ON THE LIPSCHITZ REGULARITY FOR ALMOST MINIMIZERS OF A ONE-PHASE BERNOULLI-TYPE FUNCTIONAL FOR THE p-LAPLACIAN

Serena Dipierro[♭], Fausto Ferrari[†], Nicolò Forcillo[‡] and Enrico Valdinoci[♭]

[♭]*Department of Mathematics and Statistics, University of Western Australia, 35 Stirling Hwy, Crawley WA 6009 Australia, enrico.valdinoci@uwa.edu.au, serena.dipierro@uwa.edu.au, https://www.uwa.edu.au*
[†]*Dipartimento di Matematica, Università di Bologna, Piazza di Porta S.Donato 5, 40126, Bologna, Italia, fausto.ferrari@unibo.it, https://www.unibo.it*
[‡]*Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133, Roma, Italia, nikforc6@gmail.com, http://www.uniroma2.it*

Abstract: We discuss the optimal Lipschitz continuity for almost minimizers of the one-phase Bernoulli-type energy functional

$$J_p(u,\Omega) := \int_\Omega \left(|\nabla u|^p + \chi_{\{u>0\}}\right) dx, \quad p > 1,$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $u \geq 0$.

Keywords: *almost minimizers, free boundary problems, p-Laplacian, Lipschitz regularity*
2000 AMS Subject Classification: 35R35 - 35J70

1 INTRODUCTION

In this paper, we deal with almost minimizers of the one-phase Bernoulli-type energy functional

$$J_p(u,\Omega) := \int_\Omega \left(|\nabla u|^p + \chi_{\{u>0\}}\right) dx, \quad p > 1, \tag{1}$$

with Ω a bounded domain in $\mathbb{R}^n$ and $u \geq 0$. In particular, we discuss their optimal Lipschitz regularity. This result has been recently proved by the authors in [18] under the condition $p > \max\left\{\frac{2n}{n+2}, 1\right\}$. We recall that the Lipschitz continuity in the case of $p = 2$ was already established in [15].

The case $p = 2$ has been extensively studied in the literature, before in the context of minimizers and then in the framework of almost minimizers. Regarding minimizers, they were first studied in [1], where the Lipschitz regularity of minimizers and the regularity of "flat" free boundaries were established. Later, Caffarelli, in [3, 4, 5], developed a viscosity approach to the free boundary problem associated with J_2 (see (1) above) in the two-phase setting. His work led the way to a vast literature on this problem, both in the one-phase and two-phase forms. We refer to [13] for a comprehensive survey about such literature.

Almost minimizers, when $p = 2$, were more recently attached. Specifically, in [11] and [8], David and Toro, and David, Engelstein, and Toro extended, at least in the one-phase case, using variational techniques, the pioneer results in [1] in the context of almost minimizers. The Lipschitz continuity of almost minimizers was actually proved, in [11], in the more general case of a two-phase energy functional, together with the uniform rectifiability of the free boundary, see [8]. The results in [11, 8] in the one-phase setting were also obtained by De Silva and Savin in [15], exploiting viscosity techniques. More precisely, following the ideas contained in [17], they first show that almost minimizers are "viscosity solutions" to the associated free boundary problem in a more general sense. This property roughly means that almost minimizers satisfy comparison in a neighborhood of a touching point whose size depends on the properties of the test functions. In other words, they do not necessarily solve an infinitesimal equation, but in the macro scale, there are some effective ellipticity and a mean value principle. Once the authors have proved this property, they employ the techniques developed by De Silva in [12] to study the regularity of the free boundary. We mention [9, 10, 16, 14] as further contributions on almost minimizers.

Concerning the general case $p > 1$, minimizers of J_p, as in (1), were investigated in [6], in which the regularity of the free boundary near flat points was achieved. The Lipschitz continuity for minimizers was more recently established in [19]. There, the authors also presented a proof of the Lipschitz regularity when $p = 2$, without using monotonicity formulas. We cite [20] as well, where a discrete version of the Weiss monotonicity formula for J_p was proved for p close to 2. The viscosity approach for one-phase problems governed by the p-Laplace or $p(x)$-Laplace operator was developed in [26] and [22, 21] respectively. More specifically, in [21], the authors showed the Lipschitz regularity of viscosity solutions of nonhomogeneous one-phase problems and some regularity properties of their free boundaries. We also quote [7, 24, 30, 28, 29, 31, 2, 27, 25] for regularity results in free boundary problems for the p-Laplacian.

Before the recent paper [18], see also the Ph.D. thesis of the third author [23], almost minimizers for $p \neq 2$, to the best of our knowledge, had not been fully investigated. Precisely, in [18], we have proved the optimal Lipschitz continuity of almost minimizers to J_p, with $p > \max\{\frac{2n}{n+2}, 1\}$. Our approach has been inspired by [15]. We hope that our work can encourage further investigations in this direction. For instance, it would be interesting to understand better whether or not the condition $p > \max\{\frac{2n}{n+2}, 1\}$ is essential to have a Lipschitz regularity result.

In this paper, we present our Lipschitz regularity result in [18] in Section 2. In the next and final section, we focus on the strategy of the proof and its main differences with respect to [15].

2 Main result

In this section, we state our main result concerning the optimal Lipschitz regularity for almost minimizers to J_p. We start by recalling the definition of almost minimizers, see [18].

Definition 1 *Let $\kappa \geq 0$ and $\beta > 0$. We say that $u \in W^{1,p}(\Omega)$ is an almost minimizer for J_p in Ω, with constant κ and exponent β, if $u \geq 0$ a.e. in Ω and*

$$J_p(u, B_\varrho(x)) \leq (1 + \kappa\varrho^\beta) J_p(v, B_\varrho(x)), \tag{2}$$

for every ball $B_\varrho(x)$ such that $\overline{B_\varrho(x)} \subset \Omega$ and for every $v \in W^{1,p}(B_\varrho(x))$ such that $v = u$ on $\partial B_\varrho(x)$ in the sense of the trace.

We are now ready to present our main theorem. The precise statement reads as follows, see [18].

Theorem 1 *Let $p > \max\left\{\frac{2n}{n+2}, 1\right\}$ and u be an almost minimizer for J_p in B_1 with constant κ and exponent β.*

Then,

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C\Big(\|\nabla u\|_{L^p(B_1)} + 1\Big),$$

where $C > 0$ is a constant depending on n, p, κ and β.

In addition, u is uniformly Lipschitz continuous in a neighborhood of $\{u = 0\}$, namely if $u(0) = 0$ then

$$|\nabla u| \leq C \quad \text{in } B_{r_0},$$

for some $C > 0$, depending only on n, p, κ and β, and $r_0 \in (0, 1)$, depending on n, p, κ, β and $\|\nabla u\|_{L^p(B_1)}$.

3 Strategy of the proof

In this section, we expose the strategy of the proof of Theorem 1, pointing out its main differences with respect to the proof of the correspondent result in [15].

A dichotomy-type result is the key step to obtaining the Lipschitz continuity. The dichotomy property says that either average of the energy of an almost minimizer to J_p decreases in a smaller ball or the distance between its gradient and a suitable constant vector becomes as small as we wish. In other words, either the average of the energy of an almost minimizer decreases in a smaller ball, or the almost minimizer is arbitrarily close to a linear function. Once the dichotomy is established, the next step is to show that one of two alternatives can be improved. More precisely, if an almost minimizer is close to a linear function, in a ball, in the sense of the average of the energy, it is closer to a possibly different linear function in a smaller ball. The iteration of this improvement, together with the dichotomy property, finally leads to the Lipschitz regularity of almost minimizers to J_p, when $p > \max\left\{\frac{2n}{n+2}, 1\right\}$.

The main difficulty in proving the Lipschitz continuity is the following one: if v_1 and v_2 are the p-harmonic replacements of two functions u_1 and u_2, then it is not true that $v_1 + v_2$ is the p-harmonic replacement of $u_1 + u_2$. This situation generates significant technical complications since it is intrinsically connected with the nonlinear nature of the p-Laplacian. We end up with this issue because, in [15], the authors often use harmonic replacements as competitors. The fact that this property holds, instead, for harmonic replacements allows, in [15], to achieve estimates for the average of their energy, exploiting Schauder estimates. In our case, we need to show that the sum of p-harmonic replacements taken into account solves an equation for which $C^{1,\alpha}$ estimates hold. This is true using a linearization technique, together with the assumptions of the dichotomy. We refer to [18] for the details. The difficulty we have just described appears in the proof of the improvement of dichotomy's second alternative.

Another difference with respect to [15] concerns the lower bound on p, $p > \max\left\{\frac{2n}{n+2}, 1\right\}$, for which our result holds. For this purpose, we do not prove that the result is not true for $1 < p \leq \max\left\{\frac{2n}{n+2}, 1\right\}$. We show, instead, that, with the strategy explained above, we obtain the Lipschitz continuity for $p > \max\left\{\frac{2n}{n+2}, 1\right\}$. Let us focus now on the origin of this threshold. The lower bound $p > \max\left\{\frac{2n}{n+2}, 1\right\}$ comes again from the proof of the improvement of dichotomy's second alternative. In particular, it is a consequence of an estimate of the measure of an almost minimizer zero level set, which is a nonlinear term. Technically, for $p \geq 2$ we achieve

$$|\{u = 0\} \cap B_{1/2}| \leq C_1 \varepsilon^{p+\delta},$$

with C_1 and δ universal. This inequality turns out to be crucial to get the improvement of the second alternative. The key point is that $p + \delta > p$ from the universality of δ. In the case $1 < p < 2$, the term $|\{u = 0\} \cap B_{1/2}|$ appears, instead, with exponent $p/2$. Thus, arguing as in the case $p > 2$, we have

$$|\{u = 0\} \cap B_{1/2}|^{p/2} \leq C_1 \varepsilon^{p(p+\delta)/2},$$

with $\delta = \frac{p^2}{n-p}$ since we exploit the Sobolev-Poincaré inequality. Requiring then that $p(p + \delta)/2 > p$, as in the case $p > 2$, we achieve $p > \frac{2n}{n+2}$, which yields the lower bound on p.

ACKNOWLEDGMENTS

S.D. and E.V. are members of AustMS. F.F. and N.F. are members of INdAM. F.F. and N.F. are partially supported by 2022- INDAM- GNAMPA project *Regolarità locale e globale per problemi completamente nonlineari.* E.V. is partially supported by the Australian Laureate Fellowship FL190100081 "Minimal surfaces, free boundaries and partial differential equations". The authors wish to thank Daniela De Silva and Ovidiu Savin for several valuable discussions.

REFERENCES

[1] H. W. ALT AND L. A. CAFFARELLI, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew, Math. 325 (1981), 105-144.

[2] J. E. M. BRAGA AND D. R. MOREIRA, *Existence and regularity for a minimum problem with free boundary*, Ann. Inst. H. Poincaré C Anal. Non Linéaire 31 (2014), no. 4, 823–850.

[3] L. A. CAFFARELLI, *A Harnack inequality approach to the regularity of free boundaries. I. Lipschitz free boundaries are $C^{1,\alpha}$*, Rev. Mat. Iberoamericana 3 (1987), no. 2, 139-162.

[4] L. A. CAFFARELLI, *A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz*, Comm. Pure Appl. Math. 42 (1989), no. 1, 55-78.

[5] L. A. CAFFARELLI, *A Harnack inequality approach to the regularity of free boundaries. III. Existence theory, compactness, and dependence on X*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 15 (1988), no. 4, 583-602.

[6] D. DANIELLI AND A. PETROSYAN, *A minimum problem with free boundary for a degenerate quasilinear operator*, Calc. Var. Partial Differential Equations 23 (2005), no. 1, 97-124.

[7] D. Danielli and A. Petrosyan, *Full regularity of the free boundary in a Bernoulli-type problem in two dimensions*, Math. Res. Lett. 13 (2006), no. 4, 667-681.

[8] G. David, M. Engelstein, and T. Toro, *Free boundary regularity for almost-minimizers*, Adv. Math. 350 (2019), 1109-1192.

[9] G. David, M. Engelstein, M. Smit Vega Garcia, and T. Toro, *Regularity for almost-minimizers of variable coefficient Bernoulli-type functionals*, Math. Z. 299 (2021), no. 3-4, 2131-2169.

[10] G. David, M. Engelstein, M. Smit Vega Garcia, and T. Toro, *Branch points for (almost-)minimizers of two-phase free boundary problems*, Forum Math. Sigma 11 (2023), Paper No. e1.

[11] G. David and T. Toro, *Regularity of almost minimizers with free boundary*, Calc. Var. Partial Differential Equations 54 (2015), no. 1, 455-524.

[12] D. De Silva, *Free boundary regularity for a problem with right hand side*, Interfaces Free Bound. 13 (2011), no. 2, 223-238.

[13] D. De Silva, F. Ferrari, and S. Salsa, *Recent progresses on elliptic two-phase free boundary problems*, Discrete Contin. Dyn. Syst. 39 (2019), no. 12, 6961-6978.

[14] D. De Silva, S. Jeon, and H. Shahgholian, *Almost minimizers for a singular system with free boundary*, J. Differential Equations 336 (2022), 167-203.

[15] D. De Silva and O. Savin, *Almost minimizers of the one-phase free boundary problem*, Comm. Partial Differential Equations 45 (2020), no. 8, 913-930.

[16] D. De Silva and O. Savin, *Thin one-phase almost minimizers*, Nonlinear Anal. 193 (2020), 111507, 23 pp.

[17] D. De Silva and O. Savin, *Quasi-Harnack inequality*, Amer. J. Math. 143 (2021), no. 1, 307-331.

[18] S. Dipierro, F. Ferrari, N. Forcillo, and E. Valdinoci, *Lipschitz regularity of almost minimizers in one-phase problems driven by the p-Laplace operator*, To appear in Indiana University Mathematics Journal, arXiv: 2206.03238.

[19] S. Dipierro and A. L. Karakhanyan, *Stratification of free boundary points for a two-phase variational problem*, Adv. Math. 328 (2018), 40-81.

[20] S. Dipierro and A. L. Karakhanyan, *A new discrete monotonicity formula with application to a two-phase free boundary problem in dimension two*, Comm. Partial Differential Equations 43 (2018), no. 7, 1073-1101.

[21] F. Ferrari and C. Lederman, *Regularity of Lipschitz free boundaries for a p(x)-Laplacian problem with right hand side*, Nonlinear Anal. 212 (2021), Paper No. 112444, 25 pp.

[22] F. Ferrari and C. Lederman, *Regularity of flat free boundaries for a $p(x)$-Laplacian problem with right hand side*, J. Math. Pures Appl. (9) 171 (2023), 26-74.

[23] N. Forcillo, *Regularity in degenerate elliptic and parabolic free boundary problems*, [Dissertation thesis], Alma Mater Studiorum Università di Bologna. Dottorato di ricerca in Matematica, 34 Ciclo. Posted in 2021. DOI 10.48676/unibo/amsdottorato/10001.

[24] A. L. Karakhanyan, *On the Lipschitz regularity of solutions of a minimum problem with free boundary*, Interfaces Free Bound. 10 (2008), no. 1, 79-86.

[25] A. L. Karakhanyan, *Full and partial regularity for a class of nonlinear free boundary problems*, Ann. Inst. H. Poincaré C Anal. Non Linéaire 38 (2021), no. 4, 981-999.

[26] R. Leit$\tilde{A}$o and G. Ricarte, *Free boundary regularity for a degenerate problem with right hand side*, Interfaces Free Bound. 20 (2018), no. 4, 577-595.

[27] R. Leit$\tilde{A}$o, O. S. de Queiroz, and E.V. Teixeira, *Regularity for degenerate two-phase free boundary problems*, Ann. Inst. H. Poincaré C Anal. Non Linéaire 32 (2015), no. 4, 741-762.

[28] J. L. Lewis and K. Nyström, *Regularity of Lipschitz free boundaries in two-phase problems for the p-Laplace operator*, Adv. Math. 225 (2010), no. 5, 2565-2597.

[29] J. L. Lewis and K. Nyström, *Regularity of flat free boundaries in two-phase problems for the p-Laplace operator*, Ann. Inst. H. Poincaré C Anal. Non Linéaire 29 (2012), no. 1, 83-108.

[30] S. Martínez and N. Wolanski, *A minimum problem with free boundary in Orlicz spaces*, Adv. Math. 218 (2008), no. 6, 1914-1971.

[31] D. R. Moreira and L. Wang, *Singular perturbation method for inhomogeneous nonlinear free boundary problems*, Calc. Var. Partial Differential Equations 49 (2014), no. 3-4, 1237-1261.