ARCHIVIO ISTITUZIONALE
DELLA RICERCA

Alma Mater Studiorum Università di Bologna
Archivio istituzionale della ricerca

On K-stability of some del Pezzo surfaces of Fano index 2

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

Published Version:
Liu, Y., Petracci, A. (2022). On K-stability of some del Pezzo surfaces of Fano index 2. BULLETIN OF THE LONDON MATHEMATICAL SOCIETY, 54(2), 517-525 [10.1112/blms.12581].

Availability:
This version is available at: https://hdl.handle.net/11585/865359 since: 2022-09-24

Published:

DOI: http://doi.org/10.1112/blms.12581

Terms of use:

Some rights reserved. The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (https://cris.unibo.it/).
When citing, please refer to the published version.

(Article begins on next page)

08 October 2026

This is the final peer-reviewed accepted manuscript of:

Liu, Y., & Petracci, A. (2022). On K-stability of some del pezzo surfaces of fano index 2. Bulletin of the London Mathematical Society, 54(2), 517-525

The final published version is available online at ***https://dx.doi.org/10.1112/blms.12581***

Rights / License:

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

This item was downloaded from IRIS Università di Bologna (https://cris.unibo.it/)

When citing, please refer to the published version.

ON K-STABILITY OF SOME DEL PEZZO SURFACES OF FANO INDEX 2

YUCHEN LIU AND ANDREA PETRACCI

Abstract. For every integer $a \geq 2$, we relate the K-stability of hypersurfaces in the weighted projective space $\mathbb{P}(1,1,a,a)$ of degree $2a$ with the GIT stability of binary forms of degree $2a$. Moreover, we prove that such a hypersurface is K-polystable and not K-stable if it is quasi-smooth.

1. Introduction

It is an important problem in algebraic geometry and in differential geometry to decide if a given Fano variety X admits a Kähler–Einstein (KE) metric. The Yau–Tian–Donaldson (YTD) Conjecture predicts that the existence of a KE metric on X is equivalent to the K-polystability of X. Using Cheeger–Colding–Tian theory, the YTD Conjecture was first proved when X is smooth [CDS15, Tia15, Ber16], when X is $\mathbb{Q}$-Gorenstein smoothable [LWX19, SSY16], or when X has dimension 2 [LTW21]. Later, a different method, namely the variational approach, was introduced in [BBJ21]. The analytic side of the variational approach was completed in [LTW21b, Li19] which shows that a $\mathbb{Q}$-Fano variety X, that is, a Fano variety with klt singularities, admits a KE metric if and only if X is reduced uniformly K-stable, a concept introduced in [His16] as an equivariant version of uniform K-stability (see also [XZ20]). Recently, using purely algebro-geometric methods, the work [LXZ21] establishes the equivalence between K-polystability and reduced uniform K-stability. This work, combining with the variational approach, proves the YTD Conjecture for all $\mathbb{Q}$-Fano varieties.

K-stability of del Pezzo surfaces which are quasi-smooth hypersurfaces in weighted projective 3-spaces has been studied extensively. Johnson and Kollár [JK01] classified those which are anticanonically polarised (i.e. have Fano index 1) and decided the existence of a KE metric on many of these, by using Tian's criterion which relates KE metrics to global log canonical thresholds (also called α-invariants) [Tia87, Nad90, DK01, Che08, OS12, Fuj19]. This method was applied to most of these del Pezzo surfaces by Araujo [Ara02], Boyer–Galicki–Nakamaye [BGN03], and Cheltsov–Park–Shramov [CPS10]. One case was missing and was finally solved in [CPS21] by using delta invariants (see [FO18, BJ20]).

The (non-)existence of KE metrics on many del Pezzo surfaces which are quasi-smooth hypersurfaces in weighted projective 3-spaces with Fano index ≥ 2 has been studied in [CPS10, CPS21, CS13, KW21].

In this paper, we study K-polystability of quasi-smooth degree $2a$ hypersurfaces in the weighted projective space $\mathbb{P}(1,1,a,a)$. When $a \in \{2,4\}$, such del Pezzo surfaces are $\mathbb{Q}$-Gorenstein smoothable, and their K-polystability was determined by Mabuchi–Mukai [MM93] and Odaka–Spotti–Sun [OSS16] (see Remark 5). To the authors' knowledge it is not known if they are K-polystable for an integer $a = 3$

or $a \geq 5$. In [KW21] Kim and Won conjecture that these surfaces are K-polystable and not K-stable.

Our main result relates the K-polystability (resp. K-semistability) of degree $2a$ hypersurfaces in $\mathbb{P}(1,1,a,a)$ to GIT polystability (resp. GIT semistability) of degree $2a$ binary forms (see [MFK94, Chapter 4]).

Theorem 1. *Let $a \geq 2$ be an integer and let $\mathbb{P}(1,1,a,a)$ be the weighted projective space with coordinates $[x,y,z,w]$ with weights $\deg x = \deg y = 1$ and $\deg z = \deg w = a$. Let X be a hypersurface of degree $2a$ in $\mathbb{P}(1,1,a,a)$.*

Then X is K-semistable (resp. K-polystable) if and only if, after an automorphism of $\mathbb{P}(1,1,a,a)$, the equation of X is given by $z^2+w^2+g(x,y)=0$ where $g \neq 0$ is GIT semistable (resp. GIT polystable) as a degree $2a$ binary form. Moreover, X is not K-stable.

As a consequence we prove the K-polystability of quasi-smooth hypersurfaces in $\mathbb{P}(1,1,a,a)$ of degree $2a$, hence partially confirming [KW21, Conjecture 1.3].

Corollary 2. *Let $a \geq 2$ be an integer and let X be a degree $2a$ quasi-smooth hypersurface in $\mathbb{P}(1,1,a,a)$. Then X is K-polystable and not K-stable. Moreover, X admits a KE metric.*

Recently the result of this corollary has been independently announced by Viswanathan using different methods.

It is possible to give a proof of K-polystability for a general hypersurface in $\mathbb{P}(1,1,a,a)$ of degree $2a$, when a is odd, by analysing the deformation theory of the toric surface appearing in Proposition 3 similarly to [KP21] and without using Theorem 1.

Notation and conventions. We always work over $\mathbb{C}$. A *del Pezzo surface* is a normal projective surface whose anticanonical divisor is $\mathbb{Q}$-Cartier and ample. Every toric variety we consider is normal. We do not even try to write down the definitions of K-(poly/semi)stability of Fano varieties and of log Fano pairs: we refer the reader to the excellent survey [Xu20], the paper [ADL19], and to the references therein.

Acknowledgements. The second author wishes to thank Anne-Sophie Kaloghiros for many fruitful conversations and Yuji Odaka for helpful e-mail exchanges; he is grateful also to Ivan Cheltsov and Jihun Park for useful remarks on an earlier draft of this manuscript and for sharing a preliminary version of [KW21]. The first author is partially supported by the NSF Grant DMS-2001317.

2. Proofs

In what follows a is a fixed integer greater than 1. We consider the weighted projective space $\mathbb{P}(1,1,a,a)$ with coordinates $[x,y,z,w]$ with weights $\deg x = \deg y = 1$ and $\deg z = \deg w = a$.

Proposition 3. *If Y is the hypersurface in $\mathbb{P}(1,1,a,a)$ defined by the equation $zw - x^a y^a = 0$, then Y is a K-polystable toric del Pezzo surface.*

Proof. We fix the lattice $N = \mathbb{Z}^2$ and its dual $M = \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$. Elements of N will be columns and elements of M will be rows.

Let Q be the convex hull of the points

$$(0,0),\ (0,1),\ (a^{-1},0),\ (-a^{-1},1)$$

in $M_{\mathbb{R}}$. Let Σ be the inner normal fan of Q; thus Σ is the complete normal fan in N whose rays are generated by the vectors

$$\begin{pmatrix} a \\ 1 \end{pmatrix},\ \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} -a \\ -1 \end{pmatrix},\ \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{1}$$

We want to show that Y is the toric variety associated to the fan Σ.

Provisionally, let $\mathrm{TV}(\Sigma)$ denote the toric variety associated to Σ. Consider the cone τ in $M \oplus \mathbb{Z}$ spanned by $Q \times \{1\}$. Consider the finitely generated monoid $\tau \cap (M \oplus \mathbb{Z})$ and the semigroup algebra $\mathbb{C}[\tau \cap (M \oplus \mathbb{Z})]$, which is $\mathbb{N}$-graded via the projection $M \oplus \mathbb{Z} \twoheadrightarrow \mathbb{Z}$. Toric geometry says that $\mathrm{TV}(\Sigma) = \operatorname{Proj} \mathbb{C}[\tau \cap (M \oplus \mathbb{Z})]$. One can see that the minimal set of generators of the semigroup $\tau \cap (M \oplus \mathbb{Z})$ is made up of the vectors

$$(0,0,1),\ (0,1,1),\ (1,0,a),\ (-1,a,a);$$

these vectors satisfy a unique relation:

$$a(0,0,1) + a(0,1,1) = (1,0,a) + (-1,a,a).$$

Hence the $\mathbb{N}$-graded ring $\mathbb{C}[\tau \cap (M \oplus \mathbb{Z})]$ coincides with $\mathbb{C}[x,y,z,w]/(zw - x^a y^a)$, where $\deg x = \deg y = 1$ and $\deg z = \deg w = a$. Therefore $Y = \mathrm{TV}(\Sigma)$.

The vectors in (1) are the vertices of a polytope P in N. This implies that Y is a del Pezzo surface, i.e. $-K_Y$ is $\mathbb{Q}$-Cartier and ample.

Let P° be the polar of P; thus P° is the convex hull of $(0, \pm 1)$ and $\pm(\frac{2}{a}, -1)$ in $M_{\mathbb{R}}$. The polygon P° is the moment polytope of the toric boundary of Y, which is an anticanonical divisor. Since P is centrally symmetric, also P° is centrally symmetric, thus the barycentre of P° is the origin. By [Ber16] Y is K-polystable. □

Remark 4. (1) Another way to show K-polystability of Y is by realising $Y \cong (\mathbb{P}^1 \times \mathbb{P}^1)/(\mathbb{Z}/a\mathbb{Z})$, where the $\mathbb{Z}/a\mathbb{Z}$-action on $\mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$\zeta \cdot ([u_0,u_1],[v_0,v_1]) := ([\zeta u_0, u_1], [\zeta^{-1} v_0, v_1]) \quad \text{with } \zeta = e^{\frac{2\pi i}{a}}.$$

Since the above action is free away from finitely many points, and it preserves the product of Fubini-Study metrics on $\mathbb{P}^1 \times \mathbb{P}^1$, we know that Y admits a KE metric and hence is K-polystable by [Ber16].

(2) A degree $2a$ hypersurface in $\mathbb{P}(1,1,a,a)$ is defined by an equation

$$q(z,w) + f(x,y)z + h(x,y)w + g(x,y) = 0$$

where q is a quadratic form, f and h are forms of degree a, and g is a form of degree $2a$. With an automorphism of $\mathbb{P}(1,1,a,a)$ which is induced by a linear change of the coordinates z, w, we can diagonalise the quadratic form q, so that the term zw disappears. Furthermore, if q has full rank, with an automorphism of $\mathbb{P}(1,1,a,a)$ induced by $z \mapsto z + \frac{f}{2}$ and $w \mapsto w + \frac{h}{2}$, the equation becomes

$$z^2 + w^2 + g(x,y) = 0.$$

Proof of Theorem 1. We start from the "if" part. Suppose $X \subset \mathbb{P}(1,1,a,a)$ is defined by the equation $z^2+w^2+g(x,y)=0$ with $g \neq 0$. Then the "if" part states that X is K-semistable (resp. K-polystable) if g is GIT semistable (resp. GIT polystable).

By forgetting the w-coordinate, we obtain a double cover $\pi : X \to \mathbb{P}(1,1,a)$ with branch locus $D=(z^2+g(x,y)=0)$. Thus by [LZ20, Zhu21] we know that X is K-semistable (resp. K-polystable) if and only if $(\mathbb{P}(1,1,a), \frac{1}{2}D)$ is K-semistable (resp. K-polystable).

Let us assume for the moment that g is an arbitrary degree $2a$ binary form. Denote by $D_0 := (z^2=0)$ as a divisor on $\mathbb{P}(1,1,a)$. It is clear that $\mathbb{P}(1,1,a)$ is the projective cone over $\mathbb{P}^1$ with polarization $\mathcal{O}_{\mathbb{P}^1}(a)$, and $\frac{1}{2}D_0$ is the section at infinity. Since $\mathbb{P}^1$ is Kähler–Einstein, [LL19, Proposition 3.3] shows that $(\mathbb{P}(1,1,a), (1-\frac{r}{2})\frac{1}{2}D_0)$ admits a conical KE metric, where $r \in \mathbb{Q}_{>0}$ satisfies $\mathcal{O}_{\mathbb{P}^1}(a) \sim_{\mathbb{Q}} -r^{-1}K_{\mathbb{P}^1}$, i.e. $r=\frac{2}{a}$. By computation, $(1-\frac{r}{2})\frac{1}{2} = \frac{a-1}{2a}$. Thus $(\mathbb{P}(1,1,a), \frac{a-1}{2a}D_0)$ admits a conical KE metric and hence is K-polystable. It is clear that under the $\mathbb{G}_m$-action σ on $\mathbb{P}(1,1,a)$ given by $\sigma(t)\cdot[x,y,z]=[x,y,tz]$, the log Fano pair $(\mathbb{P}(1,1,a), \frac{a-1}{2a}D)$ specially degenerates to $(\mathbb{P}(1,1,a), \frac{a-1}{2a}D_0)$ as $t \to 0$. Thus by openness of K-semistability [BLX19, Xu20b] we know that $(\mathbb{P}(1,1,a), \frac{a-1}{2a}D)$ is K-semistable.

Next, we assume that $g \neq 0$ is GIT semistable. By GIT of binary forms, we know that each linear factor in $g(x,y)$ has multiplicity at most a. In other words, the curve D has only A_{k-1}-singularities (i.e. locally analytically given by $x^2+y^k=0$) where $k \leq a$. Thus we have that $\mathrm{lct}(\mathbb{P}(1,1,a); D) \geq \frac{1}{2}+\frac{1}{a} = \frac{a+2}{2a}$. This implies that $(\mathbb{P}(1,1,a), \frac{a+2}{2a}D)$ is a log canonical log Calabi–Yau pair. Thus interpolation for K-stability [ADL19, Proposition 2.13] implies that $(\mathbb{P}(1,1,a), \frac{1}{2}D)$ is K-semistable.

Next, we assume that $g \neq 0$ is GIT polystable. There are two cases: g is strictly GIT polystable (i.e. GIT polystable but not GIT stable), or g is GIT stable. In the first case, under a suitable coordinate we may write $g(x,y)=x^a y^a$. Thus the double cover X is toric, and as shown in Proposition 3 X is K-polystable. In the second case, we know that each linear factor in $g(x,y)$ has multiplicity at most $a-1$. Thus the curve D has only A_{k-1}-singularities where $k \leq a-1$. Thus we have that $\mathrm{lct}(\mathbb{P}(1,1,a); D) \geq \frac{1}{2}+\frac{1}{a-1} > \frac{a+2}{2a}$, which implies that $(\mathbb{P}(1,1,a), \frac{a+2}{2a}D)$ is a klt log Calabi–Yau pair. Thus interpolation for K-stability [ADL19, Proposition 2.13] implies that $(\mathbb{P}(1,1,a), \frac{1}{2}D)$ is K-stable. This finishes the proof of the "if" part.

Next, we treat the "only if" part. In fact, this follows from moduli comparison arguments as in [ADL19]. Let $\mathbf{A} := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a))$ be the affine space parametrizing degree $2a$ binary forms. Let $\mathbf{A}^{\mathrm{ss}} \subset \mathbf{A}\setminus\{0\}$ be the open subset of GIT semistable binary forms. Consider the universal family of weighted hypersurfaces $\mathcal{X} \to \mathbf{A}^{\mathrm{ss}}$ where $\mathcal{X} \subset \mathbb{P}(1,1,a,a)\times \mathbf{A}^{\mathrm{ss}}$ has fibre $(z^2+w^2+g(x,y)=0)$ over each $g \in \mathbf{A}^{\mathrm{ss}}$. By the "if" part we know that each fibre of $\mathcal{X} \to \mathbf{A}^{\mathrm{ss}}$ is K-semistable. Consider the $(\mathbb{G}_m \times \mathrm{SL}_2)$-action λ on $\mathbf{A}$ given by $\lambda(t,A)\cdot g(x,y) = t^2 g(A^{-1}(x,y))$. It is clear that $\mathbf{A}^{\mathrm{ss}}$ is a $(\mathbb{G}_m \times \mathrm{SL}_2)$-invariant open subset. Then there is a $(\mathbb{G}_m \times \mathrm{SL}_2)$-action $\tilde{\lambda}$ on $\mathcal{X}$ as a lifting of λ given by

$$\tilde{\lambda}(t,A)\cdot([x,y,z,w],g) := ([A(x,y),tz,tw],\lambda(t,A)\cdot g).$$

Denote by $\mathcal{M}^{\mathrm{GIT}} := [\mathbf{A}^{\mathrm{ss}}/(\mathbb{G}_m \times \mathrm{SL}_2)]$ and $M^{\mathrm{GIT}} := \mathbf{P} /\!\!/ \mathrm{SL}_2$ where $\mathbf{P} := \mathbb{P}(\mathbf{A})$. It is clear that M^{GIT} is the good moduli space of $\mathcal{M}^{\mathrm{GIT}}$. Taking quotient of

the family $\mathcal{X} \to \mathbf{A}^{\mathrm{ss}}$ by $\tilde{\lambda}$, we obtain a $\mathbb{Q}$-Gorenstein flat family of K-semistable $\mathbb{Q}$-Fano varieties over $\mathcal{M}^{\mathrm{GIT}}$, where fibres over closed points are precisely K-polystable fibres.

From a series of important recent works [Jia20, LWX21, CP21, BX19, ABHLX20, Xu20b, BLX19, XZ20, XZ21, BHLLX21, LXZ21], we know that there exists an Artin stack of finite type $\mathcal{M}^{\mathrm{Kss}}_{2,8/a}$ parametrizing K-semistable (possibly singular) del Pezzo surfaces of degree $8/a$. Moreover, $\mathcal{M}^{\mathrm{Kss}}_{2,8/a}$ admits a projective good moduli space $M^{\mathrm{Kps}}_{2,8/a}$ parametrizing K-polystable ones. Let $\mathcal{M}^{\mathrm{K}}$ be the Zariski closure (with reduced structure) of the locally closed substack in $\mathcal{M}^{\mathrm{Kss}}_{2,8/a}$ parametrizing K-semistable degree $2a$ weighted hypersurfaces $X \subset \mathbb{P}(1,1,a,a)$. Let M^{K} be the good moduli space of $\mathcal{M}^{\mathrm{K}}$ as a closed algebraic subspace of $M^{\mathrm{Kps}}_{2,8/a}$. Then the above construction and the "if" part produces a morphism $\Phi : \mathcal{M}^{\mathrm{GIT}} \to \mathcal{M}^{\mathrm{K}}$ which descends to a morphism $\phi : M^{\mathrm{GIT}} \to M^{\mathrm{K}}$. Since a general weighted hypersurface X has the form $z^2 + w^2 + g(x,y) = 0$ in a suitable coordinate where $g \neq 0$ has no multiple linear factors, we know that Φ is dominant. The "if" part shows that Φ sends closed points to closed points. Since M^{GIT} is projective, we know that ϕ is proper and dominant, which implies that ϕ is surjective. Moreover, since SL_2 has no non-trivial characters, we have injections

$$\mathrm{Pic}(M^{\mathrm{GIT}}) = \mathrm{Pic}(\mathbf{P} /\!\!/ \mathrm{SL}_2) \hookrightarrow \mathrm{Pic}_{\mathrm{SL}_2}(\mathbf{P}^{\mathrm{ss}}) \hookrightarrow \mathrm{Pic}(\mathbf{P}^{\mathrm{ss}})$$

by [KKV89, Proposition 4.2 and Section 2.1]. It is clear that $\mathbf{P} \setminus \mathbf{P}^{\mathrm{ss}}$ has codimension at least 2 in $\mathbf{P}$. Thus we have $\mathrm{Pic}(\mathbf{P}^{\mathrm{ss}}) \cong \mathrm{Pic}(\mathbf{P}) \cong \mathbb{Z}$. In particular, the GIT quotient M^{GIT} has Picard rank 1. It is clear that M^{K} is not a single point. Thus $\phi : M^{\mathrm{GIT}} \to M^{\mathrm{K}}$ is a finite surjective morphism by Zariski's main theorem.

Next, we show that K-poly/semistability implies GIT poly/semistability. Since ϕ is surjective, a K-polystable hypersurface $X \subset \mathbb{P}(1,1,a,a)$ satisfies that $[X] = \phi([g]) \in M^{\mathrm{K}}$ for some GIT polystable binary form $g \in \mathbf{A} \setminus \{0\}$. Thus X has the form $z^2 + w^2 + g(x,y) = 0$ with $g \neq 0$ being GIT polystable. If $X \subset \mathbb{P}(1,1,a,a)$ is K-semistable, then it specially degenerates to a K-polystable point $[X_0] \in M^{\mathrm{K}}$ by [LWX21]. Clearly X_0 has the form $z^2 + w^2 + g_0(x,y) = 0$ with $g_0 \neq 0$ being GIT polystable. Since the rank of quadratic forms cannot jump up under degeneration, the quadratic terms in (z,w) of the equation of X has rank 2, which implies that $X = (z^2 + w^2 + g(x,y) = 0)$ for some g. By [Fuj19b, Corollary 1.7], we know that $(\mathbb{P}(1,1,a), \frac{1}{2}D)$ is K-semistable where $D = (z^2 + g(x,y) = 0)$. Since X carries a $\mathbb{Z}/2\mathbb{Z}$-action given by $w \mapsto -w$, we may assume that the special degeneration from X to X_0 is $\mathbb{Z}/2\mathbb{Z}$-equivariant by [LZ20, Zhu21]. In particular, this shows that $(\mathbb{P}(1,1,a), \frac{1}{2}D)$ specially degenerates to $(\mathbb{P}(1,1,a), \frac{1}{2}D_0)$ where $D_0 = (z^2 + g_0(x,y) = 0)$. By the lower semi-continuity of lct (see e.g. [DK01]), we know that $\mathrm{lct}(\mathbb{P}(1,1,a); D) \geq \mathrm{lct}(\mathbb{P}(1,1,a); D_0) \geq \frac{a+2}{2a}$ where the latter inequality was proven in the "if" part due to the fact that g_0 is GIT polystable. Thus this shows that $g \neq 0$, and each linear factor in $g(x,y)$ has multiplicity at most a. Thus we obtain the GIT semistability of g. The proof of the "only if" part is finished.

Finally, we show that any hypersurface $X \subset \mathbb{P}(1,1,a,a)$ of degree $2a$ is not K-stable. If X were K-stable, then it would have equation $z^2 + w^2 + g(x,y) = 0$, or equivalently the equation $zw + g(x,y) = 0$. It is clear that $t \cdot (z,w) = (tz, t^{-1}w)$ defines an effective action of $\mathbb{G}_{\mathrm{m}}$ on X. Thus X is not K-stable by definition. □

Proof of Corollary 2. It is clear that X is quasi-smooth if and only if, up to an automorphism of $\mathbb{P}(1,1,a,a)$, X has the equation $z^2+w^2+g(x,y)=0$ where g has no multiple linear factors. Thus by Theorem 1 we conclude that X is K-polystable and not K-stable. The existence of KE metrics on X follows from [LTW21]. □

Remark 5. For $a=2$, the del Pezzo surface X admits an embedding into $\mathbb{P}^4$ as a complete intersection of two hyperquadrics. This is induced by the linear system $|-K_X|$ which is very ample.

For $a=4$, X (as a double cover of $\mathbb{P}(1,1,4)$) appeared in [OSS16] where it lies in the exceptional divisor of Kirwan blow-up of the GIT moduli space. Hence X admits a $\mathbb{Q}$-Gorenstein smoothing to degree 2 smooth del Pezzo surfaces.

Therefore, in both cases ($a=2$ or $a=4$) our K-moduli space M^{K}, introduced in the proof of Theorem 1, form a divisor in the K-moduli spaces of $\mathbb{Q}$-Gorenstein smoothable del Pezzo surfaces of degree $\frac{8}{a}$ studied in [MM93, OSS16]. We will see in Proposition 6 what happens for $a=3$ or $a\geq 5$.

Proposition 6. *If $a=3$ or $a\geq 5$, then the locus of K-polystable degree $2a$ hypersurfaces in $\mathbb{P}(1,1,a,a)$ is a connected component of $M^{\mathrm{Kps}}_{2,8/a}$.*

Proof. We denote by Γ the connected component of $M^{\mathrm{Kps}}_{2,8/a}$ containing K-polystable degree $2a$ hypersurfaces in $\mathbb{P}(1,1,a,a)$. In the proof of Theorem 1 we showed that the locus of K-polystable degree $2a$ hypersurfaces in $\mathbb{P}(1,1,a,a)$ is closed in Γ; this locus is denoted by M^{K}. We need to prove that M^{K} coincides with Γ. We will achieve this by a dimension count. Using the notation of the proof of Theorem 1, there is a finite surjective morphism $\phi: M^{\mathrm{GIT}}\to M^{\mathrm{K}}$. Thus we have

$$\dim M^{\mathrm{K}}=\dim M^{\mathrm{GIT}}=\dim \mathbf{P}-\dim \mathrm{SL}_2=2a-3.$$

Let us now compute the dimension of Γ by analysing the deformation theory of the K-polystable toric del Pezzo surface Y introduced in Proposition 3. Note that a similar study was discussed in [MGS21].

Let $\mathscr{T}^0_Y$ denote the sheaf of derivations on Y, i.e. the dual of Ω^1_Y. Let $\mathscr{T}^{\mathrm{qG},1}_Y$ denote the sheaf of 1st order $\mathbb{Q}$-Gorenstein deformations of Y. The singular locus of Y, which consists of 4 points, contains the set-theoretic support of $\mathscr{T}^{\mathrm{qG},1}_Y$.

Since Y is a toric Fano, we have $H^1(\mathscr{T}^0_Y)=H^2(\mathscr{T}^0_Y)=0$ by [Pet19, §4.3]. Via a standard argument about the local-to-global spectral sequence for Ext, we deduce that the tangent space of the $\mathbb{Q}$-Gorenstein deformation functor of Y is $H^0(\mathscr{T}^{\mathrm{qG},1}_Y)$. The $\mathbb{Q}$-Gorenstein deformation functor of Y is unobstructed because Y is a del Pezzo surface with cyclic quotient singularities [ACC+16, Lemma 6]. Therefore the germ at the origin of the vector space $H^0(\mathscr{T}^{\mathrm{qG},1}_Y)$ is the base of the miniversal (Kuranishi) $\mathbb{Q}$-Gorenstein deformation of Y.

Consider the torus $T_N=N\otimes_{\mathbb{Z}}\mathbb{G}_{\mathrm{m}}$ acting on the toric variety Y. There is an action of T_N on the vector space $H^0(\mathscr{T}^{\mathrm{qG},1}_Y)$, hence $H^0(\mathscr{T}^{\mathrm{qG},1}_Y)$ splits into the direct sum of irreducible representations (characters) of the torus T_N.

We observe that the singularities of Y are:

- 2 points of type $\frac{1}{a}(1,-1)=A_{a-1}$, which correspond to the cones in Σ spanned by
$$\pm\begin{pmatrix}a\\1\end{pmatrix},\ \pm\begin{pmatrix}0\\1\end{pmatrix},$$

- 2 points of type $\frac{1}{a}(1,1)$, which correspond to the cones in Σ spanned by

$$\pm\begin{pmatrix} a \\ 1 \end{pmatrix},\ \mp\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $a = 3$ or $a \geq 5$, the surface singularity $\frac{1}{a}(1,1)$ is $\mathbb{Q}$-Gorenstein rigid, so it does not contribute to $H^0(\mathscr{T}_Y^{\mathrm{qG},1})$. One can see that the T_N-representation $H^0(\mathscr{T}_Y^{\mathrm{qG},1})$ is the direct sum of the 1-dimensional representation of T_N associated to the characters

$$(0,\pm 2),(0,\pm 3),\dots,(0,\pm a) \in M. \tag{2}$$

In particular $\dim H^0(\mathscr{T}_Y^{\mathrm{qG},1}) = 2a-2$, so the base of the miniversal $\mathbb{Q}$-Gorenstein deformation of Y is a smooth germ of dimension $2a-2$.

Since the weights in (2) are contained in a rank 1 sublattice of M, there exists a 1-dimensional subtorus of T_N which acts trivially on $H^0(\mathscr{T}_Y^{\mathrm{qG},1})$. More precisely one can prove that the affine quotient $H^0(\mathscr{T}_Y^{\mathrm{qG},1})/T_N$ has dimension $2a-3$.

Since every facet of the polytope P° has no interior lattice points, by [KP21, Proposition 2.6] the automorphism group of Y is $T_N \rtimes \mathrm{Aut}(P)$, where $\mathrm{Aut}(P) \subseteq \mathrm{GL}(N)$ is the finite group consisting of the lattice automorphisms which keep the polytope P invariant. Since the difference between T_N and $\mathrm{Aut}(Y)$ is just a finite group, we deduce that the affine quotient the affine quotient $H^0(\mathscr{T}_Y^{\mathrm{qG},1})/\mathrm{Aut}(Y)$ has dimension $2a-3$. By the local structure of the K-moduli space [ABHLX20, AHR20] we know that the completion of the local ring of Γ at $[Y]$ coincides with the completion at the origin of $H^0(\mathscr{T}_Y^{\mathrm{qG},1})/\mathrm{Aut}(Y)$. This proves that Γ has dimension $2a-3$ at $[Y]$. Since $\dim M^{\mathrm{K}} = 2a-3$, we know that M^{K} is an irreducible component of Γ.

Moreover, since all K-polystable del Pezzo surfaces in M^{K} have cyclic quotient singularities by Theorem 1, they have unobstructed $\mathbb{Q}$-Gorenstein deformations by [ACC+16, Lemma 6]. Thus the stack $\mathcal{M}^{\mathrm{Kss}}_{2,\frac{8}{a}}$ is smooth in an open neighbourhood of $\mathcal{M}^{\mathrm{K}}$. In particular, this implies that Γ is normal in an open neighbourhood of M^{K}. Since M^{K} is an irreducible component of Γ, we have $M^{\mathrm{K}} = \Gamma$. □

References

[ABHLX20] Jarod Alper, Harold Blum, Daniel Halpern-Leistner, and Chenyang Xu, *Reductivity of the automorphism group of K-polystable Fano varieties*, Invent. Math. **222** (2020), no. 3, 995–1032.

[ACC+16] Mohammad Akhtar, Tom Coates, Alessio Corti, Liana Heuberger, Alexander Kasprzyk, Alessandro Oneto, Andrea Petracci, Thomas Prince, and Ketil Tveiten, *Mirror symmetry and the classification of orbifold del Pezzo surfaces*, Proc. Amer. Math. Soc. **144** (2016), no. 2, 513–527.

[ADL19] Kenneth Ascher, Kristin DeVleming, and Yuchen Liu, *Wall crossing for K-moduli spaces of plane curves* (2019). arXiv:1909.04576.

[AHR20] Jarod Alper, Jack Hall, and David Rydh, *A Luna étale slice theorem for algebraic stacks*, Ann. of Math. (2) **191** (2020), no. 3, 675–738.

[Ara02] Carolina Araujo, *Kähler-Einstein metrics for some quasi-smooth log del Pezzo surfaces*, Trans. Amer. Math. Soc. **354** (2002), no. 11, 4303–4312.

[BBJ21] Robert Berman, Sébastien Boucksom, and Mattias Jonsson, *A variational approach to the Yau-Tian-Donaldson conjecture*, J. Amer. Math. Soc., to appear (2021). https://doi.org/10.1090/jams/964.

[Ber16] Robert J. Berman, *K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler-Einstein metrics*, Invent. Math. **203** (2016), no. 3, 973–1025.

[BGN03] Charles P. Boyer, Krzysztof Galicki, and Michael Nakamaye, *On the geometry of Sasakian-Einstein 5-manifolds*, Math. Ann. **325** (2003), no. 3, 485–524.

[BHLLX21] Harold Blum, Daniel Halpern-Leistner, Yuchen Liu, and Chenyang Xu, *On properness of K-moduli spaces and optimal degenerations of Fano varieties*, Selecta Math. (N.S.) **27** (2021), no. 4, Paper No. 73, 39.

[BJ20] Harold Blum and Mattias Jonsson, *Thresholds, valuations, and K-stability*, Adv. Math. **365** (2020), 107062, 57.

[BLX19] Harold Blum, Yuchen Liu, and Chenyang Xu, *Openness of K-semistability for Fano varieties* (2019). arXiv:1907.02408.

[BX19] Harold Blum and Chenyang Xu, *Uniqueness of K-polystable degenerations of Fano varieties*, Ann. of Math. (2) **190** (2019), no. 2, 609–656.

[CDS15] Xiuxiong Chen, Simon Donaldson, and Song Sun, *Kähler-Einstein metrics on Fano manifolds. I, II, III*, J. Amer. Math. Soc. **28** (2015), no. 1, 183–278.

[Che08] Ivan Cheltsov, *Log canonical thresholds of del Pezzo surfaces*, Geom. Funct. Anal. **18** (2008), no. 4, 1118–1144.

[CP21] Giulio Codogni and Zsolt Patakfalvi, *Positivity of the CM line bundle for families of K-stable klt Fano varieties*, Invent. Math. **223** (2021), no. 3, 811–894.

[CPS10] Ivan Cheltsov, Jihun Park, and Constantin Shramov, *Exceptional del Pezzo hypersurfaces*, J. Geom. Anal. **20** (2010), no. 4, 787–816.

[CPS21] ———, *Delta invariants of singular del Pezzo surfaces*, J. Geom. Anal. **31** (2021), no. 3, 2354–2382.

[CS13] Ivan Cheltsov and Constantin Shramov, *Del Pezzo zoo*, Exp. Math. **22** (2013), no. 3, 313–326.

[DK01] Jean-Pierre Demailly and János Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 4, 525–556.

[FO18] Kento Fujita and Yuji Odaka, *On the K-stability of Fano varieties and anticanonical divisors*, Tohoku Math. J. (2) **70** (2018), no. 4, 511–521.

[Fuj19a] Kento Fujita, *K-stability of Fano manifolds with not small alpha invariants*, J. Inst. Math. Jussieu **18** (2019), no. 3, 519–530.

[Fuj19b] ———, *Uniform K-stability and plt blowups of log Fano pairs*, Kyoto J. Math. **59** (2019), no. 2, 399–418.

[His16] Tomoyuki Hisamoto, *Stability and coercivity for toric polarizations* (2016). arXiv:1610.07998.

[Jia20] Chen Jiang, *Boundedness of $\mathbb{Q}$-Fano varieties with degrees and alpha-invariants bounded from below*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 4, 1235–1248.

[JK01] J. M. Johnson and J. Kollár, *Kähler-Einstein metrics on log del Pezzo surfaces in weighted projective 3-spaces*, Ann. Inst. Fourier (Grenoble) **51** (2001), no. 1, 69–79.

[KKV89] Friedrich Knop, Hanspeter Kraft, and Thierry Vust, *The Picard group of a G-variety*, Algebraische Transformationsgruppen und Invariantentheorie, 1989, pp. 77–87.

[KP21] Anne-Sophie Kaloghiros and Andrea Petracci, *On toric geometry and K-stability of Fano varieties*, Trans. Amer. Math. Soc. Ser. B **8** (2021), 548–577.

[KW21] In-Kyun Kim and Joonyeong Won, *Unstable singular del Pezzo hypersurfaces with lower index*, Comm. Algebra **49** (2021), no. 6, 2679–2688.

[Li19] Chi Li, *$\mathbb{G}$-uniform stability and Kähler-Einstein metrics on Fano varieties* (2019). arXiv:1907.09399.

[LL19] Chi Li and Yuchen Liu, *Kähler-Einstein metrics and volume minimization*, Adv. Math. **341** (2019), 440–492.

[LTW21a] Chi Li, Gang Tian, and Feng Wang, *On the Yau-Tian-Donaldson conjecture for singular Fano varieties*, Comm. Pure Appl. Math. **74** (2021), no. 8, 1748–1800.

[LTW21b] ———, *The uniform version of Yau-Tian-Donaldson conjecture for singular Fano varieties*, Peking Math. J., to appear (2021). https://doi.org/10.1007/s42543-021-00039-5.

[LWX19] Chi Li, Xiaowei Wang, and Chenyang Xu, *On the proper moduli spaces of smoothable Kähler-Einstein Fano varieties*, Duke Math. J. **168** (2019), no. 8, 1387–1459.

[LWX21] ———, *Algebraicity of the metric tangent cones and equivariant K-stability*, J. Amer. Math. Soc. **34** (2021), no. 4, 1175–1214.

[LXZ21] Yuchen Liu, Chenyang Xu, and Ziquan Zhuang, *Finite generation for valuations computing stability thresholds and applications to K-stability* (2021). arXiv:2102.09405.

[LZ20] Yuchen Liu and Ziwen Zhu, *Equivariant K-stability under finite group action* (2020). arXiv:2001.10557.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, Third, Ergebnisse der Mathematik und ihrer Grenzgebiete (2), vol. 34, Springer-Verlag, Berlin, 1994.

[MGS21] Jesus Martinez-Garcia and Cristiano Spotti, *Some observations on the dimension of Fano K-moduli* (2021). arXiv:2101.05643.

[MM93] Toshiki Mabuchi and Shigeru Mukai, *Stability and Einstein-Kähler metric of a quartic del Pezzo surface*, Einstein metrics and Yang-Mills connections (Sanda, 1990), 1993, pp. 133–160.

[Nad90] Alan Michael Nadel, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math. (2) **132** (1990), no. 3, 549–596.

[OS12] Yuji Odaka and Yuji Sano, *Alpha invariant and K-stability of $\mathbb{Q}$-Fano varieties*, Adv. Math. **229** (2012), no. 5, 2818–2834.

[OSS16] Yuji Odaka, Cristiano Spotti, and Song Sun, *Compact moduli spaces of del Pezzo surfaces and Kähler-Einstein metrics*, J. Differential Geom. **102** (2016), no. 1, 127–172.

[Pet19] Andrea Petracci, *On deformations of toric Fano varieties* (2019). arXiv:1912.01538.

[SSY16] Cristiano Spotti, Song Sun, and Chengjian Yao, *Existence and deformations of Kähler-Einstein metrics on smoothable $\mathbb{Q}$-Fano varieties*, Duke Math. J. **165** (2016), no. 16, 3043–3083.

[Tia15] Gang Tian, *K-stability and Kähler-Einstein metrics*, Comm. Pure Appl. Math. **68** (2015), no. 7, 1085–1156.

[Tia87] ———, *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$*, Invent. Math. **89** (1987), no. 2, 225–246.

[Xu20a] Chenyang Xu, *K-stability of Fano varieties: an algebro-geometric approach* (2020). arXiv:2011.10477.

[Xu20b] ———, *A minimizing valuation is quasi-monomial*, Ann. of Math. (2) **191** (2020), no. 3, 1003–1030.

[XZ20] Chenyang Xu and Ziquan Zhuang, *On positivity of the CM line bundle on K-moduli spaces*, Ann. of Math. (2) **192** (2020), no. 3, 1005–1068.

[XZ21] ———, *Uniqueness of the minimizer of the normalized volume function*, Camb. J. Math. **9** (2021), no. 1, 149–176.

[Zhu21] Ziquan Zhuang, *Optimal destabilizing centers and equivariant K-stability*, Invent. Math., to appear (2021). https://doi.org/10.1007/s00222-021-01046-0.

Department of Mathematics, Northwestern University, Evanston, IL 60208, USA
Email address: yuchenl@northwestern.edu

Institut für Mathematik, Freie Universität Berlin, Arnimallee 3, Berlin 14195, Germany
Email address: andrea.petracci@fu-berlin.de