

ARCHIVIO ISTITUZIONALE DELLA RICERCA

Alma Mater Studiorum Università di Bologna Archivio istituzionale della ricerca

Q-factorial Laurent rings

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

Published Version:

Bruzzo, U., Grassi, A. (2012). Q-factorial Laurent rings. JOURNAL OF PURE AND APPLIED ALGEBRA, 216(4), 894-896 [10.1016/j.jpaa.2011.10.016].

Availability:

This version is available at: https://hdl.handle.net/11585/683277 since: 2022-11-04

Published:

DOI: http://doi.org/10.1016/j.jpaa.2011.10.016

Terms of use:

Some rights reserved. The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (https://cris.unibo.it/). When citing, please refer to the published version.

(Article begins on next page)

This is the final peer-reviewed accepted manuscript of:

Bruzzo, U., and A. Grassi. "Q-Factorial Laurent Rings." *Journal of Pure and Applied Algebra*, vol. 216, no. 4, 2012, pp. 894-896.

The final published version is available online at: <u>https://dx.doi.org/10.1016/j.jpaa.2011.10.016</u>

Rights / License:

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (<u>https://cris.unibo.it/</u>)

When citing, please refer to the published version.

Q-FACTORIAL LAURENT RINGS

UGO BRUZZO^{§†‡} AND ANTONELLA GRASSI[¶]

§ Institut des Hautes Études Scientifiques, Le Bois-Marie, 91440 Bures-sur-Yvette, France

[†] Istituto Nazionale di Fisica Nucleare, Sezione di Trieste

I Department of Mathematics, University of Pennsylvania, David Rittenhouse Laboratory, 209 S 33rd Street, Philadelphia, PA 19104, USA

ABSTRACT. Dolgachev proves that the ring naturally associated to a generic Laurent polynomial in d variables, $d \ge 4$, is factorial [4, 5] (for any field k). We prove a sufficient condition for the ring associated to a very general complex Laurent polynomial in d = 3 variables to be Q-factorial.

1. INTRODUCTION

In [4] and Dolgachev [5] proves that the ring A_F naturally associated to generic Laurent polynomial F in d variables, $d \ge 4$, with coefficients in any field k, is factorial. The basic ingredient in Dolgachev's proof is Grothendieck's Lefschetz-type theorem ([6], Prop. 3.12) which, among other things, shows that under suitable conditions, the natural restriction map $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$, where X is a scheme and Y is subvariety corresponding to an ideal sheaf in \mathcal{O}_X , is an isomorphism. This result can be applied only when $d \ge 4$.

Date: November 7, 2018.

¹⁹⁹¹ Mathematics Subject Classification. 16S34, 14J70, 14M25.

E-mail: bruzzo@sissa.it, grassi@sas.upenn.edu.

[‡] On leave of absence from Scuola Internazionale Superiore di Studi Avanzati, Via Bonomea 265, 34136 Trieste, Italy

Support for this work was provided by the NSF Research Training Group Grant DMS-0636606, by PRIN "Geometria delle varietà algebriche e dei loro spazi dei moduli" and the INFN project PI14 "Nonperturbative dynamics of gauge theories". U.B. is a member of the VBAC group.

$\mathbb{Q}\text{-}\mathsf{FACTORIAL}$ LAURENT RINGS

In this paper we consider the case d = 3, assuming that $k = \mathbb{C}$, and prove a sufficient condition for the ring A_F to be Q-factorial (Theorem 3.1). The proof of this fact follows the lines of Dolgachev's proof, with Grothendieck's result replaced by a Noether-Lefschetz theorem for hypersurfaces in toric 3-folds (Theorem 2.5) that we proved in [2].

Acknowledgement. We thank Igor Dolgachev for interesting correspondence leading to this result and the referee for useful comments. The authors are grateful for the hospitality and support offered by the University of Pennsylvania, SISSA and IHES.

2. Preliminaries

We follow the notation in [1] and [2]. Let M be a d-dimensional lattice, $N = \text{Hom}(M, \mathbb{Z})$ and $\mathbf{T}_N = N \otimes \mathbb{C}^*$ the associated algebraic torus. Let $\Sigma \subset N_{\mathbb{R}}$ be a complete simplicial fan, and denote by X_{Σ} the corresponding complete toric variety. The torus \mathbf{T}_N naturally acts on X_{Σ} ; $\mathbf{T}_{\tau} \subset X_{\Sigma}$ denotes the orbit of a subset of X_{Σ} corresponding to a face τ of Σ under this action; the open dense orbit is denoted by \mathbf{T}_0 .

Definition 2.1. [1, Def. 4.13] A hypersurface X in X_{Σ} is nondegenerate if $X \cap \mathbf{T}_{\tau}$ is a smooth 1-codimensional subvariety of \mathbf{T}_{τ} for all faces τ in Σ .

 X_{Σ} has only abelian quotient singularities, and is therefore an orbifold.

Proposition 2.2. [1, Prop. 3.5, 4.15] Let L be a ample line bundle on X_{Σ} . The hypersurface $X \subset X_{\Sigma}$ given by the zero locus of a generic section of L is nondegenerate. Moreover, X is an orbifold.

Since X is an orbifold, its complex cohomology has a pure Hodge structure [9]. This is an essential point in the proof of our Theorem 2.5.

Definition 2.3 (The Cox Ring [3]). Consider a variable z_i for each 1-dimensional cone ς_i , i = 1, ..., n in Σ , and let $S(\Sigma)$ be the polynomial ring $\mathbb{C}[z_1, ..., z_n]$.

The Cox ring has a natural gradation given by its class group $Cl(\Sigma)$ of X_{Σ} .

Let L be an ample line bundle on X_{Σ} , and let $f \in H^0(X_{\Sigma}, L) \simeq S(\Sigma)_{\beta}$, where $\beta = \deg(L)$.

Definition 2.4. The Jacobian ring of f is the quotient $R(f) = S(\Sigma)/J(f)$, where J(f) is the ideal in $S(\Sigma)$ generated by the derivatives of f.

$\mathbb{Q}\text{-}\mathsf{FACTORIAL}$ LAURENT RINGS

The Jacobian ring R(f) inherits a natural gradation from $S(\Sigma)$.

The next theorem was proved in [2], and will be key to proving our result about Laurent rings. We assume d = 3. We recall that the Picard number is defined as the rank of the class group.

Theorem 2.5. [2] Let X_{Σ} a complete simplicial toric variety, and $X \subset X_{\Sigma}$ a very general hypersurface cut by a section f of an ample line bundle L such that the multiplication morphism

$$R(f)_{\beta} \otimes R(f)_{\beta-\beta_0} \to R(f)_{2\beta-\beta_0} \tag{1}$$

is surjective (here $\beta = \deg(L)$ and $\beta_0 = -\deg(K_{X_{\Sigma}})$, where $K_{X_{\Sigma}}$ is the canonical sheaf of X_{Σ}). Then X has the same Picard number as X_{Σ} .

Recall that a property is very general if it holds in the complement of countably many proper subvarieties.

If X is a quartic surface in \mathbb{P}^3 , or more generally a K3 surface defined by a section of the anticanonical divisor in a simplicial toric variety, then the above map is surjective [2]. It is a classical result that the map is not surjective if X is a cubic in \mathbb{P}^3 .

3. \mathbb{Q} -factorial Laurent Rings

The ring $\mathbb{C}[M]$ may be identified with the ring of regular functions on the torus $\mathbf{T}_N \simeq \mathbf{T}_0 \subset X_{\Sigma}$. An element $F \in \mathbb{C}[M]$ is called a *Laurent polynomial*; F may be regarded as a section of the ample line bundle L, and it defines a hypersurface X_F in X_{Σ} .

Let $\Delta \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ be the polytope uniquely determined by the fan Σ and L (see [8], Lemma 2.14). To each Laurent polynomial F on can associate a polytope Δ_F , called the *Newton polytope* of F. This is most easily described by choosing an isomorphism $M \simeq \mathbb{Z}^d$, writing

$$F = \sum_{i_1,\dots,i_d \in \mathbb{Z}^d} a_{i_1,\dots,i_d} t_1^{i_1} \cdots t_d^{i_d}$$

and defining

$$supp(F) = \{i_1, \dots, i_d \in \mathbb{Z}^d | a_{i_1, \dots, i_d} \neq 0\}$$

 Δ_F is then defined to be the convex hull of $\operatorname{supp}(F)$ and $\Gamma(\Delta)$ the set of all Laurent polynomials such that $\Delta_F \subset \Delta$. $\Gamma(\Delta)$ is a finite dimensional vector space over \mathbb{C} .

By results given in [7] (see also [8], Chapter 2) a Laurent polynomial F extends to a meromorphic function on X_{Σ} , which is a section of an ample line bundle L_F . Thus, F

may be regarded as an element in $S(\Sigma)_{\beta}$, where $\beta = \deg(L_F)$. Denote by A_F the ring $\mathbb{C}[M]/(F)$.

Theorem 3.1. Let d = 3, and let F be a very general Laurent polynomial in $\Gamma(\Delta)$; set $\beta = \deg(L_F)$ and $\beta_0 = -\deg(K_{X_{\Sigma}})$. If the multiplication morphism

$$R(F)_{\beta} \otimes R(F)_{\beta-\beta_0} \to R(F)_{2\beta-\beta_0} \tag{2}$$

is surjective, the ring A_F is \mathbb{Q} -factorial.

The proof that A_F is Q-factorial follows closely the proof of Theorem 1.1 in [4]. The basic idea is to formulate the problem in a geometric way:

Proof. Let $X_F \subset X_{\Sigma}$ be the hypersurface cut by F (as a section of L_F). By Proposition 2.2 the hypersurface X_F is nondegenerate, and is an orbifold.

Note that the ring A_F may be identified with the ring of regular functions on the affine part $U_F = X_F \cap \mathbf{T}_0$ of X_F . Since the Picard group of \mathbf{T}_0 is trivial, every Cartier divisor in X_{Σ} is linearly equivalent to a divisor supported in $X_{\Sigma} - \mathbf{T}_0$. By Theorem 2.5, X_F has the same Picard number as X_{Σ} , i.e., $\rho(X_F) = \rho(X_{\Sigma})$. Then any Cartier divisor in X_F is linearly equivalent modulo torsion to a divisor supported in $X_F - U_F$, so that $\operatorname{Pic}(U_F) \otimes \mathbb{Q} = 0$. Since U_F is normal (actually smooth), then $Cl(U_F) \otimes \mathbb{Q} = 0$. As $U_F \simeq \operatorname{Spec}(A_F)$, we have $Cl(A_F) \otimes \mathbb{Q} = 0$.

References

- V. V. BATYREV AND D. A. COX, On the Hodge structure of projective hypersurfaces in toric varieties, Duke Math. J., 75 (1994), pp. 293–338.
- [2] U. BRUZZO AND A. GRASSI, Picard group of hypersurfaces in toric varieties. arXiv:1011.1003 [math.AG].
- [3] D. A. Cox, The homogeneous coordinate ring of a toric variety, J. Algebraic Geom., 4 (1995), pp. 17– 50.
- [4] I. DOLGACHEV, Newton polyhedra and factorial rings, J. Pure Appl. Algebra, 18 (1980), pp. 253–258.
 Erratum, 21 (1981), pp. 9–10.
- [5] —, Erratum: Newton polyhedra and factorial rings, J. Pure Appl. Algebra, 21 (1981), pp. 9–10.
- [6] A. GROTHENDIECK, Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2), Exposé XI, Documents Mathématiques (Paris), 4, Société Mathématique de France, Paris, 2005, pp. x+208. Séminaire de Géométrie Algébrique du Bois-Marie, 1962.
- [7] A. G. KHOVANSKIĬ, Newton polyhedra and toroidal varieties, Funct. Anal. Appl., 11 (1977), pp. 289–296.

$\mathbb Q\text{-}\mathsf{FACTORIAL}$ LAURENT RINGS

- [8] T. ODA, Convex bodies and algebraic geometry. An introduction to the theory of toric varieties, vol. 15 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1988.
- [9] M. SAITO, Mixed Hodge modules, Publ. Res. Inst. Math. Sci., 26 (1990), pp. 221–333.