* $C\lfloor c_4 \rfloor = C$ *adopted by* $\mathcal{N}_{\texttt{Pri}}$ *necessarily orders* c_2 *and* c_3*, thus either the preference* $o_1 \prec o_2$ *or* $o_2 \prec o_1$ *is propagated to* c_4*, thereby violating fairness.* □

All the approaches we study in the following take into account, in a more elaborate fashion, the structure of the poset.

The first approach, called *Complete Cover propagation* (Section 6.1), is a propagation method that recursively computes the complete preference structure in a context c based on the complete preference structure in the contexts *covering* c. As we shall discuss, this approach is fair but not specific, due to the negative role played in the propagation by inactive contexts.

In an attempt to address this issue, the second approach, called *Active Cover propagation* (Section 6.2), considers the contexts covering c among those that are active. Although this method can be shown to propagate more preferences than the Complete Cover propagation, it still fails to satisfy specificity.

The last proposal, called *Object-specific Cover propagation* (Section 6.3), focuses on individual pairs of objects and, for each pair, computes the propagation based on the contexts covering c

among those that contain some ground preference regarding the given pair. This is a propagation method that turns out to be both fair and specific.

After discussing the three methods sketched above, in Section 6.4 we provide a more general view of propagation methods.

The result presented in Theorem 4 below is relevant for the following discussions and applies to all coherent propagation methods based on $\oplus$ and $\rhd\!\!\!\!\bigcirc$. The main observation is that, for all such propagation methods, the resulting indifference relation is the same. The idea behind this result is that two objects are indifferent if and only if indifference holds in all contexts whose ground preferences appear in the PC-expression; due to coherence, any such PC-expressions must include all the same contexts (i.e., the successors of the target context).

Theorem 4. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two coherent propagation methods computed, for a context $c \in C$, by PC-expressions using $\oplus$ for $+$ and $\rhd\!\!\!\!\bigcirc$ for $\rhd$. Then ${}^{\mathcal{P}_1}\!\!\approx^c = {}^{\mathcal{P}_2}\!\!\approx^c$.*

6.1 The Complete Cover Propagation

The first way of using a PC-expression for computing preference propagation, which we call CC (Complete Cover), is based on an intuitive argument: the complete preference structure in a context c can be obtained recursively by composing the ground preference structure in c, $\langle \prec^c, \approx^c \rangle$, with the complete preference structures that hold in the contexts that cover c in the context poset C. This is precisely what is done with the $\mathrm{Rec}^C(c)$ PC-expression, which makes use of the notion of cover. We remind that $\mathrm{cov}^C(c)$ indicates the set of elements in C that cover c, i.e., those contexts that are "immediately above" c in C.

Definition 16. *Let c be a context in a context poset C, with $cov^C(c) = \{c_1, \ldots, c_k\}$. The complete preference structure in c under the CC propagation, denoted $\langle {}^{CC}\!\!\prec^c, {}^{CC}\!\!\approx^c \rangle$, is computed as $Rec^C(c)$, i.e.,*

$$\begin{aligned} Rec^C(c) &= c && \text{if } cov^C(c) = \emptyset \\ Rec^C(c) &= c \rhd\!\!\!\!\bigcirc (Rec^C(c_1) \oplus \ldots \oplus Rec^C(c_k)) && \text{if } cov^C(c) \neq \emptyset \end{aligned} \tag{3}$$

By Corollary 1, when left-distributivity holds, as is the case with $\oplus$ and $\rhd\!\!\!\!\bigcirc$, the complete preference structure can equivalently be computed via the canonical expression $\mathrm{Can}^C(c)$.

Example 14. *Consider the preference configuration shown in Figure 6 for the context poset C_1 of Example 2. The gray boxes in the figure show the complete preference structures computed, for each context c_i, $i = 1, 2, 3, 4$, via $Can^{C_1}(c_i)$. By Corollary 1, these are exactly the complete preference structures $\langle {}^{CC}\!\!\prec^{c_i}, {}^{CC}\!\!\approx^{c_i} \rangle$ in c_i computed under the CC propagation.* □

In spite of the intuitive form of $\mathrm{Rec}^C(c)$, we have the following negative result.

Theorem 5. *CC propagation is coherent and fair but not specific.*

Example 15. *Let us slightly revise the preference configuration of Figure 6 by assuming that all objects are indifferent in c_2, as depicted in Figure 7. By proceeding as in Example 14, one derives that* `beef` ${}^{CC}\!\!\prec^{c_4}$ `pasta` *is the only preference in ${}^{CC}\!\!\prec^{c_4}$. In particular, since the complete preference structure in c_2 coincides with that in c_1, it is* `salad` ${}^{CC}\!\!\prec^{c_2}$ `pasta`*. Therefore the preference* `pasta` $\prec^{c_3}$ `salad` *is not propagated to c_4, which contradicts the specificity principle. Ditto for* `beef` $\prec^{c_3}$ `salad`*.* □

6.2 The Active Cover Propagation

Both Pareto and Prioritized composition have the full indifference structure $\emptyset_\approx$ as identity element, which matches the intuition that the absence of preferences in a context does not influence the

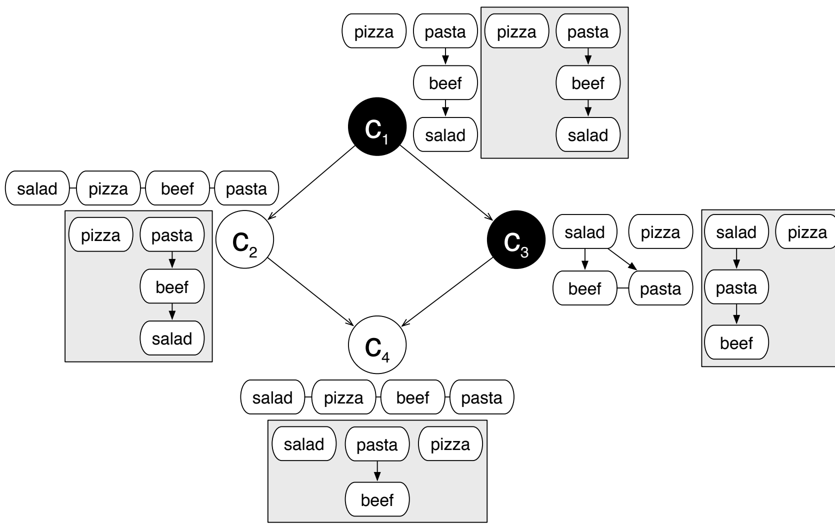


Fig. 7. A preference configuration for Example 15 using context poset C_1 from Figure 1, where context c_2 has no ground preference; each ground preference structure is shown next to the corresponding context. Complete preference structures computed under the $\mathcal{CC}$ propagation are shown in a gray box for each context c_i, $i = 1, 2, 3, 4$.

behavior of the two operators. However, Example 15 shows that inactive contexts, such as c_2, might invalidate the specificity property of the whole propagation process. In order to avoid such an undesirable behavior, we now introduce an alternative way of computing the complete preference structures that does not consider at all such contexts.

Definition 17. *Given a context poset C and a preference configuration $\langle \prec^C, \approx^C \rangle$, the* active poset *$A$ is the poset induced by the set of active contexts in $\langle \prec^C, \approx^C \rangle$.*[6]

In the following, we shall often consider the cover of a context c with respect to the poset $A \subseteq C$ of all active contexts, i.e., $\mathrm{cov}^A(c)$. However, when c is inactive, the cover of c in A is ill-defined, since c is not one of the contexts in A. Addressing this case requires considering, instead of A, a different poset, which we may denote A_c, i.e., the poset induced, through C, by the set of vertices $A \cup \{c\}$. However, in order to avoid extra notational burden, with a slight abuse of notation, we shall henceforth indicate with $\mathrm{cov}^A(c)$ the set of all the *active* contexts covering c in A_c; similarly, $A\lfloor c \rfloor$ will denote the set of successors of c in A_c.

Note that the notions of cover $\mathrm{cov}^C(c)$ and "active" cover $\mathrm{cov}^A(c)$ are different, as can be observed, e.g., by looking at poset C_1 in Figure 8, in which the cover of c_4 is $\mathrm{cov}^{C_1}(c_4) = \{c_2, c_3\}$, while its active cover is $\mathrm{cov}^A(c_4) = \{c_3\}$. Also note that there is no inclusion between cover and active cover, as can be observed by looking at poset C_3 in Figure 9a, where the cover of c is $\{c_1, c_2, c_3\}$, while its active cover is $\{c_1, c_5\}$.

We are now ready to introduce the main notion of this subsection, i.e., the *Active Cover* ($\mathcal{AC}$) *propagation*, which is defined through the recursive PC-expression $\mathrm{Rec}^A(c)$, which makes use of the contexts in the active cover $\mathrm{cov}^A(c)$ with respect to the poset $A \subseteq C$ of all active contexts, rather than the whole C.

[6]Although, technically, A depends on both the poset C and preference configuration $\langle \prec^C, \approx^C \rangle$, for readability, we omit such a dependency in the notation.

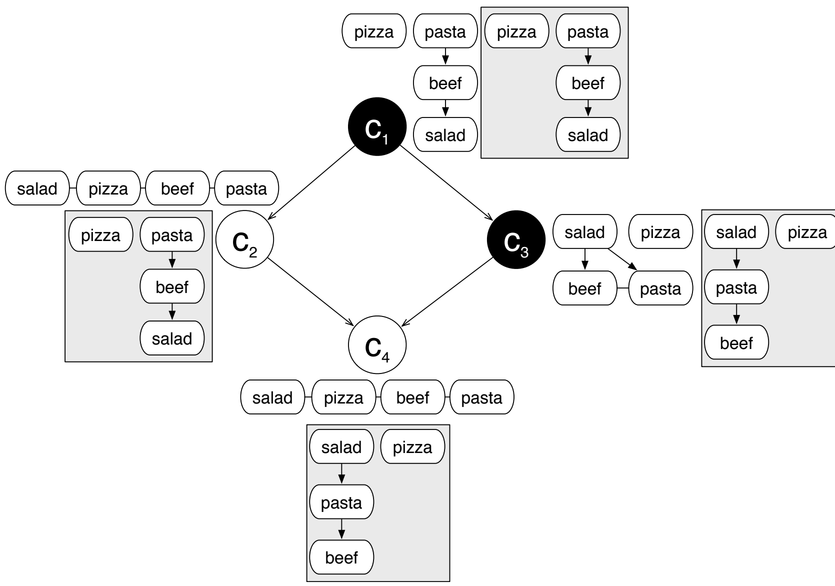


Fig. 8. Same preference configuration as Figure 7. Here, the complete preference structures are computed under the $\mathcal{AC}$ propagation, shown in a gray box for each context c_i, $i = 1, 2, 3, 4$. Notice that c_2 and c_4 are inactive.

Definition 18. *Let c be a context in a context poset C, and let $cov^A(c) = \{c_1, \ldots, c_l\}$. The complete preference structure in c under the $\mathcal{AC}$ propagation, denoted $\langle {}^{\mathcal{AC}}\!\prec^{c}, {}^{\mathcal{AC}}\!\approx^{c}\rangle$, is computed as $Rec^A(c)$, i.e.,*

$$
\begin{aligned}
Rec^A(c) &= c && \text{if } cov^A(c) = \emptyset \\
Rec^A(c) &= c \circledcirc (Rec^A(c_1) \oplus \ldots \oplus Rec^A(c_l)) && \text{if } cov^A(c) \neq \emptyset
\end{aligned}
\qquad (4)
$$

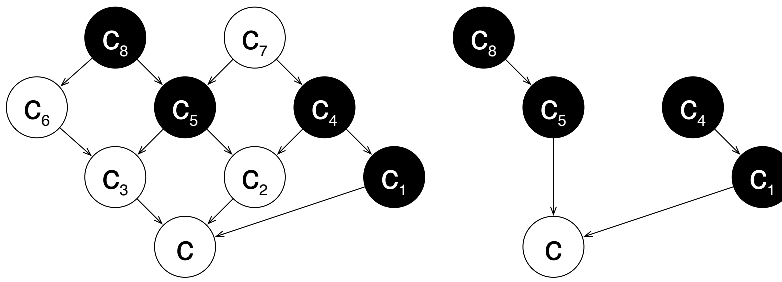
In order to characterize the relationship between CC and $\mathcal{AC}$ propagation, we introduce the following preliminary result.

Lemma 2. *Let H_1 and H_2 be two chains, such that $H_2 \subseteq H_1$. Let $\langle \prec_{1,2}, \approx_{1,2}\rangle$ be the preference structure denoted by $(\circledcirc(H_1)) \oplus (\circledcirc(H_2))$ and $\langle \prec_1, \approx_1\rangle$ the preference structure denoted by $\circledcirc(H_1)$. Then: i) $\approx_{1,2} = \approx_1$, and ii) $\prec_{1,2} \subseteq \prec_1$.*

Example 16. *Figure 8 shows how Example 15 would change according to the $\mathcal{AC}$ propagation. The ground preferences shown in Figure 8 lead to have only one maximal chain in $A\lfloor c_4 \rfloor$, i.e., $\mathcal{H}_A(c_4) = \{\langle c_3, c_1\rangle\}$. Thus, ${}^{\mathcal{AC}}\!\prec^{c_4} = \prec^{c_3} \circledcirc \prec^{c_1}$. On the other hand, we have $\mathcal{H}_C(c_4) = \{\langle c_4, c_3, c_1\rangle, \langle c_4, c_2, c_1\rangle\}$. Since c_2 and c_4 are inactive, we can discard them from the chains in $\mathcal{H}_C(c_4)$, since such contexts are irrelevant to the result of $\circledcirc(H)$ for each chain $H \in \mathcal{H}_C(c_4)$. Let us indicate as $\mathcal{H}_C^-(c_4)$ the set of such chains; we have $\mathcal{H}_C^-(c_4) = \{\langle c_3, c_1\rangle, \langle c_1\rangle\}$. By Lemma 2, it follows that ${}^{CC}\!\prec^{c_4} = \prec^{c_1} \oplus (\prec^{c_3} \circledcirc \prec^{c_1}) \subseteq \prec^{c_3} \circledcirc \prec^{c_1} = {}^{\mathcal{AC}}\!\prec^{c_4}$, that is, ${}^{CC}\!\prec^{c_4} \subseteq {}^{\mathcal{AC}}\!\prec^{c_4}$.* □

The following result allows us to extend the applicability of Lemma 2 to more complex scenarios.

Lemma 3. *Let $\langle \prec_a, \approx_a\rangle$, $\langle \prec_b, \approx_b\rangle$, $\langle \prec_c, \approx_c\rangle$, and $\langle \prec_d, \approx_d\rangle$ be four preference structures such that $\prec_a \subseteq \prec_b$, $\approx_a = \approx_b$, $\prec_c \subseteq \prec_d$ and $\approx_c = \approx_d$. Let $\langle \prec_{a,c}, \approx_{a,c}\rangle = \langle \prec_a, \approx_a\rangle \oplus \langle \prec_c, \approx_c\rangle$ and $\langle \prec_{b,d}, \approx_{b,d}\rangle = \langle \prec_b, \approx_b\rangle \oplus \langle \prec_d, \approx_d\rangle$. Then, i) $\prec_{a,c} \subseteq \prec_{b,d}$, and ii) $\approx_{a,c} = \approx_{b,d}$.*



(a) Context poset C_3 of Example 17.

(b) Same context poset as in Figure 9a with only the active contexts and c.

Fig. 9. A context poset (9a) and its active counterpart including the target context c (9b). Blank circles denote contexts that are inactive.

Example 17. *For a more complex example, consider the poset* C_3 *in Figure 9a, in which the active contexts are* c_1, c_4, c_5, *and* c_8, *while the remaining contexts (*c, c_2, c_3, c_6, *and* c_7*) are inactive. The cover* $cov^{C_3}(c)$ *of c in* C_3 *is* $\{c_1, c_2, c_3\}$*; however, if we focus on the active contexts, shown in Figure 9b along with c, then the cover of c is* $\{c_1, c_5\}$.

We therefore have

$$\mathcal{H}_A(c) = \{\langle c_5, c_8\rangle, \langle c_1, c_4\rangle\},$$

whereas

$$\mathcal{H}_C(c) = \{ \langle c, c_3, c_6, c_8\rangle, \langle c, c_3, c_5, c_8\rangle, \langle c, c_3, c_5, c_7\rangle, \langle c, c_2, c_5, c_8\rangle, \langle c, c_2, c_5, c_7\rangle, \langle c, c_2, c_4, c_7\rangle, \langle c, c_1, c_4, c_7\rangle\}.$$

Similarly to what was done in Example 16, this reduces to

$$\mathcal{H}_C^-(c) = \{\langle c_8\rangle, \langle c_5, c_8\rangle, \langle c_5\rangle, \langle c_4\rangle, \langle c_1, c_4\rangle\},$$

which also includes the non-maximal chains $\langle c_8\rangle$, $\langle c_5\rangle$ *and* $\langle c_4\rangle$.

By proceeding as in Example 16, we have

$$
\begin{aligned}
{}^{CC}{\prec^c} &= \prec^{c_8} \oplus (\prec^{c_5} \mathbin{\bigcirc\!\!\!\!\!\rhd} \prec^{c_8}) \oplus \prec^{c_5} \oplus \prec^{c_4} \oplus (\prec^{c_1} \mathbin{\bigcirc\!\!\!\!\!\rhd} \prec^{c_4}) & (5)\\
&= (\prec^{c_8} \oplus (\prec^{c_5} \mathbin{\bigcirc\!\!\!\!\!\rhd} \prec^{c_8})) \oplus (\prec^{c_5} \oplus (\prec^{c_5} \mathbin{\bigcirc\!\!\!\!\!\rhd} \prec^{c_8})) \oplus (\prec^{c_4} \oplus (\prec^{c_1} \mathbin{\bigcirc\!\!\!\!\!\rhd} \prec^{c_4})) & (6)\\
&\subseteq (\prec^{c_5} \mathbin{\bigcirc\!\!\!\!\!\rhd} \prec^{c_8}) \oplus (\prec^{c_5} \mathbin{\bigcirc\!\!\!\!\!\rhd} \prec^{c_8}) \oplus (\prec^{c_1} \mathbin{\bigcirc\!\!\!\!\!\rhd} \prec^{c_4}) & (7)\\
&= (\prec^{c_5} \mathbin{\bigcirc\!\!\!\!\!\rhd} \prec^{c_8}) \oplus (\prec^{c_1} \mathbin{\bigcirc\!\!\!\!\!\rhd} \prec^{c_4}) & (8)\\
&= {}^{\mathcal{A}C}{\prec^c} & (9)
\end{aligned}
$$

Expression (6) is obtained from (5) through idempotence, whereas (7) is derived from (6) by a repeated application of Lemma 2 and Lemma 3. Finally, (8) is obtained through idempotence again, thus yielding ${}^{CC}{\prec^c} \subseteq {}^{\mathcal{A}C}{\prec^c}$. □

The relationship between the CC and $\mathcal{A}C$ propagation methods shown in the previous examples indeed always holds, as shown in the following theorem.

Theorem 6. *Let c be a context in the context poset C. Then,* ${}^{CC}{\prec^c} \subseteq {}^{\mathcal{A}C}{\prec^c}$.

Although $\mathcal{A}C$ is insensitive to the side-effects of inactive contexts, it is still unable to guarantee specificity in all cases.

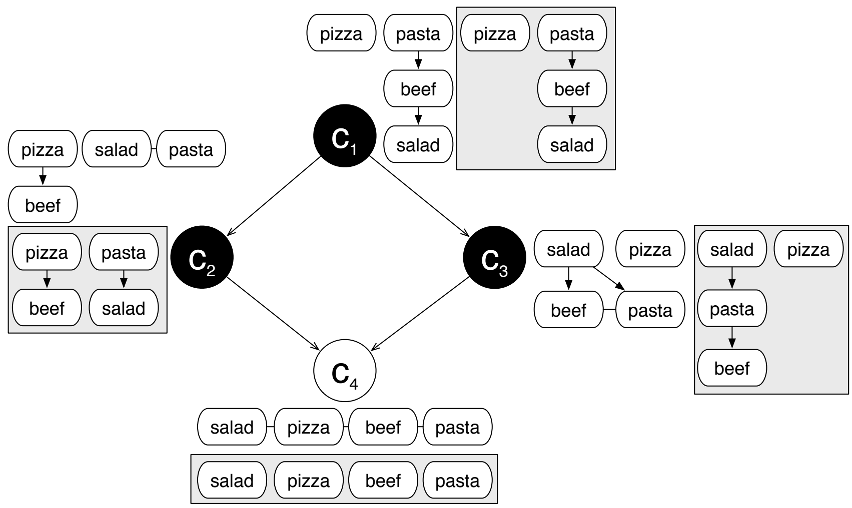


Fig. 10. A preference configuration for the poset C_1 from Figure 1. Complete preference relations are computed under the $\mathcal{AC}$ propagation, shown in a gray box for each context c_i, $i = 1, 2, 3, 4$. Notice that c_2 is now active, which makes $\mathcal{AC}$ propagation fail to comply with specificity.

Theorem 7. *$\mathcal{AC}$ propagation is coherent and fair but not specific.*

Example 18. *Consider the ground preferences in Figure 10, in which all contexts but c_4 are active. For specificity, the preference* `pasta` $\prec^{c_3}$ `salad` *should propagate to context c_4, since* `pasta` $\approx^{c_2}$ `salad`. *However, this is not the case, since context c_2 is active, thus ${}^{\mathcal{AC}}{\prec}^{c_4} = (\prec^{c_2} \rhd \prec^{c_1}) \oplus (\prec^{c_3} \rhd \prec^{c_1}) = {}^{\mathcal{AC}}{\prec}^{c_2} \oplus {}^{\mathcal{AC}}{\prec}^{c_3}$, where ${}^{\mathcal{AC}}{\prec}^{c_2}$ includes the preference* `salad` ${}^{\mathcal{AC}}{\prec}^{c_2}$`pasta`. *This entails that* `salad` *and* `pasta` *are incomparable in c_4 according to $\mathcal{AC}$ propagation (and so are all other pairs of objects).* □

6.3 The Object-Specific Cover Propagation

The rationale behind the third propagation method we introduce, called *Object-specific Cover* ($\mathcal{OC}$), is to focus on individual pairs of objects and, for each pair, compute the propagation based on the contexts covering c among those that contain some ground preference regarding the given pair. This refines the idea, used in the $\mathcal{AC}$ method, of discarding inactive contexts by ignoring, when comparing objects o_1 and o_2, also those contexts in which o_1 and o_2 are indifferent.

Definition 19. *Given a context poset C, a preference configuration $\langle \prec^C, \approx^C \rangle$, and two objects $o_1, o_2 \in O$, a context $c \in C$ is (o_1, o_2)-*active *if $o_1 \not\approx^c o_2$.*

In the $\mathcal{OC}$ propagation method, objects o_1 and o_2 are compared using the following Equation (10), in which the observation that o_1 and o_2 are either ordered or incomparable in all the (o_1, o_2)-active contexts is exploited to avoid recursion.

Definition 20. *Let c be a context in a context poset C and, for any two objects o_1, o_2 in O, let $A(o_1, o_2)$ denote the poset induced by the set of (o_1, o_2)-active contexts for a preference configuration $\langle \prec^C, \approx^C \rangle$. Let $A_c^{o_1,o_2}$ denote the poset induced by $A(o_1, o_2) \cup \{c\}$, and let $\mathrm{cov}^{A_c^{o_1,o_2}}(c) = \{c_1, \ldots, c_m\}$. The complete preference structure $\langle {}^{\mathcal{OC}}{\prec}^c, {}^{\mathcal{OC}}{\approx}^c \rangle$ in c under the $\mathcal{OC}$ propagation, denoted ${}^{\mathcal{OC}}{\prec}^c$, is*

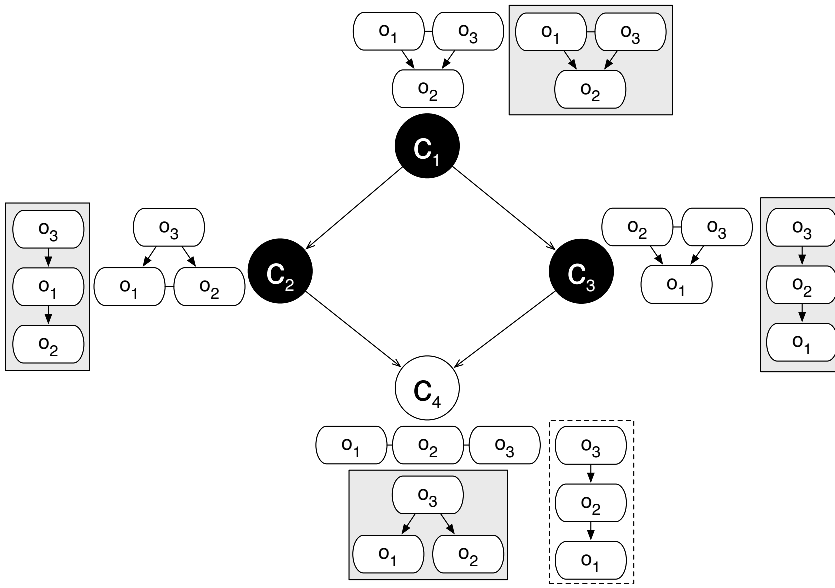


Fig. 11. A preference configuration for Example 19 using context poset C_1 from Figure 1; each ground preference structure is shown next to the corresponding context. For each context c_i, $i = 1, 2, 3, 4$, complete preference structures computed under the CC propagation are shown in a gray box (and coincide with those computed under the $\mathcal{AC}$ propagation). The complete preference structure ${}^{OC}{\prec}^{c_4}$ computed under the OC propagation for c_4 is shown in a white dashed box (for c_1, c_2, c_3 the computation is the same as under the CC propagation).

obtained, for each pair of objects o_1 and o_2, as:

$$o_1 \; {}^{OC}{\prec}^{c} o_2 \quad \Leftrightarrow \quad o_1 \prec^{c} o_2 \vee [o_1 \approx^{c} o_2 \wedge (o_1 \prec^{c_1} o_2 \wedge \cdots \wedge o_1 \prec^{c_m} o_2)] \tag{10}$$

$$o_1 \; {}^{OC}{\approx}^{c} o_2 \quad \Leftrightarrow \quad o_1 \approx^{c} o_2 \wedge cov^{A_c^{o_1,o_2}}(c) = \emptyset \tag{11}$$

Example 19. *Consider the context poset and preference configuration shown in Figure 11. We have $A(o_1, o_2) = \{c_1, c_3\}$, $A(o_2, o_3) = \{c_1, c_2\}$, and $A(o_1, o_3) = \{c_2, c_3\}$. Thus, $cov^{A_{c_4}^{o_1,o_2}}(c_4) = \{c_3\}$, $cov^{A_{c_4}^{o_2,o_3}}(c_4) = \{c_2\}$, and $cov^{A_{c_4}^{o_1,o_3}}(c_4) = \{c_2, c_3\}$. According to both $\mathcal{AC}$ and CC propagation, objects o_1 and o_2 are incomparable in context c_4, since ${}^{CC}{\prec}^{c_2} = {}^{\mathcal{AC}}{\prec}^{c_2}$ includes the preference $o_2 \prec o_1$, as inherited from c_1, whereas $o_1 \prec o_2$ is an element of ${}^{CC}{\prec}^{c_3} = {}^{\mathcal{AC}}{\prec}^{c_3}$. Instead, the OC propagation* does not *consider context c_2 for ordering o_1 and o_2, since c_2 is not (o_1, o_2)-active (i.e., $o_1 \approx^{c_2} o_2$), thus $o_1 \; {}^{OC}{\prec}^{c_4} o_2$. The complete preference relations for all the three propagation methods in context c_4 are as follows (complete preference relations coincide in the other contexts):*

$${}^{CC}{\prec}^{c_4} = {}^{\mathcal{AC}}{\prec}^{c_4} = \{o_1 \prec o_3, o_2 \prec o_3\} \subset {}^{OC}{\prec}^{c_4} = \{o_1 \prec o_2, o_2 \prec o_3, o_1 \prec o_3\}$$

□

Theorem 8 shows that OC has all the required properties for preference propagation, and thus solves Problem 1 (CFS).

Theorem 8. *OC propagation is coherent, fair and specific.*

The following result establishes a precise relationship between the propagation methods we have analyzed so far.

Theorem 9. *Let c be a context in the context poset C. Then, the complete preference structures in c under the $\mathcal{CC}$, $\mathcal{AC}$ and $\mathcal{OC}$ propagation methods satisfy the following relationships:*

$$^{\mathcal{CC}}\!\prec^{c} \subseteq\ ^{\mathcal{AC}}\!\prec^{c} \subseteq\ ^{\mathcal{OC}}\!\prec^{c} \text{ and } ^{\mathcal{CC}}\!\approx^{c} =\ ^{\mathcal{AC}}\!\approx^{c} =\ ^{\mathcal{OC}}\!\approx^{c}$$

6.3.1 PC-expressions for $\mathcal{OC}$ propagation. Apparently, $\mathcal{OC}$ propagation requires a distinct cover for each pair of objects. However, as a major result, we can show that *there exists a PC-expression, the same for all pairs of objects, that implements $\mathcal{OC}$ propagation.* The intuition behind this result is that specificity needs to avoid that a preference $o_1 \prec o_2$, for which a conflicting preference exists in a more specific context, propagates along a chain in which o_1 and o_2 are indifferent (which is the reason why both $\mathcal{CC}$ and $\mathcal{AC}$ violate specificity). Algebraically, this requires a PC-expression, which we denote $\mathrm{RG}^A(c)$, that is maximally "grouped on the right", so that this pass-through phenomenon is inhibited. The following definition provides a formal characterization of $\mathrm{RG}^A(c)$.

Definition 21 (PC-expression for $\mathcal{OC}$ propagation). *Let c' be a context in the poset of active successors $A\lfloor c\rfloor$ (so that $c <_C c'$) and let $\{c_1, \ldots, c_k\}$ be the contexts in $A\lfloor c\rfloor$ (so that $c \leqslant_C c_i$, for $1 \leqslant i \leqslant k$) that are covered by c' (i.e., $c_i \lessdot c'$). The "right-grouped" expression $RG^A(c, c')$ is recursively defined as follows:*

$$\begin{cases} RG^A(c,c) = c \\ RG^A(c,c') = (RG^A(c,c_1) \oplus \ldots \oplus RG^A(c,c_k)) \rhd c' \text{ if } c <_C c' \end{cases}$$

Let $\{\hat{c}_1, \ldots, \hat{c}_n\}$ be the set of maximal *elements in $A\lfloor c\rfloor$ (i.e., the contexts $\hat{c}_i$ in $A\lfloor c\rfloor$ such that there is no context $\tilde{c} \in A\lfloor c\rfloor$ for which $\hat{c}_i <_C \tilde{c}$). Then:*

$$RG^A(c) = RG^A(c,\hat{c}_1) \oplus \ldots \oplus RG^A(c,\hat{c}_n) \tag{12}$$

Example 20. *Consider the poset in Figure 1, and assume that all contexts are active. The PC-expression $RG^A(c_4)$ is $((c_4 \rhd c_2) \oplus (c_4 \rhd c_3)) \rhd c_1$. For convenience, this can also be more compactly rewritten, by applying the left-distributive property of $\rhd$, as $c_4 \rhd (c_2 \oplus c_3) \rhd c_1$.*

For a more complex case, consider the poset in Figure 9a, and assume that all contexts are active. The PC-expression $RG^A(c)$ is

$$\begin{gathered}[((((c \rhd c_1) \oplus (c \rhd c_2)) \rhd c_4) \oplus \\ (((c \rhd c_2) \oplus (c \rhd c_3)) \rhd c_5)) \rhd c_7] \\ \oplus \\ [((((c \rhd c_2) \oplus (c \rhd c_3)) \rhd c_5) \oplus \\ (c \rhd c_3 \rhd c_6)) \rhd c_8]\end{gathered}$$

For convenience, this can also be more compactly rewritten, by applying the left-distributive property of $\rhd$, as

$$\begin{gathered}c \rhd \{[(((c_1 \oplus c_2) \rhd c_4) \oplus \\ ((c_2 \oplus c_3) \rhd c_5)) \rhd c_7] \\ \oplus \\ [(((c_2 \oplus c_3) \rhd c_5) \oplus \\ (c_3 \rhd c_6)) \rhd c_8]\}\end{gathered} \tag{13}$$

□

Intuitively, $\mathrm{RG}^A(c)$ can be obtained from the canonical expression by first grouping chains on maximal elements and factoring them out, then recursively applying this process to the so-reduced chains until no more factors can be extracted.

THEOREM 10. *The PC-expression $RG^A(c)$ correctly computes the $\mathcal{OC}$ propagation, i.e., $o_1 \,{}^{\mathcal{OC}}\!\!\prec^c o_2$ iff $o_1 \,{}^{E}\!\!\prec o_2$, where $E = RG^A(c)$.*

Rather surprisingly, there is another expression for the semantics of $\mathcal{OC}$ propagation, which does not even need to distinguish between active and inactive contexts.

THEOREM 11. *The PC-expression $RG^C(c)$ obtained by replacing A with the complete poset C in Definition 21 is equivalent to $RG^A(c)$.*

The intuition about the equivalence between $\mathrm{RG}^A(c)$ and $\mathrm{RG}^C(c)$ is that, as shown in Example 21 below, while dropping the occurrences of inactive contexts in $\mathrm{RG}^C(c)$, one may end up with sub-expressions of the form $E_1 \oplus (E_1 \triangleright E_2)$, which reduce to $E_1 \triangleright E_2$ thanks to left-distributivity and the identity element axioms of $\oplus$ and $\triangleright$, i.e., $E_1 \oplus (E_1 \triangleright E_2) \equiv (E_1 \triangleright \bot) \oplus (E_1 \triangleright E_2) \equiv E_1 \triangleright (\bot \oplus E_2) \equiv E_1 \triangleright E_2$.

EXAMPLE 21. *Consider the poset in Figure 9a, where the blank contexts are inactive. The PC-expression $RG^C(c)$ is the same as PC-expression (13) given in Example 20. This, after simplification with the identity element axioms, becomes:*

$$
\begin{aligned}
RG^C(c) &= \bot \triangleright \{[(((c_1 \oplus \bot) \triangleright c_4) \oplus ((\bot \oplus \bot) \triangleright c_5)) \triangleright \bot] \oplus [(((\bot \oplus \bot) \triangleright c_5) \oplus (\bot \triangleright \bot)) \triangleright c_8]\} \\
&\equiv (c_1 \triangleright c_4) \oplus c_5 \oplus (c_5 \triangleright c_8) \\
&\equiv (c_1 \triangleright c_4) \oplus (c_5 \triangleright c_8) \\
&= RG^A(c)
\end{aligned}
$$

□

6.4 Classification of propagation methods

In this section we summarize the results obtained so far, and discuss why $\mathcal{OC}$ propagation can be considered the ultimate semantics for preference propagation. For what follows, it is convenient to classify the propagation methods based on how the derived PC-expressions depend on the preference configurations at hand.

DEFINITION 22. *A propagation method $\mathcal{P}$ is:*

- static *when, in order to derive a PC-expression E, $\mathcal{P}$ needs to consider only the context poset C and the target context c;*
- active-static *when, in order to derive a PC-expression E, $\mathcal{P}$ needs to consider only C, c, and the partition $\{A, C \backslash A\}$, where A is the set of active contexts in C;*
- dynamic *when $\mathcal{P}$ is neither static nor active-static.*

Dynamic methods have the ability to fully inspect a preference configuration to generate a PC-expression. However, such an inspection may generally be computationally heavy, since it depends on the number of preferences (and objects) involved. On the other hand, active-static methods yield a PC-expression without actually accessing any preferences in particular, but only discern whether a context is active or inactive. The activity/inactivity check is likely to require constant time in any reasonable representation of the preference configuration, and is thus typically negligible from a computational point of view.

Figure 12 classifies the PC-expressions implementing the propagation methods considered so far according to the properties introduced in Definition 22 and the notions of fairness and specificity. Clearly, $\mathcal{CC}$ (through $\mathrm{Rec}^C(c)$) is a static method, and so are the naive methods $\mathcal{N}_{\mathtt{Par}}$ (through $\mathrm{Par}^C(c)$) and $\mathcal{N}_{\mathtt{Pri}}$ (through $\mathrm{Pri}^C(c)$), whereas $\mathcal{AC}$ (through $\mathrm{Rec}^A(c)$) is active-static. On the other hand $\mathcal{OC}$, due to Theorems 10 and 11, admits both a static, through $\mathrm{RG}^C(c)$, and an active-static, through $\mathrm{RG}^A(c)$, PC-expression.

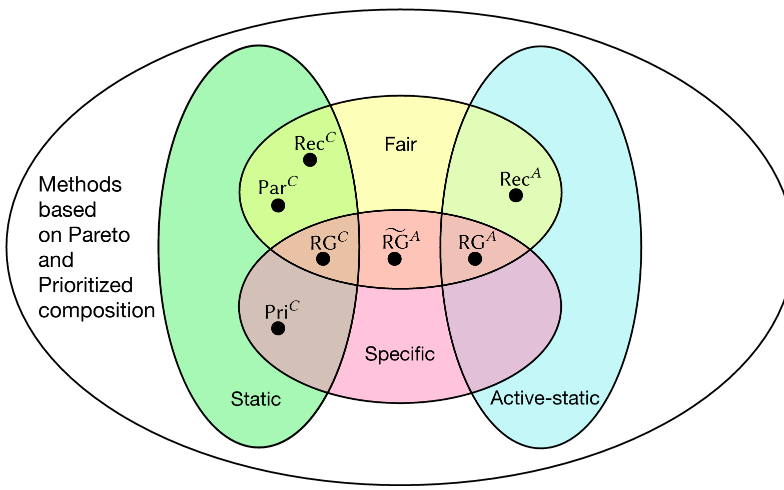


Fig. 12. Classification of PC-expressions implementing propagation methods based on Pareto and Prioritized composition. Dynamic methods, such as $\widetilde{\mathcal{OC}}$ (through a dynamic PC-expression denoted here by $\widetilde{\mathrm{RG}}^A$) defined in Example 22, are placed outside the regions enclosing Static and Active-static methods.

Ultimately, $\mathcal{OC}$ propagation is the only one that guarantees both fairness and specificity. More precisely, we have the following results.

Theorem 12. *Let $\mathcal{P}$ be a propagation method and let c be a context in a context poset C: if ${}^{\mathcal{P}}\!\prec^c \subset {}^{\mathcal{OC}}\!\prec^c$ then $\mathcal{P}$ is not specific.*

Theorem 13. *Let $\mathcal{P}$ be a coherent propagation method based on $\oplus$ and $\triangleright$ that is either static or active-static, and let c be a context in a context poset C: if ${}^{\mathcal{OC}}\!\prec^c \subset {}^{\mathcal{P}}\!\prec^c$ then $\mathcal{P}$ is not both fair and specific.*

Theorem 12 holds for any possible propagation method, including dynamic methods, methods not based on $\oplus$ and $\triangleright$, and even methods not based on PC-expressions: not propagating all the preferences that $\mathcal{OC}$ propagates leads to a violation of specificity. On the other hand, Theorem 13 only considers methods based on $\oplus$ and $\triangleright$ that are either static or active-static, thus leaving open the possibility of propagating more preferences than $\mathcal{OC}$ while violating neither fairness nor specificity. This can be done by considering either a dynamic method or non-IIO $+$ and $\triangleright$ operators (since $\oplus$ and $\triangleright$ are the only possible interpretations for IIO operators, by Theorems 2 and 3).

Notice that, if $\mathcal{OC}$ propagates neither the preference $o_1 \,{}^{\mathcal{OC}}\!\prec^c o_2$ nor $o_2 \,{}^{\mathcal{OC}}\!\prec^c o_1$, and there is at least one context in $C\lfloor c \rfloor$ for which o_1 and o_2 are not indifferent, according to Definition 20 one of the following cases occurs:

(1) there are two contexts c_i and c_j in $\mathrm{cov}^{A_c^{o_1,o_2}}(c)$ such that $o_1 \prec^{c_i} o_2$ and $o_2 \prec^{c_j} o_1$;
(2) there is a context c_i in $\mathrm{cov}^{A_c^{o_1,o_2}}(c)$ such that $o_1 \parallel^{c_i} o_2$.

Clearly, in the first case, no method can propagate a preference for o_1 and o_2 without violating fairness. In the other case, propagating $o_1 \prec o_2$ (or $o_2 \prec o_1$) could lead to a violation of the transitivity of the resulting relation, and possibly to the introduction of preference cycles, as demonstrated in Example 8. As already observed in Section 5, alternative definitions of Pareto and Prioritized compositions in which the indifference relation ($\approx$) is replaced by the unordered relation ($\sim$) do not preserve the properties of strict partial orders. Consequently, ad hoc strategies with limited applicability would be required in order to propagate more preferences than OC while yielding at the same time a strict partial order, as shown in the next example.

Example 22. *Consider a propagation method $\widetilde{OC}$ that behaves as follows: if there are only two active contexts, say c_1 and c_2, such that $c_1 \sim_C c_2$, $c <_C c_1$ and $c <_C c_2$, and, for two objects o_1 and o_2, it is $o_1 \prec^{c_1} o_2$ and $o_1 \parallel^{c_2} o_2$ and all other objects are indifferent, then $\widetilde{OC}$ yields the PC-expression $c_1 \rhd c_2$, and thus propagates $o_1 \overset{\widetilde{OC}}{\prec^c} o_2$; in all other cases, $\widetilde{OC}$ yields the same expression as OC. Note that, in the case where it differs from OC, $\widetilde{OC}$ propagates a superset of the preferences propagated by OC, but no violation of fairness or specificity occurs. Note that $\widetilde{OC}$ is dynamic, since the PC-expression it derives (call it $\widetilde{RG}^A$ by analogy with the PC-expression RG^A used by OC), depends on the presence of specific preferences in contexts c_1 and c_2. Alternatively, for aggregating preferences of c_1 and c_2, one could use a non-IIO operator $\tilde{\oplus}$ that behaves like $\oplus$, yet, in the above circumstances, it propagates the preference $o_1 \prec o_2$ to c.* □

As another example, the method described in [7] considers, for each ground preference relation to be combined using an "extended" Pareto composition operator, the *level* of an object in such a relation, and then declares as equivalent the objects that are at the same level. The level of object o in a preference relation $\prec$ is the length of the longest chain of objects that are better than o according to $\prec$. Since the complete preference relation resulting from this modified notion of equivalence is not transitive, a transitive closure operator is then applied. Although this method can indeed propagate more preferences than OC, it is applicable only when the object domain is finite and, for computational reasons, small. Also notice that changing the object domain by, e.g., adding a new object o', would influence the complete preferences of other objects (since their level can change).

7 ALGORITHMIC ASPECTS

The theoretical results we have illustrated in the previous sections lay the ground on which any system implementing the proposed framework can be built. To this end, in this section we discuss some major issues that need to be addressed from an implementation point of view, and, specifically, we study the asymptotic complexity of the main problems discussed in this paper, stated in Section 3.

We observe that, for both Problems 2 (OrdRel) and 3 (Best), the exact complexity depends on the underlying context model and on the way in which it is represented as well as on the language used for expressing preferences between objects. In order to remain parametric with respect to these aspects, we assume that two context descriptions can be compared in $O(\delta)$ time, and that $O(\gamma)$ is the complexity of determining the order relation of any two objects o_1 and o_2 according to the ground preferences in a context c'.

In light of the results obtained on Problem 1 (CFS), we shall only consider the OC propagation method.

7.1 Computing the complete preferences

Problem OrdRel can be solved by first finding the $\mathrm{RG}^A(c)$ PC-expression capturing the semantics of the OC propagation method and then using such a PC-expression against the preference configuration in order to compute the complete preferences. However, materializing the $\mathrm{RG}^A(c)$

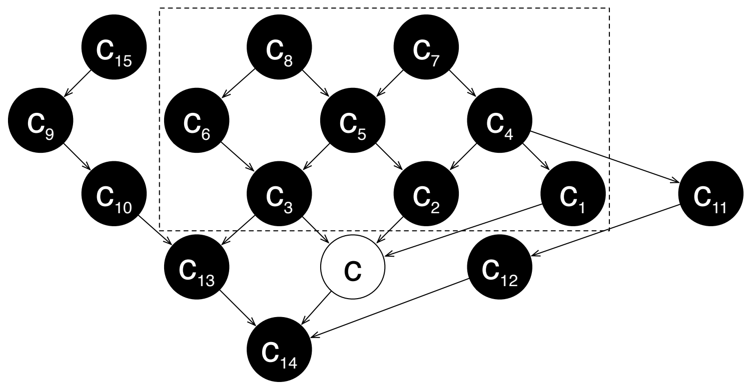


Fig. 13. Poset A given by filled circles; context c is inactive; poset $A\lfloor c \rfloor$ is enclosed in a dashed box.

PC-expression might lead to an inefficient evaluation due to repeated sub-expressions. Note that similar arguments apply if $\mathrm{RG}^C(c)$ is used instead of $\mathrm{RG}^A(c)$, with the additional step of removing the inactive contexts, which do not affect the resulting complete preference structure.

Example 23. *Consider the active poset shown in Figure 13, in which the maximal contexts are c_7, c_8, and c_{15}. In order to study preference propagation to c, the only contexts of interest are the successors of c, i.e., c_1, c_2, c_3, c_4, c_5, c_6, c_7, and c_8 (enclosed in a dashed box in the figure). Here,*

$$RG^A(c) = (c \rhd c_3 \rhd c_6 \oplus (c \rhd c_3 \oplus c \rhd c_2) \rhd c_5) \rhd c_8 \oplus \\ (c \rhd c_3 \oplus c \rhd c_2) \rhd c_5 \oplus (c \rhd c_2 \oplus c \rhd c_1) \rhd c_4) \rhd c_7.$$

Since c is inactive, we have

$$RG^A(c) \equiv (c_3 \rhd c_6 \oplus (c_3 \oplus c_2) \rhd c_5) \rhd c_8 \oplus ((c_3 \oplus c_2) \rhd c_5 \oplus (c_2 \oplus c_1) \rhd c_4) \rhd c_7,$$

in which, e.g., the sub-expression $(c_3 \oplus c_2) \rhd c_5$ is repeated twice. □

In the worst case, the size of $\mathrm{RG}^A(c)$ is *exponential* in the number of contexts in $A\lfloor c \rfloor$. Indeed, each time two contexts c_1 and c_2 in $A\lfloor c \rfloor$ cover a same context c_3, then both $\mathrm{RG}^A(c, c_1)$ and $\mathrm{RG}^A(c, c_2)$ include c_3, thus doubling the number of occurrences of $\mathrm{RG}^A(c, c_3)$ in the resulting PC-expression, which, if repeated, gives an exponential growth.

In order to circumvent this problem and also to avoid redetermining several times the order relationships between contexts, we maintain a (transitively reduced) DAG Δ representing the active poset A. We also maintain a hash table HT that associates with each context description the corresponding ground preferences as well as (a reference to) the node in Δ representing the context. For technical reasons, we equip $\Delta = \langle \Psi, \Phi \rangle$ with a unique top element $c_\top$. Then, $\Psi = A \cup \{c_\top\}$, whereas Φ includes exactly the following arcs:

- the arcs $\langle c', c'' \rangle$ connecting each context $c' \in A$ with every context $c'' \in \mathrm{cov}^A(c')$;
- the arcs $\langle c', c_\top \rangle$ connecting each maximal context $c' \in A$ with $c_\top$.

For efficiently navigating the DAG downward (as required by the $\mathcal{OC}$ propagation method), for each node $c' \in \Psi$ we maintain the list of its immediate predecessors, i.e., the contexts covered by c'.

At query time, we receive the description of a context c and first check whether $c \in A$. If this is the case, we locate, via HT, the node in Δ corresponding to c, and extract the sub-DAG Δ_c corresponding to the poset $A\lfloor c \rfloor$ of the active successors of c, in which c is, by definition, the unique bottom element. If $c \notin A$, we observe that, in order to compute the complete preferences in c, we just need to consider the active successors of c, but not c itself, as was done in Example 23; therefore,

ALGORITHM 1: `ObjectComparisonOC`.

Input: A context c; a DAG $\Delta_c = \langle \Psi_c, \Phi_c \rangle$ with top element $c_\top$; objects o_1, o_2;
a preference configuration $\langle <^A, \approx^A \rangle$.
Output: $o_1 \theta o_2$ where $\theta \in \{\prec, \succ, \approx, \|\}$

(1) **let** $D = \emptyset$ *// dictionary of pairs (visited context, order relation)*
(2) **return** `ObjectComparisonOC2`($c_\top$)

Subprocedure: `ObjectComparisonOC2`
Input: A context c'.

(3) **if** $\exists \theta \mid \langle c', \theta \rangle \in D$ **then return** θ
(4) **let** $\theta = \approx$
(5) **let** $\{c''_1, \ldots, c''_k\} = \{c'' \mid \langle c'', c' \rangle \in \Phi_c\}$ *// the contexts covered by* c'
(6) **for** $i \in \{1, \ldots, k\}$
(7)   **let** $\theta_i =$ `ObjectComparisonOC2`(c''_i)
(8)   **let** $\theta = \theta \oplus \theta_i$
(9)   **if** $\theta = \|$ **then return** $\|$
(10) **if** $\theta = \approx \wedge c' \neq c_\top$ **then let** $\theta =$ `PrefRel`(c', o_1, o_2) *// fictitious context* $c_\top$ *has no preference*
(11) **let** $D = D \cup \{\langle c', \theta \rangle\}$
(12) **return** θ

by comparing c with all the nodes in Δ, the resulting sub-DAG Δ_c has now as bottom elements the nodes corresponding to the contexts covering c.

The complexity of extracting the sub-DAG Δ_c varies depending on whether c is active or not, which can be determined in time proportional to the time required for generating a hash key from a context description. If $c \notin A$, extracting Δ_c can be done by comparing c with all the nodes in Δ and marking a node c' as belonging to Δ_c if and only if $c <_C c'$; this entails a complexity of $O(|A| \cdot \delta)$. If $c \in A$, we can proceed in a similar way. Alternatively, by also maintaining for each node $c' \in \Psi$ the list of its immediate successors (i.e., the cover of c', whose size is at most the width $w(A)$ of A), we can navigate the DAG Δ upwards, starting from c, while marking all the visited nodes as belonging to Δ_c; in this case, the complexity of extracting Δ_c is at most $O(|A| \cdot w(A))$.

Given Δ_c and a preference configuration $\langle <^A, \approx^A \rangle$, we can now compare any pair of objects o_1 and o_2 as described in Algorithm 1 so as to determine their order relation $\theta \in \{\prec, \succ, \approx, \|\}$. In order to avoid repeated computations, a dictionary D of (*visited context*, *order relation*) pairs is maintained. The algorithm starts a recursive descent of Δ_c by invoking subprocedure `ObjectComparisonOC2` on the unique top element $c_\top$, which represents a fictitious context and, thus, has no ground preferences. For each context c''_i covered by a context $c' \in \Delta_c$, we recursively invoke the subprocedure `ObjectComparisonOC2` to determine the corresponding order relation θ_i (line 7). These order relations are combined through the Pareto operator (line 8); as soon as we discover that the combined order relation θ is $\|$, we can immediately stop and return $\|$ as the overall result. Eventually, if $\theta \in \{\prec, \succ\}$, we return it. Otherwise, only if all θ_i are $\approx$ (thus θ is $\approx$), we consider the ground preference in c' concerning objects o_1 and o_2, indicated by `PrefRel`(c', o_1, o_2) (line 10). As a last step, we update the dictionary D (line 11) to avoid recomputing θ for context c' in case c' is encountered multiple times (line 3).

Theorem 14. *For each pair of objects $o_1, o_2 \in O$, each context c and each preference configuration $\langle <^A, \approx^A \rangle$, Algorithm 1 correctly computes the order relation θ of o_1 and o_2 in c according to the OC propagation.*

The complexity of Algorithm 1 is $O(|A| \cdot (w(A) + \gamma))$, since `PrefRel` is invoked at most once per element in A and, for each such element, we execute at most $w(A)$ iterations of the **for** cycle at line 6.

Therefore, Problem OrdRel can be solved in $O(|A| \cdot \delta + |A|(w(A) + \gamma)) = O(|A| \cdot (\delta + w(A) + \gamma))$ time, when c is inactive.

In order to determine $\beta_{OC_{\prec^c}}(O)$, i.e., to establish the best objects according to the complete preferences in a context c (Problem Best), in the worst case we need to execute the `ObjectComparisonOC` procedure $O(N^2)$ times for a set of N objects. In practice, far fewer tests are actually executed because of the transitivity property of preference relations (see, e.g., [8, 11, 36]).

Overall, this leads to a complexity of $O(|A| \cdot (\delta + N^2 \cdot (w(A) + \gamma)))$ for solving Problem Best, when c is inactive.

Note that, when c is active, for both OrdRel and Best, δ can be replaced by $\min\{\delta, w(A)\}$ in the resulting complexity, which leads to $O(|A| \cdot (w(A) + \gamma))$ for OrdRel and to $O(|A| \cdot N^2 \cdot (w(A) + \gamma))$ for Best.

7.2 Instantiating context and preference models

In this section, we provide specific examples of context and preference models.

7.2.1 The CMT Context Model. The model introduced in [31] and later developed in [32, 33] (henceforth CMT) can be used to concretely represent the deliberately general notion of context poset discussed in this paper. The main construct of CMT is the *contextual dimension* (henceforth dimension), such as `Time` and `Location`. Each dimension d includes a set of values, called *members*, that are partitioned into a poset L of *levels* describing d at different degrees of granularity. For instance, `July 23, 2020` and `July 2020` are possible members of the `Time` dimension occurring in the levels `Day` and `Month`, respectively, where `Day` $\leqslant_L$ `Month`. If $l_1 \leqslant_L l_2$ are levels of a dimension d, each member in l_1 maps to one value in l_2 and this induces another partial order $\leqslant_M$ on all the members of a dimension (e.g., `July 23, 2020` $\leqslant_M$ `July 2020`).

In this framework, a CMT *context* c over a set of dimensions D can be described by a tuple $\langle m_1, \ldots, m_k \rangle$ where each m_i is a member of a dimension in D. Then, a partial order $\leqslant_C$ can be easily defined over contexts as follows: $c' \leqslant_C c$ if, for each element m in c, there is an element m' in c' such that m and m' are members of the same dimension and $m' \leqslant_M m$.

Note that, assuming that testing whether $m' \leqslant_M m$ can be computed in $O(1)$ time, the complexity of establishing the order relation of two contexts, which was parametrically indicated as $O(\delta)$, is now $O(|D|)$.

Example 24. *The scenario described in Example 2 can be reproduced in the CMT model as follows:* $c_1 = \langle \texttt{Italy} \rangle$, $c_2 = \langle \texttt{Naples} \rangle$, $c_3 = \langle \texttt{summer}, \texttt{Italy} \rangle$, $c_4 = \langle \texttt{summer}, \texttt{Naples} \rangle$, *where* `Italy` *and* `Naples` *are members of the* `Location` *dimension at levels* `Country` *and* `City`*, respectively, with* `City` $\leqslant_L$ `Country` *and* `Naples` $\leqslant_M$ `Italy`*, whereas* `summer` *is a member of the* `Time` *dimension at the* `Season` *level. Then, we have for instance that* $c_4 \leqslant_C c_3$ *since* `summer` $\leqslant_M$ `summer` *and* `Naples` $\leqslant_M$ `Italy`*. The resulting context poset is, again, that of Figure 1.* □

7.2.2 Preference models. Among the many models available in the literature for expressing preferences, we consider two proposals that are explicitly designed to deal with large amounts of data.

In the algebraic language of Kießling [24, 25], the order relation of two objects can be determined in time linear in the number z of operators in the preference expression representing $\prec$, i.e., $O(\gamma) = O(z)$.

In the logical language of Chomicki [16] preferences are expressed using a first-order formula F, such that $o_1 \stackrel{F}{\prec} o_2$ iff $F(o_1, o_2)$ holds. Assume that F is in DNF and consists of m disjuncts, $F = D_1 \vee \ldots \vee D_m$. Let n be the maximum number of conjuncts in a disjunct of F, i.e., $D_i = C_{i,1} \wedge \ldots \wedge C_{i,n_i}$, $n_i \leqslant n$. According to [16], a formula F is *rational-order* if each conjunct is an atomic constraint of the form xRy or xRk, where $R \in \{=, \neq, <, >, \leqslant, \geqslant\}$, x and y are variables whose range is the domain of rational numbers, and k is a rational number. In this case, checking whether $F(o_1, o_2)$ holds requires time linear in the length of F. It follows that $O(\gamma) = O(m \cdot n)$ in case o_1 and o_2 are ordered.

A more complex procedure is however needed to distinguish between incomparable and indifferent objects. According to Definition 3.ii, two objects o_1 and o_2 are *not* indifferent iff there exists an object o in the domain O such that the following formula $F_{\not\approx}(o_1, o_2)$ is satisfiable:

$$F_{\not\approx}(o_1, o_2) = (F(o, o_1) \wedge \neg F(o, o_2)) \vee (\neg F(o, o_1) \wedge F(o, o_2)) \vee \\ (F(o_1, o) \wedge \neg F(o_2, o)) \vee (\neg F(o_1, o) \wedge F(o_2, o))$$

After distributing negation, each of the four disjuncts in the above formula can be written down as a DNF formula with $m \cdot n^m$ conjuncts, each consisting of at most $n + m$ atomic constraints. Since checking the satisfiability of a rational-order formula with q conjuncts has complexity $O(q)$ [21], it follows that checking satisfiability of $F_{\not\approx}(o_1, o_2)$ can be done in $O(m \cdot n^m(n + m))$. Therefore, determining the order relation of two unordered objects represents the worst case for the problem at hand, with $O(\gamma) = O(m \cdot n^m(n + m))$.

If F is a conjunctive formula, then $m = 1$ and $O(\gamma)$ reduces to $O(n)$ if the objects are ordered, and $O(n^2)$ otherwise.

8 RELATED WORKS

In this section we review the related literature on the combined use of preferences and contexts, with a particular emphasis on aspects related to the problem of preference propagation.

Contexts. With respect to the two aspects dealt with separately, we refer to the many existing surveys, including [39], [9] and [10], for aspects related to context modeling and reasoning in the Internet of Things (IoT), Pervasive Computing, and Data Management systems, respectively. As such surveys make clear, the approach we have adopted in this paper for context modeling is indeed quite general, since we only exploit the ability of the model to relate different contexts according to a *generic/specific* relationship (i.e., a partial order), a feature common to the vast majority of the models in the literature. In order to avoid any ambiguity, we notice that *Multi-Context Systems* (MCSs) [14], a popular framework for allowing heterogeneous knowledge sources to interoperate, are based on a notion of context quite different from the one used in this paper, since the term "context" in MCSs is used to denote a source in the architecture. Therefore, even the incorporation of preferences in MCSs [28] is not relevant to our discussion.

Preferences. Similarly to that on contexts, the literature on preferences is huge. Pigozzi et al. [40] survey the use of preferences in the Artificial Intelligence (AI) field, whereas [42] focuses on constraint satisfaction and optimization problems, and [54] provides a view from the Multicriteria decision theory area. A survey on how preferences are exploited for the purpose of database querying is available in [46]. Besides these fields, preferences are also used in a variety of diverse applications, see e.g., [1, 4, 19, 30, 37].

The majority of preference models falls into one of two distinct categories [12]: with the so-called *quantitative* preferences, a numerical score is used to assess the utility/relevance of an object from a user's point of view, whereas with *qualitative* preferences no scores are needed, and preferences are usually based on pairwise objects comparison. Typical examples of approaches based on qualitative preferences are [16, 24], which we have also considered in Section 7. As for quantitative preferences, commonly the score of an object is obtained through a so-called scoring/utility function by aggregating the attribute values of the object [23, 44]. More complex models define preferences by means of predicates with an associated degree-of-interest (*doi*) score, and then obtain the score of an object through a scoring function that aggregates all the *doi*s of the preferences satisfied by that object [26, 35]. In particular, the model in [26] also provides mechanisms to consider, for each object, only those preferences that are not overridden by more specific ones. Note that this notion of specificity is based on implication of the predicates defining the preferences (e.g., a preference for comedies with Adam Sandler is more specific than a preference just for comedies) and thus is different from ours, which applies to contexts.

Although the strength of a preference cannot be expressed, qualitative models are more general than quantitative ones from an order-theoretic point of view. The binary relation preference model that we have adopted in this paper is indeed the most common one for qualitative preferences. Note that our approach to preference propagation is applicable even when the ground preferences in each context are expressed through a quantitative preference model, in which case our propagation methods will consider only the ordering of the objects yielded by such a preference model, thus disregarding scores.

Contexts and preferences. We now detail how the combination of preferences and contexts has been considered so far in different research fields.

Method	Preference type	Context model	Propagation	Composition	Fairness	Specificity	Coherence
Agrawal *et al.* [3]	qualitative	attribute-value	no	n.a.	n.a.	n.a.	n.a.
Stefanidis et al. [45]	qualitative	set of keywords	yes	n.a.	no	no	no
van Bunningen *et al.* [50]	qualitative	description logics	yes	intersection	yes	no	yes
Stefanidis *et al.* [48]	quantitative	hierarchical	yes	scoring function	no	yes	yes
Miele *et al.* [35]	quantitative	hierarchical	yes	scoring function	no	yes	yes
Sacharidis *et al.* [43]	qualitative	attribute-value	yes	scoring function	no	no	yes

Table 2. Comparison of existing methods considering contexts and preferences in the data management field.

Data management systems. A number of papers have focused on the use and management of contextual preferences as a means to add flexibility to database queries, including [3, 35, 47, 48, 50, 51], as summarized in Table 2. The main difference with the present paper is that all of them follow a pragmatic approach based on specific heuristics and then focus on implementation issues. In particular, the works by Agrawal et al. [3] and Stefanidis et al. [45] do not explicitly address the issue of how to combine preferences defined in different contexts. However, [45] allows for a limited form of propagation, in that only preferences in one more generic context are considered if no preferences are available for the current query context. Van Bunningen et al. [50] model both contexts and preferences through description logics and propagate all preferences that are stated for contexts more generic than the target context, by taking the conjunction of all such preference specifications, i.e., the intersection of all sets of objects satisfying the preferences. This *and*-based semantics entails lack of specificity, while ensuring fairness (since intersection yields a subset of the preferences obtained through Pareto composition).

Stefanidis and Pitoura [48] consider quantitative preferences in a hierarchical context model. Preferences in a context c are computed from preferences defined in contexts that generalize c and are at "minimal distance" from c in the hierarchy. With respect to the propagation properties introduced in Section 3, we can characterize this approach as specific, whereas it is neither fair nor coherent. Miele et al. [35] also take into account numerical preferences and distances between contexts, but preferences defined on contexts at a distance from c that is not minimal are also considered, provided they are not "overwritten" by some other preference, using a notion of preference overriding similar to the one in [26]. Since c_1 having a smaller context distance to c than c_2 does not imply $c_1 \preccurlyeq_C c_2$, this approach is coherent and specific, but it is not fair. The same model is adopted in [34], where the authors focus on the problem of mining preferences. Cena et al. [15] consider a similar approach for the propagation of interests/preferences along a hierarchy of concepts. Besides the so-called *vertical* propagation from more generic to more specific concepts, they also consider a *horizontal* propagation between similar concepts, where, in both cases, a "conceptual distance" is used to determine how interests have to be propagated.

Probabilistic contextual skyline queries (p-CSQ) [43] aim to extend skyline queries (that return the Pareto-optimal tuples in a database relation) to scenarios in which preferences in a given context are not available, yet they are for other, similar contexts. Although this work is the closest in spirit to ours, it is still based on a concept of context similarity (from which preference probabilities are derived) and is only able to provide results for which their probability of being "optimal" exceeds a given threshold.

Mindolin and Chomicki [36] consider the so-called *p-skylines*, a particular case of PC-expressions in which each preference relation is used only once and is a total order over an attribute of interest. Taken together, these two restrictions simplify the problem of determining equivalence and containment of expressions, but this comes at the price of a reduced expressive power. In particular, p-skyline expressions cannot be used for arbitrary context posets and limit the kind of preferences we can define.

Artificial Intelligence. Contextual preferences could be considered as a particular case of *conditional preference networks* (CP-nets), a tool largely investigated in the AI field [13]. Behind the surface, there are however important differences between our work and that on CP-nets. With CP-nets one defines, for each attribute of interest, a set of total orders that are conditionally dependent on some other attribute(s). The resulting preferences are then defined as the transitive closure of the union of such orders, which might not be an order since cycles can arise. Conversely, we start with a set of arbitrary strict partial orders and study how to compose them in a context poset, ensuring that the result is always a strict partial order.

Aggregating preferences of multiple agents is a classical social choice problem, with Arrow's impossibility theorem stating that there exists no method that guarantees, at the same time, the properties of *unanimity*, *independence to irrelevant alternatives*, and *non-dictatorship* when preferences define a total order of the available alternatives [5]. Pini et al. have extended this result to the problem of aggregating strict partial orders [41]. Indeed, even using Pareto composition one has a form of "weak dictatorship", since if an agent (context in our scenario) prefers object o_1 to o_2, then o_2 cannot be better than o_1 in the aggregated preferences.

Recommender systems. In Context-Aware Recommender Systems (CARS) the techniques exploited by a recommender system to suggest relevant items to the user are enriched by taking into account information about the current user context [52]. This information can be exploited into one of the several stages in the recommendation process, in particular by pre-filtering items before applying the recommendation model of the system, by post-filtering items that are not relevant on the current user context, or by directly extending the recommendation model with context information [2]. Propagation of preferences through contexts is usually not considered in CARS [52].

We finally observe that, in a broad sense that goes outside the scope of this paper, the issue of preference propagation has also been studied in frameworks in which the propagation may happen through elements that are not contexts (e.g., [22]) or that are not even organized in a hierarchical structure (e.g., [53]).

9 CONCLUSION AND FUTURE WORK

In this paper we have considered the problem of how preferences propagate when they depend on the context, which is given as part of a context poset. Unlike previous approaches, which are based on heuristic arguments, we have tackled the problem in a principled way and have proposed an algebraic model for expressing preference propagation, based on two abstract operators, $+$ and $\triangleright$, that apply to unordered and, respectively, ordered context pairs. After formulating the notion of well-behaved (i.e., coherent, fair and specific) propagation, we have shown that no algebraic structure with the properties of an idempotent semiring can lead to well-behaved propagation methods. We have then abandoned right-distributivity of $\triangleright$ over $+$, thus considering idempotent left near-semirings, and have shown that the Pareto and Prioritized composition operators are the only natural possible interpretations of $+$ and $\triangleright$ that satisfy all the required algebraic properties. We have then analyzed several alternative propagation methods and shown that only the $\mathcal{OC}$ method satisfies all the desirable propagation properties. Finally, we have studied the problem of efficiently computing the best objects according to the preferences propagated through the $\mathcal{OC}$ method to a given context, and have shown that this can be done with polynomial complexity in all the involved parameters.

Throughout, we have considered operators that are *independent of irrelevant objects* (IIO), i.e., those for which the preference between any two objects o and o' *only* depends on the input preferences between o and o', while the preferences involving all other objects in the domain are immaterial. An extension of this work would be to analyze in more detail the case of non-IIO operators. While this would clearly enlarge the spectrum of alternatives, one would then have to deal with the general problem of aggregating partial orders (the intricacies of which are partly explored in [41]), as well as an increase in the computational overhead incurred when comparing objects (since the resulting preference between o and o' would also depend on all other objects in the domain).

A different line of investigation would be to analyze how the principles of preference propagation, namely coherence, fairness, and specificity, would apply to different preference models, including, e.g., those based on numerical preferences. Indeed, a major motivation for our work was to understand the possibility of using qualitative preferences based on the binary relation model in context-aware scenarios, in which most of the approaches in the literature are based on quantitative preferences. As argued in [20], approaches based on a qualitative model are more principled and, thus, deserve attention from the research community. One of the possible implications of our work is that it enables the study of hybrid approaches aiming to leverage the advantages of both quantitative and qualitative preference models. An example of the possibilities opened by this line of research is discussed in [17], where qualitative preferences expressed through constraints are imposed over numeric (quantitative) attributes.

Since the adopted context model only assumes that contexts are organized into a poset, it encompasses a variety of scenarios, including cases in which poset elements are not contexts proper, e.g., concepts [15] or (groups of) users [22]. It would then be interesting to study how our approach can meet the specific requirements of these alternative scenarios.

Furthermore, our approach could be also of interest for propagating preferences/recommendations in relational graphs, such as those that are found in social networks. For instance, consider an undirected graph in which nodes represent users and edges model a relationship over this set of users (e.g., friendship). Our framework can be applied to this scenario for computing the complete preferences for a user given their ground preferences and those of their friends by transforming the graph into a poset in which the given user is the only minimal element.

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APPENDIX

This appendix reports all the proofs of the claims included in the main body of the paper.

THEOREM 1. *No $(\mathcal{PR}_O, +, \rhd)$ structure is an idempotent semiring.*

PROOF. We prove the claim by showing that it is impossible to have both distributivity of $\rhd$ over $+$ and specificity of the $\rhd$ operator. For illustration purposes, although not strictly necessary, we later also prove the claim by showing that the same applies to fairness of the $+$ operator.

(Specificity) Consider the context poset C_1 in Figure 1 and the following PC-expression for the context c_4:

$$E = c_4 \rhd (c_2 + c_3) \rhd c_1$$

If the $\rhd$ operator distributes over $+$, this is equivalent to:

$$(c_4 \rhd c_2 \rhd c_1) + (c_4 \rhd c_3 \rhd c_1)$$

that is, the canonical expression $\text{Can}^{C_1}(c_4)$.

Now, consider a preference configuration such that all objects are indifferent in every context, except for $o' \prec^{c_1} o$ and $o \prec^{c_2} o'$.

Then, on one hand, when used to propagate preferences to c_4, expression E correctly propagates the preference $o \,{}^E\!\!\prec o'$. Indeed, since $\langle \prec^{c_3}, \approx^{c_3} \rangle = \langle \prec^{c_4}, \approx^{c_4} \rangle = \emptyset_\approx$, then E reduces to $c_2 \rhd c_1$ by the identity element axioms of $+$ and $\rhd$. Therefore $o \,{}^E\!\!\prec o'$, according to the specificity axiom of $\rhd$, since $o \prec^{c_2} o'$.

On the other hand, $\text{Can}^{C_1}(c_4)$ does not propagate any preference to context c_4. Indeed, under the same assumptions, $\text{Can}^{C_1}(c_4)$ reduces to $E_{21} + c_1$, where $E_{21} = c_2 \rhd c_1$. Now, by the specificity axiom of $\rhd$, the expression E_{21} yields $o \,{}^{E_{21}}\!\!\prec o'$. Finally, by the fairness axiom of $+$, it is neither $o \,{}^E\!\!\prec o'$ nor $o' \,{}^E\!\!\prec o$, which is absurd, because we showed that $o \,{}^E\!\!\prec o'$.

(Fairness) Consider poset C_1 and a preference configuration such that all objects are indifferent in every context, except for $o \prec^{c_2} o'$, $o' \prec^{c_3} o$, and $o' \prec^{c_1} o$. Due to the fairness axiom, no preference is propagated to c_4 if one considers the complete preference relation ${}^E\!\!\prec$ obtained with the canonical expression $E = \text{Can}^{C_1}(c_4)$, which reduces as follows:

$$
\begin{aligned}
& (c_4 \rhd c_2 \rhd c_1) + (c_4 \rhd c_3 \rhd c_1) & \\
\equiv\; & (\bot \rhd c_2 \rhd c_1) + (\bot \rhd c_3 \rhd c_1) & (\langle \prec^{c_4}, \approx^{c_4} \rangle = \emptyset_\approx) \\
\equiv\; & (c_2 \rhd c_1) + (c_3 \rhd c_1) & (\emptyset_\approx \textit{ is the identity for } \rhd) \\
\equiv\; & (c_2 \rhd c_1) + (c_1 \rhd c_1) & (\langle \prec^{c_3}, \approx^{c_3} \rangle = \langle \prec^{c_1}, \approx^{c_1} \rangle) \\
\equiv\; & (c_2 \rhd c_1) + c_1 & (\textit{idempotence of } \rhd)
\end{aligned}
$$

As before, it is neither $o \,{}^{E}\!\!\prec o'$ nor $o' \,{}^{E}\!\!\prec o$.

However, if the $\triangleright$ operator distributes over $+$, the following is also a valid rewriting of $\mathrm{Can}^{C_1}(c_4)$:

$$
\begin{array}{lll}
 & (c_2 \triangleright c_1) + c_1 & \\
\equiv & (c_2 \triangleright c_1) + (\bot \triangleright c_1) & (\emptyset_{\approx} \textit{ is the identity for } \triangleright) \\
\equiv & (c_2 + \bot) \triangleright c_1 & (\textit{right-distributivity of } \triangleright \textit{ over } +) \\
\equiv & c_2 \triangleright c_1 & (\emptyset_{\approx} \textit{ is the identity for } +).
\end{array}
$$

From the resulting expression $c_2 \triangleright c_1$ we can immediately derive that $o \,{}^{E}\!\!\prec o'$, which is absurd, because no preference was propagated to c_4. □

Lemma 1. *Prioritized composition left-distributes over Pareto composition, that is, for all objects $o_1, o_2 \in O$ and all preference structures $\langle \prec_1, \approx_1 \rangle, \langle \prec_2, \approx_2 \rangle, \langle \prec_3, \approx_3 \rangle$, it is:*

$$o_1 \prec_1 \rhd (\prec_2 \oplus \prec_3) o_2 \Leftrightarrow o_1 (\prec_1 \rhd \prec_2) \oplus (\prec_1 \rhd \prec_3) o_2$$

Prioritized composition does not right-distribute over Pareto composition, that is, there exist objects $o_1, o_2 \in O$ and preference structures $\langle \prec_1, \approx_1 \rangle, \langle \prec_2, \approx_2 \rangle, \langle \prec_3, \approx_3 \rangle$ such that:

$$o_1 (\prec_2 \oplus \prec_3) \rhd \prec_1 o_2 \not\Leftrightarrow o_1 (\prec_2 \rhd \prec_1) \oplus (\prec_3 \rhd \prec_1) o_2$$

Proof. (Left-distributivity) Let $\prec^L = \prec_1 \rhd (\prec_2 \oplus \prec_3)$ and $\prec^R = (\prec_1 \rhd \prec_2) \oplus (\prec_1 \rhd \prec_3)$. Since if $\langle \prec_1, \approx_1 \rangle = \emptyset_{\approx}$ the result immediately follows, assume $\langle \prec_1, \approx_1 \rangle \neq \emptyset_{\approx}$.
($\prec^L \subseteq \prec^R$) If $o_1 \prec^L o_2$ then either *a)* $o_1 \prec_1 o_2$, or *b)* $o_1 \approx_1 o_2$ and $o_1 \prec_2 \oplus \prec_3 o_2$. In case *a)*, due to specificity of $\rhd$, we must have both $o_1 \prec_1 \rhd \prec_2 o_2$ and $o_1 \prec_1 \rhd \prec_2 o_3$; in turn, due to fairness of $\oplus$, we must have $o_1 \prec^R o_2$. In case *b)*, assume without loss of generality $o_1 \prec_2 o_2$ and $o_1 \approx_3 o_2$, the other cases requiring similar arguments. Thus, $o_1 \prec_1 \rhd \prec_2 o_2$ whereas o_1 and o_2 are indifferent according to both $\prec_1$ and $\prec_3$. This is enough to conclude that $o_1 \prec^R o_2$.
($\prec^R \subseteq \prec^L$) The arguments to show inclusion in the other direction are almost the same. If $o_1 \prec^R o_2$ then either $o_1 \prec_1 \rhd \prec_2 o_2$ or $o_1 \prec_1 \rhd \prec_3 o_2$ (or both). If both preferences hold then either $o_1 \prec_1 o_2$, which immediately entails $o_1 \prec^L o_2$, or $o_1 \approx_1 o_2$ and both $o_1 \prec_2 o_2$ and $o_1 \prec_3 o_2$ hold. Even this case leads to conclude that $o_1 \prec^L o_2$. If only one preference holds, say $o_1 \prec_1 \rhd \prec_2 o_2$, then it is necessarily $o_1 \approx_1 o_2$ and $o_1 \approx_3 o_2$ (otherwise $o_1 \not\prec^R o_2$) and $o_1 \prec_2 o_2$. Even this case guarantees that $o_1 \prec^L o_2$.

(No right-distributivity) Consider now three preference structures such that $o_1 \prec_2 o_2$, $o_2 \prec_1 o_1$ and $\langle \prec_3, \approx_3 \rangle = \emptyset_{\approx}$, and let $\prec^L = (\prec_2 \oplus \prec_3) \rhd \prec_1$ and $\prec^R = (\prec_2 \rhd \prec_1) \oplus (\prec_3 \rhd \prec_1)$. Under these assumptions, $\prec^L$ reduces to $\prec_2 \rhd \prec_1$, since $\langle \prec_3, \approx_3 \rangle = \emptyset_{\approx}$, and thus $o_1 \prec^L o_2$ by specificity of $\rhd$. On the other hand, $\prec^R$ reduces to $(\prec_2 \rhd \prec_1) \oplus \prec_1$, since $\langle \prec_3, \approx_3 \rangle = \emptyset_{\approx}$; note that $o_1 \prec_2 \rhd \prec_1 o_2$ by specificity of $\rhd$, and thus, by fairness of $\oplus$, we have neither $o_1 \prec^R o_2$ nor $o_2 \prec^R o_1$. □

Proposition 1. *When $+$ and $\triangleright$ are interpreted as $\oplus$ and $\rhd$, respectively, $Rec^C(c)$ is equivalent to $Can^C(c)$, for each context c in a context poset C.*

Proof. The result trivially follows from the application of the left-distributivity property. □

Theorem 2. *Operator $\oplus$ is the only IIO $+$ operator.*

Proof. For any preference structure $\langle \prec_i, \approx_i \rangle$ over O, any two objects $o_a, o_b \in O$ are related in one of four possible ways, i.e., $o_a \theta_i^{a,b} o_b$, where $\theta_i^{a,b}$ is one of $\prec_i, \succ_i, \approx_i, \|_i$ ($o_a \succ_i o_b$ denotes $o_b \prec_i o_a$).

Therefore, for o_a, o_b, when combining two preference structures $\langle \prec_1, \approx_1 \rangle$ and $\langle \prec_2, \approx_2 \rangle$ with an IIO $+$ operator, we have $4 \cdot 4 = 16$ combinations of left and right argument and 4 possible outcomes in $\langle \prec_1, \approx_1 \rangle + \langle \prec_2, \approx_2 \rangle$, leading to a total of 4^{16} different interpretations of $+$. We now number these 16 combinations and show that there is only one possible outcome for each of them, coinciding with the outcome of $\oplus$.

When $o_a \prec_1 o_b$ and $o_a \prec_2 o_b$ then $o_a \prec_1 + \prec_2 o_b$, which we compactly indicate as (1): $\prec + \prec = \prec$. Indeed, if this were not the case, then, when $\prec_1$ and $\prec_2$ consist only of the above preferences, thus $\langle \prec_1, \approx_1 \rangle = \langle \prec_2, \approx_2 \rangle$, we would violate the idempotence axiom of Definition 7. Similarly, we have (2): $\succ + \succ = \succ$.

Clearly, (3): $\approx + \approx = \approx$ or else $\emptyset_\approx$ would not be the identity element of $+$, as one would have $\emptyset_\approx + \emptyset_\approx \neq \emptyset_\approx$. Similarly, (4): $\prec + \approx = \prec$, (5): $\approx + \prec = \prec$, (6): $\succ + \approx = \succ$, (7): $\approx + \succ = \succ$, (8): $\approx + \| = \|$, and (9): $\| + \approx = \|$.

By fairness, we have $\prec + \succ \in \{\approx, \|\}$. However, $\prec + \succ = \approx$ cannot be the case because $+$ is associative. Indeed, $(\prec + \prec) + \succ = \prec + \succ = \approx \neq \prec + (\prec + \succ) = \prec + \approx = \prec$. Therefore (10): $\prec + \succ = \|$. By commutativity, (11): $\succ + \prec = \|$.

We now show that (12): $\| + \prec = \|$. We have $\succ + (\prec + \prec) = \succ + \prec = \|$ and $(\succ + \prec) + \prec = \| + \prec$. Therefore, any interpretation such that $\| + \prec \neq \|$ would violate associativity. By commutativity, we also have (13): $\prec + \| = \|$. Analogously, one can prove that (14): $\succ + \| = \|$ and (15): $\| + \succ = \|$.

We have (16): $\| + \| = \|$, or else idempotence would be violated.

As shown, there is only one interpretation for $+$; this coincides with $\oplus$, as can be easily checked against Definition 13. It is known that $\oplus$ preserves strict partial orders and is both commutative and associative [25], and is also evidently idempotent and has $\emptyset_\approx$ as the identity element. □

Theorem 3. *Operator* $\rhd\!\!\!\!\bigcirc$ *is the only IIO* $\rhd$ *operator.*

Proof. As in the proof of Theorem 2, we show that the 16 combinations of left and right argument have only one possible outcome and the resulting interpretation coincides with $\rhd\!\!\!\!\bigcirc$.

By specificity of $\rhd$, we have (1): $\succ \rhd \succ = \succ$, (2): $\succ \rhd \prec = \succ$, (3): $\succ \rhd \approx = \succ$, (4): $\succ \rhd \| = \succ$, (5): $\prec \rhd \succ = \succ$, (6): $\prec \rhd \prec = \succ$, (7): $\prec \rhd \approx = \succ$, and (8): $\prec \rhd \| = \succ$.

We have (9): $\approx \rhd \succ = \succ$, or else $\emptyset_\approx$ would not be the identity element of $\rhd$, as one would have $\emptyset_\approx \rhd \{o_a \succ o_b\} \neq \{o_a \succ o_b\}$. Similarly, (10): $\approx \rhd \prec = \prec$, (11): $\approx \rhd \approx = \approx$, (12): $\approx \rhd \| = \|$, and (13): $\| \rhd \approx = \|$.

We now show that (14): $\| \rhd \prec = \|$. Consider three objects o_a, o_b, and o_c and two preference structures $\langle \prec_1, \approx_1 \rangle$, $\langle \prec_2, \approx_2 \rangle$, such that $o_a \|_1 o_b$, $o_b \|_1 o_c$, $o_a \prec_1 o_c$, $o_a \prec_2 o_b$, $o_b \prec_2 o_c$, $o_a \prec_2 o_c$. Let $\langle \prec_{12}, \approx_{12} \rangle = \langle \prec_1, \approx_1 \rangle \rhd \langle \prec_2, \approx_2 \rangle$. If it was $\| \rhd \prec = \succ$, we would have $o_a \succ_{12} o_b$, $o_b \succ_{12} o_c$ and $o_a \prec_{12} o_c$, so that $\prec_{12}$ is not a partial order. Consider now $o_a \|_1 o_b$, $o_b \|_1 o_c$, $o_a \succ_1 o_c$, $o_a \prec_2 o_b$, $o_b \prec_2 o_c$, $o_a \prec_2 o_c$. If it was $\| \rhd \prec = \prec$, we would have $o_a \prec_{12} o_b$, $o_b \prec_{12} o_c$ and $o_a \succ_{12} o_c$, so that $\prec_{12}$ is not a partial order. Finally, $\| \rhd \prec \neq \approx$, or else we would have $(\| \rhd \prec) \rhd \prec = \approx \rhd \prec = \prec \neq \| \rhd (\prec \rhd \prec) = \| \rhd \prec = \approx$, thereby violating associativity. Therefore $\| \rhd \prec = \|$, and, similarly, (15): $\| \rhd \succ = \|$.

We have (16): $\| \rhd \| = \|$, or else idempotence would be violated.

As shown, there is only one interpretation for $\rhd$; this coincides with $\rhd\!\!\!\!\bigcirc$, as can be easily checked against Definition 13. It is known that $\rhd\!\!\!\!\bigcirc$ preserves strict partial orders and is associative [25], and is also evidently idempotent and has $\emptyset_\approx$ as the identity element. □

Theorem 4. *Let* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *be two coherent propagation methods computed, for a context* $c \in C$, *by PC-expressions using* $\oplus$ *for* $+$ *and* $\rhd\!\!\!\!\bigcirc$ *for* $\rhd$. *Then* ${}^{\mathcal{P}_1}\!\approx^c = {}^{\mathcal{P}_2}\!\approx^c$.

Proof. Due to the semantics of Pareto and Prioritized composition, two objects o_1 and o_2 are indifferent in the complete preference structure in c iff this holds in all contexts whose ground preferences appear in the PC-expression for computing the complete preference structure in c. Since both $\mathcal{P}_1$ and $\mathcal{P}_2$ are coherent, the corresponding PC-expressions include all and only the contexts in $C\lfloor c \rfloor$ (except perhaps for those contexts with a full indifference structure), from which the result follows. □

Theorem 5. *CC propagation is coherent and fair but not specific.*

Proof. Coherence derives from the fact that $\mathrm{Can}^C(c)$ uses exactly the contexts in $C\lfloor c \rfloor$, so no other context can be relevant. In order to check that every context c' in $C\lfloor c \rfloor$ is relevant for c, it suffices to consider two preference configurations: $\langle \prec_1^C, \approx_1^C \rangle$, in which all objects are indifferent in all contexts, and $\langle \prec_2^C, \approx_2^C \rangle$, in which all objects are indifferent in all contexts except for $o_1 \prec_2^{c'} o_2$ in c'. Clearly, $\langle {}^{\mathcal{P}}{\prec_1}^c, {}^{\mathcal{P}}{\approx_1}^c \rangle \neq \langle {}^{\mathcal{P}}{\prec_2}^c, {}^{\mathcal{P}}{\approx_2}^c \rangle$, since, with the former, $o_1 \, {}^{\mathcal{P}}{\approx_1}^c o_2$ (via the identity element axioms of $+$ and $\triangleright$, which, thus, also hold for $\oplus$ and $\circledtriangleright$), while, with the latter, $o_1 \, {}^{\mathcal{P}}{\prec_2}^c o_2$ (as can also be checked via the identity element axioms); therefore c' is relevant for c.

To prove fairness, let c_1 and c_2 be two unordered contexts in $C\lfloor c \rfloor$, with $o_1 \prec^{c_1} o_2$ and $o_2 \prec^{c_2} o_1$, and let $o_1 \approx^{c_i} o_2 \; \forall c_i : (c \leqslant_C c_i <_C c_1) \vee (c \leqslant_C c_i <_C c_2)$. Let $c_{k,1}$ ($c_{k,2}$) be a context in $\mathrm{cov}^C(c)$ such that $c <_C c_{k,1} \leqslant_C c_1$ ($c <_C c_{k,2} \leqslant_C c_2$, respectively). Due to the semantics of $\oplus$, either o_2 is still preferable to o_1 in the complete preference relation in $c_{k,1}$, or the two objects are incomparable in this context (which may happen, e.g., if $c_{k,1} <_C c_2$). Similar arguments hold for $c_{k,2}$, from which it is derived that o_1 and o_2 are incomparable in c.

To see why CC violates specificity, consider the poset in Figure 1 and any preference configuration such that $o_1 \prec^{c_3} o_2$, $o_2 \prec^{c_1} o_1$, $o_1 \approx^{c_2} o_2$, and $o_1 \approx^{c_4} o_2$. In such a case, we have $o_1 \, {}^{CC}{\prec}^{c_3} o_2$ and $o_2 \, {}^{CC}{\prec}^{c_2} o_1$, and therefore, according to Pareto composition, we have $o_1 \, {}^{CC}{\|}^{c_4} o_2$ whereas specificity would require that $o_1 \, {}^{CC}{\prec}^{c_4} o_2$. □

Lemma 2. *Let H_1 and H_2 be two chains, such that $H_2 \subseteq H_1$. Let $\langle \prec_{1,2}, \approx_{1,2} \rangle$ be the preference structure denoted by $(\circledtriangleright(H_1)) \oplus (\circledtriangleright(H_2))$ and $\langle \prec_1, \approx_1 \rangle$ the preference structure denoted by $\circledtriangleright(H_1)$. Then: i) $\approx_{1,2} = \approx_1$, and ii) $\prec_{1,2} \subseteq \prec_1$.*

Proof. The result is a specific case of a more general result, which is the subject of the following lemma. □

Lemma 4. *Let $\langle \prec_1, \approx_1 \rangle$ be the preference structure denoted by a PC-expression E_1, whose operands are the contexts in the set $S_1 = \{c_1, \ldots, c_k\}$. Let $\langle \prec_i, \approx_i \rangle$ be similarly defined by a PC-expression E_i, over the set S_i, with $S_i \subseteq S_1$ for $2 \leqslant i \leqslant n$. Let $\langle \prec_*, \approx_* \rangle$ be the preference structure denoted by $\langle \prec_1, \approx_1 \rangle \oplus \langle \prec_2, \approx_2 \rangle \oplus \ldots \oplus \langle \prec_n, \approx_n \rangle$. Then: i) $\approx_* = \approx_1$, and ii) $\prec_* \subseteq \prec_1$.*

Proof. The first part immediately follows after observing that, for any two objects o_1 and o_2, $o_1 \approx_1 o_2$ iff o_1 and o_2 are indifferent in all contexts in S_1, thus also in those in $S_2, \ldots, S_n$. For the second part, according to Pareto composition, $o_1 \prec_* o_2$ can only occur if $o_1 \prec_1 o_2$ holds, hence the claim. Indeed, if $o_1 \approx_1 o_2$ then $o_1 \approx_i o_2$ for all i (due to part one), and then also $o_1 \approx_* o_2$. □

Theorem 6. *Let c be a context in the context poset C. Then, ${}^{CC}{\prec}^c \subseteq {}^{\mathcal{AC}}{\prec}^c$.*

Proof. Both propagation methods compute the complete preference structure in c using a PC-expression that is equivalent to the canonical form, i.e., Pareto composition of all the sub-expressions $\circledtriangleright(H_i)$, where H_i is a maximal chain of the context poset $C\lfloor c \rfloor$ in the case of CC, and of the active poset $A\lfloor c \rfloor$ in the case of $\mathcal{AC}$. We observe that each $\circledtriangleright(H_i)$ in $\mathrm{Can}^C(c)$ in Equation (1) is equivalent to an expression $\circledtriangleright(H_i^-)$, $H_i^- \subseteq H_i$, obtained by discarding inactive contexts from H_i, since such contexts are irrelevant to the result of $\circledtriangleright(H_i)$. Let $\mathcal{H}_C^-(c)$ be the set of such chains; then, the semantics of CC propagation is captured by $\mathcal{H}_C^-(c)$, i.e.:

$$\mathrm{Rec}^C(c) \equiv \circledtriangleright(H_1) \oplus \ldots \oplus \circledtriangleright(H_l) \equiv \circledtriangleright(H_1^-) \oplus \ldots \oplus \circledtriangleright(H_l^-).$$

Similarly, the semantics of $\mathcal{AC}$ propagation is captured by the set $\mathcal{H}_{A\lfloor c\rfloor}(c) = \{H'_1, \ldots, H'_k\}$ of all the maximal chains in $A\lfloor c\rfloor$, i.e.,

$$\mathrm{Rec}^A(c) \equiv \rhd(H'_1) \oplus \ldots \oplus \rhd(H'_k). \tag{14}$$

Now note that

(1) $\forall H \in \mathcal{H}^-_C(c) \quad \exists H' \in \mathcal{H}_{A\lfloor c\rfloor}(c)$ s.t. $H \subseteq H'$, i.e., $\mathcal{H}^-_C(c)$ contains only chains from $A\lfloor c\rfloor$

(2) $\mathcal{H}_{A\lfloor c\rfloor}(c) \subseteq \mathcal{H}^-_C(c)$, i.e., $\mathcal{H}^-_C(c)$ includes all the maximal chains in $A\lfloor c\rfloor$ (plus possibly some others chains that are not maximal in $A\lfloor c\rfloor$).

Let us indicate with E_i, $1 \leqslant i \leqslant k$, the PC-expression $\rhd(H''_{i,1})\oplus \ldots \oplus\rhd(H''_{i,m_i})$, where $H''_{i,1}, \ldots, H''_{i,m_i}$ are all the non-maximal subchains of H'_i occurring in $\mathcal{H}^-_C(c)$ (let $E_i = \bot$ if there are no such subchains). We then have

$$\mathrm{Rec}^C(c) \equiv (\rhd(H'_1) \oplus E_1) \oplus \ldots \oplus (\rhd(H'_k) \oplus E_k) \tag{15}$$

Let $\langle \prec_i, \approx_i\rangle$ and $\langle \prec_{i'}, \approx_{i'}\rangle$ be the preference structures resulting, respectively, from PC-expressions $(\rhd(H'_i) \oplus E_i)$ and $\rhd(H'_i)$, for $1 \leqslant i \leqslant k$. By Lemma 4, we have $\prec_i \subseteq \prec_{i'}$ and $\approx_i = \approx_{i'}$. The claim now easily follows by applying $k-1$ times Lemma 3, shown below, to compose the resulting expressions (14) and (15). □

Lemma 3. *Let $\langle \prec_a, \approx_a\rangle$, $\langle \prec_b, \approx_b\rangle$, $\langle \prec_c, \approx_c\rangle$, and $\langle \prec_d, \approx_d\rangle$ be four preference structures such that $\prec_a \subseteq \prec_b$, $\approx_a = \approx_b$, $\prec_c \subseteq \prec_d$ and $\approx_c = \approx_d$. Let $\langle \prec_{a,c}, \approx_{a,c}\rangle = \langle \prec_a, \approx_a\rangle \oplus \langle \prec_c, \approx_c\rangle$ and $\langle \prec_{b,d}, \approx_{b,d}\rangle = \langle \prec_b, \approx_b\rangle \oplus \langle \prec_d, \approx_d\rangle$. Then, i) $\prec_{a,c} \subseteq \prec_{b,d}$, and ii) $\approx_{a,c} = \approx_{b,d}$.*

Proof. If $\prec_a = \prec_b$ and $\prec_c = \prec_d$ the claim trivially holds. Assume then that $\prec_a \subset \prec_b$ and $\prec_c = \prec_d$ (the case $\prec_c \subset \prec_d$ and $\prec_a = \prec_b$ is analogous). Clearly, all the objects o_1 and o_2 with the same order relation ($\approx, \prec, \succ, \|$) in both $\langle \prec_a, \approx_a\rangle$ and $\langle \prec_b, \approx_b\rangle$ will have an identical order relation in both $\langle \prec_{a,c}, \approx_{a,c}\rangle$ and $\langle \prec_{b,d}, \approx_{b,d}\rangle$. Let then o_1 and o_2 be such that $o_1 \not\prec_a o_2$ and $o_1 \prec_b o_2$. Then, it must be $o_1 \|_a o_2$, since $\prec_a \subset \prec_b$ and $\approx_a = \approx_b$, and thus also $o_1 \|_{a,c} o_2$, which proves the first part in this case.

Assume now that $\prec_a \subset \prec_b$ and $\prec_c \subset \prec_d$. In all cases in which o_1 and o_2 have the same order relation in both $\langle \prec_a, \approx_a\rangle$ and $\langle \prec_b, \approx_b\rangle$ (or, symmetrically, in both $\langle \prec_c, \approx_c\rangle$ and $\langle \prec_d, \approx_d\rangle$), the result holds with the same argument as above. Then, we only need to consider the following cases:

(1) $o_1 \not\prec_a o_2$, $o_1 \prec_b o_2$, and $o_1 \not\prec_c o_2$, $o_1 \prec_d o_2$: here we have $o_1 \|_{a,c} o_2$ and $o_1 \prec_{b,d} o_2$, hence the claim;

(2) $o_2 \not\prec_a o_1$, $o_2 \prec_b o_1$, and $o_1 \not\prec_c o_2$, $o_1 \prec_d o_2$: here we have $o_1 \|_{a,c} o_2$ and $o_1 \|_{b,d} o_2$, hence the claim;

(3) $o_1 \not\prec_a o_2$, $o_1 \prec_b o_2$, and $o_2 \not\prec_c o_1$, $o_2 \prec_d o_1$ (this case is analogous to the previous one).

As for the second part, it suffices to observe that, for two objects to be indifferent in the result of a Pareto composition, they have to be indifferent in both sides of the composition. □

Theorem 7. *$\mathcal{AC}$ propagation is coherent and fair but not specific.*

Proof. Coherence of $\mathcal{AC}$ follows from essentially the same argument used in the proof of Theorem 5. Just consider two preference configurations, one in which all contexts are inactive, and one in which c' is the only active context in $C\lfloor c\rfloor$. In the former case, $\mathrm{Rec}^A(c) = c \equiv \bot$, while in the latter case $\mathrm{Rec}^A(c) = c \rhd c' \equiv \bot \rhd c' \equiv c'$, thus making c' relevant for c.

Fairness follows from the same arguments used in the proof of Theorem 5. The same counterexample used in the proof of Theorem 5 applies here to show that $\mathcal{AC}$ violates specificity, with the only additional hypothesis that $o_1 \approx^{c_2} o_2$, yet $\langle \prec^{c_2}, \approx^{c_2}\rangle \neq \emptyset_\approx$ (i.e., c_2 is active). □

Theorem 8. *$\mathcal{OC}$ propagation is coherent, fair and specific.*

Proof. Coherence of $\mathcal{OC}$ follows the same argument used in the proof of Theorem 7, by simply focusing on two objects o_1, o_2 and replacing the notion of active context with that of (o_1, o_2)-active context. Fairness stems directly from the definition of $\text{cov}^{A_c^{o_1,o_2}}(c)$ much in the same way as in the proof of Theorem 5. Specificity is also guaranteed, since, if $c_1 \in A_c^{o_1,o_2}\lfloor c \rfloor$ and $o_1 \prec^{c_1} o_2$ and the conditions of Definition 6 hold, then, by the definition of $\text{cov}^{A_c^{o_1,o_2}}(c)$, it is $o_1 \prec^{c_j} o_2$ for all $c_j \in \text{cov}^{A_c^{o_1,o_2}}(c)$, which entails $o_1 \, {}^{\mathcal{OC}}\!\!\prec^c o_2$. □

Theorem 9. *Let c be a context in the context poset C. Then, the complete preference relations in c under the $\mathcal{CC}$, $\mathcal{AC}$ and $\mathcal{OC}$ propagation methods satisfy the following relationships:*

$$ {}^{\mathcal{CC}}\!\!\prec^c \;\subseteq\; {}^{\mathcal{AC}}\!\!\prec^c \;\subseteq\; {}^{\mathcal{OC}}\!\!\prec^c \; \textit{and} \; {}^{\mathcal{CC}}\!\!\approx^c \;=\; {}^{\mathcal{AC}}\!\!\approx^c \;=\; {}^{\mathcal{OC}}\!\!\approx^c $$

Proof. The part about indifference is a direct consequence of Theorem 4 and the fact that $\mathcal{OC}$ is coherent (analogously to $\mathcal{AC}$).

It was shown in Theorem 6 that ${}^{\mathcal{CC}}\!\!\prec^c \;\subseteq\; {}^{\mathcal{AC}}\!\!\prec^c$. The proof that ${}^{\mathcal{AC}}\!\!\prec^c \;\subseteq\; {}^{\mathcal{OC}}\!\!\prec^c$ is analogous. Indeed, in order to propagate a preference according to $\mathcal{AC}$, all the maximal chains in A must agree on that preference. Clearly, every element in $\text{cov}^{A_c^{o_1,o_2}}(c)$, for any objects o_1 and o_2, belongs to one of these chains, and in all such chains there cannot be any context that disagrees on the preference, hence the claim. □

Theorem 10. *The PC-expression $RG^A(c)$ correctly computes the $\mathcal{OC}$ propagation, i.e., $o_1 \, {}^{\mathcal{OC}}\!\!\prec^c o_2$ iff $o_1 \, {}^{E}\!\!\prec^c o_2$, where $E = RG^A(c)$.*

Proof. *(Only if part.)*

By hypothesis, $o_1 \, {}^{\mathcal{OC}}\!\!\prec^c o_2$ holds. If $o_1 \not\approx^c o_2$ the result is obvious, since every expression of the form $\text{RG}^A(c, c')$ can be written as $c \circledgtr \ldots$. Therefore, $\text{RG}^A(c)$ can also be written as $c \circledgtr \ldots$ by using the left-distributivity property of $\circledgtr$. Hence, any ground preference in c is propagated to c by $\text{RG}^A(c)$.

Then, assume $o_1 \approx^c o_2$. In this case, by Equation (10) we have $o_1 \prec^{c_j} o_2$ for all contexts $c_j \in \text{cov}^{A_c^{o_1,o_2}}(c)$.

We first show that no other context than those in $\text{cov}^{A_c^{o_1,o_2}}(c)$ can influence the preference on o_1 and o_2 propagated to c by $\text{RG}^A(c)$. This clearly holds for all contexts c_i such that $c \leqslant_C c_i <_C c_j$, where $c_j \in \text{cov}^{A_c^{o_1,o_2}}(c)$, since in all such contexts o_1 and o_2 are indifferent. Then consider a context $c_k \notin \text{cov}^{A_c^{o_1,o_2}}(c)$ such that $o_1 \not\approx^{c_k} o_2$. By definition of $\text{cov}^{A_c^{o_1,o_2}}(c)$, there exists at least one context $c_j \in \text{cov}^{A_c^{o_1,o_2}}(c)$ such that $c_j <_C c_k$. From the definitions of $\text{RG}^A(c)$ and $\text{RG}^A(c, c_k)$, it turns out that $\text{RG}^A(c)$ includes the sub-expression $(\ldots c_j \ldots) \circledgtr c_k$, and this is the case for *every* occurrence of c_k. Since the left operand includes c_j, for which $o_1 \prec^{c_j} o_2$, the preference of c_k on these objects has no influence on objects o_1 and o_2, due to the presence of the $\circledgtr$ operator.

To complete this part, it suffices to observe that, since all contexts in $\text{cov}^{A_c^{o_1,o_2}}(c)$ agree on the preference on o_1 and o_2, such a preference is propagated by $\text{RG}^A(c)$.

(If part.)

We show that if $o_1 \, {}^{\mathcal{OC}}\!\!\prec^c o_2$ does not hold then $o_1 \, {}^{E}\!\!\prec^c o_2$ does not hold either, where $E = \text{RG}^A(c)$. Again, when $o_1 \not\approx^c o_2$ the result is obvious for the same reasons as in the only-if part.

Then, assume $o_1 \approx^c o_2$. By Equation (10), we have that $o_1 \, {}^{\mathcal{OC}}\!\!\prec^c o_2$ does not hold when for at least one context $c_j \in \text{cov}^{A_c^{o_1,o_2}}(c)$ it is $o_1 \not\prec^{c_j} o_2$. As shown in the only-if part, only the contexts in $\text{cov}^{A_c^{o_1,o_2}}(c)$ can determine the preference on o_1 and o_2 propagated by $\text{RG}^A(c)$, which is enough to prove the result.

□

Theorem 11. *The PC-expression* $\mathrm{RG}^C(c)$ *is equivalent to* $\mathrm{RG}^A(c)$.

Proof. We can proceed as in the proof of Theorem 10 (with the only care of replacing $\mathrm{RG}^A(c)$ with $\mathrm{RG}^C(c)$ and RG^A with RG^C) to show that $\mathrm{RG}^C(c)$ correctly computes the $\mathcal{OC}$ propagation. Therefore, $\mathrm{RG}^C(c)$ is equivalent to $\mathrm{RG}^A(c)$. □

Theorem 12. *Let* $\mathcal{P}$ *be a propagation method and let* c *be a context in a context poset* C*: if* ${}^{\mathcal{P}}{\prec^c} \subset {}^{\mathcal{OC}}{\prec^c}$ *then* $\mathcal{P}$ *is not specific.*

Proof. In order to prove the claim, it suffices to show that *every* preference propagated by $\mathcal{OC}$ satisfies Definition 6 of specificity. Indeed, according to (10) in Definition 20, $\mathcal{OC}$ propagates the preference $o_1\,{}^{\mathcal{OC}}{\prec^c} o_2$ if either

(1) $o_1 \prec^c o_2$, or

(2) $o_1 \approx^c o_2 \wedge (o_1 \prec^{c_1} o_2 \wedge \cdots \wedge o_1 \prec^{c_m} o_2)$, where $c_1, \ldots, c_m$ are the contexts in $\mathrm{cov}^{A_c^{o_1,o_2}}(c)$.

It is plain to see that, in both cases, the conditions of Definition 6 apply. □

Theorem 13. *Let* $\mathcal{P}$ *be a coherent propagation method based on* $\oplus$ *and* $\circledcirc$ *that is either static or active-static, and let* c *be a context in a context poset* C*: if* ${}^{\mathcal{OC}}{\prec^c} \subset {}^{\mathcal{P}}{\prec^c}$ *then* $\mathcal{P}$ *is not both fair and specific.*

Proof. We show that propagating *any* preference $o_1\,{}^{\mathcal{P}}{\prec^c} o_2$ that is not propagated by $\mathcal{OC}$ would violate either fairness or specificity. Indeed, according to (10) in Definition 20, $\mathcal{OC}$ does not propagate the preference $o_1\,{}^{\mathcal{OC}}{\prec^c} o_2$ if the following holds

$$o_1 \not\prec^c o_2 \wedge [o_1 \not\approx^c o_2 \vee (o_1 \not\prec^{c_1} o_2 \vee \cdots \vee o_1 \not\prec^{c_m} o_2)], \text{ i.e.,}$$

(1) $o_1 \not\prec^c o_2 \wedge o_1 \not\approx^c o_2$, or

(2) $o_1 \not\prec^c o_2 \wedge (o_1 \not\prec^{c_1} o_2 \vee \cdots \vee o_1 \not\prec^{c_m} o_2)$.

When Case 1 occurs, we can have either $o_2 \prec^c o_1$ or $o_1 \parallel^c o_2$. If $o_2 \prec^c o_1$ and $o_1\,{}^{\mathcal{P}}{\prec^c} o_2$, then $\mathcal{P}$ violates specificity. If $o_1 \parallel^c o_2$, we can still show that propagating $o_1\,{}^{\mathcal{P}}{\prec^c} o_2$ makes $\mathcal{P}$ a method that violates specificity. Indeed, let then $\mathcal{G}$ be a preference configuration such that $o_1 \parallel^c o_2$. Since $\mathcal{P}$ is static or active-static, $\mathcal{P}$ would produce the same PC-expression in another preference configuration $\mathcal{G}'$ that only differs from $\mathcal{G}$ for the fact that $o_2 \prec^c o_1$; as shown above, $\mathcal{P}$ violates specificity in such a case.

When Case 2 occurs, the order relation of o_1 and o_2 in c can be: *i)* $o_2 \prec^c o_1$, *ii)* $o_1 \parallel^c o_2$, or *iii)* $o_1 \approx^c o_2$. Cases *i* and *ii* lead to violations of specificity by the same argument used in Case 1. Also, if $\mathcal{P}$ is static, the PC-expression it generates in Case *iii* is the same as in Cases *i* and *ii*. Let us therefore assume, in the remainder of the proof, that $o_1 \approx^c o_2$ and $\mathcal{P}$ is active-static.

If there is only one context, c_1, in $\mathrm{cov}^{A_c^{o_1,o_2}}(c)$, then clearly specificity is violated in case $o_2 \prec^{c_1} o_1$. By proceeding as in Case 1, we can show that, if $o_1 \parallel^{c_1} o_2$, propagating $o_1\,{}^{\mathcal{P}}{\prec^c} o_2$ makes $\mathcal{P}$ a method that violates specificity. Indeed, the generated PC-expression would be the same in a preference configuration for which $o_1 \parallel^{c_1} o_2$ is replaced by $o_2 \prec^{c_1} o_1$, in which case violation of specificity occurs. Also, note that the case $o_1 \approx^{c_1} o_2$ cannot occur, since $c_1 \in \mathrm{cov}^{A_c^{o_1,o_2}}(c)$.

Assume then that $m > 1$. If $o_2 \prec^{c_i} o_1$ holds for $1 \leqslant i \leqslant m$, then $\mathcal{P}$ is clearly not specific according to Definition 6.

More generally, let $\mathcal{G}$ be a preference configuration in which $o_1 \prec o_2$ holds for no context in $\{c_1, \ldots, c_m\}$; if $\mathcal{P}$ propagates $o_1\,{}^{\mathcal{P}}{\prec^c} o_2$ in $\mathcal{G}$, then $\mathcal{P}$ is necessarily not specific. Indeed, $o_1 \prec^{c'} o_2$ must hold in some other context c', which, in the expression produced by $\mathcal{P}$, precedes all occurrences of c_i, $1 \leqslant i \leqslant m$, with a sub-expression of the form $(\ldots c' \ldots)\circledcirc(\ldots c_i \ldots)$. Then, in any preference configuration $\mathcal{G}'$ that is as $\mathcal{G}$ but in which $o_2 \prec^{c_i} o_1$ holds for $1 \leqslant i \leqslant m$, $\mathcal{P}$ would still propagate $o_1\,{}^{\mathcal{P}}{\prec^c} o_2$, thus violating specificity.

Let us then assume that $o_1 \prec o_2$ holds for at least one context (say, c_1) in $\mathrm{cov}^{A_c^{o_1,o_2}}(c)$, but not all of them. If there is a context c_i, $2 \leqslant i \leqslant m$, such that $o_2 \prec^{c_i} o_1$, then $\mathcal{P}$ violates fairness according to Definition 5.

The only case left to consider is when $o_2 \prec o_1$ never holds for $c_2, \ldots, c_m$ and $o_1 \parallel^{c_j} o_2$ holds for at least one context c_j, $2 \leqslant j \leqslant m$. Let us call $\mathcal{G}$ such a preference configuration, and consider now an almost identical preference configuration $\mathcal{G}'$ that only differs from $\mathcal{G}$ by the preference $o_2 \prec^{c_j} o_1$. The PC-expression derived by $\mathcal{P}$ is the same in $\mathcal{G}$ and $\mathcal{G}'$. Therefore, by the semantics of $\rhd$ and $\oplus$, $o_1 \,{}^{\mathcal{P}}\!\!\prec^c o_2$ is propagated also in $\mathcal{G}'$, thus violating fairness. □

Theorem 14. *For each pair of objects $o_1, o_2 \in O$, each context c and each preference configuration $\langle \prec^A, \approx^A \rangle$, Algorithm 1 correctly computes the order relation θ of o_1 and o_2 in c according to the $\mathcal{OC}$ propagation.*

Proof. In order to prove the claim it suffices to show that, for any context c', the same order result as computed by $\mathrm{RG}^A(c, c')$ (Definition 21) is returned by `ObjectComparisonOC2`.

Indeed, if `ObjectComparisonOC2` is called with input c, then the set of covered contexts is empty and the the loop at lines 6–9 is skipped. At line 10 the order relation is computed as being the one in context c, which is exactly what is computed by $\mathrm{RG}^A(c, c)$ according to Definition 21.

If `ObjectComparisonOC2` is called with input $c' \neq c$, then the contexts $c_1, \ldots, c_k$ covered by c' are computed at line 5 as $c''_1, \ldots, c''_k$. The loop at lines 6–9 combines the order relations of such contexts through $\oplus$ as computed by $\mathrm{RG}^A(c, c_i)$, $1 \leqslant i \leqslant k$, thereby reproducing the left operand of $\rhd$ in the PC-expression $\mathrm{RG}^A(c, c')$ of Definition 21. Line 10 implements the semantics of $\rhd$, i.e., the right operand affects the result only if the left operand generated $\approx$ as order relation. Overall, after line 10, θ has the same order relation as computed by $\mathrm{RG}^A(c, c')$.

Finally, observe that, when `ObjectComparisonOC2` is called by `ObjectComparisonOC`, its input is the fictitious context $c_\top$, which covers the maximal elements $\hat{c}_1, \ldots, \hat{c}_n$ in $A\lfloor c \rfloor$. The loop at lines 6–9 combines the order relations of such contexts through $\oplus$ as computed by $\mathrm{RG}^A(c, \hat{c}_i)$, $1 \leqslant i \leqslant n$, thereby reproducing the PC-expression (12) of Definition 21. □