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The cohomology rings of the unordered configuration spaces of elliptic curves

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We study the cohomology ring of the configuration space of unordered points in the two dimensional torus. In particular, we compute the mixed Hodge structure on the cohomology, the action of the mapping class group, the structure of the cohomology ring and we prove the formality over the rationals.

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Introduction

We fully describe the cohomology with rational coefficients of the configuration spaces of unordered points in an elliptic curve (frequently called torus).

Configuration spaces of points are related to physics (state spaces of non-colliding particles on a manifold), robotics (motion planning), knot theory, and topology. Configuration spaces give invariants of the homeomorphism type of the base space. In the algebraic setting, configuration spaces are open in the moduli spaces of points.

Since the literature is very extensive, we compare our work only with the main results on the (co-)homology of configuration spaces. The first computation of the cohomology algebra of configuration spaces is due to Arnol'd [1, 2] in the case of $\mathbb{R}^2$. This result has been generalized by Cohen, Lada, and May [9] to the configuration space of $\mathbb{R}^n$ and later by Goresky and MacPherson [17]. Partially additive results have been obtained: by Bödigheimer and Cohen [6] for once-punctured oriented surfaces, by the same authors and Taylor [7] for odd dimensional manifolds, and by Drummond-Cole and Knudsen [11] for surfaces in general. However there is no description of the ring structure; we provide it in the case of elliptic curves. The Betti numbers $\mathcal{C}^n(X)$ are described in the following cases: for $X = \mathbb{P}^2(\mathbb{R})$ by Wang [30], for X a sphere by Salvatore [26], for $X = \mathbb{P}^2(\mathbb{C})$ by Felix and Tanré [14] and for elliptic curves by Maguire and Schiessl [23, 27].

In this paper we improve the previous results on configuration spaces in an elliptic curve in three ways. We describe:

- the mixed Hodge structure on the cohomology (Theorem 3.3),

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- the action of the mapping class group (Theorem 3.3),
- the ring structure (Theorem 4.1).

The formality result over the rationals is proven in Theorem 4.3.

We prove these results using the Križ model [22, 28, 5, 12] and the representation theory on it [3, 4].

In Section 1 we recall the Križ model, then in Section 2 we improve the result on the decomposition of the Križ model into irreducible representations, see Theorem 2.9. Descriptions of the mixed Hodge structure and of the action of the mapping class group are obtained in Section 3 by computing the cohomology of the model. Finally, the ring structure is presented in the last section.

1 The Križ model

Let E be an elliptic curve and consider the configuration space of n ordered distinct points

$$\mathcal{F}^n(E) \stackrel{\text{def}}{=} \{\underline{p} \in E^n \mid p_i \neq p_j\}.$$

The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{F}^n(E)$ by permuting the coordinates and the quotient is the configuration space of n unordered points

$$\mathcal{C}^n(E) \stackrel{\text{def}}{=} \mathcal{F}^n(E)/\mathfrak{S}_n.$$

We also consider the space $\mathcal{M}^n(E)$, defined by

$$\mathcal{M}^n(E) \stackrel{\text{def}}{=} \{\underline{p} \in \mathcal{F}^n(E) \mid \sum p_i = 0\}.$$

Notice that there exists a non canonical isomorphism $\mathcal{F}^n(E) \cong E \times \mathcal{M}^n(E)$.

In this section we recall a rational model for the cohomology algebra of $\mathcal{F}^n(E)$. The model is a commutative differential bi-graded algebra (dga) that can be obtain in two different ways: as a specialization of the Križ model for the configuration spaces or as the second page of the Leray spectral sequence (also known as the Totaro spectral sequence) for elliptic arrangements. Our main references for the first approach are [22, 3, 4] and for the second one are [28, 12, 5]. In the following we define the models for the cohomology of $\mathcal{F}^n(E)$ and of $\mathcal{M}^n(E)$.

Let Λ be the exterior algebra over $\mathbb{Q}$ with generators

$$\{x_i, y_i, \omega_{i,j}\}_{1 \leq i < j \leq n}.$$

We set the degree of each x_i and y_i equal to $(1,0)$ and the degree of $\omega_{i,j}$ equal to $(0,1)$. Define the differential $\mathrm{d}\colon \Lambda \to \Lambda$ of bi-degree $(2,-1)$ on generators as follows: $\mathrm{d}x_i = 0$ and $\mathrm{d}y_i = 0$ for $i = 1, \dots, n$ and

$$\mathrm{d}\omega_{i,j} \overset{\text{def}}{=} (x_i - x_j)(y_i - y_j).$$

For the sake of notation we set $\omega_{i,j} := \omega_{j,i}$ for $i > j$.

We define the dga $A^{\bullet,\bullet}$ as the quotient of Λ by the following relations:

$$(x_i - x_j)\omega_{i,j} = 0 \quad \text{and} \quad (y_i - y_j)\omega_{i,j} = 0,$$
$$\omega_{i,j}\omega_{j,k} - \omega_{i,j}\omega_{k,i} + \omega_{j,k}\omega_{k,i} = 0.$$

Notice that the ideal is preserved by the differential map, thus the differential $\mathrm{d}\colon A^{\bullet,\bullet} \to A^{\bullet,\bullet}$ is well defined.

Remark 1.1 The model $A^{\bullet,\bullet}$ coincides with the Križ model $E^{\bullet}_{\bullet}$ introduced in [22] up to shifting the degrees, ie

$$A^{p,q} \cong E_q^{p+q}.$$

The dga $E^{\bullet}_{\bullet}$ is a rational model for X, as shown in [22, Theorem 1.1].

In order to study the cohomology of $A^{\bullet,\bullet}$ we need to introduce the elements $u_{i,j} = x_i - x_j$, $v_{i,j} = y_i - y_j$ and $\gamma = \sum_{i=1}^n x_i$, $\overline{\gamma} = \sum_{i=1}^n y_i \in A^{1,0}$.

We define the dga $B^{\bullet,\bullet}$ as the subalgebra of $A^{\bullet,\bullet}$ generated by $u_{i,j}, v_{i,j}$ and $\omega_{i,j}$ for $1 \leq i < j \leq n$. Let $D^{\bullet,0}$ be the subalgebra of $A^{\bullet,\bullet}$ generated by γ and $\overline{\gamma}$ endowed with the zero differential map. Notice that

$$A^{\bullet,\bullet} \cong B^{\bullet,\bullet} \otimes_{\mathbb{Q}} D^{\bullet,0} \tag{1}$$

as differential algebras and that $D^{\bullet,0}$ is the cohomology ring of the elliptic curve E.

The mixed Hodge structure on the cohomology of algebraic varieties defines a bigrading compatible with the algebra structure (see [10, p.81] or [29, Theorem 8.35]). In our case the bigrading given by the mixed Hodge structure coincides with the one given by the Leray spectral sequence as shown by Totaro [28, Theorem 3] and by Gorinov [18]. Explicitly, the subspace $A^{p,q}$ has weight $p + 2q$ and degree $p + q$.

The following result is a particular case of [5, Theorem 3.3] and of [12, Theorem 1.2].

Theorem 1.2 *The cohomology algebra of $\mathcal{F}^n(E)$ (or of $\mathcal{M}^n(E)$) with rational coefficients is isomorphic to the cohomology of the dga $A^{\bullet,\bullet}$ (respectively of $B^{\bullet,\bullet}$). Moreover, the n^2-sheeted covering*

$$\begin{aligned} E \times \mathcal{M}^n(E) &\to \mathcal{F}^n(E) \\ (q, \underline{p}) &\mapsto (p_i + q)_{i=1,\dots,n} \end{aligned}$$

induces the isomorphism of eq. (1).

2 Representation theory on the Križ model

Now we study the action of the symmetric group $\mathfrak{S}_n$ and of $SL_2(\mathbb{Q})$ on the algebras $A^{\bullet,\bullet}$ and $B^{\bullet,\bullet}$. Those actions are given by a geometric action on $\mathcal{F}^n(E)$. For general reference about the representation theory of the Lie group and of the Lie algebra we refer to [19] and to [15], respectively. The cases of $SL_2(\mathbb{C})$ and of $\mathfrak{sl}_2(\mathbb{C})$ can be found in [16].

2.1 Definition of the actions

Consider the action of $\mathfrak{S}_n$ on $\mathcal{F}^n(E)$ defined by

$$\sigma^{-1} \cdot (p_1, \dots, p_n) = (p_{\sigma(1)}, \dots, p_{\sigma(n)})$$

for all $\sigma \in \mathfrak{S}_n$. This induces an action on $A^{\bullet,\bullet}$ and on $B^{\bullet,\bullet}$ defined by

$$\begin{aligned}\sigma^{-1}(x_i) &= x_{\sigma(i)},\\ \sigma^{-1}(y_i) &= y_{\sigma(i)},\\ \sigma^{-1}(\omega_{i,j}) &= \omega_{\sigma(i),\sigma(j)}\end{aligned}$$

for all $1 \leq i < j \leq n$ and all $\sigma \in \mathfrak{S}_n$.

The mapping class group $\mathrm{MCG}(E)$ acts naturally on $\mathcal{F}^n(E)$ and on $\mathcal{C}^n(E)$.

Theorem 2.1 (Theorem 2.5 [13]) *The mapping class group* $\mathrm{MCG}(E)$ *of the torus is isomorphic to* $SL_2(\mathbb{Z})$ *and the isomorphism is given by the natural action of* $\mathrm{MCG}(E)$ *on* $H^1(E;\mathbb{Z})$*.*

Let f be an automorphism of E, the map induces the following vertical morphisms

$$\begin{tikzcd} \mathcal{F}^n(E) \arrow[r, hook] \arrow[d, "f^n_{|\mathcal{F}^n(E)}"'] & E^n \arrow[d, "f^n"] \\ \mathcal{F}^n(E) \arrow[r, hook] & E^n \end{tikzcd}$$

and by functoriality of the Leray spectral sequence it induces the action of $SL_2(\mathbb{Z})$ on $A^{\bullet,\bullet}$. We explicitly describe this action on the generators $\omega_{i,j}$, x_i, and y_i: since $f^n\colon E^n \to E^n$ fixes the divisor $\{p_i = p_j\}$, then $f \cdot \omega_{i,j} = \omega_{i,j}$. The other generators belongs to $A^{1,0} = H^1(E^n) \cong H^1(E)^{\otimes n}$. Therefore the action of $\mathrm{MCG}(E) \cong SL_2(\mathbb{Z})$ on $A^{1,0}$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x_i = ax_i + cy_i \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot y_i = bx_i + dy_i.$$

This action extends to $SL_2(\mathbb{Q})$ and since the actions of $\mathfrak{S}_n$ and of $SL_2(\mathbb{Q})$ commute, then $A^{\bullet,\bullet}$, $B^{\bullet,\bullet}$ and $D^{\bullet,0}$ become $\mathfrak{S}_n \times SL_2(\mathbb{Q})$-modules.

2.2 Decomposition into $\mathfrak{S}_n$-representations

We recall a result of [3, Theorem 3.15] on the decomposition of $A^{\bullet,\bullet}$ into $\mathfrak{S}_n$-modules. The notations used here follow the ones in [3].

Let $L_* = (\lambda_1, \dots, \lambda_t)$ be a partition of the number n, ie $\lambda_i \in \mathbb{N}_+$ and $\sum_{i=1}^t \lambda_i = n$. We mark all blocks with labels in $\{1, x, y, xy\}$, an ordered basis of $H^\bullet(E)$. The order is $1 \prec x \prec y \prec xy$.

Definition 2.2 A *marked partition* (L_*, H_*) is a partition $L_* \vdash n$ together with marks $H_* = (h_1, \dots, h_t)$, $h_i \in \{1, x, y, xy\}$ such that: if $\lambda_i = \lambda_{i+1}$ then $h_i \succeq h_{i+1}$.

Let C_k be the cyclic group of order k. For any partition $L_* \vdash n$ define C_{L_*} as the product of the cyclic groups C_{λ_i} for $i = 1, \dots, t$. It acts on $\{1, \dots, n\}$ in the natural way. Consider a marked partition (L_*, H_*) and define N_{L_*,H_*} as the group that permutes the blocks of L_* with the same labels. The group N_{L_*,H_*} is a product of symmetric groups. Call Z_{L_*,H_*} the semidirect product $C_{L_*} \rtimes N_{L_*,H_*}$.

Example 2.3 Let (L_*, H_*) be the marked partition $L_* = (5, 5, 5, 5, 1, 1, 1) \vdash 23$ and $H_* = (xy, xy, xy, 1, x, x, x)$. The group $C_{L_*} \cong (\mathbb{Z}/5\mathbb{Z})^4 < \mathfrak{S}_{23}$ is generated by $(1, 2, 3, 4, 5), (6, 7, 8, 9, 10), (11, 12, 13, 14, 15)$, and $(16, 17, 18, 19, 20)$. The subgroup $N_{L_*,H_*} \cong \mathfrak{S}_3 \times \mathfrak{S}_3$ is generated by the permutations $(1, 6)(2, 7)(3, 8)(4, 9)(5, 10)$, $(1, 11)(2, 12)(3, 13)(4, 14)(5, 15)$, $(21, 22)$, and $(21, 23)$. Finally, Z_{L_*,H_*} is a group isomorphic to $(\mathbb{Z}/5\mathbb{Z} \wr \mathfrak{S}_3) \times \mathbb{Z}/5\mathbb{Z} \times \mathfrak{S}_3$.

Given two representations V, W of two groups G and H respectively, define the tensor representation $V \boxtimes W$ of $G \times H$ by the vector space $V \otimes W$ with the action $(g, h)(v \otimes w) = g(v) \otimes h(w)$.

We define the following one-dimensional representations. Let φ_n be a faithful character of the cyclic group and φ_{L_*} the character of $C_{L_*} \cong \mathbb{Z}/\lambda_1\mathbb{Z} \times \cdots \times \mathbb{Z}/\lambda_t\mathbb{Z}$ given by

$$\varphi_{L_*} \stackrel{\text{def}}{=} \operatorname{sgn}_n|_{C_{L_*}} \cdot (\varphi_{\lambda_1} \boxtimes \cdots \boxtimes \varphi_{\lambda_r}).$$

Recall that the degree deg of $1, x, y, xy$ are respectively $0, 1, 1, 2$. Let α_{L_*,H_*} be the one dimensional representation of $N_{L_*,H_*} \cong \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_l}$ defined on generators by

$$\alpha_{L_*,H_*}(\sigma) \stackrel{\text{def}}{=} (-1)^{\lambda + \deg(h) + 1},$$

where σ is the permutation that exchange two blocks of size λ and label h. Set ξ_{L_*,H_*} to be the one dimensional representation of Z_{L_*,H_*} such that $\operatorname{Res}^{Z_{L_*,H_*}}_{C_{L_*}} \xi_{L_*,H_*} = \varphi_{L_*}$ and $\operatorname{Res}^{Z_{L_*,H_*}}_{N_{L_*,H_*}} \alpha_{L_*,H_*}$.

We define $|L_*| = n - t$ for a partition $L_* = (\lambda_1, \dots, \lambda_t)$ of n and for a mark H_* the numbers $|H_*| = \sum_{i=1}^{t} \deg(h_i)$ and $\|H_*\| = |\{i \mid h_i = x\}| - |\{i \mid h_i = y\}|$.

Definition 2.4 Let (L_*, H_*) be a marked partition and set $p = |L_*|$ and $q = |H_*|$. Define $A_{L_*,H_*} \subseteq A^{p,q}$ as the $\mathfrak{S}_n$-subrepresentation generated by the following element:

$$m_{L_*,H_*} \stackrel{\text{def}}{=} \prod_{i=1}^{t} (h_i)_{l_i+1} \prod_{j=1}^{\lambda_i - 1} \omega_{l_i+j, l_i+j+1},$$

where $l_i = \sum_{k<i} \lambda_k$.

Theorem 2.5 ([3, Theorem 3.15]) *The $\mathfrak{S}_n$-representation $A^{p,q}$ decomposes as*

$$A^{p,q} = \bigoplus_{\substack{|L_*|=q \\ |H_*|=p}} A_{L_*,H_*}.$$

Moreover:

$$A_{L_*,H_*} \otimes_{\mathbb{Q}} \mathbb{C} \cong \operatorname{Ind}_{Z_{L_*,H_*}}^{\mathfrak{S}_n} \xi_{L_*,H_*}.$$

Remark 2.6 Observe that:

- The n-Lie operad over the complex numbers is described as the $\mathfrak{S}_n$-representation induced from C_n by a faithful character.
- The direct sum over the natural numbers of all homological Križ models is isomorphic to the Chevalley-Eilenberg complex $\mathrm{CE}(\mathfrak{g}_E[1])$, where the Lie algebra $\mathfrak{g}_E$ is $H^\bullet(E;\mathbb{C}) \otimes \mathrm{Lie}_n$.

These two facts gives a conceptual explanation to Theorem 2.5. For references see [8, 21, 20].

Example 2.7 Consider the marked partition (L_*, H_*) of Theorem 2.3. We have $|L_*| = 16$, $|H_*| = 9$ and $\|H_*\| = 3$. The element generating A_{L_*,H_*} is the following:

$$m_{L_*,H_*} = x_1 y_1 x_6 y_6 x_{11} y_{11} x_{21} x_{22} x_{23} \omega_{1,2} \omega_{2,3} \omega_{3,4} \omega_{4,5} \omega_{6,7} \omega_{7,8} \omega_{8,9} \omega_{9,10} \cdot$$
$$\cdot \omega_{11,12} \omega_{12,13} \omega_{13,14} \omega_{14,15} \omega_{16,17} \omega_{17,18} \omega_{18,19} \omega_{19,20}$$

The characters associated with (L_*, H_*) are shown in the following table.

	$(1,2,3,4,5)$	$(16,17,18,19,20)$	$(1,6)(2,7)(3,8)(4,9)(5,10)$	$(21,22)$
φ	ζ_5	ζ_5		
α			1	-1
ξ	ζ_5	ζ_5	1	-1

2.3 Decomposition into $\mathfrak{S}_n \times SL_2(\mathbb{Q})$-representations

Let $T = \{H_t\} \cong \mathbb{Q}^*$ be the maximal torus in $SL_2(\mathbb{Q})$ generated by the diagonal matrices $H_t = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$. Let $\mathbb{V}_1$ be the irreducible representation $\mathbb{Q}^2$ with the standard action of matrix-vector multiplication and let $\mathbb{V}_k = S^k\mathbb{V}_1$ be the irreducible representation given by the symmetric power of $\mathbb{V}_1$. The representation $\mathbb{V}_k$ has dimension $k+1$ and can be view as $\mathbb{Q}[x,y]_k$, ie the vector space of homogeneous polynomials in two variables. The action of T on the monomials is given by $H_t \cdot x^a y^{k-a} = t^{2a-k} x^a y^{k-a}$, thus $\mathbb{V}_k$ decomposes, as representations of T

$$\mathbb{V}_k = \bigoplus_{a=0}^{k} V(2a-k), \tag{2}$$

where $V(2a-k)$ is the subspace where H_t acts with character t^{2a-k}, ie the subspace generated by $x^a y^{k-a}$. The algebraic group $SL_2(\mathbb{Q})$ is reductive, so by [24, Theorem 22.42] each representation of $SL_2(\mathbb{Q})$ is semisimple. The Fundamental Theorem [24, Theorem 22.2] says that each irreducible representation of $SL_2(\mathbb{Q})$ is isomorphic to the representation $\mathbb{V}_k$ described above, for an unique $k \in \mathbb{N}$.

As a consequence we can decompose a representation V of $SL_2(\mathbb{Q})$ using its decomposition $V = \oplus_{a\in\mathbb{Z}} V(a)^{\oplus n_a}$ as representation of T: indeed $V \cong \oplus_{k\in\mathbb{N}} \mathbb{V}_k^{\oplus m_k}$ as representation of $SL_2(\mathbb{Q})$, where $m_k = n_k - n_{k+2}$. By setting $V = \mathbb{V}_m \otimes \mathbb{V}_n$, we obtain the following formula for $n \le m$:

$$\mathbb{V}_m \otimes \mathbb{V}_n \cong \mathbb{V}_{m+n} \oplus \mathbb{V}_{m+n-2} \oplus \cdots \oplus \mathbb{V}_{m-n}.$$

As observed in Section 2.1, the group $SL_2(\mathbb{Q})$ acts trivially on $\omega_{i,j}$ for all $1 \le i < j \le n$ and the two dimensional subspace generated by x_i and y_i is isomorphic to $\mathbb{V}_1$ as representation of $SL_2(\mathbb{Q})$.

We will use the decomposition of Theorem 2.5 to obtain a decomposition of $A^{\bullet,\bullet}$ into $\mathfrak{S}_n \times SL_2(\mathbb{Q})$-modules. Define $A_a^{p,q}$ as the following direct sum:

$$A_a^{p,q} \stackrel{\text{def}}{=} \bigoplus_{\substack{|L_*|=q \\ |H_*|=p,\ \|H_*\|=a}} A_{L_*,H_*}.$$

Let $E = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in SL_2(\mathbb{Q})$ and consider the map $\iota\colon A^{p,q} \to A^{p,q}$ defined by $v \mapsto v - E\cdot v$. Notice that $\iota(x_i) = 0$, $\iota(y_i) = x_i$ and $\iota(\omega_{i,j}) = 0$, for all $i \neq j$, hence it induces a restricted map $\iota_a\colon A_a^{p,q} \to A_{a+2}^{p,q}$.

Lemma 2.8 *For all $a \ge 0$, the map ι_a is surjective.*

Proof Let W be a irreducible $SL_2(\mathbb{Q})$-representation in $A^{p,q}$ and notice that $\iota(W) \subseteq W$. By the arbitrariness of W, it is enough to prove that $W \cap A^{p,q}_{a+2}$ is contained in $\operatorname{Im} \iota_a$.

Since $W \cong \mathbb{V}_k$ for some k, we can reduce to work in $\mathbb{V}_k$. If $k < a+2$ or $k \not\equiv a \mod 2$, then the torus T acts with weight $a+2$ on $A^{p,q}_{a+2}$. The weight $a+2$ does not appear in W, therefore $W \cap A^{p,q}_{a+2} = 0$. Otherwise, $k = a+2+2b$ for some $b \in \mathbb{N}$ and $W \cap A^{p,q}_{a+2}$ is the one dimensional vector space generated by the image of $x^{a+2+b}y^b \in \mathbb{V}_k$. The identity

$$(\mathrm{Id} - E) \cdot \frac{x^{a+1+b}y^{b+1}}{b+1} = x^{a+2+b}y^b,$$

completes the proof. □

Theorem 2.9 *The algebra $A^{\bullet,\bullet}$ decomposes as $\mathfrak{S}_n \times SL_2(\mathbb{Q})$-representation in the following way:*

$$A^{p,q} \cong \bigoplus_{a=0}^{p} \ker \iota_a \boxtimes \mathbb{V}_a.$$

Moreover, $\dim \ker \iota_a$ is the difference between $\dim A^{p,q}_a$ and $\dim A^{p,q}_{a+2}$.

Proof The assertion on the dimension of $\ker \iota_a$ follows from Theorem 2.8. Let W_a be the $\mathfrak{S}_n \times SL_2(\mathbb{Q})$-subrepresentation generated by $\ker \iota_a$. It is isomorphic to $\ker \iota_a \boxtimes \mathbb{V}_a$ because all elements in $\ker \iota_a$ are of highest weight a. Thus $A^{p,q} \supseteq \sum_{a=0}^p W_a$ and the sum is direct since the $SL_2(\mathbb{Q})$-representations $\mathbb{V}_a$ are pairwise non-isomorphic. We have proven that $A^{p,q} \supseteq \oplus_{a=0}^p W_a$ and $W_a \cong \ker \iota_a \boxtimes \mathbb{V}_a$. We complete the proof by a dimensional reasoning:

$$\begin{aligned}
\dim A^{p,q} &= \sum_{a=-p}^{p} \dim A^{p,q}_a = -\dim A^{p,q}_0 + 2\sum_{a=0}^{p} \dim A^{p,q}_a \\
&= \sum_{a=0}^{p} (\dim A^{p,q}_a - \dim A^{p,q}_{a+2})(a+1) = \sum_{a=0}^{p} \dim \ker \iota_a \cdot \dim \mathbb{V}_a \\
&= \dim \bigoplus_{a=0}^{p} W_a.
\end{aligned}$$

Since $A^{p,q}$ and $\oplus_{a=0}^p W_a$ have the same dimension, they are equal. □

Define the $\mathfrak{S}_n$-invariant subalgebra of $A^{\bullet,\bullet}$ by $UA^{\bullet,\bullet}$ and of $B^{\bullet,\bullet}$ by $UB^{\bullet,\bullet}$. Obviously we have $UA^{\bullet,\bullet} = UB^{\bullet,\bullet} \otimes_{\mathbb{Q}} D^{\bullet}$. We use the previous calculation to compute $UA^{\bullet,\bullet}$

Corollary 2.10 *For $q > p+1$ we have $UA^{p,q} = 0$.*

Proof Let $\mathbb{1}_n$ be the trivial representation of $\mathfrak{S}_n$. We use Theorem 2.5 to show that

$$\langle \mathbb{1}_n, A^{p,q}\rangle_{\mathfrak{S}_n} = 0$$

for $q > p+1$. Indeed, it is enough to prove that

$$\langle \mathbb{1}_n, \mathrm{Ind}_{Z_{L_*,H_*}}^{\mathfrak{S}_n} \xi_{L_*,H_*}\rangle_{\mathfrak{S}_n} = 0$$

for all (L_*, H_*) with $|L_*| = q$ and $|H_*| = p$. By Frobenius reciprocity we have

$$\langle \mathbb{1}_n, \mathrm{Ind}_{Z_{L_*,H_*}}^{\mathfrak{S}_n} \xi_{L_*,H_*}\rangle_{\mathfrak{S}_n} = \langle \mathrm{Res}_{Z_{L_*,H_*}}^{\mathfrak{S}_n} \mathbb{1}_n, \xi_{L_*,H_*}\rangle_{Z_{L_*,H_*}}$$

Since the representations in the right hand side are one-dimensional the value of $\langle \mathbb{1}_n, \mathrm{Ind}_{Z_{L_*,H_*}}^{\mathfrak{S}_n} \xi_{L_*,H_*}\rangle_{\mathfrak{S}_n}$ is non zero if and only if $\xi_{L_*,H_*} = \mathbb{1}$.

By definition $\xi_{L_*,H_*} = \mathbb{1}$ is equivalent to $\varphi_{L_*} = \mathbb{1}$ and $\alpha_{L_*,H_*} = \mathbb{1}$. From the fact that $\varphi_k = \mathrm{sgn}_k$ only for $k = 1, 2$, $\psi_{L_*} = \mathbb{1}$ if and only if $\lambda_i = 1, 2$ for all $i = 1, \ldots, t$. The condition $\alpha_{L_*,H_*} = \mathbb{1}$ implies that the only marked blocks of (L_*, H_*) that appear more than once are the ones with $\lambda_i = 2$ and $\deg(h_i) = 1$ or the ones with $\lambda_i = 1$ and $\deg(h_i) \neq 1$.

Consequently, $\langle \mathbb{1}_n, \mathrm{Ind}_{Z_{L_*,H_*}}^{\mathfrak{S}_n} \xi_{L_*,H_*}\rangle_{\mathfrak{S}_n} \neq 0$ only if $L_* = (2^q, 1^{n-2q})$ and the degree of h_i is 1 for $i < q$, this implies $p \geq q-1$ contrary to our hypothesis. □

Corollary 2.11 *For $q > p+1$ we have $UB^{p,q} = 0$.* □

3 The additive structure of the cohomology

We compute the cohomology with rational coefficients of the unordered configuration spaces of n points, taking care of the mixed Hodge structure and of the action of $SL_2(\mathbb{Q})$. The integral cohomology groups are known only for small n in [25, Table 2], where a cellular decomposition of ordered configuration spaces is given. In this section, we use the calculation of the Betti numbers of $\mathcal{C}^n(E)$ to determine the Hodge polynomial in the Grothendieck ring of $SL_2(\mathbb{Q})$.

Observe that $H^\bullet(\mathcal{C}^n(E)) = H^\bullet(\mathcal{F}^n(E))^{\mathfrak{S}_n}$ by the Transfer Theorem. Define the series

$$T(u,v) = \frac{1+u^3v^4}{(1-u^2v^3)^2} = 1 + 2u^2v^3 + u^3v^4 + 3u^4v^6 + 2u^5v^7 + \ldots$$

and let $T_n(u,v)$ be its truncation at degree n in the variable u.

The computation of the Betti numbers of unordered configuration space of n points in an elliptic curve was done simultaneously by [11], [23], and [27] in different generality. We point to the last reference because [27, Theorem] fits exactly our generality.

Theorem 3.1 *The Poincaré polynomial of $\mathcal{C}^n(E)$ is $(1+t)^2 T_{n-1}(t,1)$.*

We use the notation $Vu^k v^h$ to denote a vector space V in degree k with a Hodge structure of weight h. The Grothendieck ring of $SL_2(\mathbb{Q})$ is the free $\mathbb{Z}$-module with basis given by $[V]$ for all finite-dimensional irreducible representations V of $SL_2(\mathbb{Q})$ and product defined by the tensor product of representations.

Definition 3.2 The Hodge polynomial of $\mathcal{C}^n(E)$ with coefficients in the Grothendieck ring of $SL_2(\mathbb{Q})$ is

$$\sum_{i=0}^{2n}\sum_{k=i}^{2i}\left[W_kH^i(\mathcal{C}^n(E);\mathbb{Q})\big/W_{k-1}H^i(\mathcal{C}^n(E);\mathbb{Q})\right]u^iv^k,$$

where $W_kH^i(\mathcal{C}^n(E);\mathbb{Q})$ is the weight filtration on $H^i(\mathcal{C}^n(E);\mathbb{Q})$. The ordinary Hodge polynomial is

$$\sum_{i=0}^{2n}\sum_{k=i}^{2i}\dim_{\mathbb{Q}}\left(W_kH^i(\mathcal{C}^n(E);\mathbb{Q})\big/W_{k-1}H^i(\mathcal{C}^n(E);\mathbb{Q})\right)u^iv^k.$$

We prove a stronger version of Theorem 3.1.

Theorem 3.3 *The Hodge polynomial of $\mathcal{C}^n(E)$ with coefficients in the Grothendieck ring of $SL_2(\mathbb{Q})$ is*

$$([\mathbb{V}_0]+[\mathbb{V}_1]uv+[\mathbb{V}_0]u^2v^2)\left(\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}[\mathbb{V}_i]u^{2i}v^{3i}+\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor-1}[\mathbb{V}_{i-1}]u^{2i+1}v^{3i+1}\right) \tag{3}$$

and the ordinary Hodge polynomial is $(1+uv)^2T_{n-1}(u,v)$.

Figure 1 represents the module $H^{\bullet,\bullet}(UB)$ that corresponds to the right factor of eq. (3).

3.1 Some elements in cohomology

Definition 3.4 Let $\alpha,\overline{\alpha}\in A^{1,1},\beta\in A^{1,2}$ be the elements

$$\alpha \stackrel{\text{def}}{=} \sum_{i,k<h}(x_i-x_k)\omega_{k,h}$$

$$\overline{\alpha} \stackrel{\text{def}}{=} \sum_{i,k<h}(y_i-y_k)\omega_{k,h}$$

$$\beta \stackrel{\text{def}}{=} \sum_{i,j,k<h}(3x_i-x_j-2x_k)(y_j-y_k)\omega_{k,h}$$

where the sum is taken over pairwise distinct indices i,j,k,h with $k<h$.

	0	1	2	3	$\cdots$	q
q						$\mathbb{V}_q$
$q-1$					$\mathbb{V}_{q-1}$	$\mathbb{V}_{q-2}$
$\vdots$				$\iddots$	$\iddots$	
2			$\mathbb{V}_2$	$\mathbb{V}_1$		
1		$\mathbb{V}_1$	$\mathbb{V}_0$			
0	$\mathbb{V}_0$					

(a) Case $n = 2q+1$ odd.

	0	1	2	3	$\cdots$	q	$q+1$
q						$\mathbb{V}_q$	$\mathbb{V}_{q-1}$
$\vdots$					$\iddots$	$\iddots$	
3				$\mathbb{V}_3$	$\mathbb{V}_2$		
2			$\mathbb{V}_2$	$\mathbb{V}_1$			
1		$\mathbb{V}_1$	$\mathbb{V}_0$				
0	$\mathbb{V}_0$						

(b) Case $n = 2q+2$ even.

Figure 1: The algebra $H^{\bullet,\bullet}(UB)$ as representation of $SL_2(\mathbb{Q})$.

Notice that the elements α and $\overline{\alpha}$ are defined only for $n > 2$ and β for $n > 3$. Remember that $\gamma, \overline{\gamma} \in D^1$ were already defined as $\sum_i x_i$ and $\sum_i y_i$.

Lemma 3.5 *The element α belongs to $UB^{1,1}$, is non-zero, and $\mathrm{d}\alpha = 0$.*

Proof First observe that $\alpha = \sum_{i,k<h} u_{i,k}\omega_{k,h} \in B^{1,1}$. For all $\sigma \in \mathfrak{S}_n$ we have

$$\sigma\alpha = \sum_{i,k<h} u_{\sigma(i),\sigma(k)}\omega_{\sigma(k),\sigma(h)} = \alpha,$$

since $u_{\sigma(i),\sigma(k)}\omega_{\sigma(k),\sigma(h)} = u_{\sigma(i),\sigma(h)}\omega_{\sigma(k),\sigma(h)}$ in A. The elements $x_i\omega_{k,h}$ and $x_k\omega_{k,h}$ are linearly independent, so it is enough to observe that the coefficient of $x_3\omega_{1,2}$ is 1. This proves that $\alpha \neq 0$. Finally, we compute $\mathrm{d}\alpha$:

$$\begin{aligned}
\mathrm{d}\alpha &= \sum_{i,k<h} x_i\,\mathrm{d}\omega_{k,h} - x_k\,\mathrm{d}\omega_{k,h} \\
&= \sum_{i,k<h} x_i(x_k - x_h)(y_k - y_h) + x_k x_h(y_k - y_h) \\
&= \sum_{i,k,h} x_i x_k y_k - x_i x_h y_k + x_k x_h y_k \\
&= -\sum_{i,k,h} x_i x_h y_k = 0,
\end{aligned}$$

where all sums are taken over pairwise distinct indices and the first two with the additional condition $k < h$. $\square$

Lemma 3.6 *The element β belongs to $UB^{1,2}$, is non-zero, and $\mathrm{d}\beta = 0$.*

Proof Observe that

$$\beta = \sum_{i,j,k<h} (u_{i,j} + 2u_{i,k}) v_{j,k} \omega_{k,h} \in B^{1,2}$$

and that $\sigma\beta = \beta$ for all $\sigma \in \mathfrak{S}_n$ by the relations $u_{j,k}\omega_{k,h} = u_{j,h}\omega_{k,h}$ and $v_{j,k}\omega_{k,h} = v_{j,h}\omega_{k,h}$. Consider the map $\varphi\colon A \to \mathbb{Q}$ defined on generators by $\varphi(\omega_{1,2}) = 1$, $\varphi(x_3) = 1$ and $\varphi(y_4) = 1$ and zero on the other generators. The map φ is well defined and $\varphi(\beta) = 3$, thus $\beta \neq 0$. Using the computation in the proof of Theorem 3.5, we can observe that $\mathrm{d}\big(\sum_{i,j,h<k} 3x_i(y_j - y_k)\omega_{k,h}\big) = 0$. The claim $\mathrm{d}\beta = 0$ follows from:

$$\begin{aligned}
\mathrm{d}(\beta) &= \mathrm{d}\Big(\sum_{j,k,h} (x_j + 2x_k)(y_j - y_k)\omega_{k,h}\Big) \\
&= \sum_{j,k<h} x_j y_j \,\mathrm{d}\omega_{k,h} + x_j y_k (x_k - x_h) y_h - 2x_k y_j x_h (y_k - y_h) - 2x_k y_k x_h y_h \\
&= \sum_{j,k,h} x_j y_j x_k y_k - x_i y_j x_h y_k + x_j y_k x_k y_h - 2x_k y_j x_h y_k - x_k y_k x_h y_h \\
&= 0,
\end{aligned}$$

where the indexes of the sums are pairwise distinct. □

Lemma 3.7 *For $n > 2q$ the element α^q is non-zero.*

Proof We show that the coefficient of the monomial $m = x_1\omega_{1,2}x_3\omega_{3,4}\dots x_{2q-1}\omega_{2q-1,2q}$ (defined for $n \geq 2q$) in α^q is non-zero for $n > 2q$. Initially, we will prove that the coefficient is

$$a_q = q! \sum_{\sigma \in \mathfrak{S}_q} \operatorname{sgn}(\sigma)(2-n)^{|\operatorname{Fix}\sigma|} 2^{q-|\operatorname{Fix}\sigma|}, \tag{4}$$

then that $a_q \neq 0$ for $n > 2q$.

We start with the following identity:

$$\alpha^q = \Big(\sum_{j<k} \sum_{i \neq j,k} (x_i - x_j)\omega_{j,k}\Big)^q = \sum_{J,K} \prod_{l=1}^{q} \Big(\sum_{i \neq j_l,k_l} (x_i - x_{j_l})\Big)\omega_{j_l,k_l},$$

where the sum is taken over all multi-indexes $J = \{j_1, \dots, j_q\}$ and $K = \{k_1, \dots, k_q\}$ with $j_l < k_l$ for all $l = 1, \dots, q$. Since, for all $l = 1, \dots, q$, the pair of indexes $(2l-1, 2l)$ appear in the monomial m, the monomial m arise only in the terms with the following property $\mathcal{P}$: there exist $\sigma \in \mathfrak{S}_q$ such that $j_l = 2\sigma(l) - 1$ and $k_l = 2\sigma(l)$ for all $l = 1, \dots, q$. We further manipulate it:

$$\begin{aligned}\sum_{(J,K)\in\mathcal{P}} \prod_{l=1}^{q} \Big(\sum_{i\neq j_l,k_l} (x_i - x_{j_l}) \Big) \omega_{j_l,k_l} &= \sum_{\sigma\in\mathfrak{S}_q} \prod_{l=1}^{q} \Big(\sum_{i\neq 2\sigma(l)-1,2\sigma(l)} (x_i - x_{2\sigma(l)-1}) \Big) \omega_{2\sigma(l)-1,2\sigma(l)} \\ &= q! \prod_{l=1}^{q} \Big(\sum_{i\neq 2l-1,2l} (x_i - x_{2l-1}) \Big) \omega_{2l-1,2l} \\ &= q! \prod_{l=1}^{q} \Big(2 \sum_{\substack{i=1,\dots,q \\ i\neq l}} x_{2i-1} + \sum_{i>2q} x_i - (n-2) x_{2l-1} \Big) \omega_{2l-1,2l},\end{aligned}$$

where the last equality is obtained using $x_j\omega_{j,k} = x_k\omega_{j,k}$. Since the variable x_i with $i > 2q$ does not appear in m, the coefficient a_q is equal to the coefficient of $x_1x_3, \dots, x_{2q-1}$ in $q! \prod_{l=1}^{q} \big(2\sum_{i\neq l} x_{2i-1} - (n-2)x_{2l-1}\big)$. We have:

$$\begin{aligned}\prod_{l=1}^{q} \big(2 \sum_{i\neq l} x_{2i-1} - (n-2) x_{2l-1}\big) &= \sum_{\sigma\in\mathfrak{S}_q} \prod_{l=1}^{q} c_{\sigma(l),l} x_{2\sigma(l)-1} \\ &= \sum_{\sigma\in\mathfrak{S}_q} \operatorname{sgn}\sigma \prod_{l=1}^{q} c_{\sigma(l),l} x_{2l-1} \\ &= \sum_{\sigma\in\mathfrak{S}_q} \operatorname{sgn}\sigma (2-n)^{|\operatorname{Fix}\sigma|} 2^{q-|\operatorname{Fix}\sigma|} \prod_{l=1}^{q} x_{2l-1},\end{aligned}$$

where $c_{k,l} = 2 - n$ if $k = l$ and $c_{k,l} = 2$ otherwise. Putting all together we obtain Equation 4.

We claim that

$$\sum_{\sigma\in\mathfrak{S}_q} \operatorname{sgn}(\sigma) x^{|\operatorname{Fix}\sigma|} = (x-1)^{q-1}(x+q-1), \tag{5}$$

since both sides are the determinant of the matrix

$$\begin{pmatrix} x & 1 & \cdots & 1 \\ 1 & x & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & x \end{pmatrix}.$$

The left hand side of eq (5) is obtained by using the Laplace formula for the determinant and the right hand side by relating the that determinant to the characteristic polynomial

$(-t)^{q-1}(q-t)$ of the matrix $\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$. We use eq (5) with $x = \frac{2-n}{2}$ to obtain:

$$a_q = q!2^q \sum_{\sigma \in \mathfrak{S}_q} \operatorname{sgn}(\sigma) \Big(\frac{2-n}{2}\Big)^{|\operatorname{Fix} \sigma|} = q!2^q \Big(\frac{-n}{2}\Big)^{q-1} \Big(\frac{2q-n}{2}\Big)$$

Thus $a_q = (-1)^q q! n^{q-1}(n-2q)$ that is non-zero for $n > 2q$. □

Lemma 3.8 *For $n > 2q+1$ the element $\alpha^{q-1}\beta$ is non-zero.*

Proof Let us rewrite β as

$$\beta = \sum_{i,j,k<h} x_i y_j \omega_{k,h} - 2(n-3) \sum_{i,k<h} (x_i y_k + x_k y_i)\omega_{k,h} - (n-3) \sum_{i,k<h} x_i y_i \omega_{k,h} + \\ + 2(n-2)(n-3) \sum_{k<h} x_k y_k \omega_{k,h}.$$

Let b_q be the coefficient in $\alpha^{q-1}\beta$ of the monomial

$$x_1\omega_{1,2} x_3 \omega_{3,4} \ldots x_{2q-1}\omega_{2q-1,2q} y_{2q+1}.$$

This monomial is defined for $n \geq 2q+1$ and we will show that $b_q \neq 0$ for $n > 2q+1$. The number b_q coincides with the coefficient of the same monomial in the product

$$\alpha^{q-1}\Big(\sum_{i,j,k<h} x_i y_j \omega_{k,h} - 2(n-3) \sum_{i,k<h} x_k y_i \omega_{k,h}\Big).$$

With further manipulation, we obtain that b_q is the coefficient of the above monomial in the expression

$$3\alpha^q y_{2q+1} + n\alpha^{q-1} \sum_{k<h} x_k \omega_{k,h} y_{2q+1}.$$

Using the computation in the proof of Theorem 3.7 we obtain

$$\begin{aligned} b_q &= 3(-1)^q q! n^{q-1}(n-2q) + nq(-1)^{q-1}(q-1)! n^{q-2}(n-2q+2) \\ &= 2(-1)^q q! n^{q-1}(n-2q-1). \end{aligned}$$

The number b_q is non zero for $n > 2q+1$. □

Proof of Theorem 3.3 It is enough to prove that the Hodge polynomial of UB in the Grothendieck ring of $SL_2(\mathbb{Q})$ is

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} [\mathbb{V}_i] u^{2i} v^{3i} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - 1} [\mathbb{V}_{i-1}] u^{2i+1} v^{3i+1}$$

Observe that $\operatorname{Im} \mathrm{d}^{q,p} = 0$ for $q > p + 1$ by Theorem 2.11. From Theorem 3.5 and Theorem 3.7 we have that the elements α^k for $2k < n$ generate as $SL_2(\mathbb{Q})$-module a subspace of dimension at least $k+1$ in $H^{k,k}(UB, \mathrm{d})$. Analogously, from Theorem 3.6 and Theorem 3.8 the elements $\alpha^{k-1}\beta$ for $2k+1 < n$ generate as $SL_2(\mathbb{Q})$-module a subspace of dimension at least k in $H^{k,k+1}(UB, \mathrm{d})$. Since the Betti numbers of $UB^{\bullet,\bullet}$ (Theorem 3.1) coincides with the above dimensions, we have that $H^{2k}(UB) \cong \mathbb{V}_k u^{2k} v^{3k}$ and $H^{2k+1}(UB) \cong \mathbb{V}_{k-1} u^{2k+1} v^{3k+1}$. □

4 The cohomology ring

In this section we determine the cup product structure in the cohomology of $\mathcal{C}^n(E)$ and we prove the formality result.

In the following we consider graded algebras with an action of $SL_2(\mathbb{Q})$. We will write $(x_i \mid i \in I)_{SL_2(\mathbb{Q})}$ for the ideal generated by the elements Mx_i for all $M \in SL_2(\mathbb{Q})$ and all $i \in I$.

Theorem 4.1 *The cohomology ring of $\mathcal{C}^n(E)$ is isomorphic to*

$$\Lambda^\bullet \mathbb{V}_1 \otimes S^\bullet \mathbb{V}_1[b] \Big/ (a^{\lfloor \frac{n+1}{2} \rfloor}, a^{\lfloor \frac{n}{2} \rfloor} b, b^2)_{SL_2(\mathbb{Q})},$$

where a is a non-zero degree-one element in $V(1) \subset \mathbb{V}_1$ and b an $SL_2(\mathbb{Q})$-invariant indeterminate of degree 3.

Proof It is enough to prove that $H^\bullet(UB) \cong S^\bullet \mathbb{V}_1[b]/(a^{\lfloor \frac{n+1}{2} \rfloor}, a^{\lfloor \frac{n}{2} \rfloor} b, b^2)_{SL_2(\mathbb{Q})}$. Define the morphism $\varphi\colon S^\bullet \mathbb{V}_1[b]/(a^{\lfloor \frac{n+1}{2} \rfloor}, a^{\lfloor \frac{n}{2} \rfloor} b, b^2)_{SL_2(\mathbb{Q})} \to H^\bullet(UB)$ that sends a, b to α, β respectively. It is well defined because $H^k(UB) = 0$ for $k \geq n$ and $\beta^2 = 0$ since it has odd degree. The map φ is surjective since $H^\bullet(UB)$ is generated by α^i and $\alpha^i\beta$ as $SL_2(\mathbb{Q})$-module by Theorem 3.3. A dimensional reasoning shows the injectivity of the map φ. □

Corollary 4.2 *The cohomology $H^\bullet(\mathcal{C}^n(E))$ is generated as an algebra in degrees one, two and three.*

Proof A minimal set of generators is given by $\alpha, \overline{\alpha}, \beta, \gamma, \overline{\gamma}$. □

Corollary 4.3 *The space $\mathcal{C}^n(E)$ is formal over the rationals.*

Proof We prove that UB is formal. Consider the subalgebra $K^{\bullet,\bullet}$ of $UB^{\bullet,\bullet}$ generated by $\alpha, \overline{\alpha}, \beta$ endowed with the zero differential. It is concentrated in degrees (i,i) and $(i, i+1)$ because $\beta^2 = 0$. Since $K \cap \operatorname{Im} \mathrm{d} = 0$ (Theorem 2.11), $K \hookrightarrow UB$ is a quasi-isomorphism. The fact that $K \cong H(UB)$ implies that the algebra UB is formal. As a consequence UA is formal. The space $\mathcal{C}^n(E)$ is formal since our model UA is equivalent to the Sullivan model. $\square$

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