Arch. Math. 114 (2020), 699–708

0003-889X/20/060699-10
published online March 12, 2020
https://doi.org/10.1007/s00013-020-01440-9

Archiv der Mathematik

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A remark on nonlocal Neumann conditions for the fractional Laplacian

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Abstract. We show how nonlocal boundary conditions of Robin type can be encoded in the pointwise expression of the fractional operator. Notably, the fractional Laplacian of functions satisfying homogeneous nonlocal Neumann conditions can be expressed as a regional operator with a kernel having logarithmic behaviour at the boundary.

Mathematics Subject Classification. 35S15, 35J99, 47G20.

Keywords. Fractional Laplacian, Nonlocal normal derivative, Nonlocal Neumann conditions, Regional Laplacian.

1. Introduction. The purpose of this short note is to put in evidence a special feature of the fractional Laplacian when coupled with nonlocal Robin boundary conditions.

By *fractional Laplacian* we mean the nonlocal operator of order $2s \in (0,2)$

$$(-\triangle)^s u(x) := c_{n,s}\,\mathrm{p.v.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy := c_{n,s}\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy, \tag{1}$$

where $c_{n,s}$ is a positive normalizing constant. We refer to Valdinoci and the author [1] for an introduction to the basic traits of this operator, with particular emphasis on the differences to the classical elliptic theory for the Laplacian, and other possible notions of the fractional Laplacian. We underline how the definition in (1) makes sense[1] only for functions defined in all of $\mathbb{R}^n$: this yields an associated boundary value problem on some domain $\Omega \subset \mathbb{R}^n$ which looks

[1]More precisely, the concerned function u must enjoy some regularity and some growth control at infinity, in order to have the integral finite: we skip here on these important details and we refer the interested reader to [1].

like

$$\begin{cases} (-\triangle)^s u = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases}$$

In this problem, the function $g : \mathbb{R}^n \backslash \Omega \to \mathbb{R}$ accounts as a boundary condition of Dirichlet sort. A Neumann boundary condition has been proposed by Dipierro, Ros-Oton, and Valdinoci [4], by means of some *nonlocal normal derivative* (for which we keep the original notation from the authors)

$$\widetilde{\mathcal{N}}_s u(x) := \left(\int_\Omega \frac{dy}{|x-y|^{n+2s}} \right)^{-1} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}} \, dy, \qquad x \in \mathbb{R}^n \backslash \Omega, \tag{2}$$

in its normalized version, see [4, equation (1.5)]. With (2), it is possible to study the elliptic boundary value problem

$$\begin{cases} (-\triangle)^s u = f & \text{in } \Omega, \\ \widetilde{\mathcal{N}}_s u = g & \text{in } \mathbb{R}^n \backslash \Omega \end{cases}$$

or also the parabolic one

$$\begin{cases} \partial_t u + (-\triangle)^s u = 0 & \text{in } \Omega \times (0, \infty), \\ \widetilde{\mathcal{N}}_s u = 0 & \text{in } \big(\mathbb{R}^n \backslash \Omega\big) \times (0, \infty), \\ u = u_0 & \text{on } \Omega \times \{0\}. \end{cases} \tag{3}$$

We show how the fractional Laplacian of a function satisfying homogeneous Neumann conditions $\widetilde{\mathcal{N}}_s u = 0$ in $\mathbb{R}^n \backslash \Omega$ can be reformulated as a regional type operator, i.e., of the form

$$(-\triangle)^s u(x) = c_{n,s} \, \text{p.v.} \int_\Omega \big(u(x) - u(y)\big) \, K(x,y) \, dy, \qquad x \in \Omega, \tag{4}$$

for some suitable measurable kernel $K : \Omega \times \Omega \to \mathbb{R}$. See equations (9) and (10) for the detailed expression of the kernel.

1.1. Notations. In the following, we will use these notations without further notice. Fixing a nonempty set $\Sigma \subset \mathbb{R}^n$, $\text{dist}(\cdot, \Sigma) : \mathbb{R}^n \to [0, +\infty)$ stands for the distance function $\text{dist}(x, \Sigma) = \inf\{|x-y| : y \in \Sigma\}$. When $\Sigma = \partial\Omega$, where Ω is the reference domain in (2), we simply write $d = \text{dist}(\cdot, \partial\Omega)$. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, we write $\text{supp}(f)$ for the support of f.

The binary operations $\wedge$ and $\vee$ will respectively denote the “min” and “max” operations between real numbers:

$$a \wedge b = \min\{a, b\}, \quad a \vee b = \max\{a, b\}, \qquad a, b \in \mathbb{R}.$$

In our computations, we will also make use of the particular choice $K(x,y) = |x-y|^{-n-2s}$ in (4), yielding the usually called *regional fractional Laplacian*

$$(-\triangle)^s_\Omega u(x) := c_{n,s} \, \text{p.v.} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}} \, dy, \qquad x \in \Omega. \tag{5}$$

Finally, for a measurable $\beta : \mathbb{R}^n \to [0,1]$, let

$$k_\beta(x,y) := \int\limits_{\mathbb{R}^n\setminus\Omega} \frac{1-\beta(z)}{|x-z|^{n+2s}|y-z|^{n+2s}\int_\Omega \frac{dw}{|z-w|^{n+2s}}}\,dz, \qquad x,y\in\Omega. \quad (6)$$

We simply write $k(x,y)$ when $\beta = 0$ in $\mathbb{R}^n\setminus\Omega$.

1.2. Main result. The precise statements go as follows.

Theorem 1.1. *Fix $\Omega\subset\mathbb{R}^n$ open, bounded, and with $C^{1,1}$ boundary. Consider a measurable $\beta : \mathbb{R}^n \to [0,1]$ such that $\beta = 0$ in Ω. Let $u \in C^{2s+\alpha}_{loc}(\Omega)\cap L^\infty(\Omega)$, for some $\alpha > 0$, satisfy*

$$\beta(x)u(x) + \big(1-\beta(x)\big)\widetilde{\mathcal{N}}_s u(x) = 0 \quad \textit{for a.e.} \;\; x\in\mathbb{R}^n\setminus\Omega. \quad (7)$$

Then, for $x\in\Omega$,

$$\begin{aligned}(-\triangle)^s u(x) &= (-\triangle)^s_\Omega u(x) - u(x)(-\triangle)^s\beta(x) + c_{n,s}\int\limits_\Omega \big(u(x)-u(y)\big)\,k_\beta(x,y)\,dy\\ &= c_{n,s}\,\text{p.v.}\int\limits_\Omega \big(u(x)-u(y)\big)\left(\frac{1}{|x-y|^{n+2s}} + k_\beta(x,y)\right)dy\\ &\quad + u(x)\,c_{n,s}\int\limits_{\mathbb{R}^n\setminus\Omega}\frac{\beta(y)}{|x-y|^{n+2s}}\,dy,\end{aligned}$$

where $k_\beta : \Omega\times\Omega\to\mathbb{R}$ is given by (6), nonnegative, symmetric, continuous, and for any fixed $x\in\Omega$, there exists $C = C(x) > 0$ such that

$$k_\beta(x,y) \le C\big(1 + \big|\ln \operatorname{dist}(y, \operatorname{supp}(1-\beta))\big|\big) \qquad \textit{for any} \;\; y\in\Omega. \quad (8)$$

A particular form of the mixed boundary condition in (7) (namely, when β is a characteristic function) has been used by Leonori, Medina, Peral, Primo, and Soria [5] to study a principal eigenvalue problem.

In the particular case when $\beta = 1$ in $\mathbb{R}^n\setminus\Omega$ in the above theorem, we have

$$(-\triangle)^s u(x) = (-\triangle)^s_\Omega u(x) + u(x)(-\triangle)^s\chi_\Omega(x),$$

where χ_Ω denotes the characteristic function of Ω. Conversely, when $\beta = 0$ in $\mathbb{R}^n$, we entail the following.

Corollary 1.2. *With the assumptions of Theorem 1.1, if u satisfies*

$$\widetilde{\mathcal{N}}_s u(x) = 0 \qquad \textit{for a.e.} \;\; x\in\mathbb{R}^n\setminus\Omega,$$

then, for $x\in\Omega$,

$$\begin{aligned}(-\triangle)^s u(x) &= (-\triangle)^s_\Omega u(x) + c_{n,s}\int\limits_\Omega \big(u(x)-u(y)\big)\,k(x,y)\,dy\\ &= c_{n,s}\,\text{p.v.}\int\limits_\Omega \big(u(x)-u(y)\big)\left(\frac{1}{|x-y|^{n+2s}} + k(x,y)\right)dy, \quad (9)\end{aligned}$$

where

$$k(x,y) := \int\limits_{\mathbb{R}^n\setminus\Omega} \frac{dz}{|x-z|^{n+2s}|y-z|^{n+2s}\int_\Omega \frac{dw}{|z-w|^{n+2s}}}, \qquad x,y\in\Omega, \tag{10}$$

is positive, symmetric, continuous, and for any fixed $x\in\Omega$*, there exists* $C=C(x)>0$ *such that*

$$k(x,y) \le C\big(1+\big|\ln \mathrm{dist}(y,\mathbb{R}^n\setminus\Omega)\big|\big) \qquad \text{for any } y\in\Omega. \tag{11}$$

Moreover, it also holds

$$k(x,y) \ge \frac{1}{C}\big(1+\big|\ln \mathrm{dist}(y,\mathbb{R}^n\setminus\Omega)\big|\big) \qquad \text{for any } y\in\Omega. \tag{12}$$

This last corollary is saying that, when homogeneous nonlocal Neumann conditions are assumed, the fractional Laplacian amounts to be a perturbation of the regional operator $(-\triangle)^s_\Omega$ defined in (5): this nice property becomes particularly interesting when thinking of its stochastic repercussions.

1.3. Stochastic heuristics. The coupling of $(-\triangle)^s$ with $\widetilde{\mathcal{N}}_s u = 0$ has indeed a precise interpretation from the stochastic perspective. We quote from [4]:

> *"The probabilistic interpretation of the Neumann problem (1.4)* [(3) in this note] *may be summarized as follows:*
> 1. $u(x,t)$ *is the probability distribution of the position of a particle moving randomly inside* Ω*.*
> 2. *When the particle exits* Ω*, it immediately comes back into* Ω*.*
> 3. *The way in which it comes back into* Ω *is the following: If the particle has gone to* $x\in\mathbb{R}^n\setminus\Omega$*, it may come back to any point* $y\in\Omega$*, the probability density of jumping from* x *to* y *being proportional to* $|x-y|^{-n-2s}$*.*
>
> *These three properties lead to the equation (1.4)* [(3) in this note]*,* u_0 *being the initial probability distribution of the position of the particle."*

We refer to [4, Section 2.1] for further details. The described process might look quite similar to a *censored process*, as introduced by Bogdan, Burdzy, and Chen [2], or at least to a *stable-like process* (following the wording of Chen and Kumagai [3]). The class of stable-like processes is the one induced by infinitesimal generators of the type (see [3, equation (1.3)])

$$\mathrm{p.v.}\int\limits_\Omega \frac{\big(u(x)-u(y)\big)\,j(x,y)}{|x-y|^{n+2s}}\,dy,$$

where the kernel j is supposed to be positive, symmetric, and bounded between two constants

$$\frac{1}{C}\le j(x,y)\le C, \qquad x,y\in\Omega.$$

In view of Corollary 1.2, and in particular of estimate (11), the process built in [4] does not fall into the stable-like class because of its singular boundary behaviour, although such singularity is rather weak.

2. Estimates. As a standing hypothesis, we consider here $\Omega \subset \mathbb{R}^n$ open, bounded, and with $C^{1,1}$ boundary satisfying the interior and the exterior sphere condition.

Lemma 2.1. *There exists $c > 0$ such that, for any $z \in \mathbb{R}^n \backslash \overline{\Omega}$,*

$$\frac{1}{c}\left(d(z)^{-2s} \wedge d(z)^{-n-2s}\right) \leq \int_{\Omega} \frac{dw}{|z-w|^{n+2s}} \leq c\left(d(z)^{-2s} \wedge d(z)^{-n-2s}\right). \quad (13)$$

Proof. As Ω is bounded, there exists $R > 0$ such that

$$\Omega \subset B_R(z) \backslash B_{d(z)}(z).$$

Therefore, using polar coordinates,

$$\begin{aligned}\int_{\Omega} \frac{dw}{|z-w|^{n+2s}} &\leq \int_{B_R(z)\backslash B_{d(z)}(z)} \frac{dw}{|z-w|^{n+2s}} \\ &= |\partial B| \int_{d(z)}^{R} t^{-1-2s}\, dt \leq \frac{|\partial B|}{2s}\, d(z)^{-2s}.\end{aligned}$$

Also

$$\int_{\Omega} \frac{dw}{|z-w|^{n+2s}} \leq \int_{\Omega} \frac{dw}{d(z)^{n+2s}} = |\Omega|\; d(z)^{-n-2s},$$

from which we conclude that there exists $c_1 > 0$ such that

$$\int_{\Omega} \frac{dw}{|z-w|^{n+2s}} \leq c_1 \left(d(z)^{-2s} \wedge d(z)^{-n-2s}\right).$$

To also get the inverse inequality, we split the analysis for $d(z)$ small and large.[2] For $d(z)$ large, one has, by Fatou's lemma,

$$\liminf_{|z|\uparrow\infty} |z|^{n+2s} \int_{\Omega} \frac{dw}{|z-w|^{n+2s}} \geq \int_{\Omega} \liminf_{|z|\uparrow\infty} \frac{|z|^{n+2s}}{|z-w|^{n+2s}}\, dw = |\Omega|,$$

so that, for some $c_2 > 0$,

$$\int_{\Omega} \frac{dw}{|z-w|^{n+2s}} \geq c_2\, d(z)^{-n-2s} \qquad \text{for } d(z) \geq 1.$$

Let us now turn to the case $d(z)$ small. Since $\partial\Omega$ is compact and smooth, d is a continuous function. For any $z \in \mathbb{R}^n \backslash \overline{\Omega}$, there exists $\pi(z) \in \partial\Omega$ such that $d(z) = |\pi(z) - z|$ and, by the interior sphere condition, there are $w_0 \in \Omega$ and $r > 0$ such that $B_r(w_0) \subset \Omega$ and $\partial B_r(w_0) \cap \partial\Omega = \{\pi(z)\}$. So

$$\int_{\Omega} \frac{dw}{|z-w|^{n+2s}} \geq \int_{B_r(w_0)} \frac{dw}{|z-w|^{n+2s}}.$$

[2]Note that $\mathbb{R}^n \backslash \overline{\Omega} \ni z \mapsto \int_{\Omega} |z-w|^{-n-2s} dw$ is a positive continuous function, thus it is locally bounded and bounded away from zero.

Up to a rotation and a translation, we can suppose $z = 0$ and $w_0 = \big(r+d(z)\big)e_1$. Then

$$\int_\Omega \frac{dw}{|z-w|^{n+2s}} \geq \int_{B_r\left((r+d(z))e_1\right)} \frac{dw}{|w|^{n+2s}} = \frac{1}{d(z)^{2s}} \int_{B_{r/d(z)}\left(\frac{r+d(z)}{d(z)}e_1\right)} \frac{d\xi}{|\xi|^{n+2s}}.$$

As $d(z) \downarrow 0$, we have the convergence

$$\int_{B_{r/d(z)}\left(\frac{r+d(z)}{d(z)}e_1\right)} \frac{d\xi}{|\xi|^{n+2s}} \longrightarrow \int_{\{\xi_1>1\}} \frac{d\xi}{|\xi|^{n+2s}},$$

so that also, for some $c_3 > 0$,

$$\int_\Omega \frac{dw}{|z-w|^{n+2s}} \geq c_3\, d(z)^{-2s} \qquad \text{for } d(z) < 1,$$

from which we conclude the validity of (13) by properly choosing the value of c. □

The following lemma computes an upper bound on k_β.

Lemma 2.2. *For a measurable $\beta : \mathbb{R}^n \setminus \Omega \to [0,1]$ and $x, y \in \Omega$, let k_β be defined as in (6). Then, for any $x \in \Omega$, there exists $C = C(x) > 0$ such that*

$$k_\beta(x,y) \leq C\big(1 + \big|\ln \operatorname{dist}(y, \operatorname{supp}(1-\beta))\big|\big) \qquad \textit{for any} \quad x, y \in \Omega. \tag{14}$$

Proof. Denote by $N := \operatorname{supp}(1-\beta)$. The term

$$\int_\Omega \frac{dw}{|z-w|^{n+2s}}, \qquad z \in N \subset \mathbb{R}^N \setminus \Omega$$

has been taken care of with Lemma 2.1. In order to prove (14), we plug (13) into (6), so that we are left with estimating

$$\int_{\mathbb{R}^n\setminus\Omega} \frac{1-\beta(z)}{|x-z|^{n+2s}|y-z|^{n+2s}\left(d(z)^{-2s} \wedge d(z)^{-n-2s}\right)}\, dz$$
$$\leq \int_N \frac{d(z)^{2s} \vee d(z)^{n+2s}}{|x-z|^{n+2s}|y-z|^{n+2s}}\, dz$$

in which we split the integration as follows

$$\int_{N\cap\{d(z)<1\}} \frac{d(z)^{2s}}{|x-z|^{n+2s}|y-z|^{n+2s}}\, dz + \int_{N\cap\{d(z)\geq 1\}} \frac{d(z)^{n+2s}}{|x-z|^{n+2s}|y-z|^{n+2s}}\, dz.$$

The second addend is obviously uniformly bounded (from above and below away from 0) in $x, y \in \Omega$. Therefore, we must concentrate on the first one: since x is fixed inside Ω, we can drop the term $|x-z|^{-n-2s}$. Then, since $d(z) < |y-z|$,

$$\begin{aligned}\int\limits_{N\cap\{d(z)<1\}} \frac{d(z)^{2s}}{|y-z|^{n+2s}}\,dz &\le \int\limits_{N\cap\{d(z)<1\}} \frac{dz}{|y-z|^{n}}\\ &\le \int\limits_{(\mathbb{R}^n\setminus B_{\mathrm{dist}(y,N)}(y))\cap\{d(z)<1\}} \frac{dz}{|y-z|^{n}}\\ &\le |\partial B|\left|\int\limits_{\mathrm{dist}(y,N)}^{\mathrm{diam}(\Omega)+2}\frac{dt}{t}\right| \le C\big(1+\big|\ln \mathrm{dist}(y,N)\big|\big)\end{aligned}$$

where $\mathrm{diam}(\Omega)=\sup\{|x-y| : x,y\in\Omega\}$. □

Finally, this lemma shows how the upper bound in (14) is optimal.

Lemma 2.3. *For a measurable $\beta:\mathbb{R}^n\setminus\Omega\to[0,1]$ and $x,y\in\Omega$, let k_β be defined as in (6). Suppose that, for some $\varepsilon>0$, $N_\varepsilon:=\{\beta<1-\varepsilon\}$ is nonempty. Then, for any $x\in\Omega$, there exists $C=C(x,\varepsilon)>0$ such that*

$$k_\beta(x,y)\ge\frac{1}{C}\big(1+\big|\ln \mathrm{dist}(y,N_\varepsilon)\big|\big). \tag{15}$$

Proof. To deduce such lower bound, remark that up to a smooth change of variable, the behaviour of the concerned integral (6) defining k_β is the same as

$$\int\limits_{\{0<z_1<1\}} \frac{z_1^{2s}}{|d_N(y)e_1+z|^{n+2s}}\,dz$$

which is easily computable as

$$\begin{aligned}&\int\limits_{\{0<z_1<1\}} \frac{z_1^{2s}}{|d_N(y)e_1+z|^{n+2s}}\,dz\\ &\quad=\int_0^1 z_1^{2s}\int\limits_{\mathbb{R}^{n-1}}\frac{dz'}{\Big(\big(\mathrm{dist}(y,N_\varepsilon)+z_1\big)^2+|z'|^2\Big)^{n/2+s}}\,dz'\,dz_1\\ &\quad=\int\limits_0^{1/\mathrm{dist}(y,N_\varepsilon)} t^{2s}\int\limits_{\mathbb{R}^{n-1}}\frac{d\xi}{\big((1+t)^2+|\xi|^2\big)^{n/2+s}}\,dt\\ &\quad=\int\limits_0^{1/\mathrm{dist}(y,N_\varepsilon)} \frac{t^{2s}}{(1+t)^{1+2s}}\int\limits_{\mathbb{R}^{n-1}}\frac{d\xi}{\big(1+|\xi|^2\big)^{n/2+s}}\,dt\\ &\quad\ge c\int\limits_1^{1/\mathrm{dist}(y,N_\varepsilon)}\frac{dt}{t}=c\,\big|\ln \mathrm{dist}(y,N_\varepsilon)\big|\end{aligned}$$

for some $c>0$. □

3. Proof of the main results.

Proof of Theorem 1.1. In the following, consider a function $u:\mathbb{R}^n\to\mathbb{R}$ satisfying (7), which, by (2), can be rewritten, for a.e. $x\in\mathbb{R}^n\backslash\Omega$, as

$$u(x)=\big(1-\beta(x)\big)\bigg(\int\limits_{\Omega}\frac{dy}{|x-y|^{n+2s}}\bigg)^{-1}\int\limits_{\Omega}\frac{u(y)}{|x-y|^{n+2s}}\,dy. \tag{16}$$

So, for a fixed $x\in\Omega$,

$$\begin{aligned}(-\triangle)^s u(x)&=c_{n,s}\ \text{p.v.}\!\int\limits_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\\&=c_{n,s}\ \text{p.v.}\!\int\limits_{\Omega}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy+c_{n,s}\int\limits_{\mathbb{R}^n\backslash\Omega}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\end{aligned}$$

and we plug (16) in the second addend obtaining

$$\begin{aligned}&(-\triangle)^s u(x)\\&\quad=(-\triangle)^s_\Omega u(x)\\&\qquad+c_{n,s}\int\limits_{\mathbb{R}^n\backslash\Omega}\frac{u(x)-\big(1-\beta(y)\big)\Big(\int_\Omega\frac{dz}{|y-z|^{n+2s}}\Big)^{-1}\int_\Omega\frac{u(z)}{|y-z|^{n+2s}}\,dz}{|x-y|^{n+2s}}\,dy\\&\quad=(-\triangle)^s_\Omega u(x)+u(x)\int\limits_{\mathbb{R}^n\backslash\Omega}\frac{\beta(y)}{|x-y|^{n+2s}}\,dy\\&\qquad+c_{n,s}\int\limits_{\mathbb{R}^n\backslash\Omega}\frac{u(x)\int_\Omega\frac{dz}{|y-z|^{n+2s}}-\int_\Omega\frac{u(z)}{|y-z|^{n+2s}}\,dz}{|x-y|^{n+2s}\int_\Omega\frac{dz}{|y-z|^{n+2s}}}\big(1-\beta(y)\big)\,dy\\&\quad=(-\triangle)^s_\Omega u(x)-u(x)(-\triangle)^s\beta(x)\\&\qquad+c_{n,s}\int\limits_{\mathbb{R}^n\backslash\Omega}\frac{\int_\Omega\frac{u(x)-u(z)}{|y-z|^{n+2s}}\,dz}{|x-y|^{n+2s}\int_\Omega\frac{dw}{|y-w|^{n+2s}}}\big(1-\beta(y)\big)\,dy \qquad\qquad (17)\\&\quad=(-\triangle)^s_\Omega u(x)-u(x)(-\triangle)^s\beta(x)\\&\qquad+c_{n,s}\int\limits_{\Omega}\big(u(x)-u(z)\big)\int\limits_{\mathbb{R}^n\backslash\Omega}\frac{1-\beta(y)}{|x-y|^{n+2s}|y-z|^{n+2s}\int_\Omega\frac{dw}{|y-w|^{n+2s}}}\,dy\,dz\\&\quad=(-\triangle)^s_\Omega u(x)-u(x)(-\triangle)^s\beta(x)+c_{n,s}\int\limits_{\Omega}\big(u(x)-u(z)\big)\,k_\beta(x,z)\,dz. \qquad (18)\end{aligned}$$

The exchange in the integration order from (17) to (18) is justified by Fubini's theorem, by noticing that for $x,z\in\Omega$ and $y\in\mathbb{R}^n\backslash\Omega$ (recall that u and β are bounded by assumption),

$$\frac{|u(x)-u(z)|\,\big(1-\beta(y)\big)}{|x-y|^{n+2s}|y-z|^{n+2s}\displaystyle\int_\Omega \frac{dw}{|y-w|^{n+2s}}} \le \frac{2\|u\|_{L^\infty(\Omega)}}{|x-y|^{n+2s}|y-z|^{n+2s}\displaystyle\int_\Omega \frac{dw}{|y-w|^{n+2s}}},$$

and

$$\int_\Omega \int_{\mathbb{R}^n\setminus\Omega} \frac{dy}{|x-y|^{n+2s}|y-z|^{n+2s}\displaystyle\int_\Omega \frac{dw}{|y-w|^{n+2s}}}\,dz$$

is finite by Lemma 2.2 (with $\beta \equiv 0$), which in turn also proves (8). This concludes the proof. □

Proof of Corollary 1.2. This is a straightforward consequence of Theorem 1.1, together with Lemma 2.3 for deducing the lower bound in (15). □

Acknowledgements. Open Access funding provided by Projekt DEAL.

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Received: 4 October 2019