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# Galerkin's Method Revisited and Corrected in the Problem of Jaworski and Dowell

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## Abstract

This paper revisits the problem first studied by Jaworsky and Dowell, namely, the free vibration of multi-step beams. Previous authors utilized approximate method of Ritz as well as the finite element method with attendant comparison with the experimental results. This study provides the exact solution for the Jaworsky and Dowell problem in terms of Krylov/Duncan functions. Additionally, the Galerkin method is applied and contrasted with the exact solution. It is shown that the straightforward implementation of the Galerkin method, as it is usually performed in the literature, does not lead to results obtained by Jaworsky and Dowell using the Ritz method. Moreover, the straightforward application of the Galerkin method does not tend to the results obtained by either exact solution or experiments. A modification of the Galerkin method is proposed by introducing generalized functions to describe both mass and stiffness of the stepped beam. Specifically, the unit step function, Dirac's delta function and the doublet functions, are utilized for this purpose. With this modification, the

Galerkin method yields result coinciding with those derived by the Ritz method, and turn out to be in close vicinity with those produced by the exact solution as well as experiments.

## Keywords

Galerkin's method; unit step function; Dirac's delta function; doublet functions; Krylov/Duncan functions; convergence

## 1. Introduction

Numerous structures, like aircraft wings, helicopter rotor blades, spacecraft antennae, robot arms can be modeled as beams. Free vibration of beams is a classic subject in structural mechanics originating at about 1735 when Daniel Bernoulli and Leonhard Euler investigated the vibration of uniform and homogeneous beams (Elishakoff, 2020). Since then, numerous papers, reviews and books have been written on this subject. **Just to cite some famous work it is important to remember the one done by Young and Felgar (1949), which provide tables with numerical solution for beam in different boundary conditions; the book devoted to structural dynamics by Gorman (1975) or by Karnovsky and Lebed (2000); Karnovsky and Lebed (2010) described also the Krylov-Duncan method as an approach to obtain the natural frequencies by solving an eigenvalue problem; Duncan W.J. (1950) provided normalized orthogonal deflection functions for beams.**

What about beams with discontinuous cross-sectional areas, i.e. stepped beams, were also investigated by means of various approaches. Specifically, Cauchy iteration method was applied by Taleb and Suppiger (1961) yielding upper bounds for natural frequencies; lower bounds were furnished by Buckens (1963) using a decomposition method; variational component method with Lagrange multipliers to satisfy geometric continuity conditions between different steps was employed by Klein (1974). Rayleigh-Ritz energy approach was used by Yuan and Dickinson (1993) in conjunction with

artificial spring constraints between beam components. This approach was utilized also by Maurizi and Belles (1993).

The above papers resorted to approximate methods. Exact solution was provided by Levinson (1976) for a single-stepped beam simply supported at its both ends. Jang and Bert (1986) summarized results for a single, centrally located stepped beam. Naguleswaran (2002) dealt with beams with up to three steps, whereas Duan and Wang (2013), Lu et al (2009), Mao (2011) and Wang X-W. and Wang Y-L (2013) dealt with the case of multiple stepped beams.

Jarowsky and Dowell (2008) conducted a thorough investigation of a beam with 13 steps using approximate methods. Namely, they applied the Rayleigh-Ritz method as well as finite element method, and to validate their results they conducted extensive experiments. For the implementation of the Rayleigh-Ritz method, the authors used the exact modes of vibration of cantilever beam, whereas they built FEM models using ANSYS and various element available in its library (BEAM4, BEAM188, SHELL93 and SOLID45). The above results were contrasted with the experimental results.

In this paper, we conduct two analyses that complement the work by Jaworsky and Dowell (2008). Namely, we first used the so-called Krylov/Duncan functions and next two versions of Galerkin method to compute the exact natural frequencies at the stepped beam. In particular, for the Galerkin method we implemented both the straightforward method, in which the basis function of the Galerkin procedure exist and are evaluated for each step of the beam and next combined, and the rigorous version of the method which is based on generalized functions of the mass and stiffness of the beam which exist over the entire beam domain. In this latter case we used the Heaviside unit step function, the Dirac's delta function and its derivative, as well as the doublet function, to formulate the characteristic equation of the free vibration problem of the stepped beam.

## **2. Basic Equations**

We are interested to evaluate the natural frequencies of a multi-step beam as shown in Fig.1.

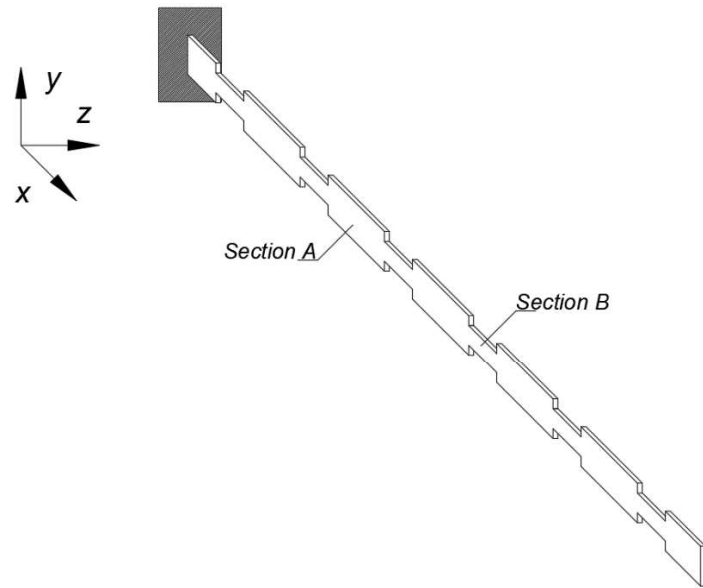
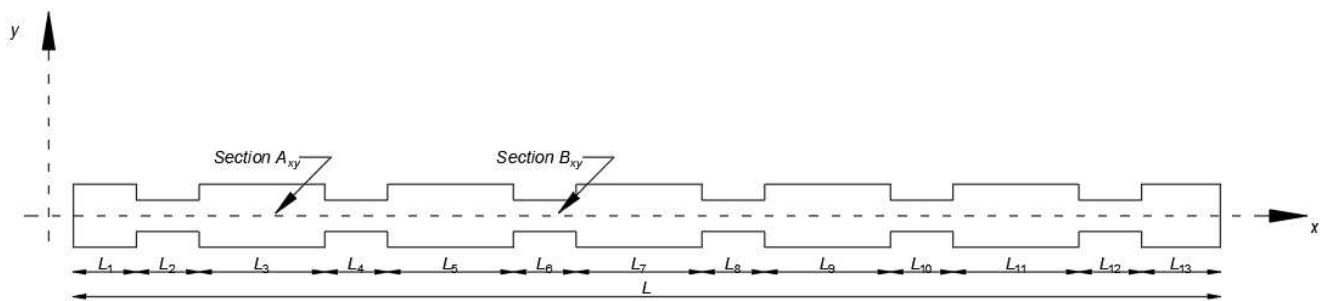


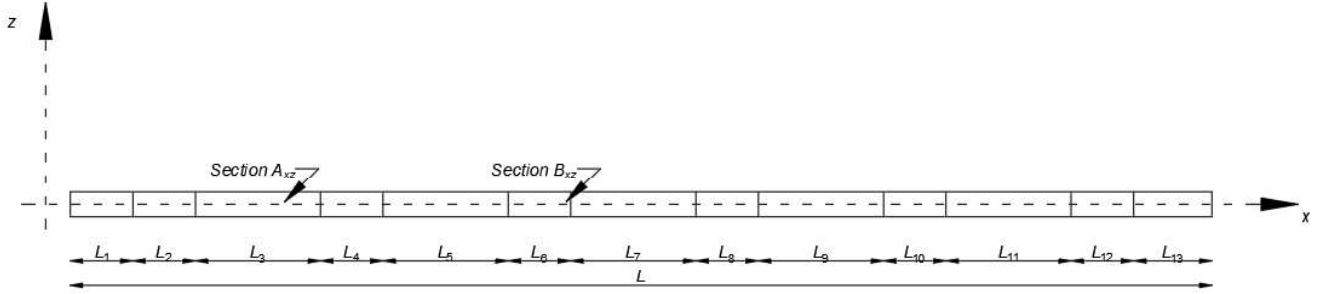
Fig.1. A 13-stepped beam of length  $L$

The beam is a cantilever made of a single material so the elastic modulus  $E$  and the mass density  $\rho$  are constants in this problem. The beam is composed by two alternating sections: the section A and the section B creating 13 components.

We study the free vibrations of this beam in both vertical x-y and horizontal x-z planes as shown in Fig. 2.



(a)



(b)

Fig. 2 schematic of the stepped beam in the two planes x-y and x-z

The Euler-Bernoulli classic differential equation governing the flexural vibrations in one principal plane of the non-uniform beam reads:

$$\frac{\partial^2}{\partial x^2} \left( E(x) I(x) \frac{\partial^2 w}{\partial x^2} \right) + \rho(x) A(x) \frac{\partial^2 w}{\partial t^2} = 0 \quad (1)$$

where  $w(x, t)$  is the vertical displacement,  $I(x)$  the moment of inertia,  $A(x)$  the cross-sectional area,  $x$  the axial coordinate and  $t$  is the time. For each segment of the stepped beam, one can write:

$$E_j I_j \frac{\partial^4 w}{\partial x^4} + \rho_j A_j \frac{\partial^2 w}{\partial t^2} = 0 \quad (2)$$

where  $j$  is an integer which identifies the component. We rewrite the vertical displacement as follows:

$$w(x, t) = W(x) \sin(\omega t) \quad (3)$$

where  $\omega$  is the sought natural frequency of the beam. Substituting Eq. (3) in the differential equations (2) we easily obtain the following set of equations valid for any time instant:

$$\frac{d^4 W}{dx^4} - \alpha_j^4 W = 0 \quad (4)$$

where  $\alpha_j$  reads:

$$\alpha_j = \sqrt[4]{\frac{\rho_j A_j \omega^2}{E_j I_j}} \quad (5)$$

As well known, the following forms of  $W_j(x)$  satisfy the differential equations in (4):

$$W_j(x) = D_{1,j} \sin(\alpha_j x) + D_{2,j} \cos(\alpha_j x) + D_{3,j} \cosh(\alpha_j x) + D_{4,j} \sinh(\alpha_j x) \quad (6)$$

where  $D_{i,j}$  are constants of integration. We now take advantage of the Krylov/Duncan functions to rewrite Eq. (6). The Krylov/Duncan functions are four functions (Krylov, 1936; Duncan, 1943) defined as follows:

$$K_1(\alpha x) = \frac{1}{2} [\cosh(\alpha x) + \cos(\alpha x)] \quad (7.a)$$

$$K_2(\alpha x) = \frac{1}{2} [\sinh(\alpha x) + \sin(\alpha x)] \quad (7.b)$$

$$K_3(\alpha x) = \frac{1}{2} [\cosh(\alpha x) - \cos(\alpha x)] \quad (7.c)$$

$$K_4(\alpha x) = \frac{1}{2} [\sinh(\alpha x) - \sin(\alpha x)] \quad (7.d)$$

One notes that:

$$K_1(0) = 1 \quad (8.a)$$

$$K_2(0) = 0 \quad (8.b)$$

$$K_3(0) = 0 \quad (8.c)$$

$$K_4(0) = 0 \quad (8.d)$$

The second property of these functions is that the first derivative of  $K_i$  is equal to  $K_{i-1}$ :

Krylov Function	$K_1(x)$	$K_2(x)$	$K_3(x)$	$K_4(x)$
First derivative	$K_4(x)$	$K_1(x)$	$K_2(x)$	$K_3(x)$
Second derivative	$K_3(x)$	$K_4(x)$	$K_1(x)$	$K_2(x)$
Third derivative	$K_2(x)$	$K_3(x)$	$K_4(x)$	$K_1(x)$

Table 1: Derivatives of Krylov/Duncan functions

We can use these functions into equation (6) in order to simplify the representation of the boundary conditions. This will lead us to the following equation:

$$W_j(x) = M_{1,j}K_1(\alpha_j x) + M_{2,j}K_2(\alpha_j x) + M_{3,j}K_3(\alpha_j x) + M_{4,j}K_4(\alpha_j x) \quad (9)$$

where  $M_{i,j}$  are constant of integration.



### 3. Exact Solution

The evaluation of the exact solution consists in the demand that not all four coefficients  $M_{i,j}$  for each component to vanish simultaneously. In our study we have 13 different segments for the multi-step beam resulting in 52 unknowns. The solution should satisfy continuity conditions between the segments and the boundary conditions at the outer sections of the beams (first and the 13<sup>th</sup> components).

For each discontinuity, we have four compatibility conditions namely continuity of vertical displacement, slope, bending moment and shear force, for a total of 48 equations of compatibility given the 12 discontinuities in the beam. In particular, they read:

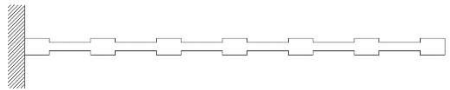
$$W_j(x = L_j) = W_{j+1}(x = L_j) \quad (10.a)$$

$$\frac{dW_j}{dx}(x = L_j) = \frac{dW_{j+1}}{dx}(x = L_j) \quad (10.b)$$

$$E_j I_j \frac{d^2 W_j}{dx^2}(x = L_j) = E_{j+1} I_{j+1} \frac{d^2 W_{j+1}}{dx^2}(x = L_j) \quad (10.c)$$

$$E_j I_j \frac{d^3 W_j}{dx^3}(x = L_j) = E_{j+1} I_{j+1} \frac{d^3 W_{j+1}}{dx^3}(x = L_j) \quad (10.d)$$

By adding the 4 boundary conditions at the extremes of the beam we can formulate a problem with 52 equations for 52 unknowns. In particular, in the following we tackle the case of the cantilever beam, to compare our results with those of Jarowsky and Dowell (2008). The boundary conditions read:

Constrain conditions		$x = 0$	$x = L$
	Cantilever	$W_1 = 0$	$E_{13} I_{13} \frac{d^2 W_{13}}{dx^2} = 0$

		$\frac{dW_1}{dx} = 0$	$E_{13}I_{13} \frac{d^3W_{13}}{dx^3} = 0$
--	--	-----------------------	---

Table 2: Boundary conditions

This system of equations has the following form:

$$\tilde{A}\bar{x} = \bar{0} \tag{11}$$

where  $\tilde{A}$  is the coefficient matrix,  $\bar{x}$  the vector of unknowns and  $\bar{0}$  the zero vector. The non trivial solutions of the homogeneous system in eq (11) lead to the natural frequencies  $\omega$  of the problem.

The matrix  $\tilde{A}$  is sparse and the non-zero terms appear around the main diagonal, as represented in Fig. 3.

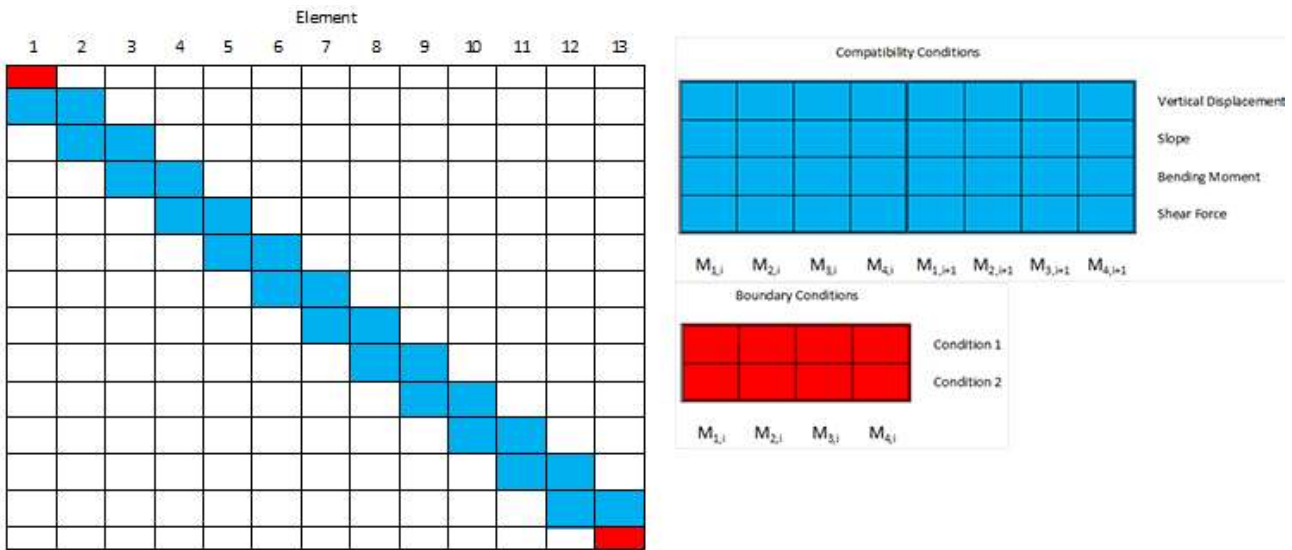


Fig. 3 Matrix structure for exact solution

#### 4. Straightforward Galerkin Method

The Galerkin method is a numerical method to solve in approximate way the differential equations:

$$E_j I_j \frac{d^4 W}{dx^4} - \rho_j A_j \omega^2 W = 0 \quad x_{j-1} < x < x_j \quad (12)$$

By introducing the axial coordinate in non-dimensional form eq (12) can be represented as:

$$E_j I_j \frac{d^4 W}{d\xi^4} - \rho_j A_j \omega^2 L^4 W = 0 \quad \xi_{j-1} < \xi < \xi_j \quad (13)$$

In order to apply the Galerkin method in its straightforward version, we have to express the vertical displacement  $W$  in terms of the so-called comparison functions  $\psi_p(\xi)$  as:

$$W(\xi) = \sum_{p=1}^n a_p \psi_p(\xi) \quad (14)$$

where  $a_p$  are unknown constants. Now we substitute the expression of  $W(\xi)$  in the differential equations obtaining residuals  $\varepsilon_j(\xi)$  since the functions  $\psi_p(\xi)$  do not necessarily satisfy the differential equations:

$$E_j I_j \sum_{p=1}^n a_k \frac{d^4 \psi_p(\xi)}{d\xi^4} - \rho_j A_j \omega^2 L^4 \sum_{p=1}^n a_p \psi_p(\xi) = \varepsilon_j(\xi) \quad \xi_{j-1} < \xi < \xi_j \quad (15)$$

We now multiply the error  $\varepsilon_j(\xi)$  by  $\psi_q(\xi)$ , we sum it up for all the components and we integrate within  $j^{\text{th}}$  span:

$$\sum_{p=1}^n \left\{ \sum_{j=1}^{13} \left[ \int_{\xi_j}^{\xi_{j+1}} E_j I_j \frac{d^4 \psi_p(\xi)}{d\xi^4} \psi_q(\xi) d\xi \right] - \omega^2 \sum_{j=1}^{13} \left[ \int_{\xi_j}^{\xi_{j+1}} \rho_j A_j L^4 \psi_p(\xi) \psi_q(\xi) d\xi \right] \right\} a_p = 0 \quad (16)$$

By defining:

$$K_{pq} = \sum_{j=1}^{13} \left[ \int_{\xi_j}^{\xi_{j+1}} E_j I_j \frac{d^4 \psi_p(\xi)}{d\xi^4} \psi_q(\xi) d\xi \right] \quad (17.a)$$

$$M_{pq} = \sum_{j=1}^{13} \left[ \int_{\xi_j}^{\xi_{j+1}} \rho_j A_j L^4 \psi_p(\xi) \psi_q(\xi) d\xi \right] \quad (17.b)$$

we obtain:

$$\sum_{p=1}^n (K_{pq} - \omega^2 M_{pq}) a_p = 0 \quad (18)$$

Eq. 18 can be rewritten in matrix notation as:

$$(K - \omega^2 M)a = 0 \quad (19)$$

where  $K$  represent the stiffness matrix of the problem,  $M$  the mass matrix of the problem and  $a$  the vector of the unknown scale factors  $a_p$ .

This non-trivial solution of eq. (19) lead to the eigenvalues  $\omega^2$  and the scale factors  $a_p$  of the problem.

## 5. Rigorous Galerkin Method

The rigorous version of the Galerkin method does require generalized functions over the entire domain of the beam length ( $0 < x < L$ ). Starting from Eq. (1) and (3) we obtain:

$$\frac{d^2}{dx^2} \left( E(x)I(x) \frac{d^2 W}{dx^2} \right) \sin(\omega t) - \omega^2 \rho(x)A(x)W(x) \sin(\omega t) = 0 \quad (20)$$

Introducing a non-dimensional axial coordinate  $\xi$  and looking for a solution true for any time value, we obtain:

$$\frac{d^2}{d\xi^2} \left( E(\xi)I(\xi) \frac{d^2 W}{d\xi^2} \right) - \omega^2 L^4 \rho(\xi)A(\xi)W(\xi) = 0 \quad (21)$$

In order to implement the rigorous Galerkin method we represent the flexural rigidity and the mass of the system as generalized functions:

$$D(\xi) = E(\xi)I(\xi) = E_1 I_1 \cdot U(\xi) + \sum_{j=1}^{12} [(E_{j+1} I_{j+1} - E_j I_j) \cdot H(\xi - \xi_j)] \quad (22.a)$$

$$M(\xi) = \rho(\xi)A(\xi) = \rho_1 A_1 \cdot U(\xi) + \sum_{j=1}^{12} [(\rho_{j+1} A_{j+1} - \rho_j A_j) \cdot H(\xi - \xi_j)] \quad (22.b)$$

where  $H(\xi - \xi_j)$  is the unit step function or Heaviside function which has the following properties:

$$H(\xi - \alpha) = \begin{cases} 1 & \text{if } \xi > \alpha \\ 0 & \text{otherwise} \end{cases} \quad (23.a)$$

$$\frac{d}{d\xi} H(\xi - \alpha) = \delta(\xi - \alpha) \quad (23.b)$$

$$\frac{d}{d\xi} \delta(\xi - \alpha) = \delta'(\xi - \alpha) \quad (23.c)$$

where  $\delta(\xi)$  is the Dirac's delta function, and  $\delta'(\xi - \alpha)$  is the doublet function. Now, rewriting the equation (21) with these considerations we obtain:

$$\frac{d^2}{d\xi^2} \left( D(\xi) \frac{d^2 W}{d\xi^2} \right) - \omega^2 L^4 M(\xi) W(\xi) = 0 \quad (24)$$

We evaluate the derivatives to get:

$$D(\xi) \frac{d^4 W}{d\xi^4} + 2 \frac{d}{d\xi} D(\xi) \frac{d^3 W}{d\xi^3} + \frac{d^2}{d\xi^2} D(\xi) \frac{d^2 W}{d\xi^2} - \omega^2 L^4 M(\xi) W(\xi) = 0 \quad (25)$$

We substitute the approximation in series of  $W(\xi)$  (Eq. (14)) arriving at:

$$\sum_{p=1}^n \left[ D(\xi) \frac{d^4 \psi_p(\xi)}{d\xi^4} + 2 \frac{d}{d\xi} D(\xi) \frac{d^3 \psi_p(\xi)}{d\xi^3} + \frac{d^2}{d\xi^2} D(\xi) \frac{d^2 \psi_p(\xi)}{d\xi^2} - \omega^2 L^4 M(\xi) \psi_p(\xi) \right] a_p = 0 \quad (26)$$

We next multiply the differential equation by  $\psi_q(\xi)$  and we integrate it from zero to one, to get:

$$\sum_{p=1}^n \left[ \int_0^1 D(\xi) \frac{d^4 \psi_p(\xi)}{d\xi^4} \psi_q(\xi) d\xi + \int_0^1 2 \frac{d}{d\xi} D(\xi) \frac{d^3 \psi_p(\xi)}{d\xi^3} \psi_q(\xi) d\xi \right. \\ \left. + \int_0^1 \frac{d^2}{d\xi^2} D(\xi) \frac{d^2 \psi_p(\xi)}{d\xi^2} \psi_q(\xi) d\xi - \omega^2 \int_0^1 L^4 M(\xi) \psi_p(\xi) \psi_q(\xi) d\xi \right] a_p = 0 \quad (27)$$

By defining:

$$K_{1,pq} = \int_0^1 D(\xi) \frac{d^4 \psi_p(\xi)}{d\xi^4} \psi_q(\xi) d\xi \quad (28.a)$$

$$K_{2,pq} = \int_0^1 2 \frac{d}{d\xi} D(\xi) \frac{d^3 \psi_p(\xi)}{d\xi^3} \psi_q(\xi) d\xi \quad (28.b)$$

$$K_{3,pq} = \int_0^1 \frac{d^2}{d\xi^2} D(\xi) \frac{d^2 \psi_p(\xi)}{d\xi^2} \psi_q(\xi) d\xi \quad (28.c)$$

$$M_{pq} = \int_0^1 L^4 M(\xi) \psi_p(\xi) \psi_q(\xi) d\xi \quad (28.d)$$

we can rewrite eq. (27) as:

$$\sum_{p=1}^n (K_{1,pq} + K_{2,pq} + K_{3,pq} - \omega^2 M_{pq}) a_p = 0 \quad (29)$$

or in more compact matrix form as:

$$(K_1 + K_2 + K_3 - \omega^2 M) a = 0 \quad (30)$$

Non-trivial solutions of the equation:

$$(K - \omega^2 M) a = 0 \quad (31)$$

where  $K = K_1 + K_2 + K_3$ , lead to the frequencies of vibration  $\omega^2$  and the scale factors  $a_p$  of the problem.

We observe that the matrix  $K_1$  coincides with the  $K$  matrix for the straightforward implementation of the method. Thus, the rigorous implementation of Galerkin method yields to two additional stiffness

matrices,  $K_2$  and  $K_3$ , which provide superior performances to the method w.r.t. its straightforward version.

## 6. Numerical investigation

### 6.1 Comparison Functions for Galerkin Method

To compare the straightforward and rigorous version of the Galerkin method with the exact solution of some beam problems, first comparison functions must be assumed. Comparison functions is supposed to represent well the solution of the differential equation while satisfying all the boundary conditions of the problem.

According to Jaworsky and Dowell (2008), good comparison functions for the problem at hand, consist in the mode shapes for the homogeneous cantilevered beam:

$$W_m(\xi) = \left( \frac{\sin(\alpha_m) - \sinh(\alpha_m)}{\cos(\alpha_m) + \cosh(\alpha_m)} \right) (\sinh(\alpha_m \xi) - \sin(\alpha_m \xi)) + (\cosh(\alpha_m \xi) - \cos(\alpha_m \xi)) \quad (32)$$

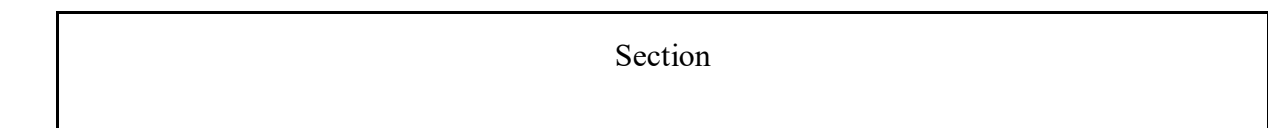
These functions, however, are not perfect candidates because for large value of  $m$  becomes numerically unstable due to the difference between large values of the hyperbolic functions arguments. To overcome this problem, some authors have proposed to use the following expressions:

$$W_m(\xi) = \sin(\alpha_m \xi) - \cos(\alpha_m \xi) + e^{-\alpha_m \xi} + (-1)^{m+1} e^{-\alpha_m(1-\xi)} + O[\varepsilon] \quad (33)$$

where the order of error is  $\varepsilon = e^{-\alpha_m}$ , which is negligible for  $n \geq 5$ .

### 6.2 Examples

We consider the cantilever beam represented in Fig.2a, composed by 13 segments, and the following geometrical and mechanical parameters:



	$A_{xy}$	$B_{xy}$	$A_{xz}$	$B_{xz}$
$E$	$6.06 \cdot 10^{10} Pa$	$6.06 \cdot 10^{10} Pa$	$6.06 \cdot 10^{10} Pa$	$6.06 \cdot 10^{10} Pa$
$\rho$	$2664 kg/m^3$	$2664 kg/m^3$	$2664 kg/m^3$	$2664 kg/m^3$
$b$	$3.175 mm$	$3.175 mm$	$25.4 mm$	$12,7 mm$
$h$	$25.4 mm$	$12,7 mm$	$3.175 mm$	$3.175 mm$
$A$	$8.0645 \cdot 10^{-5} m^2$	$4.0322 \cdot 10^{-5} m^2$	$8.0645 \cdot 10^{-5} m^2$	$4.0322 \cdot 10^{-5} m^2$
$I$	$4.33574 \cdot 10^{-9} m^4$	$5.41968 \cdot 10^{-11} m^4$	$6.7746 \cdot 10^{-11} m^4$	$3.3873 \cdot 10^{-11} m^4$

Table 3: Mechanical and geometrical data

The beam steps lengths are  $L_1 = L_2 = L_4 = L_6 = L_8 = L_{10} = 25.40mm$  and  $L_3 = L_5 = L_7 = L_9 = L_{11} = 50.80mm$ , whereas the last segment  $L_{13} = 31.75mm$ . The total length of the beam is therefore  $L = 463.55mm$ .

We show in the following the first three frequencies of vibration for the two planes x-y and x-z, computed by using the exact solution, and both the straightforward and rigorous Galerkin method.

## 8.1. Exact Solution

The first three frequencies of vibration in the  $x - y$  and x-z planes, computed by using the exact formulation in section 3, are shown in Table 4.



Exact Solution [rad/sec]		
Mode	$x - y$ Plane	$x - z$ Plane
1	342.4121	67.5133
2	2166.4943	423.9471
3	6143.9243	1191.0450

Table 4: Exact solution

## 8.2. Rigorous Galerkin Method

The rigorous Galerkin method, for 1, 2, 3, 25, 50, 75 and 100 terms, leads to the frequencies in table 5 and Table 7 for the frequencies of vibration in the x-y and x-z plane, respectively.

Frequencies [rad/s]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	532.3005	525.2584	525.2059	385.5929	362.8573	353.8210	352.2366
2	-	3303.9364	3302.5800	2590.9190	2296.1458	2238.5734	2229.1627
3	-	-	9288.0450	7360.6548	6511.9947	6377.2805	6322.6071

Table 5: Frequencies of vibration for the x-y plane obtained with rigorous Galerkin method

The relative error between the natural frequencies computed via the Galerkin method and the exact ones, computed as:

$$\varepsilon = \frac{\omega_{Galerkin} - \omega_{Exact}}{\omega_{Exact}} \cdot 100\% \quad (34)$$

is reported in Table 6.

Relative error [%]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	53.41%	53.40%	53.38%	12.61%	5.97%	3.33%	2.87%
2	-	52.50%	52.44%	19.59%	5.98%	3.33%	2.89%
3	-	-	51.17%	19.80%	5.99%	3.80%	2.91%

Table 6: Relative error between the rigorous Galerkin method and the exact solution for the frequencies of vibration in the x-y plane.

Frequencies [rad/s]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	71.1399	71.1379	71.1364	68.1456	67.9620	67.7800	67.7035
2	-	446.1477	446.0782	427.7656	426.7094	425.6417	425.1656
3	-	-	1250.6176	1208.3126	1199.1534	1195.9843	1194.5526

Table 7: Frequencies of vibration for the x-z plane obtained with rigorous Galerkin method

Relative error [%]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	5.37%	5.37%	5.37%	0.94%	0.66%	0.39%	0.28%
2	-	5.24%	5.22%	0.90%	0.65%	0.40%	0.29%
3	-	-	5.00%	1.45%	0.68%	0.41%	0.29%

Table 8: Relative error between the rigorous Galerkin method and the exact solution for the frequencies of vibration in the x-z plane.

### 8.3. Straightforward Galerkin Method

Similarly, we computed the frequencies of vibration for the straightforward implementation of the Galerkin method for 1, 2, 3, 25, 50, 75 and 100 terms. The frequencies of vibration in the x-y and x-z plane are reported in Table 9 and Table 11, respectively, whereas the relative errors with respect to the exact solutions are reported in Tables 10 and 12.

Frequencies [rad/s]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	531.0601	530.9327	530.7799	462.7421	427.8476	420.8202	419.3807
2	-	3335.5952	3334.2992	2884.2579	2697.5635	2652.9731	2645.0811
3	-	-	9357.7550	8163.1046	7594.1873	7490.8281	7450.9256

Table 9: Frequencies of vibration for the x-y plane obtained with straightforward Galerkin method

Relative error [%]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	55.09%	55.06%	55.01%	35.14%	24.95%	22.90%	22.48%
2	-	53.96%	53.90%	33.13%	24.51%	22.45%	22.09%
3	-	-	52.31%	32.86%	23.60%	21.92%	21.27%

Table 10: Relative error between the straightforward Galerkin method and the exact solution for the frequencies of vibration in the x-y plane.

Frequencies [rad/s]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms

1	71.5207	71.5207	71.5207	71.5207	71.5207	71.5207	71.5207
2	-	448.2448	448.2448	448.2448	448.2448	448.2448	448.2448
3	-	-	1255.2248	1255.2248	1255.2248	1255.2248	1255.2248

Table 11: Frequencies of vibration for the x-z plane obtained with straightforward Galerkin method

Relative error [%]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	5.94%	5.94%	5.94%	5.94%	5.94%	5.94%	5.94%
2	-	5.73%	5.73%	5.73%	5.73%	5.73%	5.73%
3	-	-	5.39%	5.39%	5.39%	5.39%	5.39%

Table 12: Relative error between the straightforward Galerkin method and the exact solution for the frequencies of vibration in the x-z plane.

## 8.4 Final comparison

Table 13 reports the frequencies of vibration computed on the previous subsections, in Hertz, along with those obtained experimentally by Jaworsky and Dowell (2008).

Frequency		Exact	Rigorous	Straightforward	Experimental
		Solution	Galerkin Method (100 Terms)	Galerkin Method (100 Terms)	Results, Jaworsky and Dowell (2008)
x-y Plane	1	54.4965	56.0601	66.7464	49.38
	2	344.8075	354.7814	420.9767	-
	3	977.8336	1006.2718	1185.8489	-
x-z Plane	1	10.7451	10.7753	11.3828	10.63
	2	67.4731	67.6670	71.3402	66.75

	3	189.5603	190.1185	199.7748	-
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Table 13: Frequencies obtained by three methods compared with those proposed in Ref. Jaworsky and Dowell (2008)

The relative error between the first three columns and the fourth, evaluated with formula (35), is collected in Table 14:

$$\varepsilon = \frac{\omega - \omega_{exact}}{\omega_{exact}} \cdot 100\% \quad (35)$$

Frequency		Exact Solution vs Paper by Jaworsky/Dowell	Rigorous Galerkin Method vs Paper by Jaworsky/Dowell	Straightforward Galerkin Method vs Paper by Jaworsky/Dowell
In Plane	1	10.36%	13.53%	35.17%
Out of Plane	1	1.08%	1.37%	7.08%
	2	1.08%	1.37%	6.88%

Table 14: Relative errors between all the methods

The tables 14 show that the rigorous Galerkin method tends to the experimental study carried out by Jaworsky and Dowell (2008), moreover tables 6 and 8 prove that the cited approach tends to the exact solution too. This aspect is demonstrated also by Figure 4 which shows that the error decreases with increase of the number of terms. On the other hand, straightforward Galerkin method does not converge to the exact solution. The experimental frequencies reported by Jaworsky and Dowell (2008) are close to the results obtained with Krylov/Duncan functions and rigorous Galerkin method but not with its straightforward implementation.

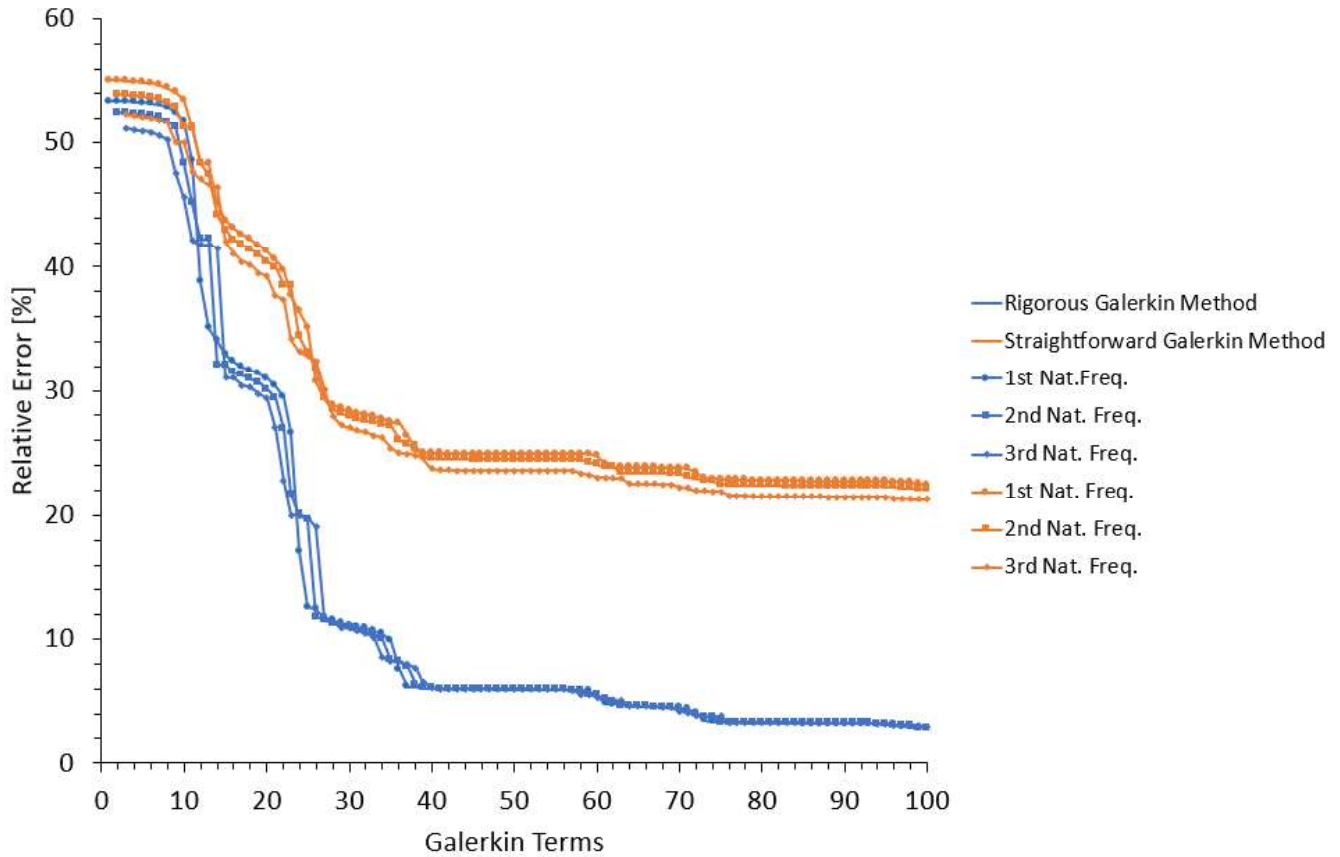


Fig.4: Rigorous vs Straightforward Galerkin method – Relative error

## 9. Conclusions

In this study we have analyzed the free vibrations of a cantilever homogeneous non-uniform beam with different methods. In particular, we have compared two different versions of Galerkin method, namely the straightforward version which is generally used in literature and the rigorous one proposed in this work, with the exact solution based on the Krylov/Duncan functions. While the straightforward approach considers basis functions with domain existing only over each segment, the rigorous implementation is based on two generalized functions, one for the stiffness and one for the mass of the beam, existing over the entire length of the beam. In this case, the abrupt changes of cross-sections are taken into account via the Heaviside function and its derivatives. The study shows that the proposed

rigorous implementation does convergence to the derived exact solutions whereas the straightforward version does not.

## 10. Dedication

This paper is dedicated to the blessed memory of Professor Simon G. Braun, the former Editor-in-Chief of this journal and former colleague of one of the authors (I.E.), at the Technion--Israel Institute of Technology, Haifa, Israel.

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