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A note about the invariance of the basic reproduction number for stochastically perturbed SIS models

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Abstract

In [4] a susceptible-infected-susceptible (SIS) stochastic differential equation, obtained via a suitable random perturbation of the disease transmission coefficient in the classic SIS model, has been studied. Such random perturbation enters via an informal manipulation of stochastic differentials and leads to an Itô's type SDE. The authors identify a stochastic reproduction number, which differs from the standard one for the presence of those additional parameters that describe the employed random perturbation, and show that, similarly to the deterministic case, the stochastic reproduction number rules the asymptotic behaviour of the solution.
Aiming to make that random perturbation rigorous, we suggest an alternative approach based on a Wong-Zakai approximation argument thus arriving at a different stochastic model corresponding to the Stratonovich version of the Itô equation analysed in [4]. Rather surprisingly, the asymptotic behaviour of this alternative model turns out to be governed by the same reproduction number as the deterministic SIS equation. In other words, the random perturbation does not modify the threshold for extinction and persistence of the disease.

Key words and phrases: SIS epidemic model, Itô and Stratonovich stochastic differential equations, Wong-Zakai approximation, extinction, persistence.

AMS 2000 classification: 60H10, 60H30, 92D30.

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1 Introduction

To introduce environmental stochasticity in the evolution of some interacting populations one usually replaces deterministic systems with stochastic differential equations (SDEs) where the presence of a term driven by noise should account for such randomness. Since there is no canonical way to perform such replacement, different approaches have been considered in the literature.
One of them is the construction described in [1] (see also [2]): here one starts with a discrete Markov chain whose transition probabilities reflect the dynamical behaviour of the deterministic model; then, via a suitable scaling on the one-step transition probability, one obtains a forward Fokker-Planck equation which is canonically associated with a stochastic differential equation.
Another common method for introducing stochasticity is the so-called parameter perturbation approach: it amounts at perturbing one of the parameters of the model equation with a suitable source of randomness, usually a Gaussian white noise. See the classical reference [12], more recent paper [3] and references quoted there.
It is also worth mentioning that, the possibility of choosing between several reasonable stochastic integration theories (mainly Itô and Stratonovich ones) leaves the identification of the *right* SDE somewhat undetermined. And this in turn conditions the importance of the conclusions derived by the investigation of those models. We refer the reader to [3] for a nice account of this long lasting issue.
The aim of the present paper is to highlight a remarkable instance of such phenomenon. More precisely, in the paper [4] the authors propose, through a parameter perturbation technique, a stochastic equation aiming at introducing environmental stochasticity in the classic susceptible-infected-susceptible model. Due to this perturbation, the usual basic reproduction number, which is responsible for determining the asymptotic regimes of the solution, is replaced by a stochastic reproduction number whose expression involves a parameter describing the random perturbation. The SDE considered in [4] is of Itô's type and is derived via manipulations of *infinitesimal increments*.
We discover that, by suitably formalizing their parameter perturbation technique, one is lead to the Stratonovich version of the SDE from [4] and that its asymptotic regime is now independent of the parameter describing the stochastic perturbation.
In order to better describe the details of our investigation we briefly recall the main features of the classic susceptible-infected-susceptible (SIS) model and summarize the random parameter perturbation's technique employed in [4] to derive their SIS stochastic differential equation.

The susceptible-infected-susceptible (SIS) model is a simple mathematical model that describes, under suitable assumptions, the spread of diseases with no permanent immunity (see e.g. [2],[5]). In such models an individual starts being susceptible to a disease, at some point of time gets infected and then recovers after some other time interval, becoming susceptible again. If $S(t)$ and $I(t)$ denote the number of susceptibles and infecteds at time t, respectively, then the differential equations describing the spread of the disease are

$$\begin{cases} \frac{dS(t)}{dt} = \mu N - \beta S(t)I(t) + \gamma I(t) - \mu S(t), & S(0) = s_0 > 0; \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (\mu+\gamma)I(t), & I(0) = i_0 > 0. \end{cases} \tag{1.1}$$

Here, $N := s_0 + i_0$ is the initial size of the population amongst whom the disease is spreading, μ denotes the per capita death rate, γ is the rate at which infected individuals become cured and β stands for the disease transmission coefficient. Note that

$$\frac{d}{dt}(S(t)+I(t)) = \mu(N - (S(t)+I(t))), \quad S(0)+I(0) = N,$$

and hence

$$S(t)+I(t) = S(0)+I(0) = N, \qquad \text{for all } t \geq 0.$$

Therefore, system (1.1) reduces to the differential equation

$$\frac{dI(t)}{dt} = \beta I(t)(N - I(t)) - (\mu+\gamma)I(t), \quad I(0) = i_0 \in]0, N[, \tag{1.2}$$

with $S(t) := N - I(t)$, for $t \geq 0$. Equation (1.2) can be solved explicitly as

$$I(t) = \frac{i_0 e^{[\beta N - (\mu+\gamma)]t}}{1 + \beta \int_0^t i_0 e^{[\beta N - (\mu+\gamma)]s} ds}, \quad t \geq 0, \tag{1.3}$$

and one finds that

$$\lim_{t \to +\infty} I(t) = \begin{cases} 0, & \text{if } R_0 \leq 1; \\ N(1 - 1/R_0), & \text{if } R_0 > 1, \end{cases}$$

where

$$R_0 := \frac{\beta N}{\mu + \gamma}. \tag{1.4}$$

This ratio is known as *basic reproduction number* of the infection and determines whether the disease will become extinct, i.e. $I(t)$ will tend to zero as t goes to infinity, or will be persistent, i.e. $I(t)$ will tend to a positive limit as t increases.

1.1 The stochastic model from [4]

With the aim of examining the effect of environmental stochasticity, Gray et al. [4] have proposed a stochastic version of (1.2) which is obtained via a suitable perturbation of the parameter β. More precisely, they write equation (1.2) in the differential form

$$dI(t) = \beta I(t)(N - I(t))dt - (\mu+\gamma)I(t)dt, \quad I(0) = i_0 \in]0, N[, \tag{1.5}$$

and *formally* replace the infinitesimal increment βdt with $\beta dt + \sigma dB(t)$, where σ is a new positive parameter and $\{B(t)\}_{t\geq 0}$ denotes a standard one dimensional Brownian motion. This perturbation transforms the deterministic differential equation (1.2) into the stochastic differential equation

$$dI(t) = [\beta I(t)(N - I(t)) - (\mu + \gamma)I(t)]dt + \sigma I(t)(N - I(t))dB(t), \tag{1.6}$$

which the authors interpret in the Itô's sense. Equation (1.6) is then investigated and the authors prove the existence of a unique global strong solution living in the interval $]0, N[$ with probability one for all $t \geq 0$. Moreover, they identify a *stochastic reproduction number*

$$R_0^S := R_0 - \frac{\sigma^2 N^2}{2(\mu + \gamma)}, \tag{1.7}$$

which characterizes the following asymptotic behaviour:

- if $R_0^S < 1$ and $\sigma^2 < \frac{\beta}{N}$ or $\sigma^2 > \max\{\frac{\beta}{N}, \frac{\beta^2}{2(\mu+\gamma)}\}$, then the *disease will become extinct*, i.e.

$$\lim_{t\to+\infty} I(t) = 0;$$

- if $R_0^S > 1$, then *the disease will be persistent*, i.e.

$$\liminf_{t\to+\infty} I(t) \leq \xi \leq \limsup_{t\to+\infty} I(t),$$

  where $\xi := \frac{1}{\sigma^2}\left(\sqrt{\beta^2 - 2\sigma^2(\mu + \gamma)} - (\beta - \sigma^2 N)\right)$.

It is worth mentioning that Xu [13] refined the above description as follows:

- if $R_0^S < 1$, then $I(t)$ tends to zero, as t tends to infinity, almost surely;
- if $R_0^S \geq 1$, then $I(t)$ is recurrent on $]0, N[$.

We also refer the reader to the recent papers [11], which investigate the basic reproduction number of more general models obtained via the random parameter's perturbation proposed in [4], and [10], resorting to estimation techniques for the reproduction number in several discrete models.

1.2 The stochastic model from [4] revised

We already mentioned that the Itô equation

$$dI(t) = [\beta I(t)(N - I(t)) - (\mu + \gamma)I(t)]dt + \sigma I(t)(N - I(t))dB(t),$$

proposed in [4] is derived from

$$\frac{dI(t)}{dt} = \beta I(t)(N - I(t)) - (\mu + \gamma)I(t) \tag{1.8}$$

via the formal substitution

$$\beta dt \mapsto \beta dt + \sigma dB(t)$$

in

$$dI(t) = \beta I(t)(N - I(t))dt - (\mu + \gamma)I(t)dt.$$

It is important to remark that the non differentiability of the Brownian paths prevents us from implementing the formal transformation

$$\beta \mapsto \beta + \sigma \frac{dB(t)}{dt} \tag{1.9}$$

for equation (1.8). We now start from this simple observation and try to make such procedure rigorous.
Fix $T > 0$ and, for a partition π of the interval $[0, T]$, let $\{B^\pi(t)\}_{t\in[0,T]}$ be the polygonal approximation of the Brownian motion $\{B(t)\}_{t\in[0,T]}$, relative to the partition π. This means that $\{B^\pi(t)\}_{t\in[0,T]}$ is a continuous piecewise linear random function converging to $\{B(t)\}_{t\in[0,T]}$ almost surely and uniformly on $[0, T]$, as the mesh of the partition tends to zero. Now, substituting $\{B(t)\}_{t\in[0,T]}$ with $\{B^\pi(t)\}_{t\in[0,T]}$ in (1.9) we get a well defined transformation

$$\beta \mapsto \beta + \sigma \frac{dB^\pi(t)}{dt},$$

which in connection with (1.8) leads to the random ordinary differential equation

$$\frac{dI^\pi(t)}{dt} = [\beta I^\pi(t)(N - I^\pi(t)) - (\mu + \gamma)I^\pi(t)] + \sigma I^\pi(t)(N - I^\pi(t))\frac{dB^\pi(t)}{dt}.$$

According to the celebrated Wong-Zakai theorem [14], the solution of the previous equation converges, as the mesh of π tends to zero, to the solution $\{\mathtt{I}(t)\}_{t\in[0,T]}$ of the Stratonovich-type stochastic differential equation

$$d\mathtt{I}(t) = [\beta \mathtt{I}(t)(N - \mathtt{I}(t)) - (\mu + \gamma)\mathtt{I}(t)]dt + \sigma \mathtt{I}(t)(N - \mathtt{I}(t)) \circ dB(t), \tag{1.10}$$

which is equivalent to the Itô-type equation

$$\begin{aligned} d\mathtt{I}(t) = &\left[\beta \mathtt{I}(t)(N - \mathtt{I}(t)) - (\mu + \gamma)\mathtt{I}(t) + \frac{\sigma^2}{2}\mathtt{I}(t)(N - \mathtt{I}(t))(N - 2\mathtt{I}(t))\right] dt \\ &+ \sigma \mathtt{I}(t)(N - \mathtt{I}(t))dB(t) \end{aligned} \tag{1.11}$$

(see e.g. [7] for the definition of Stratonovich integral and Itô-Stratonovich correction term). Therefore, the model equation obtained via this procedure differs from the one proposed in [4] for the presence in the drift coefficient of the additional term

$$\frac{\sigma^2}{2}\mathtt{I}(t)(N - \mathtt{I}(t))(N - 2\mathtt{I}(t)).$$

Surprisingly, the *stochastic reproduction number* for the corrected model (1.11) coincides with $R_0 = \frac{\beta N}{\mu+\gamma}$. In other words, the stochastic perturbation of β doesn't affect the basic reproduction number.

Theorem 1.1. *Equation (1.11) possesses a unique global strong solution* $\{\mathtt{I}(t)\}_{t\geq 0}$ *which lives in the interval* $]0, N[$ *for all* $t \geq 0$ *with probability one. Such solution can be explicitly represented as*

$$\mathtt{I}(t) = \frac{i_0 \mathcal{E}(t)}{1 + \frac{i_0}{N}(\mathcal{E}(t) - 1) + i_0 \frac{\mu+\gamma}{N} \int_0^t \mathcal{E}(s)ds}, \quad t \geq 0, \tag{1.12}$$

where

$$\mathcal{E}(t) := e^{(\beta N - (\mu+\gamma))t + N\sigma B(t)}.$$

Moreover,

- *if* $R_0 < 1$*, then* $\mathtt{I}(t)$ *tends to zero, as* t *tends to infinity, almost surely;*
- *if* $R_0 \geq 1$*, then* $\mathtt{I}(t)$ *is recurrent on* $]0, N[$.

The paper is organized as follows: in Section 2 we develop a general framework to study existence, uniqueness and sufficient conditions for extinction and persistence for a large class of equations encompassing the model equation (1.6) and its revised version (1.11); Section 3 contains the proof of Theorem 1.1. The last section contains a discussion of our findings, some possible related generalizations and a list of figures illustrating our main result.

2 A general approach

The aim of this section is to propose a general method for studying existence and uniqueness of global strong solutions, as well as conditions for their extinction or persistence, for a large class of equations, which includes (1.6) and (1.11) as particular cases. Namely, we consider stochastic differential equations of the form

$$\begin{cases} dX(t) = [f(X(t)) - h(X(t))]dt + \sum_{i=1}^m g_i(X(t))dB_i(t), \quad t > 0; \\ X(0) = x_0 \in]0, N[, \end{cases} \tag{2.1}$$

where the coefficients satisfy only those fairly general assumptions needed to derive the desired properties (see Theorem 2.3 below for the detailed assumptions). Our method allows for a great flexibility in the choice of the coefficients while preserving the essential features of (1.6) and (1.11). In particular, we allow the diffusion coefficients to vanish on arbitrary intervals, thus ruling out the techniques based on Feller's test for explosions (see for instance Chapter 5 in [7]). Also the method based on the Lyapunov function, which is successfully applied in [4] doesn't seem to be appropriate for the greater generality considered here. Our approach relies instead on two general theorems of the theory of stochastic differential equations, which we now restate for the readers' convenience as Theorem 2.1 and Theorem 2.2 here below.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with an m-dimensional standard Brownian motion $\{(B_1(t), ..., B_m(t))\}_{t\geq 0}$ and denote by $\{\mathcal{F}_t^B\}_{t\geq 0}$ its augmented natural filtration. In the sequel we will be working with one dimensional Itô's type stochastic differential equations driven by the m-dimensional Brownian motion $\{(B_1(t), ..., B_m(t))\}_{t\geq 0}$.

Theorem 2.1. *Let $\{X(t)\}_{t\geq 0}$ be the unique global strong solution of the stochastic differential equation*

$$\begin{cases} dX(t) = \mu(X(t))dt + \sum_{i=1}^{m} \sigma_i(X(t))dB_i(t), \quad t > 0; \\ X(0) = x_0 \in \mathbb{R}, \end{cases}$$

where the coefficients $\mu, \sigma_1, ..., \sigma_m : \mathbb{R} \to \mathbb{R}$ are assumed to be globally Lipschitz continuous. If we set

$$\Lambda := \{x \in \mathbb{R} : \mu(x) = \sigma_1(x) = \cdots = \sigma_m(x) = 0\}$$

and assume $x_0 \notin \Lambda$, then

$$\mathbb{P}\left(X(t) \notin \Lambda, \text{ for all } t \geq 0\right) = 1.$$

Proof. See the theorem in [8]. □

Theorem 2.2. *Let $\{X(t)\}_{t\geq 0}$ be the unique global strong solution of the stochastic differential equation*

$$\begin{cases} dX(t) = \mu_1(X(t))dt + \sum_{i=1}^{m} \sigma_i(X(t))dB_i(t), \quad t > 0; \\ X(0) = z \in \mathbb{R}, \end{cases}$$

and $\{Y(t)\}_{t\geq 0}$ be the unique global strong solution of the stochastic differential equation

$$\begin{cases} dY(t) = \mu_2(Y(t))dt + \sum_{i=1}^{m} \sigma_i(Y(t))dB_i(t), \quad t > 0; \\ Y(0) = z \in \mathbb{R}, \end{cases}$$

where the coefficients $\mu_1, \mu_2, \sigma_1, ..., \sigma_m : \mathbb{R} \to \mathbb{R}$ are assumed to be globally Lipschitz continuous. If $\mu_1(z) \leq \mu_2(z)$, for all $z \in \mathbb{R}$, then

$$\mathbb{P}\left(X(t) \leq Y(t), \text{ for all } t \geq 0\right) = 1.$$

Proof. See Proposition 2.18, Chapter 5 in [7], where the proof is given for $m = 1$. The extension to several Brownian motions is immediate. See also Theorem 1.1, Chapter VI in [6]. □

2.1 Existence, uniqueness and support

We are now ready to state our existence and uniqueness result.

Theorem 2.3. *For $i \in \{1, ..., m\}$, let $f, g_i, h : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz-continuous functions such that*

1. *$f(0) = g_i(0) = 0$ and $f(N) = g_i(N) = 0$, for some $N > 0$;*
2. *$h(0) = 0$ and $h(x) > 0$, when $x > 0$.*

Then, the stochastic differential equation

$$\begin{cases} dX(t) = [f(X(t)) - h(X(t))]dt + \sum_{i=1}^m g_i(X(t))dB_i(t), \quad t > 0; \\ X(0) = x_0 \in]0, N[, \end{cases} \tag{2.2}$$

admits a unique global strong solution, which satisfies $\mathbb{P}(0 < X(t) < N) = 1$, for all $t \geq 0$.

Remark 2.4. *It is immediate to verify that equations (1.6) and (1.11) fulfill the assumptions of Theorem 2.3.*

Proof. The local Lipschitz-continuity of the coefficients entails pathwise uniqueness for equation (2.2), see for instance Theorem 2.5, Chapter 5 in [7]. Now, we consider the modified equation

$$\begin{cases} d\mathcal{X}(t) = [\bar{f}(\mathcal{X}(t)) - \hat{h}(\mathcal{X}(t))]dt + \sum_{i=1}^m \bar{g}_i(\mathcal{X}(t))dB_i(t), \quad t > 0; \\ \mathcal{X}(0) = x_0 \in]0, N[, \end{cases} \tag{2.3}$$

where

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \in [0, N]; \\ 0, & \text{if } x \notin [0, N], \end{cases} \quad \text{and} \quad \bar{g}_i(x) = \begin{cases} g_i(x), & \text{if } x \in [0, N]; \\ 0, & \text{if } x \notin [0, N], \end{cases}$$

while

$$\hat{h}(x) = \begin{cases} 0, & \text{if } x < 0; \\ h(x), & \text{if } x \in [0, N]; \\ h(N), & \text{if } x > N. \end{cases}$$

The coefficients of equation (2.3) are bounded and globally Lipschitz-continuous; this implies the existence of a unique global strong solution $\{\mathcal{X}(t)\}_{t\geq 0}$ for (2.3). Moreover, the drift and diffusion coefficients vanish at $x = 0$. Therefore, according to Theorem 2.1, the solution never visits the origin, unless it starts from there. Since $\mathcal{X}(0) = x_0 \in]0, N[$, we deduce that $\mathcal{X}(t) > 0$, for all $t \geq 0$, almost surely. Recalling the assumption $h(x) > 0$ for $x > 0$, we can rewrite equation (2.3) as

$$\begin{cases} d\mathcal{X}(t) = [\bar{f}(\mathcal{X}(t)) - \hat{h}(\mathcal{X}(t))^+]dt + \sum_{i=1}^m \bar{g}_i(\mathcal{X}(t))dB_i(t), \quad t > 0; \\ \mathcal{X}(0) = x_0 \in]0, N[, \end{cases} \tag{2.4}$$

where $x^+ := \max\{x, 0\}$. We now compare the solution of the previous equation with the one of

$$\begin{cases} d\mathcal{Y}(t) = \bar{f}(\mathcal{Y}(t))dt + \sum_{i=1}^m \bar{g}_i(\mathcal{Y}(t))dB_i(t), \quad t > 0; \\ \mathcal{Y}(0) = x_0 \in]0, N[, \end{cases} \tag{2.5}$$

which also possesses a unique global strong solution $\{\mathcal{Y}(t)\}_{t\geq 0}$. Systems (2.4) and (2.5) have the same initial condition and diffusion coefficients; moreover, the drift in (2.5) is greater than the drift in (2.4). By Theorem 2.2 we conclude that

$$\mathcal{X}(t) \leq \mathcal{Y}(t), \qquad \text{for all } t \geq 0,$$

almost surely. Moreover, both the drift and diffusion coefficients in (2.5) vanish at $x = N$. Therefore, invoking once more Theorem 2.1, the solution never visits N, unless it starts from there. Since $\mathcal{Y}(0) = x_0 \in]0, N[$, we deduce that $\mathcal{Y}(t) < N$, for all $t \geq 0$, almost surely. Combining all these facts, we conclude that

$$0 < \mathcal{X}(t) < N, \qquad \text{for all } t \geq 0,$$

almost surely. This in turn implies

$$\bar{f}(\mathcal{X}(t)) = f(\mathcal{X}(t)), \quad \bar{g}_i(\mathcal{X}(t)) = g_i(\mathcal{X}(t)), \quad \hat{h}(\mathcal{X}(t)) = h(\mathcal{X}(t)),$$

and that $\{\mathcal{X}(t)\}_{t\geq 0}$ solves equation

$$\begin{cases} d\mathcal{X}(t) = [f(\mathcal{X}(t)) - h(\mathcal{X}(t))]dt + \sum_{i=1}^m g_i(\mathcal{X}(t))dB(t), \quad t > 0; \\ \mathcal{X}(0) = x_0 \in]0, N[, \end{cases}$$

which coincides with (2.2). The uniqueness of the solution completes the proof. $\square$

2.2 Extinction

We now investigate the asymptotic behaviour of the solution of (2.2); here we are interested in sufficient conditions for extinction.

Theorem 2.5. *Under the same assumptions of Theorem 2.3 assuming in addition,*

$$\sup_{x\in]0,N[} \left\{ \frac{f(x) - h(x)}{x} - \frac{1}{2}\sum_{i=1}^m \frac{g_i^2(x)}{x^2} \right\} < 0, \tag{2.6}$$

the solution $\{X(t)\}_{t\geq 0}$ of equation (2.2) tends to zero exponentially, as t tends to infinity, almost surely. More precisely,

$$\limsup_{t\to+\infty} \frac{\ln(X(t))}{t} \leq \sup_{x\in]0,N[} \left\{ \frac{f(x) - h(x)}{x} - \frac{1}{2}\sum_{i=1}^m \frac{g_i^2(x)}{x^2} \right\} < 0, \qquad \textit{almost surely,}$$

Remark 2.6. *The function inside the supremum in* (2.6) *corresponds to the drift of the stochastic process* $\ln(X(t))$*; therefore, if this function is negative, then the process* $X(t)$ *is controlled, modulo small stochastic oscillations, by an exponential function with negative exponent.*

Proof. We follow the proof of Theorem 4.1 in [4]. First of all, we observe that the local Lipschitz-continuity of f implies the existence of a constant L_N such that

$$|f(x) - f(0)| \leq L_N|x - 0|, \quad \text{for all } x \in [0, N].$$

In particular, using the equality $f(0) = 0$, we can rewrite the previous condition as

$$\left|\frac{f(x)}{x}\right| \leq L_N, \quad \text{for all } x \in [0, N].$$

Since the same reasoning applies also to h and g_i, for $i \in \{1, ..., m\}$, we deduce that the supremum in (2.6) is always finite.
Now, let $\{X(t)\}_{t\geq 0}$ be the unique global strong solution of equation (2.2). An application of the Itô formula gives

$$\begin{aligned} \ln(X(t)) = &\ln(x_0) + \int_0^t \left[\frac{f(X(s)) - h(X(s))}{X(s)} - \frac{1}{2}\sum_{i=1}^m \frac{g_i^2(X(s))}{X(s)^2}\right] ds \\ &+ \sum_{i=1}^m \int_0^t \frac{g_i(X(s))}{X(s)} dB_i(s). \end{aligned} \tag{2.7}$$

Note that the boundedness of the function $x \mapsto \frac{g_i(x)}{x}$ on $]0, N[$ mentioned above entails that the stochastic process

$$t \mapsto \sum_{i=1}^m \int_0^t \frac{g_i(X(s))}{X(s)} dB_i(s), \quad t \geq 0,$$

is an $(\{\mathcal{F}_t^B\}_{t\geq 0}, \mathbb{P})$-martingale. Therefore, from the strong law of large numbers for martingales (see e.g. Theorem 3.4, Chapter 1 in [9]) we conclude that

$$\lim_{t\to+\infty} \sum_{i=1}^m \frac{1}{t}\int_0^t \frac{g_i(X(s))}{X(s)} dB_i(s) = 0,$$

almost surely. This fact, combined with (2.7) gives

$$\begin{aligned} \limsup_{t\to+\infty} \frac{\ln(X(t))}{t} &\leq \limsup_{t\to+\infty} \frac{1}{t}\int_0^t \left[\frac{f(X(s)) - h(X(s))}{X(s)} - \frac{1}{2}\sum_{i=1}^m \frac{g_i^2(X(s))}{X(s)^2}\right] ds \\ &\leq \limsup_{t\to+\infty} \sup_{x\in]0,N[} \left\{\frac{f(x) - h(x)}{x} - \frac{1}{2}\sum_{i=1}^m \frac{g_i^2(x)}{x^2}\right\} < 0, \end{aligned}$$

almost surely. The proof is complete. $\square$

2.3 Persistence

We now search for conditions ensuring the persistence for the solution $\{X(t)\}_{t\geq 0}$ of (2.2).

Theorem 2.7. *Under the same assumptions of Theorem 2.3, if inequality*

$$\sup_{x\in]0,N[}\left\{\frac{f(x)-h(x)}{x}-\frac{1}{2}\sum_{i=1}^{m}\frac{g_i^2(x)}{x^2}\right\}>0, \tag{2.8}$$

holds and moreover the function

$$x\mapsto \frac{f(x)-h(x)}{x}-\frac{1}{2}\sum_{i=1}^{m}\frac{g_i^2(x)}{x^2} \tag{2.9}$$

is strictly decreasing on the interval $]0,N[$*, then the solution* $\{X(t)\}_{t\geq 0}$ *of the stochastic differential equation (2.2) verifies*

$$\limsup_{t\to+\infty} X(t)\geq \xi \qquad \text{and} \qquad \liminf_{t\to+\infty} X(t)\leq \xi, \tag{2.10}$$

almost surely. Here, ξ *is the unique zero of the function (2.9) in the interval* $[0,N]$*.*

Proof. We follow the proof of Theorem 5.1 in [4]. To ease the notation we set

$$\eta(x):=\frac{f(x)-h(x)}{x}-\frac{1}{2}\sum_{i=1}^{m}\frac{g_i^2(x)}{x^2}, \quad x\in[0,N].$$

First of all, we note that $\eta(N)=-\frac{h(N)}{N}<0$; this gives, in combination with (2.8) and the strict monotonicity of η, the existence and uniqueness of ξ. Now, assume the first inequality in (2.10) to be false. This implies the existence of $\varepsilon>0$ such that

$$\mathbb{P}\left(\limsup_{t\to+\infty} X(t)\leq \xi-2\varepsilon\right)>\varepsilon. \tag{2.11}$$

In particular, for any $\omega\in A:=\{\limsup_{t\to+\infty} X(t)\leq\xi-2\varepsilon\}$, there exists $T(\omega)$ such that

$$X(t,\omega)\leq\xi-\varepsilon, \qquad \text{for all } t\geq T(\omega),$$

which implies

$$\eta(X(t,\omega))\geq\eta(\xi-\varepsilon)>0, \qquad \text{for all } \omega\in A \text{ and } t\geq T(\omega).$$

Therefore, for $\omega\in A$ and $t>T(\omega)$ we can write

$$\frac{\ln(X(t))}{t}=\frac{\ln(x_0)}{t}+\frac{1}{t}\int_0^t\eta(X(s))ds+\sum_{i=1}^{m}\frac{1}{t}\int_0^t\frac{g_i(X(s))}{I(s)}dB_i(s)$$

$$
\begin{aligned}
=&\frac{\ln(x_0)}{t} + \frac{1}{t}\int_0^{T(\omega)} \eta(X(s))ds + \frac{1}{t}\int_{T(\omega)}^t \eta(X(s))ds \\
&+ \sum_{i=1}^m \frac{1}{t}\int_0^t \frac{g_i(X(s))}{X(s)} dB_i(s) \\
\geq&\frac{\ln(x_0)}{t} + \frac{1}{t}\int_0^{T(\omega)} \eta(X(s))ds + \frac{t-T(\omega)}{t}\eta(\xi-\varepsilon) \\
&+ \sum_{i=1}^m \frac{1}{t}\int_0^t \frac{g_i(X(s))}{X(s)} dB_i(s).
\end{aligned}
$$

Hence, recalling that the strong law of large numbers for martingales gives

$$
\lim_{t\to+\infty} \sum_{i=1}^m \frac{1}{t}\int_0^t \frac{g_i(X(s))}{X(s)} dB_i(s) = 0 \quad \text{almost surely,}
$$

we conclude that

$$
\liminf_{t\to+\infty} \frac{\ln(X(t))}{t} \geq \eta(\xi-\varepsilon) > 0, \quad \text{on the set } A,
$$

which implies

$$
\lim_{t\to+\infty} X(t) = +\infty, \quad \text{on the set } A.
$$

This contradicts (2.11) and hence prove the first inequality in (2.10).
The second inequality in (2.10) is proven similarly; if the thesis is not true, then

$$
\mathbb{P}\left(\liminf_{t\to+\infty} X(t) \geq \xi + 2\varepsilon\right) > \varepsilon. \tag{2.12}
$$

for some positive ε. In particular, for any $\omega \in B := \{\liminf_{t\to+\infty} X(t) \geq \xi + 2\varepsilon\}$, there exists $S(\omega)$ such that

$$
X(t,\omega) \geq \xi + \varepsilon, \qquad \text{for all } t \geq S(\omega),
$$

which implies

$$
\eta(X(t,\omega)) \leq \gamma(\xi+\varepsilon) < 0, \qquad \text{for all } t \geq S(\omega).
$$

Therefore, for $\omega \in B$ and $t > S(\omega)$ we can write

$$
\begin{aligned}
\frac{\ln(X(t))}{t} =&\frac{\ln(x_0)}{t} + \frac{1}{t}\int_0^t \eta(X(s))ds + \sum_{i=1}^m \frac{1}{t}\int_0^t \frac{g_i(X(s))}{X(s)} dB_i(s) \\
=&\frac{\ln(x_0)}{t} + \frac{1}{t}\int_0^{S(\omega)} \eta(X(s))ds + \frac{1}{t}\int_{S(\omega)}^t \eta(X(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\frac{1}{t}\int_0^t \frac{g_i(X(s))}{X(s)}dB_i(s)\\
\leq&\frac{\ln(x_0)}{t}+\frac{1}{t}\int_0^{S(\omega)}\eta(X(s))ds+\frac{t-S(\omega)}{t}\eta(\xi+\varepsilon)\\
&+\sum_{i=1}^{m}\frac{1}{t}\int_0^t \frac{g_i(X(s))}{X(s)}dB_i(s)
\end{aligned}
$$

Therefore,

$$
\limsup_{t\to+\infty}\frac{\ln(X(t))}{t}\leq \eta(\xi+\varepsilon)<0, \quad \text{on the set } B,
$$

which implies

$$
\lim_{t\to+\infty} X(t)=0, \quad \text{on the set } B.
$$

This contradicts (2.12) and hence proves the second inequality in (2.10). □

3 Proof of Theorem 1.1

We are now ready to prove our main theorem.

3.1 Existence, uniqueness, extinction and persistence

It is immediate to verify that the SDE

$$
\begin{aligned}
d\mathtt{I}(t)=&\left[\beta\mathtt{I}(t)(N-\mathtt{I}(t))-(\mu+\gamma)\mathtt{I}(t)+\frac{\sigma^2}{2}\mathtt{I}(t)(N-\mathtt{I}(t))(N-2\mathtt{I}(t))\right]dt\\
&+\sigma\mathtt{I}(t)(N-\mathtt{I}(t))dB(t),
\end{aligned}
\tag{3.1}
$$

with initial condition $\mathtt{I}(0)=i_0\in]0,N[$ fulfills the assumptions of Theorem 2.3 if we set $m=1$,

$$
f(x):=\beta x(N-x)+\frac{\sigma^2}{2}x(N-x)(N-2x), \quad h(x)=(\mu+\gamma)x, \quad g(x):=\sigma x(N-x).
$$

Therefore, equation (3.1) possesses a unique global strong solution which lives in the interval $]0,N[$ for all $t\geq 0$ with probability one.
Let us now observe that

$$
\begin{aligned}
\eta(x):=&\frac{f(x)-h(x)}{x}-\frac{1}{2}\frac{g^2(x)}{x^2}\\
=&\beta(N-x)+\frac{\sigma^2}{2}(N-x)(N-2x)-(\mu+\gamma)-\frac{1}{2}\sigma^2(N-x)^2
\end{aligned}
$$

$$= \left(\frac{\sigma^2}{2}x - \beta\right)(x - N) - (\mu + \gamma),$$

and hence

$$\eta(0) = \beta N - (\mu + \gamma) \quad \text{and} \quad \eta(N) = -(\mu + \gamma).$$

This gives:

- if $\beta N - (\mu + \gamma) < 0$, that is $\frac{\beta N}{\mu+\gamma} < 1$, then the assumptions of Theorem 2.5 are satisfied (γ is a convex second order polynomial which takes negative values on the boundaries of $[0, N]$); therefore, $\mathtt{I}(t)$ will be extinct as t tends to infinity;
- if $\beta N - (\mu + \gamma) > 0$, that is $\frac{\beta N}{\mu+\gamma} > 1$, then the assumptions of Theorem 2.7 are satisfied (γ is a convex second order polynomial which takes a positive value at 0 and a negative value at N); therefore, $\mathtt{I}(t)$ will be persistent as t tends to infinity.

3.2 Explicit representation of the solution

We observe that the solution of the deterministic equation

$$\frac{dI(t)}{dt} = \beta(t)I(t)(N - I(t)) - (\mu + \gamma)I(t), \quad I(0) = i_0 \in]0, N[, \tag{3.2}$$

where $t \mapsto \beta(t)$ is now a continuous function of t, can be written as

$$I(t) = \frac{i_0 e^{\int_0^t N\beta(s)ds - (\mu+\gamma)t}}{1 + \int_0^t \beta(s) i_0 e^{\int_0^s N\beta(r)dr - (\mu+\gamma)s} ds}, \quad t \geq 0. \tag{3.3}$$

If we set $\beta(t) := \beta + \sigma \dot{B}^\pi(t)$, where $\dot{B}^\pi(t)$ stands for $\frac{d}{dt}B^\pi(t)$, then equation (3.2) and formula (3.3) become respectively

$$\frac{dI^\pi(t)}{dt} = \beta I^\pi(t)(N - I^\pi(t)) - (\mu + \gamma)I^\pi(t) + \sigma I^\pi(t)(N - I^\pi(t))\dot{B}^\pi(t), \tag{3.4}$$

with initial condition $I^\pi(0) = i_0 \in]0, N[$, and

$$I^\pi(t) = \frac{i_0 e^{\int_0^t N\left(\beta + \sigma \dot{B}^\pi(s)\right)ds - (\mu+\gamma)t}}{1 + \int_0^t (\beta + \sigma \dot{B}^\pi(s)) i_0 e^{\int_0^s N(\beta + \sigma \dot{B}^\pi(r))dr - (\mu+\gamma)s} ds}. \tag{3.5}$$

We recall that according to the Wong-Zakai theorem the stochastic process $\{I^\pi(t)\}_{t \geq 0}$ converges, as the mesh of the partition π tends to zero, to the unique strong solution of the Stratonovich SDE

$$d\mathtt{I}(t) = [\beta \mathtt{I}(t)(N - \mathtt{I}(t)) - (\mu + \gamma)\mathtt{I}(t)]dt + \mathtt{I}(t)\sigma(N - \mathtt{I}(t)) \circ dB(t), \quad \mathtt{I}(0) = i_0 \in]0, N[,$$

which is equivalent to the Itô-type equation

$$d\mathtt{I}(t) = \left[\beta\mathtt{I}(t)(N - \mathtt{I}(t)) - (\mu+\gamma)\mathtt{I}(t) + \frac{\sigma^2}{2}\mathtt{I}(t)(N-\mathtt{I}(t))(N-2\mathtt{I}(t))\right]dt$$
$$+\sigma\mathtt{I}(t)(N-\mathtt{I}(t))dB(t), \quad \mathtt{I}(0) = i_0 \in]0, N[. \tag{3.6}$$

We now simplify the expression in (3.5) and compute its limit as the mesh of the partition π tends to zero: this will give us an explicit representation for the solution of (3.6). To ease the notation we set

$$\mathcal{E}^\pi(t) := e^{\int_0^t N\left(\beta+\sigma\dot{B}^\pi(s)\right)ds-(\mu+\gamma)t} = e^{N\beta t+N\sigma B^\pi(t)-(\mu+\gamma)t} = e^{\delta t+N\sigma B^\pi(t)},$$

where $\delta := N\beta - (\mu+\gamma)$, and rewrite (3.5) as

$$\begin{aligned} I^\pi(t) &= \frac{i_0\mathcal{E}^\pi(t)}{1+i_0\int_0^t(\beta+\sigma\dot{B}^\pi(s))\mathcal{E}^\pi(s)ds} \\ &= \frac{i_0\mathcal{E}^\pi(t)}{1+i_0\beta\int_0^t\mathcal{E}^\pi(s)ds+i_0\sigma\int_0^t\dot{B}^\pi(s)\mathcal{E}^\pi(s)ds}. \end{aligned} \tag{3.7}$$

Note that $\delta \geq 0$ if and only if $R_0 = \frac{\beta N}{\mu+\gamma} \geq 1$. Now, consider the second integral in the denominator above: an integration by parts gives

$$\begin{aligned} \int_0^t \dot{B}^\pi(s)\mathcal{E}^\pi(s)ds &= \int_0^t \dot{B}^\pi(s)e^{\delta s+N\sigma B^\pi(s)}ds \\ &= \int_0^t \dot{B}^\pi(s)e^{N\sigma B^\pi(s)}e^{\delta s}ds \\ &= \frac{1}{N\sigma}\left(e^{N\sigma B^\pi(t)}e^{\delta t}-1\right) - \frac{\delta}{N\sigma}\int_0^t e^{N\sigma B^\pi(s)}e^{\delta s}ds \\ &= \frac{1}{N\sigma}(\mathcal{E}^\pi(t)-1) - \frac{\delta}{N\sigma}\int_0^t\mathcal{E}^\pi(s)ds. \end{aligned}$$

Therefore, inserting the last expression in (3.7) we get

$$\begin{aligned} I^\pi(t) &= \frac{i_0\mathcal{E}^\pi(t)}{1+i_0\beta\int_0^t\mathcal{E}^\pi(s)ds+i_0\sigma\int_0^t\dot{B}^\pi(s)\mathcal{E}^\pi(s)ds} \\ &= \frac{i_0\mathcal{E}^\pi(t)}{1+i_0\beta\int_0^t\mathcal{E}^\pi(s)ds+\frac{i_0}{N}(\mathcal{E}^\pi(t)-1)-\frac{i_0\delta}{N}\int_0^t\mathcal{E}^\pi(s)ds} \\ &= \frac{i_0\mathcal{E}^\pi(t)}{1+\frac{i_0}{N}(\mathcal{E}^\pi(t)-1)+i_0\left(\beta-\frac{\delta}{N}\right)\int_0^t\mathcal{E}^\pi(s)ds} \\ &= \frac{i_0\mathcal{E}^\pi(t)}{1+\frac{i_0}{N}(\mathcal{E}^\pi(t)-1)+i_0\frac{\mu+\gamma}{N}\int_0^t\mathcal{E}^\pi(s)ds}. \end{aligned}$$

We can now let the mesh of the partition π tend to zero and get

$$\begin{aligned}
\mathtt{I}(t) = \lim_{|\pi|\to 0} I^\pi(t) &= \lim_{|\pi|\to 0} \frac{i_0 \mathcal{E}^\pi(t)}{1 + \frac{i_0}{N}(\mathcal{E}^\pi(t) - 1) + i_0 \frac{\mu+\gamma}{N} \int_0^t \mathcal{E}^\pi(s) ds} \\
&= \frac{i_0 \mathcal{E}(t)}{1 + \frac{i_0}{N}(\mathcal{E}(t) - 1) + i_0 \frac{\mu+\gamma}{N} \int_0^t \mathcal{E}(s) ds},
\end{aligned}$$

with

$$\mathcal{E}(t) := e^{\delta t + N\sigma B(t)}.$$

3.3 Recurrence

To prove the recurrence of $\mathtt{I}(t)$ in the case $R_0 \geq 1$, we need to exploit the specific structure of equation (3.1). In particular, we will follow the approach utilized in [13] which is based on the Feller's test for explosion (see for instance Chapter 5.5 C in [7]). Let $\varphi(x) := \ln\left(\frac{x}{N-x}\right)$ and apply the Itô formula to $\varphi(\mathtt{I}(t))$; this gives

$$\begin{aligned}
d\varphi(\mathtt{I}(t)) &= \left(\beta N - (\mu+\gamma) - (\mu+\gamma)\frac{\mathtt{I}(t)}{N - \mathtt{I}(t)}\right) dt + \sigma N dB(t) \\
&= \left(\beta N - (\mu+\gamma) - (\mu+\gamma) e^{\varphi(\mathtt{I}(t))}\right) dt + \sigma N dB(t),
\end{aligned}$$

and, setting $\mathtt{J}(t) := \varphi(\mathtt{I}(t))$, we can write

$$d\mathtt{J}(t) = \left(\beta N - (\mu+\gamma) - (\mu+\gamma) e^{\mathtt{J}(t)}\right) dt + \sigma N dB(t).$$

Now, the scale function for this process is

$$\psi(x) = \int_0^x \theta(y) dy$$

where

$$\begin{aligned}
\theta(y) &= \exp\left\{-\frac{2}{\sigma^2 N^2} \int_0^y \beta N - (\mu+\gamma) - (\mu+\gamma) e^z dz\right\} \\
&= \exp\left\{-\frac{2(\beta N - (\mu+\gamma))}{\sigma^2 N^2} y + \frac{2(\mu+\gamma)}{\sigma^2 N^2}(e^y - 1)\right\}.
\end{aligned}$$

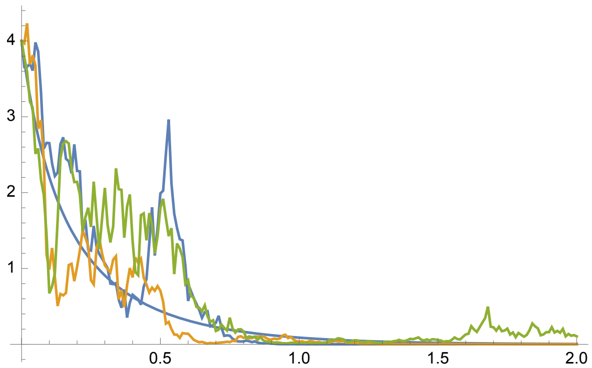
It is clear that $\psi(+\infty) = +\infty$; moreover, for $\beta N - (\mu+\gamma) \geq 0$, that means $R_0 \geq 1$, we get $\psi(-\infty) = -\infty$. These two facts together with Proposition 5.22, Chapter 5 in [7] imply that $\{\mathtt{J}(t)\}_{t\geq 0}$ is recurrent on $]-\infty, +\infty[$ and hence that $\{\mathtt{I}(t)\}_{t\geq 0}$ is recurrent on $]0, N[$.

4 Discussion

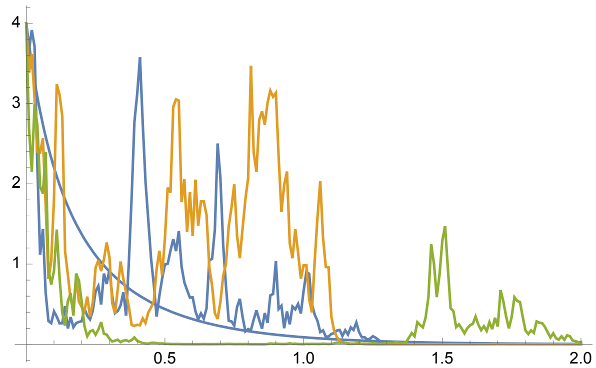
In this paper we have proposed a different and more rigorous derivation of the popular SDE presented in [4] which results from a suitable stochastic perturbation of the disease transmission coefficient in the classical SIS model. We have shown that according to our approach the *correct* interpretation of the resulting stochastic equation should be the Stratonovich one, contrary to the Itô choice made in [4]. This different interpretation has a crucial implication: the reproduction number for the Stratonovich SDE coincides with the one of the deterministic equation, thus making the stochastic perturbation invisible to the asymptotic regimes of the solution. This is crucially in contrast with the discoveries made in [4] for the Itô's type equation, where a new (stochastic) reproduction number is identified as responsible for the asymptotic behaviour of the solution.
We believe that our result may facilitate a better understanding of the stochastic parameter perturbation technique for SIS and more general models; our contribution also provides an additional instance of the complexity related to the construction of stochastic models (see [3] and [12]). We are planning to investigate in future works the extent to which similar results are valid also for other models; in particular, we would like to identify those systems for which the invariance of the asymptotic regimes under stochastic parameter perturbation takes place.
We conclude this section with several illustrations of our discoveries. In Figure 1, the solution (1.3) to the classical deterministic SIS model (1.2) is compared with three simulated paths of (1.12), which solves the revised stochastic SIS model (1.11). Here, we have chosen $N = 5$, $i_0 = 4$, $\mu + \gamma = 8$ and

- $\beta = 1$ and $\sigma^2 = 0.5$ in Figure 1a (producing extinction);
- $\beta = 1$ and $\sigma^2 = 1$ in Figure 1b (producing extinction);
- $\beta = 2$ and $\sigma^2 = 0.5$ in Figure 1c (producing persistence);
- $\beta = 2$ and $\sigma^2 = 1$ in Figure 1b (producing persistence).

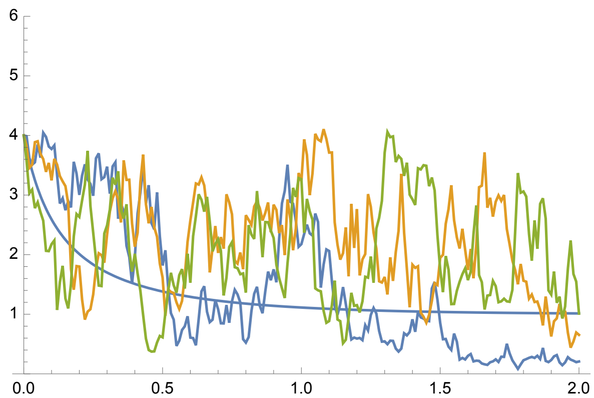
The figures show that the asymptotic behaviour of (1.12) doesn't depend on σ and agrees with the one of (1.3).
In Figure 2 we compared (1.12), solution to (1.11) (red), with the solution to (1.6) (blue); the parameters are $N = 5$, $\beta = 2$, $\mu + \gamma = 8$, $\sigma^2 = 0.5$ and $i_0 = 4$, for Figure 2a, and $i_0 = 2$, for Figure 2b. With such choices, the reproduction number R_0 in (1.4) is greater than one (producing persistence in 1.12) while the stochastic reproduction number R_0^S in (1.7) is smaller than one (producing extinction for (1.6)). Hence, the two models are governed by different threshold values.
Lastly, in Figure 3 we compared once more the process (1.12), solution to (1.11) (red), with the solution to (1.6) (blue); now, the parameters $N = 5$, $\beta = 2$, $\mu+\gamma = 8$, $\sigma^2 = 0.05$ and $i_0 = 4$, for Figure 3a, and $i_0 = 2$, for Figure 3b, entail for both R_0 and R_0^S values smaller than one. This yields extinction in both cases.



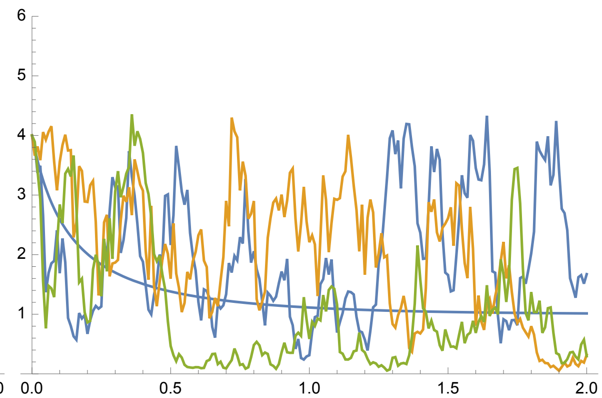
(a) $R_0 < 1$ and $\sigma^2 = 0.5$



(b) $R_0 < 1$ and $\sigma^2 = 1$

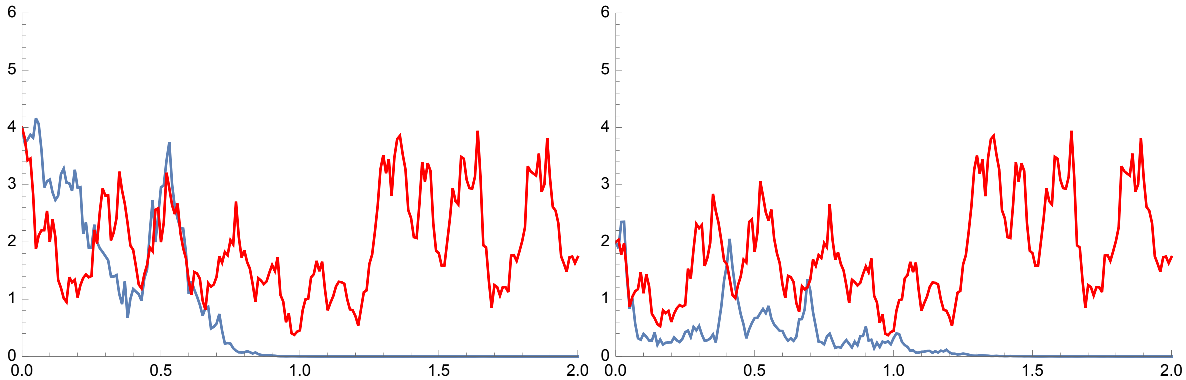


(c) $R_0 > 1$ and $\sigma^2 = 0.5$



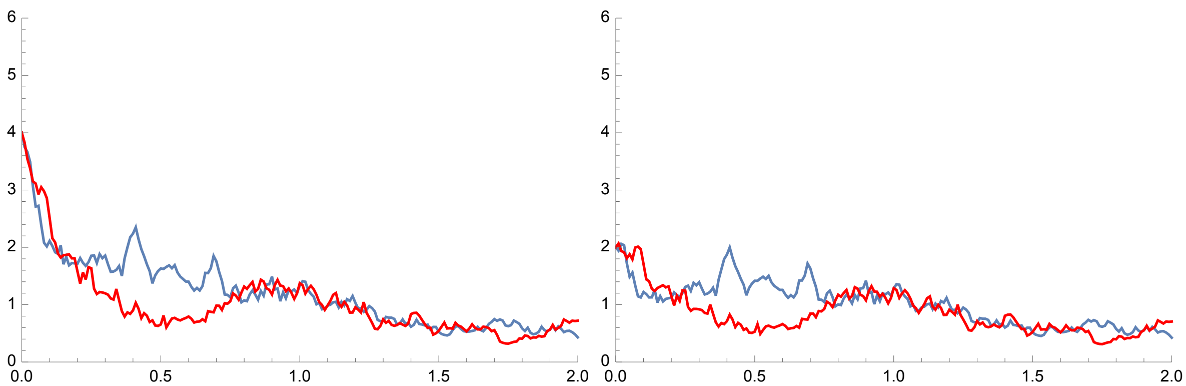
(d) $R_0 > 1$ and $\sigma^2 = 1$

Figure 1: Comparison between (1.3), solution to (1.2), and three simulated paths of (1.12), solution to (1.11)



(a) $R_0 > 1$, $R_0^S < 1$ and $i_0 = 4$ (b) $R_0 > 1$, $R_0^S < 1$ and $i_0 = 2$

Figure 2: Comparison between one simulated path of (1.12), solution to (1.11) (red) and one simulated path of the solution to (1.6) (blue)



(a) $R_0 < 1$, $R_0^S < 1$ and $i_0 = 4$ (b) $R_0 < 1$, $R_0^S < 1$ and $i_0 = 2$

Figure 3: Comparison between one simulated path of (1.12), solution to (1.11) (red) and one simulated path of the solution to (1.6) (blue)

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