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Absolute continuity and Fokker-Planck equation for the law of Wong-Zakai approximations of Itô's stochastic differential equations

Alberto Lanconelli*

Abstract

We investigate the regularity of the law of Wong-Zakai-type approximations for Itô stochastic differential equations. These approximations solve random differential equations where the diffusion coefficient is Wick-multiplied by the smoothed white noise. Using a criteria based on the Malliavin calculus we establish absolute continuity and a Fokker-Planck-type equation solved in the distributional sense by the density. The parabolic smoothing effect typical of the solutions of Itô equations is lacking in this approximated framework; therefore, in order to prove absolute continuity, the initial condition of the random differential equation needs to possess a density itself.

Key words and phrases: stochastic differential equations, Wong-Zakai approximation, Malliavin calculus

AMS 2000 classification: 60H10; 60H07; 60H30

1 Introduction and statement of the main result

The celebrated Wong-Zakai theorem [23],[24], extended to the multidimensional case by Stroock and Varadhan [21], provides a crucial insight in the theory of stochastic differential equations. It asserts that, given a suitable smooth approximation $\{B_t^\varepsilon\}_{t\in[0,T]}$ of the Brownian motion $\{B_t\}_{t\in[0,T]}$, the solution $\{X_t^\varepsilon\}_{t\in[0,T]}$ of the random ordinary differential equation

$$\dot{X}_t^\varepsilon = b(t, X_t^\varepsilon) + \sigma(t, X_t^\varepsilon) \cdot \dot{B}_t^\varepsilon, \tag{1.1}$$

converges, as ε goes to zero, to the solution of the Stratonovich stochastic differential equation (SDE, for short)

$$dX_t = b(t, X_t)dt + \sigma(t, X_t) \circ dB_t$$

instead to the more popular Itô's interpretation of the corresponding stochastic equation,

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t.$$

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The understanding behind this phenomenon is not the way the Brownian motion $\{B_t\}_{t\in[0,T]}$ is approximated but rather the way we multiply the diffusion coefficient $\sigma(t, X_t^\varepsilon)$ with the smoothed white noise $\dot{B}_t^\varepsilon$ in (1.1). The following example clarifies this point.

Example 1.1 *Consider the random ordinary differential equation*

$$\dot{X}_t^\varepsilon = X_t^\varepsilon \cdot \dot{B}_t^\varepsilon, \quad X_0^\varepsilon = x$$

which corresponds to (1.1) for $b(t,x) = 0$ and $\sigma(t,x) = x$. Its solution

$$X_t^\varepsilon = x \exp\{B_t^\varepsilon\}, \quad t \in [0,T]$$

converges to

$$X_t := x \exp\{B_t\}, \quad t \in [0,T] \tag{1.2}$$

whenever for each $t \in [0,T]$ the random variable B_t^ε converges to B_t in probability as ε tends to zero. A direct verification shows that (1.2) is the unique solution of

$$dX_t = X_t/2dt + X_t dB_t, \quad X_0 = x$$

which is equivalent to

$$dX_t = X_t \circ dB_t, \quad X_0 = x.$$

In fact, it is well known (see for instance Karatzas and Shreve [14]) that the Stratonovich SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t) \circ dB_t, \quad X_0 = x$$

is equivalent to the Itô SDE

$$dX_t = \Big[b(t, X_t) + \sigma(t, X_t)\partial_x\sigma(t, X_t)/2\Big]dt + \sigma(t, X_t)dB_t, \quad X_0 = x.$$

Therefore, in order to recover the Itô interpretation of the limiting SDE one may start with the modified equation

$$\dot{Y}_t^\varepsilon = \Big[b(t, Y_t^\varepsilon) - \sigma(t, Y_t^\varepsilon)\partial_y\sigma(t, Y_t^\varepsilon)/2\Big] + \sigma(t, Y_t^\varepsilon) \cdot \dot{B}_t^\varepsilon, \quad Y_0^\varepsilon = x$$

to obtain in the limit

$$dY_t = \Big[b(t, Y_t) - \sigma(t, Y_t)\partial_x\sigma(t, Y_t)/2\Big]dt + \sigma(t, Y_t) \circ dB_t, \quad Y_0 = x$$

which corresponds to

$$dY_t = b(t, Y_t)dt + \sigma(t, Y_t)dB_t, \quad Y_0 = x.$$

However, this procedure has some drawbacks. For instance, certain probabilistic properties of the exact solution may be lost in the approximated solution.

Example 1.2 *Suppose we wish to approximate*

$$dY_t = Y_t dB_t, \quad Y_0 = x \tag{1.3}$$

according to the previous procedure; then, we should consider the random ordinary differential equation

$$\dot{Y}_t^\varepsilon = -Y_t^\varepsilon/2 + Y_t^\varepsilon \cdot \dot{B}_t^\varepsilon, \quad Y_0^\varepsilon = x$$

whose solution is

$$Y_t^\varepsilon = x \exp\left\{B_t^\varepsilon - t/2\right\}, \quad t \in [0,T].$$

However, on one hand we have

$$E[Y_t] = x \quad \textit{for all } t \in [0,T]$$

while on the other

$$E[Y_t^\varepsilon] \neq x \quad \textit{unless} \quad E[(B_t^\varepsilon)^2] = t \textit{ for all } \varepsilon > 0. \tag{1.4}$$

The problem of finding a version of equation (1.1) having in the limit the Itô interpretation of the SDE (by-passing the Stratonovich interpretation involved in the procedure described above) was partially solved by Hu and Øksendal [12]: they proved that the solution of

$$\dot{Y}_t^\varepsilon = b(t, Y_t^\varepsilon) + \sigma(t) Y_t^\varepsilon \diamond \dot{B}_t^\varepsilon, \tag{1.5}$$

where $\diamond$ stands for the Wick product, converges as ε goes to zero to the solution of the Itô SDE

$$dY_t = b(t, Y_t)dt + \sigma(t) Y_t dB_t. \tag{1.6}$$

Here σ is a deterministic function; the assumption of a linear diffusion coefficient is utilized in connection with a reduction method to solve equation (1.5).

Example 1.3 *Referring to Example 1.2, we now utilize (1.5) to approximate equation (1.3), i.e.*

$$\dot{Y}_t^\varepsilon = Y_t^\varepsilon \diamond \dot{B}_t^\varepsilon, \quad Y_0^\varepsilon = x.$$

The solution to the previous equation can be computed using the Wick calculus as

$$Y_t^\varepsilon = x \exp^\diamond\{B_t^\varepsilon\} = x \exp\left\{B_t^\varepsilon - E[|B_t^\varepsilon|^2]/2\right\}.$$

Therefore,

$$\lim_{\varepsilon \to 0} Y_t^\varepsilon = x \exp\left\{B_t - t/2\right\} \quad \textit{as long as} \quad B_t^\varepsilon \to B_t \textit{ in } L^2(\Omega)$$

which is the solution of equation (1.3). Moreover, in this case

$$E[Y_t^\varepsilon] = E[Y_t] = x \quad \textit{for any } t \in [0,T]$$

independently of the particular approximation of $\{B_t\}_{t\in[0,T]}$ *utilized (in contrast with (1.4)).*

Roughly speaking, the reason for choosing equation (1.5) as the correct approximated version of the Itô SDE (1.6) can be traced back to the identity

$$\int_0^T \gamma_t dB_t = \int_0^T \gamma_t \diamond \dot{B}_t dt \tag{1.7}$$

which connects the Itô-Skorohod integral of a stochastic process $\{\gamma_t\}_{t\in[0,T]}$ (on the left hand side) with a *standard* integral of the Wick product between γ_t and the white noise $\dot{B}_t$. We refer the reader to the book Holden et al. [8] for a proof of this identity (see also the book by Kuo [16] for a discussion in a white noise framework).

The aim of the present paper is to investigate the probabilistic properties of the Wong-Zakai-type approximation (1.5), as suggested in [12], of the Itô SDE (1.6). In particular, we will focus on the absolute continuity of the law of the solution Y_t^ε finding sufficient conditions for the existence of a density with respect to the one dimensional Lebesgue measure. We will also write a Fokker-Planck-type equation solved in the distributional sense by the density of Y_t^ε. This analogy with exact solutions of Itô SDEs is however conditioned by a restriction on the initial data of equation (1.5) that is required to possess a density. Our approach relies on the criteria for absolute continuity based on the Malliavin calculus. We stress that Wong-Zakai approximations, contrary to many other approximation schemes like for instance the Euler discretization (see e.g. Bally and Talay [1] for an investigation on the convergence of the densities of such approximations), are defined in terms of non adapted stochastic processes.
There is a vast literature on Wong-Zakai approximations for Stratonovich stochastic (partial) differential equations driven by different types of noise: one may look at Brezniak and Flandoli [4], Gyöngy and A. Shmatkov [6], Hu et al. [10], Hu and Nualart [11], Konecny [15], Tessitore and Zabczyk [22] just to mention a few. We also mention the remarkable paper Hairer and Pardoux [7] where a Wong-Zakai theorem for a general nonlinear Itô-type stochastic heat equation driven by a space-time white noise is proved. Wong-Zakai approximations for Itô SDEs are quite rare in the literature. As the insight of [12] shows, one has to deal in this case with equations involving the Wick product, which corresponds to treat Skorohod SDEs (see Section 2). This type of equations possesses a global solution only in some particular cases making even the existence of the Wong-Zakai approximation a tough issue. We mention the paper Da Pelo et al. [5] dealing with Wong-Zakai approximations for Itô-Stratonovich interpolations and Ben Ammou and Lanconelli [2] investigating the rate of convergence for Wong-Zakai approximations of Itô SDEs in the spirit of the present paper.

To state our main result we need to introduce a few notation. Let $(\Omega, \mathcal{F}, P)$ be the classical Wiener space over the time interval $[0,T]$, where T is an arbitrary positive constant, and denote by $\{B_t\}_{t\in[0,T]}$ the coordinate process, i.e.

$$\begin{aligned} B_t : \Omega &\to \mathbb{R} \\ \omega &\mapsto B_t(\omega) = \omega(t). \end{aligned}$$

By construction, the process $\{B_t\}_{t\in[0,T]}$ is under the measure P a one dimensional Brownian motion. Now, let π be a finite partition of the interval $[0,T]$, that means $\pi = \{t_0, t_1, ..., t_{n-1}, t_n\}$ with

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$$

and consider the *polygonal* approximation of the Brownian motion $\{B_t\}_{t\in[0,T]}$ relative to the partition π:

$$B_t^\pi := \left(1 - \frac{t-t_k}{t_{k+1}-t_k}\right) B_{t_k} + \frac{t-t_k}{t_{k+1}-t_k} B_{t_{k+1}} \quad \text{if } t \in [t_k, t_{k+1}[\tag{1.8}$$

and $B_T^\pi := B_T$. It is well known that for any $\varepsilon > 0$ and $p \geq 1$ there exists a positive constant $C_{p,T,\varepsilon}$ such that

$$\left(E\left[\sup_{t\in[0,T]} |B_t^\pi - B_t|^p \right] \right)^{1/p} \leq C_{p,T,\varepsilon} |\pi|^{1/2-\varepsilon}$$

where $|\pi| := \max_{k\in\{0,\dots,n-1\}}(t_{k+1} - t_k)$ stands for the mesh of the partition π. We refer the reader to Lemma 2.1 in Hu et al. [10] and Lemma 11.8 in Hu [9] for sharper estimates. We assume that the finite partition π is fixed throughout the present paper.
Given the partition π, let h^π be a function in $L^2([0,T])$ such that $h^\pi \neq 0$ almost everywhere and

$$\frac{1}{t_{k+1}-t_k} \int_{t_k}^{t_{k+1}} h^\pi(u)du = 0 \quad \text{for all } k \in \{0,\dots,n-1\}.$$

The crucial role of the function h^π will be made clear in Proposition 2.4 below. The following is our main result: the symbols $\diamond$, $\mathbb{D}^{2,p}$ and D_{h^π} denote the Wick product, the Sobolev-Malliavin space and the directional Malliavin derivative in the direction h^π, respectively. Definitions and useful properties are postponed to Section 2.

Theorem 1.4 *Let $\{X_t^\pi\}_{t\in[s,T]}$ be the unique solution of the random Cauchy problem*

$$\begin{cases} \dot{X}_t^\pi = b(t, X_t^\pi) + \sigma(t) X_t^\pi \diamond \dot{B}_t^\pi, \quad t \in]s,T] \\ X_s^\pi = Y \end{cases} \tag{1.9}$$

where $s \in [0,T[$ and we assume that

- *$b : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with bounded first and second partial derivatives with respect to the second variable;*
- *$\sigma : [0,T] \to \mathbb{R}$ belongs to $L^2([0,T])$;*
- *$Y \in \mathbb{D}^{2,p}$ for all $p \geq 1$ and $E[|D_{h^\pi} Y|^{-q}]$ is finite for some $q > 4$.*

Then,

1. *for any $t \in [s,T]$ the law of X_t^π is absolutely continuous with respect to the one dimensional Lebesgue measure, with a bounded and continuous density;*
2. *the density $(t,x) \mapsto p^\pi(t,x)$ of the random variable X_t^π solves in the sense of distributions the Fokker-Planck equation $(\partial_t + \mathcal{L}_x)^* u(t,x) = 0$, where $(\partial_t + \mathcal{L}_x)^*$ stands for the formal adjoint of the operator $\partial_t + \mathcal{L}_x$ and*

$$\mathcal{L}_x := b(t,x)\partial_x + \sigma(t) x g(t,x) \partial_{xx}, \tag{1.10}$$

for a suitable measurable function $g : [s,T] \times \mathbb{R} \to \mathbb{R}$. Moreover,

$$\int_{\mathbb{R}} |g(t,x)|^q p^\pi(t,x)dx \text{ is finite for all } q \geq 1 \text{ and } t \in [s,T].$$

Remark 1.5 *We observe that the assumptions on Y entail the absolute continuity of its law with respect to the one dimensional Lebesgue measure. In fact, contrary to the assumptions usually adopted for the study of the absolute continuity for exact solutions of SDEs, here we need the initial condition to have a density. The stochastic equation (1.9), being an approximated version of an Itô SDE, does not possess the same smoothing properties of the original equation.*

Remark 1.6 *The way we prove the existence of the function g appearing in the operator $\mathcal{L}_x$ does not give us information about its regularity and sign (see formula (2.16) below). Therefore, we don't know whether the existence of a density for the law of X_t^π can be deduced from the properties of the operator $\mathcal{L}_x$ in (1.10). From this point of view, the criterion for absolute continuity based on the Malliavin calculus as utilized in this paper turns out to be crucial.*

The paper is organized as follows: in Section 2 we begin recalling definitions and auxiliary results from the Malliavin calculus and analysis on the Wiener space which will be employed in the proof of Theorem 1.4; Section 2.1 deals with the proof of the absolute continuity of the law of the Wong-Zakai approximation while Section 2.2 addresses the derivation of the Fokker-Planck-type equation solved by the density. Here, an anticipating Itô formula for the Wong-Zakai approximation (see Theorem 2.5 below) plays a major role. An example illustrating the results previously obtained closes the section.

2 Proof of Theorem 1.4

The proof of our main theorem will be divided in two parts: the existence of the density in Section 2.1 and the Fokker-Planck equation in Section 2.2. We first set the notations and recall few auxiliary results from the Malliavin Calculus. For more details we refer the reader to one of the books Bogachev [3], Hu [9], Janson [13] and Nualart [20]. Here we adopt the presentation of [20].

Let $\mathcal{S}$ denote the class of *smooth* random variables of the form

$$F = \varphi\left(\int_0^T h_1(u)dB_u, ..., \int_0^T h_n(u)dB_u\right), \tag{2.1}$$

where $\varphi \in C_{\mathcal{P}}^{\infty}(\mathbb{R}^n)$ (the space of infinitely differentiable functions having, together with all their partial derivatives, polynomial growth), the functions h_1, ...,h_n are elements of $L^2([0,T])$ and $n \geq 1$. In the sequel to ease the notation we will set $H := L^2([0,T])$ and denote by $\|\cdot\|_p$ the norm in $L^p(\Omega)$. We note that $\mathcal{S}$ is dense in $L^p(\Omega)$ for all $p \geq 1$.
The *derivative* of a smooth random variable F of the form (2.1) is the H-valued random variable DF given by

$$D_tF = \sum_{i=1}^n \partial_i\varphi\left(\int_0^T h_1(u)dB_u, ..., \int_0^T h_n(u)dB_u\right)h_i(t), \quad t \in [0,T].$$

It is easy to see that for any $h \in H$ and smooth random variables F and G we have the following integration-by-parts formula

$$E[GD_hF] = E\left[-FD_hG + FG\int_0^T h(t)dB_t\right].$$

Here $E[\cdot]$ denotes the expectation on the probability space $(\Omega, \mathcal{F}, P)$ while $D_h Z := \int_0^T D_u Z \cdot h(u)du$ stands for the *directional derivative* of Z in the direction $h \in H$.
By means of the previous identity one can prove that the operator D is closable from $L^p(\Omega)$ to $L^p(\Omega; H)$ for any $p \geq 1$. Therefore, one can define the space $\mathbb{D}^{1,p}$ as the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = \left(E\left[|F|^p\right] + E\left[|DF|_H^p\right]\right)^{\frac{1}{p}}.$$

Iterating the action of the operator D in such a way that for a smooth random variable F the iterated derivative $D^k F$ is a random variable with values in $H^{\otimes k}$, we introduce on $\mathcal{S}$, for every $p \geq 1$ and any natural number k, the seminorm defined by

$$\|F\|_{k,p} = \left(E\left[|F|^p\right] + \sum_{j=1}^{k} E\left[|D^j F|_{H^{\otimes j}}^p\right]\right)^{\frac{1}{p}}.$$

We will denote by $\mathbb{D}^{k,p}$ the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{k,p}$.

We now introduce the *divergence operator* δ which is the adjoint of the operator D. It is a closed and unbounded operator on $L^2(\Omega; H)$ with values in $L^2(\Omega)$ such that:

- the domain of δ is the set of H-valued square integrable random variables $u \in L^2(\Omega; H)$ such that

  $$|E[\langle DF, u\rangle_H]| \leq c\|F\|_2,$$

  for all $F \in \mathbb{D}^{1,2}$, where c is a constant depending only on u

- if u belongs to the domain of δ, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by

  $$E[F\delta(u)] = E[\langle DF, u\rangle_H]$$

  for any $F \in \mathbb{D}^{1,2}$.

One can prove that $\mathbb{D}^{1,2}(H)$ is included in the domain of δ. In particular, if $F \in \mathbb{D}^{1,2}$ and $h \in H$, then $Fh \in \mathbb{D}^{1,2}(H)$ and

$$F \diamond \int_0^T h(u)dB_u := \delta(Fh) \in L^2(\Omega) \tag{2.2}$$

is called the *Wick product* of F and $\int_0^T h(u)dB_u$. This definition can be generalized to include more general random variables in the place of $\int_0^T h(u)dB_u$. Moreover, it follows from the properties of D and δ that

$$F \diamond \int_0^T h(u)dB_u = F \cdot \int_0^T h(u)dB_u - D_h F.$$

Remark 2.1 *There are different ways to define the Wick product of two random variables: via the Wiener-Itô chaos expansion or specifying its action on stochastic exponentials (see below) or through the so-called S-transform. Here, we preferred to use (2.2) since it clearly shows the duality between Wick product and Malliavin derivative. Other approaches can be found in [8], [9], [16]. We also remark that, in addition to its key role in the theory of Itô-Skorohod integration (1.7), the Wick product has important probabilistic interpretations in the Gaussian and Poissonian analysis. See [18], [19], [17] and the references quoted there.*

For $h \in H$ we define the *stochastic exponential* to be a random variable of the form

$$\mathcal{E}(h) := \exp\left\{\int_0^T h(u)dB_u - \frac{1}{2}\int_0^T h^2(u)du\right\}.$$

We remark that the span of such family of elements is dense in $L^p(\Omega)$ and $\mathbb{D}^{k,p}$ for any $p \geq 1$ and $k \in \mathbb{N}$. For $g \in H$ we also define the *translation operator* $\mathcal{T}_g$ as the operator that shifts the Brownian path by the function $\int_0^{\cdot} g(u)du$; more precisely, the action of $\mathcal{T}_g$ on a stochastic exponential is given by

$$\mathcal{T}_g\mathcal{E}(h) := \mathcal{E}(h) \cdot \exp\{\langle h, g\rangle_H\}.$$

There is a close relationship between Wick product and translation operators; it is the so called Gjessing formula:

$$F \diamond \mathcal{E}(h) = \mathcal{T}_{-h}X \cdot \mathcal{E}(h) \tag{2.3}$$

which is valid for any $h \in H$ and $F \in L^p(\Omega)$ for some $p > 1$. We refer to Holden et al. [8] and Janson [13] for more details on Wick product and translation operators.

We complete this preliminary part observing that the polygonal approximation $\{B_t^\pi\}_{t\in[0,T]}$ of the Brownian motion $\{B_t\}_{t\in[0,T]}$ defined in (1.8) can be written also in the compact form

$$B_t^\pi = \int_0^T K_t^\pi(u)dB_u \tag{2.4}$$

with

$$K_t^\pi(u) := \sum_{k=0}^{n-1}\left(1_{[0,t_k[}(u) + \frac{t-t_k}{t_{k+1}-t_k}1_{[t_k,t_{k+1}[}(u)\right)1_{[t_k,t_{k+1}[}(t). \tag{2.5}$$

It is straightforward to see that $0 \leq K_t^\pi(u) \leq 1$ for all $t, u \in [0,T]$ and

$$\dot{B}_t^\pi = \frac{B_{t_{k+1}} - B_{t_k}}{t_{k+1}-t_k} \quad \text{if } t \in [t_k, t_{k+1}[.$$

Moreover, in analogy with (2.4) we set

$$\dot{B}_t^\pi = \int_0^T \partial_t K_t^\pi(u)dB_u \quad \text{with} \quad \partial_t K_t^\pi(u) = \sum_{k=0}^{n-1}\frac{1}{t_{k+1}-t_k}1_{[t_k,t_{k+1}[}(u)1_{[t_k,t_{k+1}[}(t). \tag{2.6}$$

2.1 Existence of the density

The aim of the present section is to prove the first statement in Theorem 1.4, i.e. the absolute continuity of the law of X_t^π with respect to the one dimensional Lebesgue measure. We will assume that the function σ in the diffusion coefficient of equation (1.9) is identically equal to one. The general case can be recovered with straightforward modifications.
As a first step we investigate the Malliavin regularity of X_t^π and write an equation for DX_t^π.

Theorem 2.2 *For any $t \in [s,T]$ the random variable X_t^π belongs to $\mathbb{D}^{2,p}$ for all $p \geq 1$. Moreover,*

$$\begin{aligned} D_u X_t^\pi &= D_u Y + \int_s^t b_x(r, X_r^\pi) D_u X_r^\pi dr + \int_s^t D_u X_r^\pi \diamond \dot{B}_r^\pi dr \\ &\quad + \int_s^t X_r^\pi \partial_r K_r^\pi(u) dr. \end{aligned} \tag{2.7}$$

Remark 2.3 *The need for checking that X_t^π belongs to $\mathbb{D}^{2,p}$ in order to write the equation for $D_u X_t^\pi$ is due to the fact that integrand in*

$$\int_s^t D_u X_r^\pi \diamond \dot{B}_r^\pi dr$$

requires a control on the second order Malliavin derivative of X_r^π to be well defined. In fact, from (2.2) we have

$$D_u X_r^\pi \diamond \dot{B}_r^\pi = \delta(D_u X_r^\pi \partial_r K_r^\pi(\cdot))$$

implying that $D_u X_r^\pi \in \mathbb{D}^{1,2}$ is a sufficient conditions for the membership of $D_u X_r^\pi \partial_r K_r^\pi(\cdot)$ to the domain of δ (see Proposition 1.3.1 in [20]).

Proof. Employing a standard reduction method in combination with the properties of the Wick product, we can represent the solution $\{X_t^\pi\}_{t\in[s,T]}$ of equation (1.9) as $X_t^\pi = Z_t^\pi \diamond \mathcal{E}(K_{s,t}^\pi)$ where $K_{s,t}^\pi(\cdot) := K_t^\pi(\cdot) - K_s^\pi(\cdot)$ and $\{Z_t^\pi\}_{t\in[s,T]}$ is the unique solution of the equation

$$\begin{cases} \dot{Z}_t^\pi = b(t, Z_t^\pi \diamond \mathcal{E}(K_{s,t}^\pi)) \diamond \mathcal{E}(-K_{s,t}^\pi) \\ Z_s^\pi = Y \end{cases} \tag{2.8}$$

(see [2] for the details of this technique). Moreover, according to the Gjessing formula (2.3) we have

$$Z_t^\pi \diamond \mathcal{E}(K_{s,t}^\pi) = \left(\mathcal{T}_{-K_{s,t}^\pi} Z_t^\pi\right) \cdot \mathcal{E}(K_{s,t}^\pi).$$

Therefore, since $\mathcal{E}(K_{s,t}^\pi)$ belongs to $\mathbb{D}^{k,p}$ for all $p \geq 1$ and $k \in \mathbb{N}$ and $\mathcal{T}_{-K_{s,t}^\pi}$ maps $\mathbb{D}^{k,p}$ into $\mathbb{D}^{k,q}$ for any $q < p$ and $k \in \mathbb{N}$, the first part of the statement will follow from Proposition 1.5.6 in [20] once we prove that Z_t^π belongs to $\mathbb{D}^{2,p}$ for all $p \geq 1$.
Resorting once more to the Gjessing formula we can rewrite equation (2.8) as

$$Z_t^\pi = Y + \int_s^t b(r, Z_r^\pi \cdot \mathcal{E}(-K_{s,r}^\pi)^{-1}) \cdot \mathcal{E}(-K_{s,r}^\pi) dr. \tag{2.9}$$

We first derive an upper bound for $|Z_t^\pi|$ which will be useful to control the norms in $\mathbb{D}^{2,p}$:

$$\begin{aligned} |Z_t^\pi| &\leq |Y| + \int_s^t |b(r, Z_r^\pi \cdot \mathcal{E}(-K_{s,r}^\pi)^{-1})| \cdot \mathcal{E}(-K_{s,r}^\pi) dr \\ &\leq |Y| + M \int_s^t (1 + |Z_r^\pi| \cdot \mathcal{E}(-K_{s,r}^\pi)^{-1}) \cdot \mathcal{E}(-K_{s,r}^\pi) dr \end{aligned}$$

$$
\begin{aligned}
&= |Y| + M\int_s^t \mathcal{E}(-K^\pi_{s,r})dr + M\int_s^t |Z^\pi_r|dr \\
&\leq |Y| + M\int_s^T \mathcal{E}(-K^\pi_{s,r})dr + M\int_s^t |Z^\pi_r|dr.
\end{aligned}
$$

The positive constant M utilized above comes from the inequality

$$
|b(t,x)| \leq M(1+|x|), \quad t \in [0,T], x \in \mathbb{R}
$$

which follows from the assumptions on b (in particular the boundedness of the first partial derivative with respect to x). By the Gronwall inequality we deduce that

$$
|Z^\pi_t| \leq \left(|Y| + M\int_s^T \mathcal{E}(-K^\pi_{s,r})dr\right) e^{M(t-s)}
$$

and hence

$$
\sup_{t\in[s,T]} |Z^\pi_t| \leq \left(|Y| + M\int_s^T \mathcal{E}(-K^\pi_{s,r})dr\right) e^{M(T-s)}.
$$

Computing the $L^p(\Omega)$-norms we get

$$
\begin{aligned}
\left\| \sup_{t\in[s,T]} |Z^\pi_t| \right\|_p &\leq \left(\|Y\|_p + M\int_s^T \|\mathcal{E}(-K^\pi_{s,r})\|_p dr\right) e^{M(T-s)} \\
&= \left(\|Y\|_p + M\int_s^T \exp\{(p-1)|K^\pi_{s,r}|^2_H/2\}dr\right) e^{M(T-s)} \\
&\leq \left(\|Y\|_p + C\right) e^{M(T-s)} \qquad (2.10)
\end{aligned}
$$

where C is a positive constant depending on p, T and M (recall that $0 \leq K^\pi_t(u) \leq 1$ and hence that $|K^\pi_{s,t}| \leq 1$). We are now ready to compute the Malliavin derivative of Z^π_t; from equation (2.9) we obtain

$$
\begin{aligned}
D_u Z^\pi_t &= D_u Y + \int_s^t D_u[b(r, Z^\pi_r \cdot \mathcal{E}(-K^\pi_{s,r})^{-1}) \cdot \mathcal{E}(-K^\pi_{s,r})]dr \\
&= D_u Y + \int_s^t b_x(r, Z^\pi_r \cdot \mathcal{E}(-K^\pi_{s,r})^{-1})(D_u Z^\pi_r + Z^\pi_r K^\pi_{s,r}(u))dr \\
&\quad - \int_s^t b(r, Z^\pi_r \cdot \mathcal{E}(-K^\pi_{s,r})^{-1}) \cdot \mathcal{E}(-K^\pi_{s,r}) K^\pi_{s,r}(u)dr
\end{aligned}
$$

which gives

$$
\begin{aligned}
|D_u Z^\pi_t| &\leq |D_u Y| + L\int_s^t |D_u Z^\pi_r|dr + L\int_s^t |Z^\pi_r||K^\pi_{s,r}(u)|dr \\
&\quad + M\int_s^t (1 + |Z^\pi_r \cdot \mathcal{E}(-K^\pi_{s,r})^{-1}|) \cdot \mathcal{E}(-K^\pi_{s,r})|K^\pi_{s,r}(u)|dr \\
&\leq |D_u Y| + L\int_s^t |D_u Z^\pi_r|dr + L\int_s^t |Z^\pi_r|dr
\end{aligned}
$$

$$
\begin{aligned}
&+M\int_s^t \mathcal{E}(-K^\pi_{s,r})dr + M\int_s^t |Z^\pi_r|dr \\
\leq\ & |D_uY| + L\int_s^t |D_uZ^\pi_r|dr + (L+M)\int_s^T |Z^\pi_r|dr \\
&+M\int_s^T \mathcal{E}(-K^\pi_{s,r})dr.
\end{aligned}
$$

Here, L stands for the Lipschitz constant of b with respect to the variable x. Then, by the Gronwall inequality we obtain

$$
|D_uZ^\pi_t| \leq \left(|D_uY| + (L+M)\int_s^T |Z^\pi_r|dr + M\int_s^T \mathcal{E}(-K^\pi_{s,r})dr\right) e^{L(t-s)},
$$

which in turn implies

$$
\begin{aligned}
|DZ^\pi_t|_H &\leq e^{L(t-s)}\left(|DY|_H + \sqrt{T}(L+M)\int_s^T |Z^\pi_r|dr + \sqrt{T}M\int_s^T \mathcal{E}(-K^\pi_{s,r})dr\right) \\
&\leq e^{L(t-s)}\left(|DY|_H + \sqrt{T}(L+M)(T-s)\sup_{r\in[s,T]}|Z^\pi_r| + \sqrt{T}M\int_s^T \mathcal{E}(-K^\pi_{s,r})dr\right).
\end{aligned}
$$

With the help of estimate (2.10) we can conclude that

$$
\||DZ^\pi_t|_H\|_p \leq C_1\|Y\|_{\mathbb{D}^{1,p}} + C_2,
$$

where C_1 and C_2 are positive constants depending on p, T, M and L. This proves that Z^π_t belongs to $\mathbb{D}^{1,p}$. For the second order Malliavin derivative of Z^π_t, one proceeds as before differentiating twice the identity (2.9) and resorting to the Grownwall inequality for the estimation of the norm of $D_vD_uZ^\pi_t$.
To get identity (2.7) one has simply to differentiate the equality

$$
X^\pi_t = Y + \int_s^t b(r, X^\pi_r)dr + \int_s^t X^\pi_r \diamond \dot{B}^\pi_r dr
$$

in connection with the following chain rule for the Wick product:

$$
\begin{aligned}
D_u(X^\pi_r \diamond \dot{B}^\pi_r) &= (D_uX^\pi_r)\diamond \dot{B}^\pi_r + X^\pi_r \diamond D_u\dot{B}^\pi_r \\
&= (D_uX^\pi_r)\diamond \dot{B}^\pi_r + X^\pi_r \diamond \partial_r K^\pi_r(u) \\
&= (D_uX^\pi_r)\diamond \dot{B}^\pi_r + X^\pi_r \cdot \partial_r K^\pi_r(u)
\end{aligned}
$$

The proof is complete. ∎

Equation (2.7) shows that $D_uX^\pi_r$ solves a linear differential equation containing the non homogeneous term

$$
\int_s^t X^\pi_r \partial_r K^\pi_r(u)dr.
$$

In the next proposition we will find a direction $h^\pi \in H$ along which the quantity above will be identically zero and this will produce an explicit expression for the resulting directional derivative of X^π_r.

Proposition 2.4 *There exists a function $h^\pi \in H$ such that $h^\pi \neq 0$ almost everywhere and $\int_0^T \partial_r K_r^\pi(u) h^\pi(u) du = 0$ for all $r \in [0,T]$. Moreover,*

$$D_{h^\pi} X_t^\pi = \mathcal{T}_{-K_{s,t}^\pi} D_{h^\pi} Y \cdot \exp\left\{ \int_s^t b_x(r, \mathcal{T}_{-K_{s,t}^\pi} \mathcal{T}_{K_{s,r}^\pi} X_r^\pi) dr \right\} \cdot \mathcal{E}(K_{s,t}^\pi). \tag{2.11}$$

PROOF. Let h^π be a function in H such that $h^\pi \neq 0$ almost everywhere and

$$\frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} h^\pi(u) du = 0 \quad \text{for all } k \in \{0, ..., n-1\}.$$

Then, according to the second equation in (2.6) we have

$$\begin{aligned} \int_0^T \partial_r K_r^\pi(u) h^\pi(u) du &= \sum_{k=0}^{n-1} \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} h^\pi(u) du \cdot 1_{[t_k, t_{k+1}[}(r) \\ &= 0, \quad \text{for all } r \in [0,T]. \end{aligned}$$

If we multiply now both sides of equation (2.7) by the function h^π and integrate with respect to u between zero and T, we see that the last term is zero and we are left with

$$D_{h^\pi} X_t^\pi = D_{h^\pi} Y + \int_s^t b_x(r, X_r^\pi) D_{h^\pi} X_r^\pi dr + \int_s^t D_{h^\pi} X_r^\pi \diamond \dot{B}_r^\pi dr. \tag{2.12}$$

Equation (2.12) is linear and homogeneous in $D_{h^\pi} X_\cdot^\pi$. We find now its unique solution. First of all, observe that employing the reduction method mentioned above we can write

$$D_{h^\pi} X_t^\pi = V_t^\pi \diamond \mathcal{E}(K_{s,t}^\pi)$$

where $\{V_t^\pi\}_{t \in [s,T]}$ satisfies

$$V_t^\pi = D_{h^\pi} Y + \int_s^t (b_x(r, X_r^\pi)(V_r^\pi \diamond \mathcal{E}(K_{s,r}^\pi))) \diamond \mathcal{E}(-K_{s,r}^\pi) dr \tag{2.13}$$

We note that applying the Gjessing formula (2.3) twice we can write

$$\begin{aligned} (b_x(r, X_r^\pi)(V_r^\pi \diamond \mathcal{E}(K_{s,r}^\pi))) \diamond \mathcal{E}(-K_{s,r}^\pi) &= (b_x(r, X_r^\pi) \cdot \mathcal{T}_{-K_{s,r}^\pi} V_r^\pi \cdot \mathcal{E}(K_{s,r}^\pi)) \diamond \mathcal{E}(-K_{s,r}^\pi) \\ &= \mathcal{T}_{K_{s,r}^\pi}(b_x(r, X_r^\pi) \cdot \mathcal{T}_{-K_{s,r}^\pi} V_r^\pi \cdot \mathcal{E}(K_{s,r}^\pi)) \cdot \mathcal{E}(-K_{s,r}^\pi) \\ &= b_x(r, \mathcal{T}_{K_{s,r}^\pi} X_r^\pi) \cdot V_r^\pi \cdot \mathcal{T}_{K_{s,r}^\pi} \mathcal{E}(K_{s,r}^\pi) \cdot \mathcal{E}(-K_{s,r}^\pi) \\ &= b_x(r, \mathcal{T}_{K_{s,r}^\pi} X_r^\pi) \cdot V_r^\pi. \end{aligned}$$

Therefore, equation (2.13) now reads

$$\begin{aligned} V_t^\pi &= D_{h^\pi} Y + \int_s^t (b_x(r, X_r^\pi)(V_r^\pi \diamond \mathcal{E}(K_{s,r}^\pi))) \diamond \mathcal{E}(-K_{s,r}^\pi) dr \\ &= D_{h^\pi} Y + \int_s^t b_x(r, \mathcal{T}_{K_{s,r}^\pi} X_r^\pi) \cdot V_r^\pi dr \end{aligned}$$

implying that

$$V_t^\pi = D_{h^\pi} Y \exp\left\{ \int_s^t b_x(r, \mathcal{T}_{K_{s,r}^\pi} X_r^\pi) dr \right\}.$$

Therefore,

$$\begin{aligned} D_{h^\pi} X_t^\pi &= V_t^\pi \diamond \mathcal{E}(K_{s,t}^\pi) \\ &= \left(D_{h^\pi} Y \exp\left\{ \int_s^t b_x(r, \mathcal{T}_{K_{s,r}^\pi} X_r^\pi) dr \right\} \right) \diamond \mathcal{E}(K_{s,t}^\pi) \\ &= \mathcal{T}_{-K_{s,t}^\pi} D_{h^\pi} Y \cdot \mathcal{T}_{-K_{s,t}^\pi} \exp\left\{ \int_s^t b_x(r, \mathcal{T}_{K_{s,r}^\pi} X_r^\pi) dr \right\} \cdot \mathcal{E}(K_{s,t}^\pi) \\ &= \mathcal{T}_{-K_{s,t}^\pi} D_{h^\pi} Y \cdot \exp\left\{ \int_s^t b_x(r, \mathcal{T}_{-K_{s,t}^\pi} \mathcal{T}_{K_{s,r}^\pi} X_r^\pi) dr \right\} \cdot \mathcal{E}(K_{s,t}^\pi). \end{aligned}$$

The proof is complete. ∎

We are now ready to prove that the law of X_t^π is absolutely continuous with respect to the Lebesgue measure. Our strategy is based on the following criteria that generalizes to some extent Proposition 2.1.1 in [20]:

Suppose that $X \in \mathbb{D}^{1,2}$ and let $h \in H$ be such that $D_h X \neq 0$ almost surely and $h/D_h X$ belongs to the domain of δ. Then, X possesses a bounded and continuous density.

First of all we note that the assumption

$$E[|D_{h^\pi} Y|^{-q}] \text{ is finite for some } q > 4$$

from Theorem 1.4 implies that $D_{h^\pi} Y \neq 0$ almost surely (and also that $\mathcal{T}_{-K_{s,t}^\pi} D_{h^\pi} Y \neq 0$ since Cameron-Martin shifts preserve negligible sets). This fact combined with Proposition 2.4 entails that $D_{h^\pi} X_t^\pi \neq 0$ almost surely. Hence, we are left with the verification that $h^\pi / D_{h^\pi} X_t^\pi$ belongs to the domain of δ; a sufficient condition for that is $h^\pi / D_{h^\pi} X_t^\pi \in \mathbb{D}^{1,2}(H)$.
We observe that

$$E[|h^\pi / D_{h^\pi} X_t^\pi|_H^2] = |h^\pi|_H^2 E[1/|D_{h^\pi} X_t^\pi|^2]$$

and

$$\begin{aligned} E[|D(h^\pi / D_{h^\pi} X_t^\pi)|_{H\otimes H}^2] &= |h^\pi|_H^2 E[|D D_{h^\pi} X_t^\pi|_H^2 / |D_{h^\pi} X_t^\pi|^4] \\ &\leq |h^\pi|_H^2 E[|D D_{h^\pi} X_t^\pi|_H^{2p}]^{1/p} \cdot E[1/|D_{h^\pi} X_t^\pi|^{4q}]^{1/q}, \end{aligned}$$

where we applied the Hölder inequality with $1/p + 1/q = 1$. Therefore, $h^\pi / D_{h^\pi} X_t^\pi \in \mathbb{D}^{1,2}(H)$ if $D_{h^\pi} X_t^\pi \in \mathbb{D}^{1,p}$ for all $p \geq 1$ and

$$E[1/|D_{h^\pi} X_t^\pi|^q] \text{ is finite for some } q > 4. \tag{2.14}$$

The first condition follows from the fact that $X_t^\pi \in \mathbb{D}^{2,p}$ (see Theorem 2.2). We now verify the validity of (2.14). Let $q > 4$ be such that $E[|D_{h^\pi} Y|^{-q}]$ is finite and let $\alpha > 1$ and $\varepsilon > 0$ be such that $\tilde{q} := (q - \varepsilon)/\alpha > 4$; then, recalling the identity (2.11) we can write

$$|D_{h^\pi} X_t^\pi|^{-\tilde{q}} = |\mathcal{T}_{-K_{s,t}^\pi} D_{h^\pi} Y|^{-\tilde{q}} \cdot \exp\left\{ -\tilde{q} \int_s^t b_x(r, \mathcal{T}_{-K_{s,t}^\pi} \mathcal{T}_{K_{s,r}^\pi} X_r^\pi) dr \right\} \cdot |\mathcal{E}(K_{s,t}^\pi)|^{-\tilde{q}}$$

$$
\begin{aligned}
&\leq e^{\tilde{q}L(t-s)}|\mathcal{T}_{-K^{\pi}_{s,t}} D_{h^\pi} Y|^{-\tilde{q}} \cdot |\mathcal{E}(K^{\pi}_{s,t})|^{-\tilde{q}} \\
&= e^{\tilde{q}L(t-s)}\mathcal{T}_{-K^{\pi}_{s,t}}|D_{h^\pi} Y|^{-\tilde{q}} \cdot |\mathcal{E}(K^{\pi}_{s,t})|^{-\tilde{q}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E[|D_{h^\pi} X^{\pi}_t|^{-\tilde{q}}] &\leq e^{\tilde{q}L(t-s)} E[\mathcal{T}_{-K^{\pi}_{s,t}}|D_{h^\pi} Y|^{-\tilde{q}} \cdot |\mathcal{E}(K^{\pi}_{s,t})|^{-\tilde{q}}] \\
&\leq e^{\tilde{q}L(t-s)} E[\mathcal{T}_{-K^{\pi}_{s,t}}|D_{h^\pi} Y|^{-q+\varepsilon}]^{1/\alpha} \cdot E[|\mathcal{E}(K^{\pi}_{s,t})|^{-\tilde{q}\beta}]^{1/\beta} \\
&\leq C E[|D_{h^\pi} Y|^{-q}]^{(q-\varepsilon)/q\alpha}
\end{aligned}
$$

where $1/\alpha + 1/\beta = 1$ and C is a positive constant depending on L, α, ε, q and T. In the last inequality we utilized the fact that $\mathcal{T}_{-K^{\pi}_{s,t}}$ maps $L^q(\Omega)$ into $L^r(\Omega)$ for all $r < p$ and the membership of $\mathcal{E}(K^{\pi}_{s,t})$ to all the spaces $L^p(\Omega)$ for $p \geq 1$. The assumption on Y completes the proof of the claim (2.14) which in turn implies the absolute continuity of the law of X^{π}_t with respect to the one dimensional Lebesgue measure with a bounded and continuous density.

2.2 Fokker-Planck equation

We now prove the second part of Theorem 1.4. We will assume as before that the function σ in the diffusion coefficient of equation (1.9) is identically equal to one.
As for exact solutions, the key ingredient, to relate the law of the solution of the stochastic equation (1.9) with a Fokker-Planck-type equation, is the Itô formula.

Theorem 2.5 (Itô formula) *Let $\{X^{\pi}_t\}_{t\in[s,T]}$ be the unique solution of equation (1.9) and let $\varphi \in C^{1,2}([0,T]\times\mathbb{R})$ be such that $\partial_x\varphi$ and $\partial_{xx}\varphi$ have at most polynomial growth at infinity. Then, for $s \leq t \leq T$ we have*

$$
\begin{aligned}
\varphi(t, X^{\pi}_t) - \varphi(s, Y) &= \int_s^t [\partial_t\varphi(r, X^{\pi}_r) + \partial_x\varphi(r, X^{\pi}_r) b(r, X^{\pi}_r)]\, dr \\
&\quad + \int_s^t \partial_{xx}\varphi(r, X^{\pi}_r) X^{\pi}_r D_{\partial_r K^{\pi}_r} X^{\pi}_r dr \\
&\quad + \int_s^t (\partial_x\varphi(r, X^{\pi}_r) X^{\pi}_r) \diamond \dot{B}^{\pi}_r dr.
\end{aligned}
$$

Proof. First of all we observe that the condition on the growth at infinity of $\partial_x\varphi$ and $\partial_{xx}\varphi$ together with the membership of X^{π}_r to $\mathbb{D}^{2,p}$ for all $p \geq 1$ ensure that

$$
(\partial_x\varphi(r, X^{\pi}_r) X^{\pi}_r) \diamond \dot{B}^{\pi}_r = \delta(\partial_x\varphi(r, X^{\pi}_r) X^{\pi}_r \partial_r K^{\pi}_r)
$$

is well defined since $\partial_x\varphi(r, X^{\pi}_r) X^{\pi}_r \in \mathbb{D}^{1,2}$. Now, using equation (1.9) we get

$$
\begin{aligned}
\varphi(t, X^{\pi}_t) - \varphi(s, Y) &= \int_s^t \left[\partial_t\varphi(r, X^{\pi}_r) + \partial_x\varphi(r, X^{\pi}_r)\dot{X}^{\pi}_r\right] dr \\
&= \int_s^t [\partial_t\varphi(r, X^{\pi}_r) + \partial_x\varphi(r, X^{\pi}_r) b(r, X^{\pi}_r)]\, dr \\
&\quad + \int_s^t \partial_x\varphi(r, X^{\pi}_r) \cdot (X^{\pi}_r \diamond \dot{B}^{\pi}_r) dr. \qquad (2.15)
\end{aligned}
$$

Moreover, according to Proposition 1.3.3 in [20] we can write

$$\begin{aligned}
\partial_x\varphi(r,X_r^\pi)\cdot(X_r^\pi\diamond\dot{B}_r^\pi) &= \partial_x\varphi(r,X_r^\pi)\cdot\delta(X_r^\pi\partial_rK_r^\pi)\\
&= \delta(\partial_x\varphi(r,X_r^\pi)X_r^\pi\partial_rK_r^\pi)\\
&\quad+\int_0^T(D_u\partial_x\varphi(r,X_r^\pi))X_r^\pi\partial_rK_r^\pi(u)du\\
&= (\partial_x\varphi(r,X_r^\pi)X_r^\pi)\diamond\dot{B}_r^\pi\\
&\quad+\int_0^T\partial_{xx}\varphi(r,X_r^\pi)(D_uX_r^\pi)X_r^\pi\partial_rK_r^\pi(u)du\\
&= (\partial_x\varphi(r,X_r^\pi)X_r^\pi)\diamond\dot{B}_r^\pi\\
&\quad+\partial_{xx}\varphi(r,X_r^\pi)X_r^\pi D_{\partial_rK_r^\pi}X_r^\pi.
\end{aligned}$$

The substitution of the integrand in (2.15) with the last member of the previous chain of equalities provides the desired formula. ∎

Now, let $\varphi\in C_0^\infty([s,T]\times\mathbb{R})$; then, by Theorem 2.5 we obtain

$$\begin{aligned}
0 &= \varphi(T,X_T^\pi)-\varphi(s,Y)\\
&= \int_s^T[\partial_t\varphi(r,X_r^\pi)+\partial_x\varphi(r,X_r^\pi)b(r,X_r^\pi)]\,dr\\
&\quad+\int_s^T\partial_{xx}\varphi(r,X_r^\pi)X_r^\pi D_{\partial_rK_r^\pi}X_r^\pi dr\\
&\quad+\int_s^T(\partial_x\varphi(r,X_r^\pi)X_r^\pi)\diamond\dot{B}_r^\pi dr.
\end{aligned}$$

Taking the expectation, recalling that for all $X\in\mathbb{D}^{1,2}$

$$E[X\diamond\dot{B}_r^\pi]=E[\delta(X\partial_rK_r^\pi)]=0$$

and denoting by $p^\pi(r,x)$ the density of the random variable X_r^π, we get

$$\begin{aligned}
0 &= E\left[\int_s^T\partial_t\varphi(r,X_r^\pi)+\partial_x\varphi(r,X_r^\pi)b(r,X_r^\pi)dr\right]\\
&\quad+E\left[\int_s^T\partial_{xx}\varphi(r,X_r^\pi)X_r^\pi D_{\partial_rK_r^\pi}X_r^\pi dr\right]\\
&= \int_s^T E\left[\partial_t\varphi(r,X_r^\pi)+\partial_x\varphi(r,X_r^\pi)b(r,X_r^\pi)\right]dr\\
&\quad+\int_s^T E\left[\partial_{xx}\varphi(r,X_r^\pi)X_r^\pi D_{\partial_rK_r^\pi}X_r^\pi\right]dr\\
&= \int_s^T\int_{\mathbb{R}}(\partial_t\varphi(r,x)+\partial_x\varphi(r,x)b(r,x))p^\pi(r,x)dxdr\\
&\quad+\int_s^T E\left[\partial_{xx}\varphi(r,X_r^\pi)X_r^\pi E[D_{\partial_rK_r^\pi}X_r^\pi|\sigma(X_r^\pi)]\right]dr\\
&= \int_s^T\int_{\mathbb{R}}(\partial_t\varphi(r,x)+\partial_x\varphi(r,x)b(r,x))p^\pi(r,x)dxdr
\end{aligned}$$

$$+\int_s^T \int_{\mathbb{R}} \partial_{xx}\varphi(r,x)xg(r,x)p^\pi(r,x)dxdr,$$

where $g : [s,T] \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$g(r,x) = E[D_{\partial_r K_r^\pi} X_r^\pi | \sigma(X_r^\pi)]|_{X_r^\pi = x}. \tag{2.16}$$

We observe that g is uniquely defined up to sets that are negligible with respect to the law of X_r^π and hence negligible with respect to the Lesbegue measure. According to this construction, the function g is barely measurable with only some integrability properties against the density $p^\pi(t,x)$. In fact, since $D_{\partial_r K_r^\pi} X_r^\pi \in L^q(\Omega)$ for all $q \geq 1$ we get the bound

$$\begin{aligned}\int_{\mathbb{R}} |g(r,x)|^q p^\pi(r,x)dx &= E\left[|E[D_{\partial_r K_r^\pi} X_r^\pi | \sigma(X_r^\pi)]|^q\right] \\ &\leq E\left[|D_{\partial_r K_r^\pi} X_r^\pi|^q\right].\end{aligned}$$

We have therefore proved for any $\varphi \in C_0^\infty([s,T] \times \mathbb{R})$ the identity

$$\int_s^T \int_{\mathbb{R}} (\partial_t \varphi(r,x) + \partial_x \varphi(r,x) b(r,x) + \partial_{xx}\varphi(r,x)xg(r,x))p^\pi(r,x)dxdr = 0,$$

which is equivalent to the statement that $p^\pi(t,x)$ is a distributional solution of the equation

$$(\partial_t + b(t,x)\partial_x + xg(t,x)\partial_{xx})^* u(t,x) = 0.$$

Example 2.6 *Consider the case where $b(t,x) = 0$ and $\sigma(t) = 1$. Then, equation (1.9) reads*

$$\begin{cases} \dot{X}_t^\pi = X_t^\pi \diamond \dot{B}_t^\pi, \quad t \in]s,T] \\ X_s^\pi = Y \end{cases} \tag{2.17}$$

The solution to this equation is given by the formula

$$X_t^\pi = Y \diamond \mathcal{E}(K_{s,t}^\pi). \tag{2.18}$$

If we take $Y = X_s$ where $\{X_t\}_{t\in[0,T]}$ is the solution of

$$\begin{cases} dX_t = X_t dB_t, \quad t \in]0,T] \\ X_0 = x \in \mathbb{R} \end{cases} \tag{2.19}$$

(i.e. the SDE we are approximating with (2.17)) then $X_s = x \cdot \mathcal{E}(1_{[0.s[})$ and by the properties of the Wick product we can write (2.18) as

$$\begin{aligned} X_t^\pi &= X_s \diamond \mathcal{E}(K_{s,t}^\pi) \\ &= x \cdot \mathcal{E}(1_{[0.s[}) \diamond \mathcal{E}(K_{s,t}^\pi) \\ &= x \cdot \mathcal{E}(1_{[0.s[} + K_{s,t}^\pi). \end{aligned}$$

We now aim at finding an explicit expression for g from formula (2.16) and hence for the Fokker-Planck-type equation associated to (2.17). First of all we compute the Malliavin derivative of X_t^π:

$$D_u X_t^\pi = D_u(x \cdot \mathcal{E}(1_{[0.s[} + K_{s,t}^\pi))$$

$$= \quad X_t^\pi \cdot (1_{[0.s[}(u) + K_{s,t}^\pi(u)).$$

Then, we get

$$\begin{aligned} g(t,x) &= E[D_{\partial_t K_t^\pi} X_t^\pi | \sigma(X_t^\pi)]|_{X_t^\pi = x} \\ &= E\left[X_t^\pi \int_0^T (1_{[0.s[}(u) + K_{s,t}^\pi(u)) \partial_t K_t^\pi(u) du \Big| \sigma(X_t^\pi) \right] \Big|_{X_t^\pi = x} \\ &= x \int_0^T (1_{[0.s[}(u) + K_{s,t}^\pi(u)) \partial_t K_t^\pi(u) du. \end{aligned}$$

Denoting

$$\xi^\pi(t) := \int_0^T (1_{[0.s[}(u) + K_{s,t}^\pi(u)) \partial_t K_t^\pi(u) du,$$

we can write the Fokker-Planck-type equation for (2.17) as

$$(\partial_t + x^2 \xi^\pi(t) \partial_{xx})^* p^\pi(t,x) = 0$$

to be compared with

$$(\partial_t + (x^2/2)\partial_{xx})^* p(t,x) = 0$$

which is the one for the exact equation (2.19).

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