






The Model Matching Problem for Max-Plus Linear Systems: A Geometric Approach

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Abstract—Linear systems over the max-plus algebra provide a suitable formalism to model discrete-event systems where synchronization, without competition, is involved. In this article, we consider a formulation of the model matching problem for systems of such class, in which the output of a given system, called the plant, is forced, by a suitable input, to track exactly that of a given model. A necessary and sufficient condition for its solvability is obtained by making a suitable use of geometric methods in the framework of systems over the max-plus algebra.

Index Terms—Discrete-event systems, geometric methods, manufacturing, max-plus systems.

I. INTRODUCTION

Linear systems over the max-plus algebra \mathbb{R}_{\max} were introduced in [1] as a suitable formalism to model discrete-event systems where synchronization, without competition, is involved (an in-depth discussion can be found in [2], while a good summary can be found in [3]). This class of systems coincides with timed event graphs, which are Petri nets where all the places have only one upstream and one downstream transition.

Control problems involving systems over the max-plus algebra are important in many applications, especially in industrial engineering [4]–[6], and a number of analysis and control techniques have been developed in the last years [7]–[10]. In particular, the development of a structural geometric approach for linear systems over the max-plus algebra has been indicated as a promising research direction in [3] and, since then, several results have been obtained along that line [9], [11], [12] and [13].

One of the problems considered in the literature consists in searching for a suitable control law that forces a given plant, modeled as a linear system over the max-plus algebra \mathbb{R}_{\max} , to behave accordingly to a given model of the same kind. Different formulations of this problem, usually referred to as the model matching problem (MMP), have been given in relation to max-plus systems. In particular, in [14], it is required that the output of the plant is forced to be greater than or equal to that of the model in a suitable ordering and the problem is dealt with by exploiting the properties of monotone formal series over a dioid.

Manuscript received 2 November 2021; revised 12 April 2022; accepted 26 June 2022. Date of publication 18 July 2022; date of current version 29 May 2023. Recommended by Associate Editor Bart De Schutter. (*Corresponding author: David Scaradozzi.*)

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Digital Object Identifier 10.1109/TAC.2022.3191362

In [15], the matching condition consists in minimizing the distance, in a suitable sense, between the output of the plant and that of a reference model while delaying as much as possible the control action. The tools used for solving the problem are based on residuation theory and some properties of the Kleene star operator. A similar formulation is considered in [16], where the problem is tackled by developing a three-block control structure that implements a precompensation and a feedback action. In a similar vein, we consider here the problem of forcing the plant to generate an output that equals that of a given model, giving in this way a formulation of the problem that more closely mimics the classical MMP for linear systems [17].

The contribution of our study consists in deriving a novel, simple structural necessary and sufficient condition for the solvability of the problem that can be practically checked under suitable conditions.

The rest of this article is organized as follows. In Section II, we introduce notions and tools of the max-plus algebra and we show by an example how they can be used to model a class of discrete-event systems as linear systems over \mathbb{R}_{\max} . Moreover, we formally state the MMP, we want to study. Section III contains the main result of this work that is a structural geometric necessary and sufficient solvability condition for the considered MMP. An example is provided in Section IV. Finally, Section V concludes this article.

II. BACKGROUND AND PROBLEM STATEMENT

In this section, we recall some basic concepts and the corresponding notation for max-plus linear systems. An example is also provided in order to show how a manufacturing system can be modeled using such formalism.

The max-plus algebra \mathbb{R}_{\max} consists of the set $\mathbb{R} \cup \{-\infty\}$, equipped with two operations denoted, respectively, by \oplus and by \otimes and defined by $a \oplus b = \max\{a, b\}$ for $a, b \in \mathbb{R}_{\max}$ and by $a \otimes b = a + b$ if a, b belongs to \mathbb{R} or by $(-\infty) \otimes a = a \otimes (-\infty) = -\infty$ for any $a \in \mathbb{R}_{\max}$. The neutral element for \oplus and for \otimes are denoted, respectively, by ϵ and by e and we have $\epsilon = -\infty$ and $e = 0 \in \mathbb{R}$. As \otimes distributes over \oplus , \mathbb{R}_{\max} is a semiring.

Semimodules over semirings are analogous to vector spaces over fields and the aspects of their theory that are considered in this article can be dealt with by standard tools of linear algebra, provided the fact that the elements of \mathbb{R}_{\max} , except ϵ , have no inverse with respect to \oplus is kept into account. An introduction to the theory of semimodules over \mathbb{R}_{\max} with application to the study of max-plus systems can be found in [9], [11], and [18]. By \mathbb{R}_{\max}^n , we denote the semimodule over \mathbb{R}_{\max} consisting of the set of all the n -tuples of elements of \mathbb{R}_{\max} , or vector, equipped with the component-wise and the scalar operations defined in a standard way in terms of \oplus and \otimes . If v and w are vectors of the same dimension, the relation $v \geq w$ is understood to hold component-wise. By abuse of notations, $\epsilon \in \mathbb{R}_{\max}^n$ will be used to denote the vector whose components are all equals to ϵ , also referred to as the null vector. Subsemimodules of \mathbb{R}_{\max}^n are subsets of elements that are closed with

respect to the component-wise and the scalar operations defined in terms of \oplus and \otimes . Subsemimodules will be denoted by script letters, as $\mathcal{V} \subseteq \mathbb{R}_{\max}^n$. A set of vectors $\{v_1, \dots, v_n\} \in \mathcal{V} \subseteq \mathbb{R}_{\max}^n$ is a set of generators for \mathcal{V} if any element $x \in \mathcal{V}$ can be written as a linear combination of the v_i s, namely $x = \bigoplus_{i=1}^n a_i \otimes v_i$. Note that not all the subsemimodules of \mathbb{R}_{\max}^n have a finite set of generators and those which enjoy this property are said to be finitely generated.

In particular, $(n \times m)$ -matrices with entries in \mathbb{R}_{\max} can be viewed as elements of $\mathbb{R}_{\max}^{n \times m}$, so that, given $A, B \in \mathbb{R}_{\max}^{n \times m}$ and using subscripts to denote, in a standard way, the elements of a matrix, their sum $A \oplus B \in \mathbb{R}_{\max}^{n \times m}$ is defined by $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$. Given $A \in \mathbb{R}_{\max}^{n \times m}$ and $B \in \mathbb{R}_{\max}^{m \times p}$, their product $A \otimes B \in \mathbb{R}_{\max}^{n \times p}$, or simply AB , is defined by $(A \otimes B)_{ij} = \bigoplus_{r=1}^m A_{ir} \otimes B_{rj}$. Analogously, given $A \in \mathbb{R}_{\max}^{n \times m}$ and $a \in \mathbb{R}_{\max}$, the scalar product $a \otimes A \in \mathbb{R}_{\max}^{n \times m}$, or simply aA , is defined by $(a \otimes A)_{ij} = a \otimes A_{ij}$. If no confusion arises, ϵ will also be used to indicate a matrix whose entries are all equal to ϵ . The identity matrix, whose diagonal elements are all equal to e and whose off-diagonal elements are all equal to ϵ , is denoted by I_n , where the subscript denotes the dimension, or simply by I whenever its dimension is clear from the context. If A is an $n \times m$ matrix the semimodule $\text{Im } A$ is the subsemimodule of \mathbb{R}_{\max}^n generated by the columns of A .

In modeling a situation in which events of n different types may occur, we can consider a so-called n -dimensional dater function $d(\cdot) : \mathbb{N} \rightarrow \mathbb{R}_{\max}^n$, whose value at $k \in \mathbb{N}$ is an n -dimensional vector $d(k) = (d_1(k), \dots, d_n(k))^T$ in which the i th component $d_i(k)$ indicates the time instant at which an event of the i th type occurs for the k th time. Dater functions must be nondecreasing (i.e., such that $d(k+1) \geq d(k)$ for each $k \in \mathbb{N}$) in order to have physical meaning.

Using the abovementioned notions, we can see a max-plus linear system Σ as a dynamical object whose evolution is defined by the equations

$$\Sigma \equiv \begin{cases} x(k+1) = Ax(k) \oplus Bu(k+1) \\ y(k) = Cx(k) \\ x(0) = \epsilon \end{cases} \quad (1)$$

where $k \in \mathbb{N}$ is the event instance index, $x(\cdot) : \mathbb{N} \rightarrow \mathcal{X} = \mathbb{R}_{\max}^n$ is the dater of internal events, $u(\cdot) : \mathbb{N} \rightarrow \mathcal{U} = \mathbb{R}_{\max}^m$ is the dater of input events, and $y(\cdot) : \mathbb{N} \rightarrow \mathcal{Y} = \mathbb{R}_{\max}^p$ is the dater of output events, $A \in \mathbb{R}_{\max}^{n \times n}$, $B \in \mathbb{R}_{\max}^{n \times m}$, and $C \in \mathbb{R}_{\max}^{p \times n}$. Coherently with the usual terminology for dynamical systems, the semimodules \mathcal{X} , \mathcal{U} , and \mathcal{Y} are called, respectively, the state module, the input module, and the output module of the system. Note that in Σ , we have n types of internal events that correspond to the n components of x , m types of input events that correspond to the m components of u and p types of output events that correspond to the p components of y . The vector $x(k) = (x_1(k), \dots, x_n(k))^T \in \mathbb{R}_{\max}^n$ indicates that the k th internal event of type i occurs at time $x_i(k)$ and a similar interpretation holds for $u(k)$ and for $y(k)$. A sequence $\{u(k+1)\}_{k \in \mathbb{N}}$ is viewed as an input to Σ , while a sequence $\{y(k+1)\}_{k \in \mathbb{N}}$ is viewed as an output of Σ .

Example 1: Two types of components, R1 and R2, are used in a manufacturing plant. These raw parts are processed by the machines M1 and M2, respectively, in order to obtain the semifinished parts S1 and S2. These intermediate parts are then assembled together by the machine M3, in order to obtain a unit of final product F1. The machines M1, M2, and M3 require 1, 2, and 4 time units, respectively, to execute their processing cycles. The plant is schematized, from an operational point of view, as shown in Fig. 1. We assume that all the machines involved can process only one part at a time, and that buffers of infinite capacity are present at each stage of the plant. Clearly, these assumptions can be suitably modified, still obtaining a max-plus linear model.

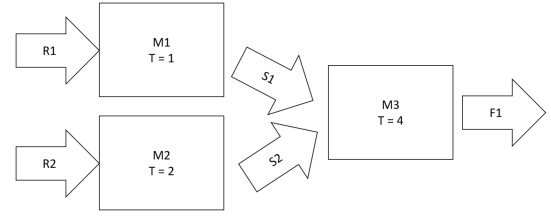


Fig. 1. Scheme of the plant.

TABLE I
DATERS AND EVENTS

$u_1(k)$	arrival time of the k th component of type R1
$u_2(k)$	arrival time of the k th component of type R2
$x_1(k)$	completion time of the k th cycle on the machine M1
$x_2(k)$	completion time of the k th cycle on the machine M2
$x_3(k)$	completion time of the k th cycle on the machine M3
$y(k)$	

The only type of event that is triggered internally by the system, and is visible from outside it, is the “completion of a cycle by the machine M3,” we will refer to this event as an output event. Two types of events are triggered by outside of the system: “arrival of a component of type R1” and “arrival of a component of type R2,” we refer to them as input events. We can consider as internal events the ones of type “completion of a cycle by the machine M1” and “completion of a cycle by the machine M2.” We can associate a dater function to each of these events, as reported in Table I.

Clearly, the values $u_1(k)$ and $u_2(k)$ need to be provided by some external source, while the values of the other variables can be computed using appropriate rules. Events of type “completion of a cycle on the machine M1” can occur only after the previous activity on M1 has been completed and one unit of time has elapsed after a new input raw part of type R1 has been made available. Assuming that all the activity starts as soon as possible, we can express this statement by the following equation:

$$\begin{aligned} x_1(k+1) &= \max \{x_1(k), u_1(k+1)\} + 1 \\ &= \max \{x_1(k) + 1, u_1(k+1) + 1\} \end{aligned}$$

and, using similar arguments, we have

$$\begin{aligned} x_2(k+1) &= \max \{x_2(k) + 2, u_2(k+1) + 2\} \\ x_3(k+1) &= \max \{x_1(k+1) + 4, x_2(k+1) \\ &\quad + 4, x_3(k) + 4\} \\ y(k) &= x_3(k). \end{aligned}$$

The two abovementioned equations are linear in the max-plus algebra and they can be written as

$$\begin{cases} x_1(k+1) = 1 \otimes x_1(k) \oplus 1 \otimes u_1(k+1) \\ x_2(k+1) = 2 \otimes x_2(k) \oplus 2 \otimes u_2(k+1) \\ x_3(k+1) = 4 \otimes x_1(k+1) \oplus \\ \quad 4 \otimes x_2(k+1) \oplus 4 \otimes x_3(k) \\ y(k) = x_3(k) \end{cases}$$

or, using a matrix notation and omitting the multiplication operator as usual, as

$$\begin{cases} x(k+1) = A_0 x(k+1) \oplus A_1 x(k) \oplus \\ \quad B'u(k+1) \\ y(k) = Cx(k) \end{cases} \quad (2)$$

with

$$A_0 = \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ 4 & 4 & \epsilon \end{pmatrix}, A_1 = \begin{pmatrix} 1 & \epsilon & \epsilon \\ \epsilon & 2 & \epsilon \\ \epsilon & \epsilon & \epsilon \end{pmatrix}, B' = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 2 \\ \epsilon & \epsilon \end{pmatrix}, C = (\epsilon \ \epsilon \ \epsilon).$$

The first equation of (2), due to the presence of the term $A_0x(k+1)$, describes an implicit relation that has to be made explicit in order to obtain an expression of the form (1). In the max-plus algebra, the least solution of the implicit equation $x = Ax \oplus b$ (that is: $x = \max\{Ax, b\}$) can be expressed, by means of the so-called Kleene star $A^* = \bigoplus_{n \in \mathbb{N}} A_0^n$, as $x = A^*b$, whenever A^* can be given a meaning. In our case, thanks to the fact that A_0 is lower triangular, we have $A_0^i = \epsilon$ for all $i \geq \dim A_0$ (note that this holds in general, provided that in the plant to be modeled there are no events that cannot be triggered because of mutual dependence or, in other words, if there are no cyclic paths in the graph describing the sequence of the plant's operations), and

therefore, we can compute A_0^* getting $A_0^* = \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ 4 & 4 & \epsilon \end{pmatrix}$. Then, we obtain

the following representation of the form (1) of the considered plant:

$$\begin{cases} x(k+1) = A_0^* A_1 x(k) \oplus A_0^* B' u(k+1) \\ \quad = Ax(k) \oplus Bu(k+1) \\ y(k) = Cx(k) \\ x(0) = \epsilon \end{cases}$$

with $A = \begin{pmatrix} 1 & \epsilon & \epsilon \\ \epsilon & 2 & \epsilon \\ 5 & 6 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 2 \\ 5 & 6 \end{pmatrix}$, $C = (\epsilon \ \epsilon \ \epsilon)$, and the condition $x(0) = \epsilon$ is taken accordingly to the derivation of the explicit form from the least solution of the implicit equation.

Assuming, for sake of illustration that raw parts of type R1 and R2 arrive together at the time instant 0 and, again, at the instant 1 and simulating the evolution of the system, we get

$$\begin{aligned} u(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} &\rightarrow x(1) = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} &y(1) = 6 \\ u(2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} &\rightarrow x(2) = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix} &y(2) = 10. \end{aligned}$$

Remark 1: In the physical world, the $(k+1)$ th occurrence of an event cannot anticipate the k th one and so, using dater representations, any input $\{u(k+1)\}_{k \in \mathbb{N}}$ is as a nondecreasing sequence. The same must hold for the sequence $\{x(k+1)\}_{k \in \mathbb{N}}$ that describes the state evolution and this implies that the dynamic matrix A in the defining (1) is greater than or equal to the identity matrix I_n , i.e., $A \geq I_n$, where I_n is the $n \times n$ -matrix with all elements on the diagonal equal to e and all other elements equal to ϵ . Decreasing inputs or systems with a dynamic matrix that is not greater than the identity matrix, or anticipative systems, are, therefore, of little interest and they will not be considered here.

We now have the tools to formalize the problem, we tackle in this article.

Problem 1 (MMP): Given a linear max-plus system

$$\Sigma_P \equiv \begin{cases} x_P(k+1) = A_P x_P(k) \oplus B_P u_P(k+1) \\ y_P(k) = C_P x_P(k) \\ x_P(0) = \epsilon \end{cases} \quad (3)$$

of the form (1), called the plant, and a linear max-plus system

$$\Sigma_M \equiv \begin{cases} x_M(k+1) = A_M x_M(k) \oplus B_M u_M(k+1) \\ y_M(k) = C_M x_M(k) \\ x_M(0) = \epsilon \end{cases} \quad (4)$$

of the form (1), called the model, with $x_P : \mathbb{N} \rightarrow \mathbb{R}_{\max}^{n_P}$, $x_M : \mathbb{N} \rightarrow \mathbb{R}_{\max}^{n_M}$, $u_P : \mathbb{N} \rightarrow \mathbb{R}_{\max}^{m_P}$, $u_M : \mathbb{N} \rightarrow \mathbb{R}_{\max}^{m_M}$, and $y_P, y_M : \mathbb{N} \rightarrow \mathbb{R}_{\max}^{p_P}$, the MMP consists in finding, for all possible nondecreasing input sequences $\{u_M(k+1)\}_{k \in \mathbb{N}}$ of the model, an appropriate nondecreasing control input sequence $\{u_P(k+1)\}_{k \in \mathbb{N}}$ for the plant, such that the output $\{y_P(k+1)\}_{k \in \mathbb{N}}$ of this latter equals the output $\{y_M(k+1)\}_{k \in \mathbb{N}}$ of the model, i.e., $y_P(k+1) = y_M(k+1)$ for all $k \in \mathbb{N}$.

A more restrictive formulation of the MMP is obtained by requiring that the control signal can be computed as a linear function of the state of the plant and of the state and the input of the model, which can be viewed as a feedback map.

Problem 2 (Feedback Model Matching Problem (FMMP)): Given a plant of the form (3) and a model of the form (4), the FMMP consists in finding, for all possible nondecreasing input sequences $\{u_M(k+1)\}_{k \in \mathbb{N}}$ of the model, two appropriate matrices $F \in \mathbb{R}_{\max}^{m_P \times (n_P + n_M)}$ and $G \in \mathbb{R}_{\max}^{m_P \times m_M}$ such that the control input sequence $\{u_P(k+1)\}_{k \in \mathbb{N}}$ defined by

$$u_P(k+1) = F \begin{pmatrix} x_P(k) \\ x_M(k) \end{pmatrix} \oplus G u_M(k+1) \text{ for } k \geq 0 \quad (5)$$

is a solution for the corresponding MMP.

Remark 2: Since in the given formulation of the FMMP the entries of F and G are not required to be positive real numbers, it may happen that some components of $u_P(k+1)$ computed by (5) are smaller than some component of $u_M(k+1)$, or even that $u_P(k+1) < u_M(k+1)$. According to the meaning of the daters $u_P(\cdot)$ and $u_M(\cdot)$, this implies that, in order to satisfy the matching requirement, the plant must receive (some components of) its $(k+1)$ -input before the model has received (some components of) its $(k+1)$ -input. Therefore, since the knowledge of $u_M(k+1)$ is necessary to compute $u_P(k+1)$, the control law constructed by means of (5) is physically implementable only if the time at which the model receives (each component of) its k -input is appropriately known in advance. For instance, this happens in practice when the sequence of input events of the model is scheduled in advance to satisfy some specific requirements or to comply with some specific policy. In such cases, the value of $x_P(k)$ and $x_M(k)$ can be obtained by simulating the joint dynamics, using the previous states and the scheduled value for $u_M(k)$.

III. SOLUTION OF THE PROBLEMS

Given the plant Σ_P described by (3) and the model Σ_M described by (4), let us consider the joint internal event dater $x_E(\cdot) = \begin{pmatrix} x_P(\cdot) \\ x_M(\cdot) \end{pmatrix} : \mathbb{N} \rightarrow \mathbb{R}_{\max}^{(n_P + n_M)}$ and the related joint dynamics, which is described by the equation

$$x_E(k+1) = A_E x_E(k) \oplus B_1 u_P(k+1) \oplus B_2 u_M(k+1) \quad (6)$$

with $A_E = \begin{pmatrix} A_P & \epsilon \\ \epsilon & A_M \end{pmatrix}$, $B_1 = \begin{pmatrix} B_P \\ \epsilon \end{pmatrix}$, $B_2 = \begin{pmatrix} \epsilon \\ B_M \end{pmatrix}$ and $x_E(0) = \epsilon$.

It is possible, then, to reformulate the control problem expressed in Problem 1 as that of finding, for any input $\{u_M(k+1)\}_{k \in \mathbb{N}}$, a control input $\{u_P(k+1)\}_{k \in \mathbb{N}}$ that keeps $x_E(k)$ inside the *output equalizer* subsemimodule $\mathcal{K} \subseteq \mathbb{R}_{\max}^{(n_P + n_M)}$ defined by

$$\mathcal{K} = \{x_E = \begin{pmatrix} x_P \\ x_M \end{pmatrix} \in \mathbb{R}_{\max}^{(n_P + n_M)}, \text{ s. t. } C_P x_P = C_M x_M\}. \quad (7)$$

In the framework of max-plus systems, control problems of this type, in which the control objective is that of constraining $x(k)$ inside a given subsemimodule, have been considered and dealt with by employing a structural geometric approach in [9] and [19].

Remark 3: Viewing $\{u_M(k+1)\}_{k \in \mathbb{N}}$ as a disturbance input and $\{u_P(k+1)\}_{k \in \mathbb{N}}$ as a control input, the abovementioned formulation essentially reduces the MMP to a disturbance decoupling problem, as it happens in the case of classical linear systems with coefficients in a field or in a ring. In those cases, it is possible to consider the difference between the output of the plant and that of the model and to describe the resulting control problem as that of keeping the state inside the kernel of such function [20]. Differences cannot be computed in the max-plus algebra and, in general, it is not possible to find a matrix C_E such that the *output equalizer* subsemimodule \mathcal{K} defined in (7) is the kernel of C_E , neither using the classical definition of kernel (i.e., $\text{Ker } A = \{x \in \mathbb{R}_{\max}^{n_r}, \text{ such that } Ax = \epsilon\}$, where n_r is the number of rows of A), nor the alternative definition, more convenient in the max-plus case, given by some authors (see, for instance, [12]) in terms of congruence (i.e., $\text{Ker} = \{(x, y) \in \mathbb{R}_{\max}^{2n_r}, \text{ such that } Ax = Ay\}$). This means that it is not possible to associate to the dynamics (6) a linear output map that represents the difference between the output of the plant and that of the model. Note, however, that the *output equalizer* subsemimodule \mathcal{K} defined in (7) is the pull-back, in the category of modules over \mathbb{R}_{\max} , of the pair of maps (C_P, C_M) and, as such, it generalizes the notion of kernel of the difference between these two maps [21]. Note that it would be possible to proceed differently by introducing the operator \odot , defined over \mathbb{R}_{\max} by $a \odot b = a - b$ for $a, b \in \mathbb{R}$, by $\epsilon \odot a = a \odot \epsilon = a$ for any $a \in \mathbb{R}_{\max}$ and by $\epsilon \odot \epsilon = e$. This, actually makes it possible to write

$$y_E(k) = y_P(k) \odot y_M(k) = C_P x(k) \odot C_M x_M(k) \quad (8)$$

and the MMP reduces to find, for any disturbance input $\{u_M(k+1)\}_{k \in \mathbb{N}}$, a control input $\{u_P(k+1)\}_{k \in \mathbb{N}}$ such that $y_E(k) = e$ for all $k \in \mathbb{N}$. However, the output (8) is nonlinear and it cannot be conveniently handled in this context. For such reason, we do not follow this approach.

In order to deal with the MMP in the formulation that consists in keeping $x_E(k)$ inside the output equalizer subsemimodule \mathcal{K} , it is convenient to make use of a structural geometric approach that extends those developed for classical linear systems with coefficients in a field in [22] and [23]. Extensions of such approach to systems with coefficients in a ring [24]–[26] and, then, in a semiring [3], [9], [11]–[13] have been considered by several authors. Here, we recall the basic notion of controlled invariant subsemimodule for max-plus linear systems given in [9] and some related results.

Definition 1: Given a max-plus linear system Σ of the form (1), a subsemimodule $\mathcal{V} \subseteq \mathbb{R}_{\max}^n$ is said to be an (A, B) -controlled invariant subsemimodule if for all $v \in \mathcal{V}$ there exists $u \in \mathbb{R}_{\max}^m$ such that $(Av \oplus Bu)$ belongs to \mathcal{V} .

Given a max-plus linear system Σ of the form (1) and a subsemimodule of its state semimodule $\mathcal{K} \subseteq \mathbb{R}_{\max}^n$, the set of all the (A, B) -controlled invariant subsemimodules contained in \mathcal{K} is a semilattice with respect to inclusion and sum of semimodules, so a maximum element, denoted $\mathcal{V}^*(\mathcal{K})$, exists.

The sequence of semimodules \mathcal{V}_k defined by

$$\begin{aligned} \mathcal{V}_0 &= \mathcal{K} \\ \mathcal{V}_k &= \mathcal{V}_{k-1} \cap A^{-1}(\mathcal{V}_{k-1} \oplus \text{Im } B) \end{aligned} \quad (9)$$

where $A^{-1}(\mathcal{V}) = \{v \in \mathbb{R}_{\max}^n, \text{ such that } Av \in \mathcal{V}\}$ and $\mathcal{V}_{k-1} \oplus \text{Im } B = \{x \in \mathbb{R}_{\max}^n, \text{ for which there exists } u \in \mathbb{R}_{\max}^m \text{ such that } x \oplus Bu \in \mathcal{V}_{k-1}\}$, is such that $\mathcal{V}_{k+1} \subseteq \mathcal{V}_k$ for all $k \in \mathbb{N}$. If we define $\mathcal{V}_\infty = \lim_{k \rightarrow \infty} \mathcal{V}_k = \bigcap_{k \in \mathbb{N}} \mathcal{V}_k$, then every (A, B) -invariant semimodule contained in \mathcal{K}

is also contained in \mathcal{V}_∞ . Moreover, $\mathcal{V}_{k+1} = \mathcal{V}_k$ if and only if \mathcal{V}_k is (A, B) -invariant and in such case $\mathcal{V}_\infty = \mathcal{V}_k = \mathcal{V}^*(\mathcal{K})$. As it happens for systems with coefficients in a ring, the sequence (9) does not necessarily converge in a finite number of steps, and therefore, it does not provide, in general, an algorithm for the computation of $\mathcal{V}^*(\mathcal{K})$, as its counterpart does for systems with coefficients in a field [23].

Remark 4: If \mathcal{K} is finitely generated, then the semimodules \mathcal{V}_k are finitely generated for all $k \in \mathbb{N}$. In fact, given some finitely generated semimodules \mathcal{Z} and \mathcal{Y} , the semimodules $\mathcal{Y} \oplus \mathcal{Z}$, $A^{-1}(\mathcal{Y})$, and $\mathcal{Y} \cap \mathcal{Z}$ are all finitely generated (see [18, Corollary 86]). As explained in [9, Remark 1], their generators can be obtained as the set of solutions of appropriate systems of linear equations over the max-plus algebra \mathbb{R}_{\max} .

Definition 2: Given a max-plus linear system Σ of the form (1), a subsemimodule $\mathcal{V} \subseteq \mathbb{R}_{\max}^n$ is said to be an (A, B) -controlled invariant subsemimodule of feedback type if there exists a matrix $F \in \mathbb{R}_{\max}^{m \times n}$ such that $(A \oplus BF)v$ belongs to \mathcal{V} for all $v \in \mathcal{V}$.

Clearly, (A, B) -controlled invariance of feedback type implies (A, B) -controlled invariance. In the framework of systems with coefficients in a field, the two properties are known to be equivalent [22], [23], but this is not true in the case of systems with coefficients in a ring [24], [25] or in a semiring [9] and, in particular, in the case of the max-plus systems considered here.

We can now state the main result of this work that is a necessary and sufficient condition for the existence of a solution to the MMP.

Theorem 1: Given a plant Σ_P of the form (3) and a model Σ_M of the form (4), assume that $A_P \geq I_{n_P}$ and $A_M \geq I_{n_M}$. Then, the related MMP is solvable if and only if for all $x \in \text{Im } B_2 = \text{Im} \begin{pmatrix} \epsilon \\ B_M \end{pmatrix}$ there

exists $y \in \text{Im } B_1 = \text{Im} \begin{pmatrix} B_P \\ \epsilon \end{pmatrix}$ such that $x \oplus y$ belongs to $\mathcal{V}^*(\mathcal{K})$, where $\mathcal{V}^*(\mathcal{K})$ is the maximum (A_E, B_1) -invariant semimodule contained in the output equalizer semimodule \mathcal{K} defined by (7).

Proof: *If.* By (A_E, B_1) -controlled invariance of $\mathcal{V}^*(\mathcal{K})$, it follows that given $x_E(k) \in \mathcal{V}^*(\mathcal{K})$, there exists $u_1(k+1) \in \mathbb{R}_{\max}^{m_P}$ such that $A_E x_E(k) \oplus B_1 u_1(k+1)$ belongs to $\mathcal{V}^*(\mathcal{K})$. Moreover, by hypothesis, given $u_M(k+1) \in \mathbb{R}_{\max}^{m_M}$, there exists $u_2(k+1) \in \mathbb{R}_{\max}^{m_P}$ such that $B_1 u_2(k+1) \oplus B_2 u_M(k+1)$ belongs to $\mathcal{V}^*(\mathcal{K})$. We can then construct recursively a control input $\{u_P(k+1)\}_{k \in \mathbb{N}}$ for the dynamics (6) as

$$u_P(k+1) = \begin{cases} u_2(1) & \text{for } k = 0 \\ u_1(k+1) \oplus u_2(k+1) \oplus u_P(k) & \text{for } k > 0. \end{cases} \quad (10)$$

More precisely, we start by taking $u_2(1)$ such that $B_1 u_2(1) \oplus B_2 u_M(1) \in \mathcal{V}^*(\mathcal{K})$ and we set $u_P(1) = u_2(1)$. Then, we compute $x_E(1)$ by means of (6), $x_E(0)$, $u_P(1)$, and $u_M(1)$, and we take $u_1(2)$ and $u_2(2)$ such that $A_E x_E(1) \oplus B_1 u_1(2) \in \mathcal{V}^*(\mathcal{K})$ and $B_1 u_2(2) \oplus B_2 u_M(2) \in \mathcal{V}^*(\mathcal{K})$. We set $u_P(2) = u_1(2) \oplus u_2(2) \oplus u_P(1)$ and we iterate the same procedure increasing by 1 the index k at each step. Note that the sequence $\{u_P(k+1)\}_{k \in \mathbb{N}}$, thanks to the presence of the term $u_P(k)$ in the second equation of (10), is nondecreasing and it gives rise to the following state evolution:

$$x_E(k+1) = \begin{cases} A_E x_E(0) \oplus B_1 u_1(1) \oplus B_2 u_M(1) & \text{for } k = 0 \\ A_E x_E(k) \oplus B_1 u_1(k+1) \oplus (B_1 u_2(k+1) \oplus B_2 u_M(k+1)) \oplus B_1 u_P(k) & \text{for } k > 0 \end{cases} \quad (11)$$

In (11), the term $B_1 u_P(k)$ is irrelevant, since, by induction, we have $x_E(k) \geq B_1 u_P(k)$ and hence, thanks to the assumption $A_E \geq I$, also $A_E x_E(k) \geq B_1 u_P(k)$. Then, disregarding this last term, we can show by induction that the state evolution $\{x_E(k)\}_{k \in \mathbb{N}}$ given in (11) is contained in $\mathcal{V}^*(\mathcal{K})$. In fact, $x_E(0) = \epsilon$ belongs to $\mathcal{V}^*(\mathcal{K})$. Moreover, by the definition of $u_1(\cdot)$, it follows that the summand

$(A_E x_E(k) \oplus B_1 u_1(k+1))$ in the right-hand term of (11) is contained in $\mathcal{V}^*(\mathcal{K})$ if $x_E(k)$ is contained in $\mathcal{V}^*(\mathcal{K})$. Finally, the second summand $(B_1 u_2(k+1) \oplus B_2 u_M(k+1))$ in the right-hand term of (11) is contained in $\mathcal{V}^*(\mathcal{K})$ by the definition of $u_2(k+1)$. Since $\mathcal{V}^*(\mathcal{K}) \subseteq \mathcal{K}$, by the definition of \mathcal{K} given in (7), it follows that the output $\{y_P(k+1)\}_{k \in \mathbb{N}}$ of the plant generated by the input $\{u_P(k+1)\}_{k \in \mathbb{N}}$ defined by (10) is equal to the output $\{y_M(k+1)\}_{k \in \mathbb{N}}$ of the model generated by the input $\{u_M(k+1)\}_{k \in \mathbb{N}}$ and the MMP is solved.

Only if: If the condition of the theorem does not hold, there exists an input vector \bar{u}_M such that $B_2 \bar{u}_M \oplus B_1 u_P \notin \mathcal{V}^*(\mathcal{K})$ for any $u_P \in \mathbb{R}_{\max}^{m_P}$. Then, for the constant input $\{u_M(k+1)\}_{k \in \mathbb{N}}$ with $u_M(k+1) = \bar{u}_M$ for $k \in \mathbb{N}$, we have, from (6) that $x_E(1) = A_E x_E(0) \oplus B_1 u_P(1) \oplus B_2 u_M(1) = A_E \epsilon \oplus B_1 u_P(1) \oplus B_2 \bar{u}_M = B_1 u_P(1) \oplus B_2 \bar{u}_M$ does not belong to $\mathcal{V}^*(\mathcal{K})$ for any value $u_P(1) \in \mathbb{R}_{\max}^{m_P}$ and also $x_E(1) \geq B_2 \bar{u}_M$. The latter inequality, thanks to the assumption $A_E \geq I$, implies recursively $x_E(k+1) = A_E x_E(k) \oplus B_1 u_P(k+1) \oplus B_2 u_M(k+1) = A_E x_E(k) \oplus B_1 u_P(k+1) \oplus B_2 \bar{u}_M = A_E x_E(k) \oplus B_1 u_P(k+1)$, while the fact that $x_E(1)$ does not belong to $\mathcal{V}^*(\mathcal{K})$ implies that for any input $\{u_P(k+1)\}_{k \in \mathbb{N}}$ there exists $\bar{k} \in \mathbb{Z}$ such that $x_E(\bar{k}+1) = A_E x_E(\bar{k}) \oplus B_1 u_P(\bar{k}+1) \notin \mathcal{K}$. In other words, $x_E(k+1)$ cannot be kept indefinitely inside the subsemimodule \mathcal{K} and, as a consequence, the MMP cannot be solved. \square

Remark 5: The condition expressed by Theorem 1 can be equivalently written, using the \ominus operator introduced in (9), as $\text{Im} B_2 \subseteq \mathcal{V}^*(\mathcal{K}) \ominus \text{Im} B_1$ and it can be practically checked using the same techniques described in Remark 4.

For the feedback version stated in Problem 2, we can state the following solvability condition.

Theorem 2: Given a plant Σ_P of the form (3) and a model Σ_M of the form (4) as in Theorem 1, the related FMMP is solvable if and only if there exists an (A_E, B_1) -invariant subsemimodule \mathcal{V} of feedback type contained in the output equalizer subsemimodule \mathcal{K} such that for all $x \in \text{Im} B_2 = \text{Im} \begin{pmatrix} \epsilon \\ B_M \end{pmatrix}$ there exists $y \in \text{Im} B_1 = \text{Im} \begin{pmatrix} B_P G \\ \epsilon \end{pmatrix}$ with $x \oplus y \in \mathcal{V}$.

Proof: Let $\mathcal{V} \subseteq \mathcal{K}$ be an (A_E, B_1) -invariant subsemimodule of feedback type for which the condition of the theorem holds. Then, by hypothesis, there exists a matrix F such that for each $x_E(k) \in \mathcal{V}$, $(A_E \oplus B_1 F)x_E(k)$ belongs to \mathcal{V} and a matrix G such that the columns of the matrix $\begin{pmatrix} \epsilon \\ B_M \end{pmatrix} I_{m_M} \oplus \begin{pmatrix} B_P G \\ \epsilon \end{pmatrix} = \begin{pmatrix} B_P G \\ B_M \end{pmatrix}$ belong to \mathcal{V} . We can then construct recursively a control input $\{u_P(k+1)\}_{k \in \mathbb{N}}$, of the form (5), for the dynamics (6) as

$$u_P(k+1) = F x_E(k) \oplus G u_M(k+1). \quad (12)$$

More precisely, we start by taking $u_P(1) = F x_E(0) \oplus G u_M(1) = G u_M(1)$. Then, we compute $x_E(1)$ by means of (6), $x_E(0)$, $u_P(1)$ and $u_M(1)$, and we take $u_P(2) = F x_E(1) \oplus G u_M(2)$. We iterate the same procedure increasing by 1 the index k at each step. The sequence $\{x_E(k+1)\}_{k \in \mathbb{N}}$ that is generated, together with $\{u_P(k+1)\}_{k \in \mathbb{N}}$, during the iteration, is nondecreasing, since $A_E \geq I_{n_P+n_M}$. Since also $\{u_M(k+1)\}_{k \in \mathbb{N}}$ is assumed to be nondecreasing, the input $\{u_P(k+1)\}_{k \in \mathbb{N}}$ turns out to be nondecreasing. The resulting state evolution

$$x_E(k+1) = (A_E \oplus B_1 F)x_E(k) \oplus \begin{pmatrix} B_P G \\ B_M \end{pmatrix} u_M(k+1) \quad (13)$$

clearly evolves in $\mathcal{V} \subseteq \mathcal{K}$ and, hence, the MMP is solved.

Only if. Assume that the FMMP is solved by a control law of the form (5). Then, the set of reachable states for the dynamics (13) is an (A, B) -invariant subsemimodule of feedback type contained in \mathcal{K} that contains

all the columns of the matrix $\begin{pmatrix} B_P G \\ B_M \end{pmatrix} = \begin{pmatrix} \epsilon \\ B_M \end{pmatrix} I_{m_M} \oplus \begin{pmatrix} B_P \\ \epsilon \end{pmatrix} G$. This clearly implies the condition of the theorem. \square

Remark 6: The solvability condition for the FMMP given in Theorem 2 is stronger than the solvability condition for the MMP given in Theorem 1, since any (A, B) -controlled invariant of feedback type \mathcal{V} contained in \mathcal{K} is also contained in $\mathcal{V}^*(\mathcal{K})$ due to the maximality of the latter, and $\mathcal{V}^*(\mathcal{K})$ is not necessarily of feedback type. Therefore, the solvability of the FMMP implies the solvability of the MMP and any solution of the first is a solution also of the second. In particular, if the solvability condition given in Theorem 2 is satisfied, the solution $\{u_P(k+1)\}_{k \in \mathbb{N}}$ given by (12), being nondecreasing, can be expressed as

$$u_P(k+1) = \begin{cases} G u_M(1) & \text{for } k = 0 \\ F x_E(k) \oplus G u_M(k+1) \oplus u_P(k) & \text{for } k > 0 \end{cases} \quad (14)$$

Letting $u_1(k+1) = F x_E(k)$ and $u_2(k+1) = G u_M(k+1)$, (14) shows that the procedure indicated in the proof of Theorem 2 provides the same control input we can have from (10) in the proof of Theorem 1.

Remark 7: In practice, the condition of Theorem 1 can be checked and the elements $u_1(k+1)$, $u_2(k+1)$ that are needed in (10) to construct the control input $\{u_P(k+1)\}_{k \in \mathbb{N}}$ can be found by solving systems of linear equations over the max-plus algebra \mathbb{R}_{\max} that involve the matrices A_E, B_1, B_2 and the generators of $\mathcal{V}^*(\mathcal{K})$. The same holds for the condition of Theorem 1 and for the matrices F, G that are needed in (12). Since inverses with respect to \oplus do not exist in \mathbb{R}_{\max} , the general systems of linear equations in the vector indeterminate ξ over \mathbb{R}_{\max} one has to deal with takes the bilateral form

$$M \xi \oplus \alpha = N \xi \oplus \beta \quad (15)$$

where M and N are known matrices and α and β are known vectors of compatible dimensions with elements in \mathbb{R}_{\max} (see [2]). Computational problems that arise in dealing with such systems of equations are illustrated in [9] and [18]. Elimination methods can be applied to find solutions, as described in [9] and [27], possibly with the aid of a dedicated Scilab toolbox [28]. An example of how to deal in practice with the computations involved in solving the FMMP is given in the following section.

IV. EXAMPLE

In this section, we provide an example to illustrate the previous results.

Let us consider a plant Σ_P of the form (3) described by

$$\Sigma_P \equiv \begin{cases} x_P(k+1) = \begin{pmatrix} e & e \\ 2 & e \end{pmatrix} x_P(k) \oplus \begin{pmatrix} 2 \\ e \end{pmatrix} u_P(k+1) \\ y_P(k) = (e \ e) x_P(k) \\ x_P(0) = \epsilon \end{cases}.$$

and a model Σ_M of the form (4) described by

$$\Sigma_M \equiv \begin{cases} x_M(k+1) = 3x_M(k) \oplus 4u_M(k+1) \\ y_M(k) = x_M(k) \\ x_M(0) = \epsilon \end{cases}.$$

The related joint dynamics is given by

$$x_E(k+1) = A_E x_E(k) \oplus B_1 u_P(k+1) \oplus B_2 u_M(k+1)$$

with $x_E = \begin{pmatrix} x_P \\ x_M \end{pmatrix}$ and

$$A_E = \begin{pmatrix} e & e & \epsilon \\ 2 & e & \epsilon \\ \epsilon & \epsilon & 3 \end{pmatrix}, B_1 = \begin{pmatrix} 2 \\ e \\ \epsilon \end{pmatrix}, B_2 = \begin{pmatrix} \epsilon \\ \epsilon \\ 4 \end{pmatrix}.$$

The output equalizer subsemimodule \mathcal{K} is easily seen to be given by the set of vectors in \mathbb{R}_{\max}^3 whose last component is equal to the maximum of the first two components, namely

$$\mathcal{K} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}_{\max}^3 \text{ with } \max\{x_1, x_2\} = x_3 \right\} \quad (16)$$

and we also have $\mathcal{K} = \text{Im } K = \text{Im} \begin{pmatrix} e & e \\ e & e \end{pmatrix}$. Note that \mathcal{K} is not A_E -invariant, since, e.g., $\begin{pmatrix} 2 & e & e \\ e & e & e \end{pmatrix} \begin{pmatrix} e \\ e \\ e \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ does not belong to \mathcal{K} , but it can be proved to be (A_E, B_1) -invariant of feedback type and, hence, $\mathcal{V}^*(\mathcal{K}) = \mathcal{K}$.

To show that \mathcal{K} is (A_E, B_1) -invariant of feedback type, we need to solve the two sided linear max-plus system of equations

$$(A_E \oplus B_1 F)K = KQ \quad (17)$$

whose set of solutions (F, Q) is a finitely generated max-plus set [9]. This can be done, first, by rewriting (17) as a system of max-plus linear equations in the vector indeterminate ξ of the form (15) and, then, by solving it by implementing the technique presented in [27]. To accomplish the first step, let us consider separately each column K_i , with $i = 1, 2$, of K in the left-hand side of (17), so obtaining a set of equations of the form

$$(A_E \oplus B_1 F)K_i = A_E K_i \oplus B_1 F K_i = K Q_i \text{ for } i = 1, 2. \quad (18)$$

Now, for the max-plus product $F K_i$, we can write $F K_i = \Delta(K_1^\top, K_2^\top)f$ where f is a vector in $\mathbb{R}_{\max}^{(n_M+n_P)m_P}$ that consists of all the columns of F stacked on top of each other in lexicographic order and $\Delta(K_1^\top, K_2^\top)$ is a block-diagonal matrix in $\mathbb{R}_{\max}^{n_P \times ((n_P+n_M)m_P)}$ whose diagonal blocks are all equal to K_i^\top . Substituting in (18), we get

$$B_1 \Delta(K_1^\top, K_2^\top)f \oplus A_E K_i = K Q_i \text{ for } i = 1, 2 \quad (19)$$

that is an equation of the form (15) with $\xi = \begin{pmatrix} f \\ Q_i \end{pmatrix} \in \mathbb{R}_{\max}^{(n_P+n_M)m_P+2}$,

$M = (B_1 \Delta(K_1^\top, K_2^\top) \epsilon) \in \mathbb{R}_{\max}^{(n_P+n_M) \times ((n_P+n_M)m_P+2)}$, $\alpha = A_E K_i \in \mathbb{R}_{\max}^{n_P+n_M}$, $N = (\epsilon K)$, $\beta = \epsilon$, and N and β have the same dimensions of M and α , respectively.

Using elimination methods, as mentioned in Remark 7, it is possible to find a solution of (19) and then a solution

$$(F, Q) = \left((1 \ 1 \ \epsilon), \begin{pmatrix} 3 & 3 \\ 2 & 1 \end{pmatrix} \right)$$

of (17) [we do not report here the single elementary steps of the computation, but the reader can easily verify that the pair (F, Q) given above solves (17)]. In particular, the control law $u_P(k+1) = F x_E(k)$ keeps inside \mathcal{K} any state evolution, which starts in \mathcal{K} .

The condition of Theorem 2 is satisfied for $\mathcal{V} = \mathcal{V}^*(\mathcal{K}) = \mathcal{K}$. In

fact, any $x \in \text{Im } B_2$ is of the form $x = B_2 a = \begin{pmatrix} e & e \\ e & e \end{pmatrix} a = \begin{pmatrix} e & e \\ 4 & 4+a \end{pmatrix}$ with

$a \in \mathbb{R}_{\max}$ and, taking $y = B_1 b = \begin{pmatrix} 2 & 2+b \\ e & e \end{pmatrix} b = \begin{pmatrix} b & b \end{pmatrix}$ with $b = 2a \in$

\mathbb{R}_{\max} , we have that $x \oplus y = \begin{pmatrix} 2+a & 2+a \\ 4+a & 4+a \end{pmatrix}$ belongs to \mathcal{K} . In particular, the

columns of the matrix $\begin{pmatrix} e & e \\ B_M \end{pmatrix} I_{m_M} \oplus \begin{pmatrix} B_P \\ e \end{pmatrix} G = \begin{pmatrix} B_P G \\ B_M \end{pmatrix} = \begin{pmatrix} 2 & G \\ G \\ 4 \end{pmatrix}$

belongs to \mathcal{K} for $G = (2)$. Hence, according to the proof of Theorem 2, a solution to the FMMP turns out to be given by the control law

$$\begin{aligned} u_P(k+1) &= F x_E(k) \oplus G u_M(k+1) \\ &= (1 \ 1 \ \epsilon) x_E(k) \oplus 2 u_M(k+1) \\ &= 1 x_{P1}(k) \oplus 1 x_{P2}(k) \oplus 2 u_M(k+1) \end{aligned} \quad (20)$$

that is of the form (5). In fact, by substituting $u_P(k+1)$ with the abovementioned expression in the joint dynamics (6) we get

$$\begin{pmatrix} x_{P1}(k+1) \\ x_{P2}(k+1) \\ x_M(k+1) \end{pmatrix} = \begin{pmatrix} 3 & 3 & \epsilon \\ 2 & 1 & \epsilon \\ \epsilon & \epsilon & 3 \end{pmatrix} \begin{pmatrix} x_{P1}(k) \\ x_{P2}(k) \\ x_M(k) \end{pmatrix} \oplus \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} u_M(k+1)$$

and

$$\begin{aligned} y_P(k+1) &= x_{P1}(k+1) \oplus x_{P2}(k+1) \\ &= 3 x_{P1}(k) \oplus 3 x_{P2}(k) \oplus 4 u_M(k+1) \\ &= 3 y_P(k) \oplus 4 u_M(k+1) \\ y_M(k+1) &= x_M(k+1) \\ &= 3 x_M(k) \oplus 4 u_M(k+1) \\ &= 3 y_M(k) \oplus 4 u_M(k+1). \end{aligned}$$

From the last expression, since $x_P(0) = x_M(0) = \epsilon$, it is easy to see by induction that $\{y_P(k+1)\}_{k \in \mathbb{N}}$ is equal to $\{y_M(k+1)\}_{k \in \mathbb{N}}$. Obviously, the found solution of the FMMP is a solution also of the MMP and, as explained in Remark 6, the control law (20) satisfies the requirement of Problem 1.

V. CONCLUSION

A natural formulation of the matching problem for max-plus systems has been given and the problem has been tackled in the framework of the geometric approach. This provides solvability conditions that basically extend those originally found in the framework of classical linear systems, showing, in particular, the validity and the potential of the geometric approach also for this class of systems. Future work will consider the extension of the methods and of the results to max-plus systems that have a switching structure and can, therefore, model the behaviour of real plants in varying operational conditions.

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