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On the action of the Koszul map on the enveloping algebra of the general linear Lie algebra

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Abstract

We describe a linear *equivariant isomorphism* \mathcal{K} from the enveloping algebra $\mathbf{U}(gl(n))$ to the algebra $\mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n))$ of polynomials in the entries of a “generic” square matrix of order n .

The isomorphism \mathcal{K} maps any *Capelli bitableau* $[S|T]$ in $\mathbf{U}(gl(n))$ to the (*determinantal*) *bitableau* $(S|T)$ in $\mathbb{C}[M_{n,n}]$ and any *Capelli *-bitableau* $[S|T]^*$ in $\mathbf{U}(gl(n))$ to the (*permanental*) **-bitableau* $(S|T)^*$ in $\mathbb{C}[M_{n,n}]$.

These results are far-reaching generalizations of the pioneering result of J.-L. Koszul [19] on the Capelli determinant in $\mathbf{U}(gl(n))$ (see, e.g. [24], [27]).

We introduce *column* Capelli bitableaux and **-bitableaux* in Section 6; since they are mapped by the isomorphism \mathcal{K} to *monomials* in $\mathbb{C}[M_{n,n}]$, this isomorphism can be regarded as a sharpened version of the PBW isomorphism for the enveloping algebra $\mathbf{U}(gl(n))$.

Since the center $\zeta(n)$ of $\mathbf{U}(gl(n))$ equals the subalgebra of invariants $\mathbf{U}(gl(n))^{Ad_{gl(n)}}$, then

$$\mathcal{K}[\zeta(n)] = \mathbb{C}[M_{n,n}]^{ad_{gl(n)}}.$$

Keyword: Enveloping algebras; Young tableaux; Lie superalgebras; central elements; Capelli determinants.

1 Introduction

The starting points of the present work are ([5], [6]):

- The linear operator $\mathcal{B} : \mathbb{C}[M_{n,n}] \rightarrow \mathbf{U}(gl(n))$ that maps any (*determinantal*) *bitableau* $(S|T)$ in $\mathbb{C}[M_{n,n}]$ to the *Capelli bitableau* $[S|T]$ in $\mathbf{U}(gl(n))$.
- The linear operator $\mathcal{B}^* : \mathbb{C}[M_{n,n}] \rightarrow \mathbf{U}(gl(n))$ that maps any (*permanental*) *bitableau* $(S|T)^*$ in $\mathbb{C}[M_{n,n}]$ to the *Capelli *-bitableau* $[S|T]^*$ in $\mathbf{U}(gl(n))$.

The map

$$\mathcal{K} : \mathbf{U}(gl(n)) \rightarrow \mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n))$$

introduced by Koszul in 1981 [19] is proved to be the inverse of both \mathcal{B} and \mathcal{B}^* . Then \mathcal{B} , \mathcal{B}^* , \mathcal{K} are vector space isomorphisms and $\mathcal{B} = \mathcal{B}^*$.

Since the set of *standard* bitableaux is a basis of $\mathbb{C}[M_{n,n}]$ ([16], [15], [14], [17]), then the set of *standard* Capelli bitableaux is a basis of $\mathbf{U}(gl(n))$. Since the set of *costandard* *-bitableaux is a basis of $\mathbb{C}[M_{n,n}]$, then the set of *costandard* Capelli *-bitableaux is a basis of $\mathbf{U}(gl(n))$.

Some of these topics were treated in a sketchy way in the present author's notes [5], [6] (in the more general setting of superalgebras), in a rather cumbersome notation and almost without proofs. The main novelty of the present approach is the major role played by *column Capelli bitableaux* and *column Capelli *-bitableaux*; although they are far from being "monomials" in the enveloping algebra $\mathbf{U}(gl(n))$, their images with respect to the Koszul isomorphism \mathcal{K} are indeed monomials in the polynomial algebra $\mathbb{C}[M_{n,n}]$. Therefore, column Capelli bitableaux and column Capelli *-bitableaux play the same role in $\mathbf{U}(gl(n))$ that monomials play in $\mathbb{C}[M_{n,n}]$ and this leads to a new and transparent presentation.

The expressions of column Capelli bitableaux and column Capelli *-bitableaux as elements of $\mathbf{U}(gl(n))$ can be simply computed.

Capelli bitableaux and Capelli *-bitableaux expand - up to a global sign - into column Capelli bitableaux just in the same way as *determinantal* bitableaux and *permanental* *-bitableaux expand into the corresponding monomials in $\mathbb{C}[M_{n,n}]$ (Laplace expansions).

The isomorphism $\mathcal{B} = \mathcal{K}^{-1}$ maps any *right symmetrized bitableau* $(S|\overline{T}) \in \mathbb{C}[M_{n,n}]$ ([3], [4]) to the *right Young-Capelli bitableau* $[S|\overline{T}]$ in $\mathbf{U}(gl(n))$. The basis of *standard* right Young-Capelli bitableaux acts in a remarkable way on the *Gordan-Capelli basis* of *standard* right symmetrized bitableaux. Moreover, the elements of the *Schur-Sahi-Okounkov* basis of the center $\zeta(n)$ of $\mathbf{U}(gl(n))$ (*quantum immanants* [25], [21], [22], [23]) admit quite effective presentations as linear combinations of right Young-Capelli bitableaux as well as of Capelli immanants [8] and [7].

The Koszul map \mathcal{K} is proved to be an *equivariant* isomorphism with respect to the adjoint representations of $gl(n)$ on $\mathbf{U}(gl(n))$ and $\mathbb{C}[M_{n,n}]$ (*polarization action*), respectively. Since the center $\zeta(n)$ of $\mathbf{U}(gl(n))$ is the subalgebra of $Ad_{gl(n)}$ -invariants of $\mathbf{U}(gl(n))^{Ad_{gl(n)}}$, then

$$\mathcal{K}[\zeta(n)] = \mathbb{C}[M_{n,n}]^{ad_{gl(n)}},$$

where $\mathbb{C}[M_{n,n}]^{ad_{gl(n)}}$ is the subalgebra of $ad_{gl(n)}$ -invariants of $\mathbb{C}[M_{n,n}]$.

2 Determinantal Young bitableaux, permanental Young *-bitableaux and right symmetrized bitableaux in the polynomial algebra $\mathbb{C}[M_{n,n}]$

Let

$$\mathbb{C}[M_{n,n}] = \mathbb{C}[(i|j)]_{i,j=1,\dots,n}$$

be the polynomial algebra in the (commutative) "generic" entries $(i|j)$ of the matrix:

$$M_{n,n} = [(i|j)]_{i,j=1,\dots,n} = \begin{pmatrix} (1|1) & \dots & (1|n) \\ \vdots & & \vdots \\ (n|1) & \dots & (n|n) \end{pmatrix}.$$

Given the *standard basis* $\{e_{ij}; i, j = 1, 2, \dots, n\}$ of the *general linear Lie algebra* $gl(n)$, the map $e_{ij} \mapsto (i|j)$ induces an isomorphism $\mathbf{Sym}(gl(n)) \cong \mathbb{C}[M_{n,n}]$.

Let $\omega = i_1 i_2 \dots i_p$, $\varpi = j_1 j_2 \dots j_p$ be words on the alphabet $\{1, 2, \dots, n\}$.

Following [17] and [3], the *biprodut* of the two words ω and ϖ

$$(\omega|\varpi) = (i_1 i_2 \dots i_p | j_1 j_2 \dots j_p) \quad (1)$$

is the *signed minor*:

$$(\omega|\varpi) = (-1)^{\binom{p}{2}} \det \left((i_r | j_s) \right)_{r,s=1,2,\dots,p} \in \mathbb{C}[M_{n,n}].$$

Let $S = (\omega_1, \omega_2, \dots, \omega_p)$ and $T = (\varpi_1, \varpi_2, \dots, \varpi_p)$ be Young tableaux on $\{1, 2, \dots, n\}$ of the same shape λ .

Following again [17] and [3], the (determinantal) *Young bitableau*

$$(S|T) = \left(\begin{array}{c|c} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{array} \right) \quad (2)$$

is the *signed product* of the biproduts of the pairs of corresponding rows:

$$(S|T) = \mathbf{s} (\omega_1|\varpi_1)(\omega_2|\varpi_2) \dots (\omega_p|\varpi_p), \quad (3)$$

where

$$\mathbf{s} = (-1)^{\ell(\omega_2)\ell(\varpi_1) + \ell(\omega_3)(\ell(\varpi_1) + \ell(\varpi_2)) + \dots + \ell(\omega_p)(\ell(\varpi_1) + \ell(\varpi_2) + \dots + \ell(\varpi_{p-1}))}, \quad (4)$$

and the symbol $\ell(w)$ denotes the length of the word w .

The **-biprodut* of the two words ω and ϖ

$$(\omega|\varpi)^* = (i_1 i_2 \dots i_p | j_1 j_2 \dots j_p)^* \quad (5)$$

is the *permanent*:

$$(\omega|\varpi)^* = \text{per} \left((i_r | j_s) \right)_{r,s=1,2,\dots,p} \in \mathbb{C}[M_{n,n}].$$

Let $S = (\omega_1, \omega_2, \dots, \omega_p)$ and $T = (\varpi_1, \varpi_2, \dots, \varpi_p)$ be Young tableaux on $\{1, 2, \dots, n\}$ of the same shape λ .

Following again [17] and [3], the (permanental) *Young *-bitableau*

$$(S|T)^* = \left(\begin{array}{c|c} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{array} \right)^* \quad (6)$$

is the product of the *-biproduts of the pairs of corresponding rows:

$$(S|T)^* = (\omega_1|\varpi_1)^*(\omega_2|\varpi_2)^* \cdots (\omega_p|\varpi_p)^*. \quad (7)$$

A *column* Young tableau of *depth* h is a tableau of shape (1^h) . Then for a column Young bitableau, we have:

$$\left(\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right) = (-1)^{\binom{h}{2}} (i_1|j_1)(i_2|j_2) \cdots (i_h|j_h) \quad (8)$$

and for a column Young *-bitableau, we have:

$$\left(\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right)^* = (i_1|j_1)(i_2|j_2) \cdots (i_h|j_h). \quad (9)$$

We recall the definition of the *right symmetrized bitableau* $(S|\boxed{T})$ (see, e.g. [3]):

$$(S|\boxed{T}) = \sum_{\bar{T}} (S|\bar{T}), \quad (10)$$

where the sum is extended over *all* \bar{T} column permuted of T (hence, repeated entries in a column give rise to multiplicities).

Example 2.1.

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{array} \right) = 2 \left(\begin{array}{cc|cc} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{array} \right) + 2 \left(\begin{array}{cc|cc} 1 & 3 & 1 & 3 \\ 2 & 4 & 1 & 2 \end{array} \right).$$

We recall same elementary definitions and notational conventions. Given a partition (shape) $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p) \vdash n$, let $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_q) \vdash n$ denote its *conjugate* partition, where $\tilde{\lambda}_s = \#\{t; \lambda_t \geq s\}$. Similarly, given a Young tableau S of shape $sh(S) = \lambda$, let \tilde{S} denote its conjugate (dual) Young tableau. In plain words, \tilde{S} is the tableau whose rows are the columns of S and whose shape is $sh(\tilde{S}) = \tilde{\lambda}$. A Young tableau X on the (linearly ordered) set $L = \{1 < 2 < \cdots < n\}$ is said to be *standard* whenever its rows are increasing

from left to right and its columns are non decreasing downwards. In a dual way, a Young tableau Y is said to be *costandard* whenever its conjugate Young tableau \tilde{Y} is standard.

We recall the basis theorems for *standard determinantal bitableaux* (see, e.g. [16], [15], [14]), *costandard permanental *-bitableaux* [17] and *right symmetrized bitableaux* [3], respectively.

Proposition 2.2. *The sets*

- $\{(S|T); sh(S) = sh(T) = \lambda, \lambda_1 \leq n, S, T \text{ standard}\},$
- $\{(U|V)^*; sh(U) = sh(V) = \mu, \tilde{\mu}_1 \leq n, U, V \text{ costandard}\},$
- $\{(S|\boxed{T}); sh(S) = sh(T) = \lambda, \lambda_1 \leq n, S, T \text{ standard}\}$

are linear bases of $\mathbb{C}[M_{n,n}]$.

3 Polarization operators and Lie algebra representations of $gl(n)$ on $\mathbb{C}[M_{n,n}]$ and $U(gl(n))$

Given $i, j = 1, 2, \dots, n$, the *left polarization operator* (of j to i) D_{ij}^l is the linear operator from $\mathbb{C}[M_{n,n}]$ to itself defined by the conditions:

- D_{ij}^l is a derivation
- $D_{ij}^l((h|k)) = \delta_{jh}(i|k)$ for every k .

Similarly, the *right polarization operator* (of i to j) D_{ji}^r is the linear operator from $\mathbb{C}[M_{n,n}]$ to itself defined by the conditions:

- D_{ji}^r is a derivation
- $D_{ji}^r((h|k)) = \delta_{ik}(h|j)$ for every h .

In the following, we consider *three* Lie algebra representations

$$gl(n) \rightarrow \text{End}_{\mathbb{C}}[\mathbb{C}[M_{n,n}]]$$

and the corresponding Lie modules.

1. The *left* (covariant) representation ρ^l is defined by setting

$$\rho^l : e_{ij} \mapsto D_{ij}^l.$$

2. The *right* (contravariant) representation ρ^r is defined by setting

$$\rho^r : e_{ij} \mapsto -D_{ji}^r.$$

3. Notice that $D_{ij}^l D_{hk}^r = D_{hk}^r D_{ij}^l$. The *adjoint* representation $ad_{gl(n)}$ is defined by setting

$$ad_{gl(n)} : e_{ij} \mapsto D_{ij}^l - D_{ji}^r.$$

Given $i, j = 1, 2, \dots, n$, consider the linear operator T_{ij} from $\mathbf{U}(gl(n))$ to itself defined by setting

$$T_{ij}(\mathbf{M}) = e_{ij}\mathbf{M} - \mathbf{M}e_{ij},$$

for every $\mathbf{M} \in \mathbf{U}(gl(n))$.

We recall that T_{ij} is the unique derivation of $\mathbf{U}(gl(n))$ such that

$$T_{ij}(e_{st}) = \delta_{js}e_{it} - \delta_{it}e_{sj} = [e_{ij}, e_{st}]$$

for every $s, t = 1, 2, \dots, n$. Hence

$$T_{ij} \circ T_{hk} - T_{hk} \circ T_{ij} = \delta_{jh}T_{ik} - \delta_{ik}T_{hj}.$$

The Lie algebra representation

$$Ad_{gl(n)} : gl(n) \rightarrow \text{End}_{\mathbb{C}}[\mathbf{U}(gl(n))]$$

$$e_{ij} \mapsto T_{ij}$$

is the *adjoint* representation of $\mathbf{U}(gl(n))$ on itself.

4 The superalgebraic approach to the enveloping algebra $\mathbf{U}(gl(n))$

In this Section, we provide a synthetic presentation of the *superalgebraic method of virtual variables* for $gl(n)$.

This method was developed by the present authors for the general linear Lie superalgebras $gl(m|n)$ [18], in the series of notes [1], [2], [3], [4], [5], [6].

The technique of virtual variables is an extension of Capelli's method of *variabili ausiliarie* (Capelli [12], see also Weyl [27]).

Capelli introduced the technique of *variabili ausiliarie* in order to manage symmetrizer operators in terms of polarization operators and to simplify the study of some skew-symmetrizer operators (namely, the famous central Capelli operator).

Capelli's idea was well suited to treat symmetrization, but it did not work in the same efficient way while dealing with skew-symmetrization.

One had to wait the introduction of the notion of *superalgebras* (see, e.g. [26], [18]) to have the right conceptual framework to treat symmetry and skew-symmetry in one and the same way. To the best of our knowledge, the first mathematician who intuited the connection between Capelli's idea and superalgebras was Koszul in 1981 [19]. In particular, Koszul proved that the classical determinantal Capelli operator can be rewritten - in a much simpler way - by

adding to the symbols to be dealt with an extra auxiliary symbol that obeys to different commutation relations.

The superalgebraic method of virtual variables allows us to express remarkable classes of elements in $\mathbf{U}(gl(n))$ as images - with respect to the *Capelli devirtualization epimorphism* - of simple *monomials* and to obtain transparent combinatorial descriptions of their actions on irreducible $gl(n)$ -modules.

This method is very well suited for the study of the *polarization action* of $\mathbf{U}(gl(n))$ on $\mathbb{C}[M_{n,n}]$ and for the study of the *center* of $\mathbf{U}(gl(n))$.

4.1 The superalgebras $\mathbb{C}[M_{m_0|m_1+n,n}]$ and $gl(m_0|m_1+n)$

4.1.1 The general linear Lie super algebra $gl(m_0|m_1+n)$

Given a vector space V_n of dimension n , we will regard it as a subspace of a \mathbb{Z}_2 -graded vector space $W = W_0 \oplus W_1$, where

$$W_0 = V_{m_0}, \quad W_1 = V_{m_1} \oplus V_n.$$

The vector spaces V_{m_0} and V_{m_1} (we assume that $\dim(V_{m_0}) = m_0$ and $\dim(V_{m_1}) = m_1$ are “sufficiently large”) are called the *positive virtual (auxiliary) vector space*, the *negative virtual (auxiliary) vector space*, respectively, and V_n is called the *(negative) proper vector space*.

The inclusion $V_n \subset W$ induces a natural embedding of the ordinary general linear Lie algebra $gl(n)$ of V_n into the *auxiliary* general linear Lie *superalgebra* $gl(m_0|m_1+n)$ of $W = W_0 \oplus W_1$ (see, e.g. [18], [26]).

Let $A_0 = \{\alpha_1, \dots, \alpha_{m_0}\}$, $A_1 = \{\beta_1, \dots, \beta_{m_1}\}$, $L = \{1, 2, \dots, n\}$ denote *fixed bases* of V_{m_0} , V_{m_1} and V_n , respectively; therefore $|\alpha_s| = 0 \in \mathbb{Z}_2$, and $|\beta_t| = |i| = 1 \in \mathbb{Z}_2$.

Let

$$\{e_{a,b}; a, b \in A_0 \cup A_1 \cup L\}, \quad |e_{a,b}| = |a| + |b| \in \mathbb{Z}_2$$

be the standard \mathbb{Z}_2 -homogeneous basis of the Lie superalgebra $gl(m_0|m_1+n)$ provided by the elementary matrices. The elements $e_{a,b} \in gl(m_0|m_1+n)$ are \mathbb{Z}_2 -homogeneous of \mathbb{Z}_2 -degree $|e_{a,b}| = |a| + |b|$.

The superbracket of the Lie superalgebra $gl(m_0|m_1+n)$ has the following explicit form:

$$[e_{a,b}, e_{c,d}] = \delta_{bc} e_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{ad} e_{c,b},$$

$a, b, c, d \in A_0 \cup A_1 \cup L$.

In the following, the elements of the sets A_0, A_1, L will be called *positive virtual symbols*, *negative virtual symbols* and *negative proper symbols*, respectively.

4.1.2 The supersymmetric algebra $\mathbb{C}[M_{m_0|m_1+n,n}]$

We regard the commutative algebra $\mathbb{C}[M_{n,n}]$ as a subalgebra of the “*auxiliary*” *supersymmetric algebra*

$$\mathbb{C}[M_{m_0|m_1+n,n}] = \mathbb{C}[(\alpha_s|j), (\beta_t|j), (i|j)]$$

generated by the (\mathbb{Z}_2 -graded) variables $(\alpha_s|j), (\beta_t|j), (i|j), j = 1, 2, \dots, n$, where

$$|(\alpha_s|j)| = |\alpha_s| = 1 \in \mathbb{Z}_2, \quad |(\beta_t|j)| = |\beta_t| + 1 = 0 \in \mathbb{Z}_2$$

and $|(i|j)| = |i| + |j| = 0$, subject to the commutation relations:

$$(a|h)(b|k) = (-1)^{|(a|h)||(\beta_t|k)|} (b|k)(a|h),$$

for $a, b \in \{\alpha_1, \dots, \alpha_{m_0}\} \cup \{\beta_1, \dots, \beta_{m_1}\} \cup \{1, 2, \dots, n\}$.

In plain words, all the variables commute each other, with the exception of pairs of variables $(\alpha_s|j), (\alpha_t|j)$ that skew-commute:

$$(\alpha_s|j)(\alpha_t|j) = -(\alpha_t|j)(\alpha_s|j).$$

In the standard notation of multilinear algebra, we have:

$$\begin{aligned} \mathbb{C}[M_{m_0|m_1+n,n}] &\cong \Lambda[W_0 \otimes P_n] \otimes \text{Sym}[W_1 \otimes P_n] \\ &= \Lambda[V_{m_0} \otimes P_n] \otimes \text{Sym}[(V_{m_1} \oplus V_n) \otimes P_n] \end{aligned}$$

where $P_n = (P_n)_1$ denotes the trivially \mathbb{Z}_2 -graded vector space with distinguished basis $\{j; |j| = 1, j = 1, 2, \dots, n\}$.

The algebra $\mathbb{C}[M_{m_0|m_1+n,n}]$ is a supersymmetric \mathbb{Z}_2 -graded algebra (super-algebra), whose \mathbb{Z}_2 -graduation is inherited by the natural one in the exterior algebra.

4.1.3 Left superderivations and left superpolarizations

A *left superderivation* D^l (\mathbb{Z}_2 -homogeneous of degree $|D^l|$) (see, e.g. [26], [18]) on $\mathbb{C}[M_{m_0|m_1+n,n}]$ is an element of the superalgebra $\text{End}_{\mathbb{C}}[\mathbb{C}[M_{m_0|m_1+n,n}]]$ that satisfies "Leibniz rule"

$$D^l(\mathbf{p} \cdot \mathbf{q}) = D^l(\mathbf{p}) \cdot \mathbf{q} + (-1)^{|D^l||\mathbf{p}|} \mathbf{p} \cdot D^l(\mathbf{q}),$$

for every \mathbb{Z}_2 -homogeneous of degree $|\mathbf{p}|$ element $\mathbf{p} \in \mathbb{C}[M_{m_0|m_1+n,n}]$.

Given two symbols $a, b \in A_0 \cup A_1 \cup L$, the *left superpolarization* $D_{a,b}^l$ of b to a is the unique *left superderivation* of $\mathbb{C}[M_{m_0|m_1+n,n}]$ of \mathbb{Z}_2 -degree $|D_{a,b}^l| = |a| + |b| \in \mathbb{Z}_2$ such that

$$D_{a,b}^l((c|j)) = \delta_{bc} (a|j), \quad c \in A_0 \cup A_1 \cup L, \quad j = 1, \dots, n.$$

Informally, we say that the operator $D_{a,b}^l$ *annihilates* the symbol b and *creates* the symbol a .

4.1.4 The superalgebra $\mathbb{C}[M_{m_0|m_1+n,n}]$ as a $\mathbf{U}(\mathfrak{gl}(m_0|m_1+n))$ -module

Since

$$D_{a,b}^l D_{c,d}^l - (-1)^{(|a|+|b|)(|c|+|d|)} D_{c,d}^l D_{a,b}^l = \delta_{b,c} D_{a,d}^l - (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{a,d} D_{c,b}^l,$$

the map

$$e_{a,b} \mapsto D_{a,b}^l, \quad a, b \in A_0 \cup A_1 \cup L$$

is a Lie superalgebra morphism from $gl(m_0|m_1+n)$ to $End_{\mathbb{C}}[\mathbb{C}[M_{m_0|m_1+n,n}]]$ and, hence, it uniquely defines a representation:

$$\varrho : \mathbf{U}(gl(m_0|m_1+n)) \rightarrow End_{\mathbb{C}}[\mathbb{C}[M_{m_0|m_1+n,n}]].$$

In the following, we always regard the superalgebra $\mathbb{C}[M_{m_0|m_1+n,n}]$ as a $\mathbf{U}(gl(m_0|m_1+n))$ -supermodule, with respect to the action induced by the representation ϱ :

$$e_{a,b} \cdot \mathbf{p} = D_{a,b}^l(\mathbf{p}),$$

for every $\mathbf{p} \in \mathbb{C}[M_{m_0|m_1+n,n}]$.

We recall that $\mathbf{U}(gl(m_0|m_1+n))$ -module $\mathbb{C}[M_{m_0|m_1+n,n}]$ is a semisimple module, whose simple submodules are - up to isomorphism - *Schur supermodules* (see, e.g. [3], [4], [1]. For a more traditional presentation, see also [13]).

Clearly, $\mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n))$ is a subalgebra of $\mathbf{U}(gl(m_0|m_1+n))$ and the subalgebra $\mathbb{C}[M_{n,n}]$ is a $\mathbf{U}(gl(n))$ -submodule of $\mathbb{C}[M_{m_0|m_1+n,n}]$.

4.2 The virtual algebra $Virt(m_0+m_1, n)$ and the virtual presentations of elements in $\mathbf{U}(gl(n))$

We say that a product

$$e_{a_m b_m} \cdots e_{a_1 b_1} \in \mathbf{U}(gl(m_0|m_1+n)), \quad a_i, b_i \in A_0 \cup A_1 \cup L, \quad i = 1, \dots, m$$

is an *irregular expression* whenever there exists a right subword

$$e_{a_i, b_i} \cdots e_{a_2, b_2} e_{a_1, b_1},$$

$i \leq m$ and a virtual symbol $\gamma \in A_0 \cup A_1$ such that

$$\#\{j; b_j = \gamma, j \leq i\} > \#\{j; a_j = \gamma, j < i\}. \quad (11)$$

The meaning of an irregular expression in terms of the action of $\mathbf{U}(gl(m_0|m_1+n))$ by left superpolarization on the algebra $\mathbb{C}[M_{m_0|m_1+n,n}]$ is that there exists a virtual symbol γ and a right subsequence in which the symbol γ is *annihilated* more times than it was already *created* and, therefore, the action of an irregular expression on the algebra $\mathbb{C}[M_{n,n}]$ is *zero*.

Example 4.1. Let $\gamma \in A_0 \cup A_1$ and $x_i, x_j \in L$. The product

$$e_{\gamma, x_j} e_{x_i, \gamma} e_{x_j, \gamma} e_{\gamma, x_i}$$

is an irregular expression. □

Let **Irr** be the *left ideal* of $\mathbf{U}(gl(m_0|m_1+n))$ generated by the set of irregular expressions.

Proposition 4.2. *The superpolarization action of any element of \mathbf{Irr} on the subalgebra $\mathbb{C}[M_{n,n}] \subset \mathbb{C}[M_{m_0|m_1+n,n}]$ - via the representation ϱ - is identically zero.*

Proposition 4.3. ([5], [2]) *The sum $\mathbf{U}(gl(0|n)) + \mathbf{Irr}$ is a direct sum of vector subspaces of $\mathbf{U}(gl(m_0|m_1+n))$.*

Proposition 4.4. ([5], [2]) *The direct sum vector subspace $\mathbf{U}(gl(0|n)) \oplus \mathbf{Irr}$ is a subalgebra of $\mathbf{U}(gl(m_0|m_1+n))$.*

The subalgebra

$$Virt(m_0 + m_1, n) = \mathbf{U}(gl(0|n)) \oplus \mathbf{Irr} \subset \mathbf{U}(gl(m_0|m_1+n)).$$

is called the *virtual algebra*.

The proof of the following proposition is immediate from the definitions.

Proposition 4.5. *The left ideal \mathbf{Irr} of $\mathbf{U}(gl(m_0|m_1+n))$ is a two sided ideal of $Virt(m_0 + m_1, n)$.*

The *Capelli devirtualization epimorphism* is the surjection

$$\mathfrak{p} : Virt(m_0 + m_1, n) = \mathbf{U}(gl(0|n)) \oplus \mathbf{Irr} \twoheadrightarrow \mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n))$$

with $Ker(\mathfrak{p}) = \mathbf{Irr}$.

Any element in $\mathbf{M} \in Virt(m_0 + m_1, n)$ defines an element in $\mathbf{m} \in \mathbf{U}(gl(n))$ - via the map \mathfrak{p} - and \mathbf{M} is called a *virtual presentation* of \mathbf{m} .

Since the map \mathfrak{p} a surjection, any element $\mathbf{m} \in \mathbf{U}(gl(n))$ admits several virtual presentations. In the sequel, we even take virtual presentations as the *true definition* of special elements in $\mathbf{U}(gl(n))$, and this method will turn out to be quite effective.

Recall that $\mathbf{U}(gl(m_0|m_1+n))$ is a Lie module with respect to the adjoint representation $Ad_{gl(m_0|m_1+n)}$. Since $gl(n) = gl(0|n)$ is a Lie subalgebra of $gl(m_0|m_1+n)$, $\mathbf{U}(gl(m_0|m_1+n))$ is a $gl(n)$ -module with respect to the adjoint action $Ad_{gl(n)}$ of $gl(n)$.

The following results follow from the definitions.

Proposition 4.6. *The virtual algebra $Virt(m_0 + m_1, n)$ is a submodule of $\mathbf{U}(gl(m_0|m_1+n))$ with respect to the adjoint action $Ad_{gl(n)}$ of $gl(n)$.*

Proposition 4.7. *The Capelli epimorphism*

$$\mathfrak{p} : Virt(m_0 + m_1, n) \twoheadrightarrow \mathbf{U}(gl(n))$$

is an $Ad_{gl(n)}$ -equivariant map.

Corollary 4.8. *The isomorphism \mathfrak{p} maps any $Ad_{gl(n)}$ -invariant element $\mathbf{m} \in Virt(m_0 + m_1, n)$ to a central element of $\mathbf{U}(gl(n))$.*

Balanced monomials are elements of the algebra $\mathbf{U}(gl(m_0|m_1+n))$ of the form:

- $e_{i_1, \gamma_{p_1}} \cdots e_{i_k, \gamma_{p_k}} \cdot e_{\gamma_{p_1}, j_1} \cdots e_{\gamma_{p_k}, j_k},$
- $e_{i_1, \theta_{q_1}} \cdots e_{i_k, \theta_{q_k}} \cdot e_{\theta_{q_1}, \gamma_{p_1}} \cdots e_{\theta_{q_k}, \gamma_{p_k}} \cdot e_{\gamma_{p_1}, j_1} \cdots e_{\gamma_{p_k}, j_k},$
- and so on,

where $i_1, \dots, i_k, j_1, \dots, j_k \in L$, i.e., the $i_1, \dots, i_k, j_1, \dots, j_k$ are k proper (negative) symbols, and the $\gamma_{p_1}, \dots, \gamma_{p_k}, \dots, \theta_{q_1}, \dots, \theta_{q_k}, \dots$ are virtual symbols. In plain words, a balanced monomial is product of two or more factors where the rightmost one *annihilates* (by superpolarization) the k proper symbols j_1, \dots, j_k and *creates* (by superpolarization) some virtual symbols; the leftmost one *annihilates* all the virtual symbols and *creates* the k proper symbols i_1, \dots, i_k ; between these two factors, there might be further factors that annihilate and create virtual symbols only.

Proposition 4.9. ([3], [4], [1], [2]) *Every balanced monomial belongs to $\text{Virt}(m_0 + m_1, n)$. Hence, the Capelli epimorphism \mathfrak{p} maps balanced monomials to elements of $\mathbf{U}(\mathfrak{gl}(n))$.*

Let S and T be the Young tableaux

$$S = \begin{pmatrix} i_{p_1} & \cdots & i_{p_{\lambda_1}} \\ i_{q_1} & \cdots & i_{q_{\lambda_2}} \\ \cdots & & \cdots \\ i_{r_1} & \cdots & i_{r_{\lambda_m}} \end{pmatrix}, \quad T = \begin{pmatrix} j_{s_1} & \cdots & j_{s_{\lambda_1}} \\ j_{t_1} & \cdots & j_{t_{\lambda_2}} \\ \cdots & & \cdots \\ j_{v_1} & \cdots & j_{v_{\lambda_m}} \end{pmatrix}. \quad (12)$$

To the pair (S, T) , we associate the *bitableau monomial*:

$$e_{S,T} = e_{i_{p_1}, j_{s_1}} \cdots e_{i_{p_{\lambda_1}}, j_{s_{\lambda_1}}} e_{i_{q_1}, j_{t_1}} \cdots e_{i_{q_{\lambda_2}}, j_{t_{\lambda_2}}} \cdots e_{i_{r_1}, j_{v_1}} \cdots e_{i_{r_{\lambda_p}}, j_{v_{\lambda_p}}}$$

in $\mathbf{U}(\mathfrak{gl}(m_0 | m_1 + n))$.

Let $\beta_1, \dots, \beta_{\lambda_1} \in A_1, \alpha_1, \dots, \alpha_p \in A_0$ be sets of negative and positive *virtual symbols*, respectively. Set

$$D_\lambda = \begin{pmatrix} \beta_1 & \cdots & \beta_{\lambda_1} \\ \beta_1 & \cdots & \beta_{\lambda_2} \\ \cdots & & \cdots \\ \beta_1 & \cdots & \beta_{\lambda_p} \end{pmatrix}, \quad C_\lambda = \begin{pmatrix} \alpha_1 & \cdots & \alpha_1 \\ \alpha_2 & \cdots & \alpha_2 \\ \cdots & & \cdots \\ \alpha_p & \cdots & \alpha_p \end{pmatrix}$$

The tableaux D_λ and C_λ are called the *virtual Deruyts and Coderuyts tableaux* of shape λ , respectively.

Given a pair of Young tableaux S, T of the same shape λ on the proper alphabet L , consider the elements

$$e_{S, C_\lambda} e_{C_\lambda, T} \in \mathbf{U}(\mathfrak{gl}(m_0 | m_1 + n)), \quad (13)$$

$$e_{S, \widetilde{D}_\lambda} e_{\widetilde{D}_\lambda, T} \in \mathbf{U}(\mathfrak{gl}(m_0 | m_1 + n)), \quad (14)$$

$$e_{S, C_\lambda} e_{C_\lambda, D_\lambda} e_{D_\lambda, T} \in \mathbf{U}(\mathfrak{gl}(m_0 | m_1 + n)). \quad (15)$$

Since elements (13), (14) and (15) are balanced monomials in $\mathbf{U}(gl(m_0|m_1+n))$, they belong to the subalgebra $Virt(m_0+m_1, n)$.

We set

$$\mathfrak{p}(e_{S, C_\lambda} e_{C_\lambda, T}) = [S|T] \in \mathbf{U}(gl(n)),$$

and call the element $[S|T]$ a *Capelli bitableau* [5], [6].

We set

$$\mathfrak{p}(e_{S, \widetilde{D}_\lambda} e_{\widetilde{D}_\lambda, T}) = [S|T]^* \in \mathbf{U}(gl(n)),$$

and call the element $[S|T]^*$ a *Capelli *-bitableau* [5], [6].

We set

$$\mathfrak{p}(e_{S, C_\lambda} e_{C_\lambda, D_\lambda} e_{D_\lambda, T}) = [S|\boxed{T}] \in \mathbf{U}(gl(n)).$$

and call the element $[S|\boxed{T}]$ a *right Young-Capelli bitableau* [4].

5 The *bitableaux correspondence* maps \mathcal{B} and \mathcal{B}^* and the Koszul map \mathcal{K}

Theorem 5.1. *The bitableaux correspondence map*

$$\mathcal{B} : (S|T) \mapsto [S|T]$$

uniquely extends to a linear map

$$\mathcal{B} : \mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n)) \rightarrow \mathbf{U}(gl(n)).$$

Proof. We recall that bitableaux and Capelli bitableaux satisfy the **same** (determinantal) *straightening laws* in $\mathbb{C}[M_{n,n}]$ and $\mathbf{U}(gl(n))$, respectively ([5], Proposition 7). The straightening laws imply that standard (determinantal) bitableaux span $\mathbb{C}[M_{n,n}]$ (see, e.g. [16], [14], [15]); furthermore, standard bitableaux are linearly independent. Then, the map \mathcal{B} is a uniquely defined linear operator. \square

Theorem 5.2. *The *-bitableaux correspondence map*

$$\mathcal{B}^* : (S|T)^* \mapsto [S|T]^*$$

uniquely extends to a linear map

$$\mathcal{B}^* : \mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n)) \rightarrow \mathbf{U}(gl(n)).$$

Proof. The proof is essentially the same as the proof of Theorem 5.1, just by replacing the determinantal straightening laws with the permanental straightening laws, and standard (determinantal) bitableaux with costandard (permanental) bitableaux. Notice that both arguments are special cases of the superalgebraic version of the straightening laws and of the standard basis theorem ([17], [1]). \square

Given $i, j = 1, 2, \dots, n$, let

$$\rho_{ij} : \mathbb{C}[M_{n,n}] \rightarrow \mathbb{C}[M_{n,n}]$$

be the linear operator

$$\rho_{ij}(\mathbf{p}) = D_{ij}^l(\mathbf{p}) + (i|j) \cdot \mathbf{p}, \quad \text{for every } \mathbf{p} \in \mathbb{C}[M_{n,n}].$$

Proposition 5.3. *We have:*

$$[\rho_{ij}, \rho_{hk}] = \rho_{ij}\rho_{hk} - \rho_{hk}\rho_{ij} = \delta_{jh}\rho_{ik} - \delta_{ik}\rho_{hj}.$$

□

By the universal property of $\mathbf{U}(gl(n))$, Proposition 5.3 implies

Proposition 5.4. *The map*

$$e_{ij} \mapsto \rho_{ij}, \quad e_{ij} \in gl(n)$$

defines an associative algebra morphism

$$\tau : \mathbf{U}(gl(n)) \rightarrow \text{End}_{\mathbb{C}}[\mathbb{C}[M_{n,n}]].$$

□

Let ε_1 be the linear map *evaluation at 1*

$$\varepsilon_1 : \text{End}_{\mathbb{C}}[\mathbb{C}[M_{n,n}]] \rightarrow \mathbb{C}[M_{n,n}],$$

$$\varepsilon_1(\rho) = \rho(1) \in \mathbb{C}[M_{n,n}], \quad \text{for every } \rho \in \text{End}_{\mathbb{C}}[\mathbb{C}[M_{n,n}]].$$

The *Koszul map* [19] is the (*linear*) composition map

$$\mathcal{K} : \mathbf{U}(gl(n)) \rightarrow \mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n)),$$

$$\mathcal{K} = \varepsilon_1 \circ \tau.$$

Proposition 5.5. *We have:*

1. $\mathcal{K}(e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_h j_h}) = \rho_{i_1 j_1} \rho_{i_2 j_2} \cdots \rho_{i_h j_h}(1)$, $e_{i_p j_p} \in gl(n)$, $p = 1, 2, \dots, h$.
2. $\mathcal{K}(e_{ij} \mathbf{P}) = \rho_{ij}(\mathcal{K}(\mathbf{P}))$, for every $\mathbf{P} \in \mathbf{U}(gl(n))$, $e_{ij} \in gl(n)$.

□

6 Expansion formulae for *column Capelli bitableaux* and *column Capelli *-bitableaux*

Consider the *column Capelli bitableau*

$$\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right] = \mathbf{p}(e_{i_1 \alpha_1} \cdots e_{i_h \alpha_h} e_{\alpha_1 j_1} \cdots e_{\alpha_h j_h}) \in \mathbf{U}(gl(n)),$$

(where $\alpha_1, \dots, \alpha_h$ are arbitrary *distinct positive virtual* symbols) and the *column Capelli *-bitableau*

$$\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* = \mathbf{p} \left(e_{i_1 \beta_1} \cdots e_{i_h \beta_h} e_{\beta_1 j_1} \cdots e_{\beta_h j_h} \right) \in \mathbf{U}(\mathfrak{gl}(n))$$

(where β_1, \dots, β_h are arbitrary *distinct negative virtual* symbols).

Remember that the *proper symbols* $i_1, \dots, i_h, j_1, \dots, j_h \in L = \{1, 2, \dots, n\}$ are assumed to be *negative*.

From the definitions, it follows

$$\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right] = (-1)^{\binom{n}{2}} \left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^*. \quad (16)$$

From the definitions, we infer

Proposition 6.1. *Column Capelli bitableaux and column Capelli *-bitableaux are row-commutative as elements of $\mathbf{U}(\mathfrak{gl}(n))$:*

1.

$$\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right] = \left[\begin{array}{c|c} i_{\sigma(1)} & j_{\sigma(1)} \\ i_{\sigma(2)} & j_{\sigma(2)} \\ \vdots & \vdots \\ i_{\sigma(h)} & j_{\sigma(h)} \end{array} \right], \quad \sigma \in \mathcal{S}_h,$$

2.

$$\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* = \left[\begin{array}{c|c} i_{\sigma(1)} & j_{\sigma(1)} \\ i_{\sigma(2)} & j_{\sigma(2)} \\ \vdots & \vdots \\ i_{\sigma(h)} & j_{\sigma(h)} \end{array} \right]^*, \quad \sigma \in \mathcal{S}_h,$$

□

We provide two basic expansion formulae that describe the effect of picking out (on the left hand side) the first row of column Capelli bitableaux and column Capelli *-bitableaux. These formulae play a crucial role in the theory of the Koszul map \mathcal{K} , and provide a simple way to compute the *actual* forms of column Capelli bitableaux and column Capelli *-bitableaux as elements of $\mathbf{U}(\mathfrak{gl}(n))$.

Proposition 6.2. *We have:*

1.

$$\begin{aligned} & \left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right] = \\ & = (-1)^{h-1} e_{i_1 j_1} \left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right] + (-1)^{h-2} \sum_{k=2}^h \delta_{i_k j_1} \left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_1 & j_k \\ \vdots & \vdots \\ i_h & j_h \end{array} \right] \in \mathbf{U}(\mathfrak{gl}(n)). \end{aligned}$$

2.

$$\begin{aligned} & \left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* = \\ & = e_{i_1 j_1} \left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* - \sum_{k=2}^h \delta_{i_k j_1} \left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_1 & j_k \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \in \mathbf{U}(\mathfrak{gl}(n)). \end{aligned}$$

Proof. By definition,

$$\begin{aligned} & \left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_{h-1} & j_{h-1} \\ i_h & j_h \end{array} \right] = \\ & = \mathbf{p} [e_{i_1, \alpha_1} e_{i_2, \alpha_2} \cdots e_{i_{h-1}, \alpha_{h-1}} e_{i_h, \alpha_h} \cdot e_{\alpha_1, j_1} e_{\alpha_2, j_2} \cdots e_{\alpha_{h-1}, j_{h-1}} e_{\alpha_h, j_h}] = \\ & = \mathbf{p} [- e_{i_1, \alpha_1} e_{i_2, \alpha_2} \cdots e_{i_{h-1}, \alpha_{h-1}} e_{\alpha_1, j_1} e_{i_h, \alpha_h} \cdot e_{\alpha_2, j_2} \cdots e_{\alpha_{h-1}, j_{h-1}} e_{\alpha_h, j_h} \\ & \quad + e_{i_1, \alpha_1} e_{i_2, \alpha_2} \cdots e_{i_{h-1}, \alpha_{h-1}} \cdot \delta_{i_h, j_1} e_{\alpha_1, \alpha_h} e_{\alpha_2, j_2} \cdots e_{\alpha_{h-1}, j_{h-1}} e_{\alpha_h, j_h}] = \\ & = \mathbf{p} [- e_{i_1, \alpha_1} e_{i_2, \alpha_2} \cdots e_{i_{h-1}, \alpha_{h-1}} e_{\alpha_1, j_1} e_{i_h, \alpha_h} \cdot e_{\alpha_2, j_2} \cdots e_{\alpha_{h-1}, j_{h-1}} e_{\alpha_h, j_h} \\ & \quad + e_{i_1, \alpha_1} e_{i_2, \alpha_2} \cdots e_{i_{h-1}, \alpha_{h-1}} \cdot \delta_{i_h, j_1} e_{\alpha_2, j_2} \cdots e_{\alpha_{h-1}, j_{h-1}} e_{\alpha_1, j_h}]. \end{aligned}$$

Notice that

$$\delta_{i_h, j_1} e_{i_1, \alpha_1} e_{i_2, \alpha_2} \cdots e_{i_{h-1}, \alpha_{h-1}} \cdot e_{\alpha_2, j_2} \cdots e_{\alpha_{h-1}, j_{h-1}} e_{\alpha_1, j_h} =$$

$$\delta_{i_h, j_1} (-1)^{h-2} e_{i_1, \alpha_1} e_{i_2, \alpha_2} \cdots e_{i_{h-1}, \alpha_{h-1}} \cdot e_{\alpha_1, j_h} e_{\alpha_2, j_2} \cdots e_{\alpha_{h-1}, j_{h-1}}$$

as elements of the algebra $\mathbf{U}(gl(m_0|m_1+n))$.

Therefore, the summand

$$\mathfrak{p} [e_{i_1, \alpha_1} e_{i_2, \alpha_2} \cdots e_{i_{h-1}, \alpha_{h-1}} \cdot \delta_{i_h, j_1} e_{\alpha_2, j_2} \cdots e_{\alpha_{h-1}, j_{h-1}} e_{\alpha_1, j_h}]$$

equals

$$(-1)^{h-2} \delta_{i_h, j_1} \left[\begin{array}{c|c} i_1 & j_h \\ i_2 & j_2 \\ \vdots & \vdots \\ i_{h-1} & j_{h-1} \end{array} \right].$$

By repeating the above procedure of moving left the element e_{α_1, j_1} - using the commutator identities in $\mathbf{U}(gl(m_0|m_1+n))$ - we finally get

$$\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_{h-1} & j_{h-1} \\ i_h & j_h \end{array} \right] =$$

$$\begin{aligned} &= \mathfrak{p} [(-1)^{h-1} e_{i_1, \alpha_1} e_{\alpha_1, j_1} e_{i_2, \alpha_2} \cdots e_{i_{h-1}, \alpha_{h-1}} e_{i_h, \alpha_h} \cdot e_{\alpha_2, j_2} \cdots e_{\alpha_{h-1}, j_{h-1}} e_{\alpha_h, j_h} \\ &\quad + \sum_{i=0}^{h-2} (-1)^i e_{i_1, \alpha_1} \cdots \delta_{i_{h-i}, j_1} \widehat{e_{i_{h-i}, \alpha_{h-i}} e_{\alpha_1, \alpha_{h-i}}} \cdots e_{i_h, \alpha_h} \cdot e_{\alpha_2, j_2} \cdots e_{\alpha_h, j_h}] \\ &= \mathfrak{p} [(-1)^{h-1} e_{i_1, \alpha_1} e_{\alpha_1, j_1} e_{i_2, \alpha_2} \cdots e_{i_{h-1}, \alpha_{h-1}} e_{i_h, \alpha_h} \cdot e_{\alpha_2, j_2} \cdots e_{\alpha_{h-1}, j_{h-1}} e_{\alpha_h, j_h} \\ &\quad + \sum_{i=0}^{h-2} (-1)^i e_{i_1, \alpha_1} \cdots \delta_{i_{h-i}, j_1} \cdots e_{i_h, \alpha_h} \cdot e_{\alpha_2, j_2} \cdots e_{\alpha_1, j_{h-i}} \cdots e_{\alpha_h, j_h}]. \end{aligned}$$

Notice that the summand

$$(-1)^i \delta_{i_{h-i}, j_1} e_{i_1, \alpha_1} \cdots \delta_{i_{h-i}, j_1} \cdots e_{i_h, \alpha_h} \cdot e_{\alpha_2, j_2} \cdots e_{\alpha_1, j_{h-i}} \cdots e_{\alpha_h, j_h}$$

equals

$$(-1)^i \delta_{i_{h-i}, j_1} (-1)^{h-i-2} \times e_{i_1, \alpha_1} \cdots \widehat{e_{i_{h-i}, \alpha_{h-i}}} \cdots e_{i_h, \alpha_h} \cdot e_{\alpha_1, j_{h-i}} e_{\alpha_2, j_2} \cdots \widehat{e_{\alpha_{h-i}, j_{h-i}}} \cdots e_{\alpha_h, j_h}$$

as elements of the algebra $\mathbf{U}(gl(m_0|m_1+n))$.

Hence

$$\mathfrak{p} [(-1)^i \delta_{i_{h-i}, j_1} e_{i_1, \alpha_1} \cdots \widehat{e_{i_{h-i}, \alpha_{h-i}}} e_{\alpha_1, \alpha_{h-i}} \cdots e_{i_h, \alpha_h} \cdot e_{\alpha_2, j_2} \cdots e_{\alpha_h, j_h}]$$

equals

$$(-1)^{h-2} \delta_{i_{h-i}, j_1} \left[\begin{array}{c|c} i_1 & j_{h-i} \\ i_2 & j_2 \\ \vdots & \vdots \\ i_{h-i-1} & j_{h-i-1} \\ \hline i_{h-i} & j_{h-i} \\ i_{h-i+1} & j_{h-i+1} \\ i_h & j_h \end{array} \right].$$

Furthermore

$$\begin{aligned} \mathfrak{p} \left[(-1)^{h-1} e_{i_1, \alpha_1} e_{\alpha_1, j_1} e_{i_2, \alpha_2} \cdots e_{i_{h-1}, \alpha_{h-1}} e_{i_h, \alpha_h} \cdot e_{\alpha_2, j_2} \cdots e_{\alpha_{h-1}, j_{h-1}} e_{\alpha_h, j_h} \right] = \\ = (-1)^{h-1} e_{i_1, j_1} \left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_{h-1} & j_{h-1} \\ i_h & j_h \end{array} \right]. \end{aligned}$$

Since column Capelli bitableaux are row-commutative, by setting $k = h - i$ we proved the first expansion identity. The second expansion identity can be proved in a similar way. \square

Example 6.3.

$$\begin{aligned} \left[\begin{array}{c|c} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 2 & 3 \end{array} \right] &= -e_{12} \left[\begin{array}{c|c} 2 & 3 \\ 3 & 4 \\ 2 & 3 \end{array} \right] + \left[\begin{array}{c|c} 1 & 3 \\ 3 & 4 \\ 2 & 3 \end{array} \right] + \left[\begin{array}{c|c} 2 & 3 \\ 3 & 4 \\ 1 & 3 \end{array} \right] \\ &= -e_{12} \left[\begin{array}{c|c} 2 & 3 \\ 3 & 4 \\ 2 & 3 \end{array} \right] + 2 \left[\begin{array}{c|c} 1 & 3 \\ 3 & 4 \\ 2 & 3 \end{array} \right] \\ &= -e_{12} \left(e_{23} \left[\begin{array}{c|c} 3 & 4 \\ 2 & 3 \end{array} \right] - \left[\begin{array}{c|c} 2 & 4 \\ 2 & 3 \end{array} \right] \right) \\ &\quad + 2 \left(e_{13} \left[\begin{array}{c|c} 3 & 4 \\ 2 & 3 \end{array} \right] - \left[\begin{array}{c|c} 1 & 4 \\ 2 & 3 \end{array} \right] \right) \\ &= e_{12} e_{23} e_{34} e_{23} - e_{12} e_{24} e_{23} - 2e_{13} e_{34} e_{23} + 2e_{14} e_{23} \in \mathbf{U}(\mathfrak{gl}(4)). \end{aligned}$$

\square

Notice that, for $h = 1$, $[i|j] = [i|j]^* = e_{ij}$. Then, from Proposition 6.2, it follows

Corollary 6.4. *The family of column Capelli bitableaux ($*$ -bitableaux) is a system of linear generators of $\mathbf{U}(\mathfrak{gl}(n))$.*

7 Main results

Proposition 7.1.

$$\begin{aligned} \mathcal{K}\left(\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array}\right]\right) &= \left(\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array}\right) \\ &= (-1)^{\binom{h}{2}}(i_1|j_1)(i_2|j_2)\dots(i_h|j_h) \in \mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n)). \end{aligned}$$

Proof.

$$\begin{aligned} \mathcal{K}\left(\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array}\right]\right) &= \\ &= (-1)^{h-1}\mathcal{K}(e_{i_1j_1}\left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array}\right]) + (-1)^{h-2}\mathcal{K}\left(\sum_{k=2}^h \delta_{i_kj_1}\left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_1 & j_k \\ \vdots & \vdots \\ i_h & j_h \end{array}\right]\right) \\ &= (-1)^{h-1}\rho_{i_1j_1}(\mathcal{K}\left(\left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array}\right]\right)) + (-1)^{h-2}\mathcal{K}\left(\sum_{k=2}^h \delta_{i_kj_1}\left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_1 & j_k \\ \vdots & \vdots \\ i_h & j_h \end{array}\right]\right) \\ &= (-1)^{h-1}D_{i_1j_1}^1\left(\left(\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array}\right)\right) + (-1)^{h-1}(i_1|j_1)\left(\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array}\right) \\ &\quad + (-1)^{h-2}\sum_{k=2}^h \delta_{i_kj_1}\left(\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_1 & j_k \\ \vdots & \vdots \\ i_h & j_h \end{array}\right) \\ &= (-1)^{h-1}(i_1|j_1)\left(\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array}\right) = \left(\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array}\right). \end{aligned}$$

□

Example 7.2. Consider the column Capelli bitableau

$$\left[\begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{array} \right] = e_{12} \left[\begin{array}{c|c} 2 & 1 \\ 3 & 1 \end{array} \right] - \left[\begin{array}{c|c} 1 & 1 \\ 3 & 1 \end{array} \right] = -e_{12}e_{21}e_{31} + e_{11}e_{31} \in \mathbf{U}(gl(n)).$$

We have

$$\begin{aligned} \mathcal{K} \left(\left[\begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{array} \right] \right) &= \mathcal{K}(-e_{12}e_{21}e_{31} + e_{11}e_{31}) \\ &= \left(\begin{array}{c|c} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{array} \right) \\ &= -(1|2)(2|1)(3|1) \in \mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n)). \end{aligned}$$

□

Proposition 7.3.

$$\begin{aligned} \mathcal{K} \left(\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right) &= \left(\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right)^* \\ &= (i_1|j_1)(i_2|j_2) \dots (i_h|j_h) \in \mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n)). \end{aligned}$$

Proof.

$$\begin{aligned} \mathcal{K} \left(\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right) &= \\ &= \mathcal{K} \left(e_{i_1 j_1} \left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right) - \mathcal{K} \left(\sum_{k=2}^h \delta_{i_k j_1} \left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_1 & j_k \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right) \end{aligned}$$

$$\begin{aligned}
&= \rho_{i_1 j_1} \left(\mathcal{K} \left(\left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right) \right) - \mathcal{K} \left(\sum_{k=2}^h \delta_{i_k j_1} \left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_1 & j_k \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right) \\
&= D_{i_1 j_1}^l \left(\left(\left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right) \right) + (i_1 | j_1) \left(\left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right) \\
&\quad - \sum_{k=2}^h \delta_{i_k j_1} \left(\left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_1 & j_k \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right) \\
&= (i_1 | j_1) \left(\left[\begin{array}{c|c} i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right) = \left(\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right).
\end{aligned}$$

□

Notice that Theorem 5.1 specializes to

$$\mathcal{B} \left(\left(\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right) \right) = \left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right] \quad (17)$$

and, Theorem 5.2 specializes to

$$\mathcal{B}^* \left(\left(\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^* \right) \right) = \left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right]^*. \quad (18)$$

Theorem 7.4. *We have:*

1. $\mathcal{B} = \mathcal{K}^{-1}$,
2. $\mathcal{B}^* = \mathcal{K}^{-1}$,

3. \mathcal{B} , \mathcal{B}^* , \mathcal{K} are linear isomorphisms,

4. $\mathcal{B} = \mathcal{B}^*$.

Proof. From Corollary 6.4 and Eqs. (17) and (18), it follows that the operators \mathcal{B} and \mathcal{B}^* are surjective. Since column bitableaux span $\mathbb{C}[M_{n,n}]$, Propositions 7.1 and 7.3 imply that \mathcal{B} and \mathcal{B}^* are injective and $\mathcal{B} = \mathcal{K}^{-1}$ and $\mathcal{B}^* = \mathcal{K}^{-1}$. Then $\mathcal{B} = \mathcal{B}^*$. \square

By combining Theorems 5.1 and 5.2 with Theorem 7.4, it follows

Corollary 7.5. *We have:*

- $\mathcal{K} : [S|T] \mapsto (S|T)$,
- $\mathcal{K} : [S|T]^* \mapsto (S|T)^*$.

\square

The Koszul isomorphism \mathcal{K} well-behaves with respect to *right symmetrized bitableaux* and *right Young-Capelli bitableaux*.

Proposition 7.6. *We have:*

$$\mathcal{K} : [S|\boxed{T}] \mapsto (S|\boxed{T}).$$

Proof. We notice that

$$[S|\boxed{T}] = \sum_{\bar{T}} [S|\bar{T}], \quad (S|\boxed{T}) = \sum_{\bar{T}} (S|\bar{T}),$$

where the sum is extended over *all* \bar{T} column permuted of T (hence, repeated entries in a column give rise to multiplicities). The proof of the first equality easily follows from the definition, by applying the commutator identities in the superalgebra $\mathbf{U}(gl(m_0|m_1+n))$. The second equality is the definition of the right symmetrized bitableau $(S|\boxed{T})$, Eq. (10). \square

From Proposition 2.2, Corollary 7.5 and Proposition 7.6, it follows

Corollary 7.7. *The sets of standard Capelli bitableaux, of costandard Capelli *-bitableaux and of standard Young-Capelli bitableaux:*

- $\left\{ [S|T]; sh(S) = sh(T) = \lambda, \lambda_1 \leq n, S, T \text{ standard} \right\}$,
- $\left\{ [U|V]^*; sh(U) = sh(V) = \mu, \widetilde{\mu}_1 \leq n, U, V \text{ costandard} \right\}$,
- $\left\{ [S|\boxed{T}]; sh(S) = sh(T) = \lambda, \lambda_1 \leq n, S, T \text{ standard} \right\}$

are linear bases of $\mathbf{U}(gl(n))$.

Furthermore, we have

Theorem 7.8. *The Koszul isomorphism \mathcal{K} is equivariant with respect to the adjoint representations $(Ad_{gl(n)}, \mathbf{U}(gl(n)))$ and $(ad_{gl(n)}, \mathbb{C}[M_{n,n}])$.*

Proof. We recall that the action of $e_{hk} \in gl(n)$ on $\mathbf{U}(gl(n))$ through the adjoint representation $Ad_{gl(n)}$ is implemented by the derivation T_{hk} such that $T_{hk}(e_{st}) = \delta_{ks}e_{it} - \delta_{ht}e_{sj}$. From the definition of column Capelli bitableau and Proposition 4.7, we infer

$$T_{hk} \left(\left[\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right] \right) = \sum_{p=1}^h \delta_{k,i_p} \left[\begin{array}{c|c} i_1 & j_1 \\ \vdots & \vdots \\ h & j_p \\ \vdots & \vdots \\ i_h & j_h \end{array} \right] - \sum_{p=1}^h \delta_{j_p,h} \left[\begin{array}{c|c} i_1 & j_1 \\ \vdots & \vdots \\ i_p & k \\ \vdots & \vdots \\ i_h & j_h \end{array} \right].$$

We recall that the action of e_{hk} on $\mathbb{C}[M_{n,n}]$ through the adjoint representation $ad_{gl(n)}$ is implemented by the derivation $D_{hk}^l - D_{kh}^r$. Then

$$\begin{aligned} (D_{hk}^l - D_{kh}^r) \left(\left(\begin{array}{c|c} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{array} \right) \right) &= \sum_{p=1}^h \delta_{k,i_p} \left(\begin{array}{c|c} i_1 & j_1 \\ \vdots & \vdots \\ h & j_p \\ \vdots & \vdots \\ i_h & j_h \end{array} \right) \\ &\quad - \sum_{p=1}^h \delta_{j_p,h} \left(\begin{array}{c|c} i_1 & j_1 \\ \vdots & \vdots \\ i_p & k \\ \vdots & \vdots \\ i_h & j_h \end{array} \right). \end{aligned}$$

Since column Capelli bitableaux span $\mathbf{U}(gl(n))$ and column bitableaux span $\mathbb{C}[M_{n,n}]$, the assertion follows from Proposition 7.1. \square

Since

$$\zeta(n) = \mathbf{U}(gl(n))^{Ad_{gl(n)}},$$

the preceding Theorem implies:

Corollary 7.9. *We have*

$$\mathcal{K}[\zeta(n)] = \mathbb{C}[M_{n,n}]^{ad_{gl(n)}}.$$

In the left representation $(\rho^l, \mathbb{C}[M_{n,n}])$ (i.e. $\rho^l : e_{ij} \mapsto D_{ij}^l$), standard Young-Capelli bitableaux $[S \boxed{T}]$, $sh(S) = sh(T) = \lambda \vdash k$, act on right symmetrized bitableaux $(U \boxed{V})$, $sh(U) = sh(V) = \mu \vdash h$, in a quite remarkable way.

Proposition 7.10. [4] *We have:*

- If $h < k$, the action is zero.
- If $h = k$ and $\lambda \neq \mu$, the action is zero.
- If $h = k$ and $\lambda = \mu$, the action is nondegenerate triangular (with respect to a suitable linear order on standard tableaux of the same shape).

For details and proof, see [1] Theorem 10.1. Clearly, similar results hold for the right and the adjoint representations.

8 Laplace expansions

8.1 Laplace expansions in $\mathbb{C}[M_{n,n}]$

Recall that

$$(i_1 i_2 \cdots i_h | j_1 j_2 \cdots j_h) = (-1)^{\binom{h}{2}} \det[(i_s | j_t)]_{s,t=1,2,\dots,h} \in \mathbb{C}[M_{n,n}],$$

and, therefore, the biproduct $(i_1 i_2 \cdots i_h | j_1 j_2 \cdots j_h) \in \mathbb{C}[M_{n,n}]$ expands into column bitableaux as follows:

$$\begin{aligned} (i_1 i_2 \cdots i_h | j_1 j_2 \cdots j_h) &= \sum_{\sigma \in \mathbf{S}_h} (-1)^{|\sigma|} \left(\begin{array}{c|c} i_{\sigma(1)} & j_1 \\ i_{\sigma(2)} & j_2 \\ \vdots & \vdots \\ i_{\sigma(h)} & j_h \end{array} \right) \\ &= \sum_{\sigma \in \mathbf{S}_h} (-1)^{|\sigma|} \left(\begin{array}{c|c} i_1 & j_{\sigma(1)} \\ i_2 & j_{\sigma(2)} \\ \vdots & \vdots \\ i_h & j_{\sigma(h)} \end{array} \right). \end{aligned}$$

Notice that, in the passage from monomials to column bitableaux, the sign $(-1)^{\binom{h}{2}}$ disappears.

Recall that

$$(i_1 i_2 \cdots i_h | j_1 j_2 \cdots j_h)^* = \text{per}[(i_s | j_t)]_{s,t=1,2,\dots,h} \in \mathbb{C}[M_{n,n}],$$

and, therefore, the *-biproduct $(i_1 i_2 \cdots i_h | j_1 j_2 \cdots j_h)^* \in \mathbb{C}[M_{n,n}]$ expands into column *-bitableaux as follows:

$$\begin{aligned} (i_1 i_2 \cdots i_h | j_1 j_2 \cdots j_h)^* &= \sum_{\sigma \in \mathbf{S}_h} \left(\begin{array}{c|c} i_{\sigma(1)} & j_1 \\ i_{\sigma(2)} & j_2 \\ \vdots & \vdots \\ i_{\sigma(h)} & j_h \end{array} \right)^* \\ &= \sum_{\sigma \in \mathbf{S}_h} \left(\begin{array}{c|c} i_1 & j_{\sigma(1)} \\ i_2 & j_{\sigma(2)} \\ \vdots & \vdots \\ i_h & j_{\sigma(h)} \end{array} \right)^*. \end{aligned}$$

The preceding arguments extend to bitableaux and to *-bitableaux of any shape λ , $\lambda_1 \leq n$. Given the Young tableaux

$$S = \begin{pmatrix} i_{p_1} \cdots \cdots \cdots i_{p_{\lambda_1}} \\ i_{q_1} \cdots \cdots \cdots i_{q_{\lambda_2}} \\ \cdots \cdots \cdots \\ i_{r_1} \cdots \cdots \cdots i_{r_{\lambda_m}} \end{pmatrix}, \quad T = \begin{pmatrix} j_{s_1} \cdots \cdots \cdots j_{s_{\lambda_1}} \\ j_{t_1} \cdots \cdots \cdots j_{t_{\lambda_2}} \\ \cdots \cdots \cdots \\ j_{v_1} \cdots \cdots \cdots j_{v_{\lambda_m}} \end{pmatrix}.$$

From a simple sign computation, it follows

Proposition 8.1.

$$\begin{aligned} (S|T) &= \sum_{\sigma_1, \dots, \sigma_m} (-1)^{\sum_{k=1}^m |\sigma_k|} \begin{pmatrix} i_{p_{\sigma_1(1)}} & | & j_{s_1} \\ \cdot & & \cdot \\ i_{p_{\sigma_1(\lambda_1)}} & & j_{s_{\lambda_1}} \\ \vdots & & \vdots \\ i_{r_{\sigma_m(1)}} & & j_{v_1} \\ \cdot & & \cdot \\ i_{r_{\sigma_m(\lambda_m)}} & & j_{v_{\lambda_m}} \end{pmatrix}, \\ &= \sum_{\sigma_1, \dots, \sigma_m} (-1)^{\sum_{k=1}^m |\sigma_k|} \begin{pmatrix} i_{p_1} & | & j_{s_{\sigma_1(1)}} \\ \cdot & & \cdot \\ i_{p_{\lambda_1}} & & j_{s_{\sigma_1(\lambda_1)}} \\ \vdots & & \vdots \\ i_{r_1} & & j_{v_{\sigma_m(1)}} \\ \cdot & & \cdot \\ i_{r_{\lambda_m}} & & j_{v_{\sigma_m(\lambda_m)}} \end{pmatrix}, \end{aligned}$$

where the multiple sums range over all permutations $\sigma_1 \in \mathbf{S}_{\lambda_1}, \dots, \sigma_m \in \mathbf{S}_{\lambda_m}$.

Notice that, in the expansions with respect to column bitableaux, only the signs of permutations will remain.

Similarly, we have

Proposition 8.2.

$$(S|T)^* = \sum_{\sigma_1, \dots, \sigma_m} \begin{pmatrix} i_{p_{\sigma_1(1)}} & | & j_{s_1} \\ \cdot & & \cdot \\ i_{p_{\sigma_1(\lambda_1)}} & & j_{s_{\lambda_1}} \\ \vdots & & \vdots \\ i_{r_{\sigma_m(1)}} & & j_{v_1} \\ \cdot & & \cdot \\ i_{r_{\sigma_m(\lambda_m)}} & & j_{v_{\lambda_m}} \end{pmatrix}^*$$

$$= \sum_{\sigma_1, \dots, \sigma_m} \left(\begin{array}{c|c} i_{p_1} & j_{s_{\sigma_1(1)}} \\ \cdot & \cdot \\ i_{p_{\lambda_1}} & j_{s_{\sigma_1(\lambda_1)}} \\ \vdots & \vdots \\ i_{r_1} & j_{v_{\sigma_m(1)}} \\ \cdot & \cdot \\ i_{r_{\lambda_m}} & j_{v_{\sigma_m(\lambda_m)}} \end{array} \right)^*$$

8.2 Laplace expansions in $U(gl(n))$

Let S and T be the Young tableaux

$$S = \left(\begin{array}{c} i_{p_1} \cdots i_{p_{\lambda_1}} \\ i_{q_1} \cdots i_{q_{\lambda_2}} \\ \cdots \\ i_{r_1} \cdots i_{r_{\lambda_m}} \end{array} \right), \quad T = \left(\begin{array}{c} j_{s_1} \cdots j_{s_{\lambda_1}} \\ j_{t_1} \cdots j_{t_{\lambda_2}} \\ \cdots \\ j_{v_1} \cdots j_{v_{\lambda_m}} \end{array} \right).$$

Propositions 8.1 and 8.2 and Theorems 5.1 and 5.2 imply to the following *Laplace expansions* of Capelli bitableaux into column Capelli bitableaux and of Capelli *-bitableaux into column Capelli *-bitableaux.

Corollary 8.3. *We have*

$$\begin{aligned} [S|T] &= \sum_{\sigma_1, \dots, \sigma_m} (-1)^{\sum_{k=1}^m |\sigma_k|} \left[\begin{array}{c|c} i_{p_{\sigma_1(1)}} & j_{s_1} \\ \cdot & \cdot \\ i_{p_{\sigma_1(\lambda_1)}} & j_{s_{\lambda_1}} \\ \vdots & \vdots \\ i_{r_{\sigma_m(1)}} & j_{v_1} \\ \cdot & \cdot \\ i_{r_{\sigma_m(\lambda_m)}} & j_{v_{\lambda_m}} \end{array} \right] \\ &= \sum_{\sigma_1, \dots, \sigma_m} (-1)^{\sum_{k=1}^m |\sigma_k|} \left[\begin{array}{c|c} i_{p_1} & j_{s_{\sigma_1(1)}} \\ \cdot & \cdot \\ i_{p_{\lambda_1}} & j_{s_{\sigma_1(\lambda_1)}} \\ \vdots & \vdots \\ i_{r_1} & j_{v_{\sigma_m(1)}} \\ \cdot & \cdot \\ i_{r_{\lambda_m}} & j_{v_{\sigma_m(\lambda_m)}} \end{array} \right]. \end{aligned}$$

Corollary 8.4. *We have*

$$\begin{aligned}
[S|T]^* &= \sum_{\sigma_1, \dots, \sigma_m} \left[\begin{array}{c|c} \dot{i}_{p_{\sigma_1(1)}} & \dot{j}_{s_1} \\ \cdot & \cdot \\ \dot{i}_{p_{\sigma_1(\lambda_1)}} & \dot{j}_{s_{\lambda_1}} \\ \vdots & \vdots \\ \dot{i}_{r_{\sigma_m(1)}} & \dot{j}_{v_1} \\ \cdot & \cdot \\ \dot{i}_{r_{\sigma_m(\lambda_m)}} & \dot{j}_{v_{\lambda_m}} \end{array} \right]^* \\
&= \sum_{\sigma_1, \dots, \sigma_m} \left[\begin{array}{c|c} \dot{i}_{p_1} & \dot{j}_{s_{\sigma_1(1)}} \\ \cdot & \cdot \\ \dot{i}_{p_{\lambda_1}} & \dot{j}_{s_{\sigma_1(\lambda_1)}} \\ \vdots & \vdots \\ \dot{i}_{r_1} & \dot{j}_{v_{\sigma_m(1)}} \\ \cdot & \cdot \\ \dot{i}_{r_{\lambda_m}} & \dot{j}_{v_{\sigma_m(\lambda_m)}} \end{array} \right]^* .
\end{aligned}$$

By combining the expansions of Corollaries 8.3 and 8.4 with the results of Proposition 6.2, one gets explicit expansions of Capelli bitableaux and of Capelli *-bitableaux as elements of $\mathbf{U}(gl(n))$.

Example 8.5. The Capelli bitableau (of shape $\lambda = (2, 2)$)

$$\left[\begin{array}{cc|cc} 1 & 2 & 2 & 3 \\ 2 & 4 & 3 & 4 \end{array} \right] \in \mathbf{U}(gl(4)) \tag{19}$$

equals

$$\left[\begin{array}{c|c} 1 & 2 \\ 2 & 3 \\ 2 & 3 \\ 4 & 4 \end{array} \right] - \left[\begin{array}{c|c} 1 & 3 \\ 2 & 2 \\ 2 & 3 \\ 4 & 4 \end{array} \right] - \left[\begin{array}{c|c} 1 & 2 \\ 2 & 3 \\ 2 & 4 \\ 4 & 3 \end{array} \right] + \left[\begin{array}{c|c} 1 & 3 \\ 2 & 2 \\ 2 & 4 \\ 4 & 3 \end{array} \right],$$

where

$$\begin{aligned} \left[\begin{array}{c|c} 1 & 2 \\ 2 & 3 \\ 2 & 3 \\ 4 & 4 \end{array} \right] &= e_{12}e_{23}e_{23}e_{44} - 2e_{13}e_{23}e_{44}, \\ \left[\begin{array}{c|c} 1 & 3 \\ 2 & 2 \\ 2 & 3 \\ 4 & 4 \end{array} \right] &= e_{13}e_{22}e_{23}e_{44} - e_{13}e_{23}e_{44}, \\ \left[\begin{array}{c|c} 1 & 2 \\ 2 & 3 \\ 2 & 4 \\ 4 & 3 \end{array} \right] &= e_{12}e_{23}e_{24}e_{43} - e_{12}e_{23}e_{23} - e_{13}e_{24}e_{43} + e_{13}e_{23} - e_{23}e_{14}e_{43} + e_{23}e_{13}, \\ \left[\begin{array}{c|c} 1 & 3 \\ 2 & 2 \\ 2 & 4 \\ 4 & 3 \end{array} \right] &= e_{13}e_{22}e_{24}e_{43} - e_{13}e_{22}e_{23} - e_{13}e_{24}e_{43} + e_{13}e_{23}. \end{aligned}$$

This example can be used to enlighten the difference between the PBW Theorem and Theorem 7.4.

The PBW Theorem establishes an isomorphism ϕ from the *graded algebra*

$$Gr[\mathbf{U}(gl(n))] = \bigoplus_{h \in \mathbb{Z}^+} \frac{\mathbf{U}^{(h)}(gl(n))}{\mathbf{U}^{(h-1)}(gl(n))}$$

associated to the *filtered algebra* $\mathbf{U}(gl(n))$ to the algebra $\mathbf{Sym}(gl(n)) \cong \mathbb{C}[M_{n,n}]$. Clearly, the isomorphism ϕ maps the projection to $Gr[\mathbf{U}(gl(n))]$ of the Capelli bitableau (19) - as an element of the quotient space $\frac{\mathbf{U}^{(4)}(gl(4))}{\mathbf{U}^{(3)}(gl(4))}$ - to the product determinants

$$\mathbf{det} \left(\begin{array}{cc} (1|2) & (1|3) \\ (2|2) & (2|3) \end{array} \right) \times \mathbf{det} \left(\begin{array}{cc} (2|3) & (2|4) \\ (4|3) & (4|4) \end{array} \right) \in \mathbb{C}[M_{4,4}]. \quad (20)$$

The Koszul isomorphism \mathcal{K} (injectively) maps the Capelli bitableau (19) - as an element of $\mathbf{U}(gl(4))$ - to the product of determinants (20). Similarly, the isomorphism \mathcal{K} maps the Capelli *-bitableau

$$\left[\begin{array}{cc|cc} 1 & 2 & 2 & 3 \\ 2 & 4 & 3 & 4 \end{array} \right]^* \in \mathbf{U}(gl(4))$$

to the product permanents

$$\mathbf{per} \left(\begin{array}{cc} (1|2) & (1|3) \\ (2|2) & (2|3) \end{array} \right) \times \mathbf{per} \left(\begin{array}{cc} (2|3) & (2|4) \\ (4|3) & (4|4) \end{array} \right) \in \mathbb{C}[M_{4,4}].$$

□

In the following, we will discuss some implications of Corollary 7.9.

Proposition 8.6. (Koszul [19]) *Consider the row Capelli bitableau*

$$[n \cdots 21 | 12 \cdots n] \in \mathbf{U}(\mathfrak{gl}(n)).$$

We have:

1.

$$[n \cdots 21 | 12 \cdots n] = \mathbf{cdet} \begin{pmatrix} e_{11} + (n-1) & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} + (n-2) & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{pmatrix},$$

the Capelli column determinant¹ in $\mathbf{U}(\mathfrak{gl}(n))$.

2.

$$\mathcal{K}([n \cdots 21 | 12 \cdots n]) = \mathbf{det} \begin{pmatrix} (1|1) & \cdots & (1|n) \\ \vdots & & \vdots \\ (n|1) & \cdots & (n|n) \end{pmatrix} \in \mathbb{C}[M_{n,n}].$$

Proof. We have

$$\begin{aligned} [n \cdots 21 | 12 \cdots n] &= \sum_{\sigma \in \mathbf{S}_n} (-1)^{|\sigma|} \left[\begin{array}{c|c} \sigma(n) & 1 \\ \sigma(n-1) & 2 \\ \vdots & \vdots \\ \sigma(1) & n \end{array} \right] \\ &= \sum_{\sigma \in \mathbf{S}_n} (-1)^{|\sigma|} \times \\ &\quad \left((-1)^{n-1} e_{\sigma(n)1} \left[\begin{array}{c|c} \sigma(n-1) & 2 \\ \sigma(n-2) & 3 \\ \vdots & \vdots \\ \sigma(1) & n \end{array} \right] \right. \\ &\quad \left. + (-1)^{n-2} \sum_{k=2}^n \delta_{\sigma(n-k+1)1} \left[\begin{array}{c|c} \sigma(n-1) & 2 \\ \vdots & \vdots \\ \sigma(n) & k \\ \vdots & \vdots \\ \sigma(1) & n \end{array} \right] \right) \\ &= (-1)^{n-1} \sum_{\sigma \in \mathbf{S}_n} (-1)^{|\sigma|} (e_{\sigma(n)1} + (n-1)\delta_{\sigma(n)1}) \left[\begin{array}{c|c} \sigma(n-1) & 2 \\ \sigma(n-2) & 3 \\ \vdots & \vdots \\ \sigma(1) & n \end{array} \right], \end{aligned}$$

¹The symbol \mathbf{cdet} denotes the column determinat of a matrix $A = [a_{ij}]$ with noncommutative entries: $\mathbf{cdet}(A) = \sum_{\sigma} (-1)^{|\sigma|} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$.

from Proposition 6.2.

By iterating the same argument,

$$\begin{aligned}
& [n \cdots 21|12 \cdots n] = \\
& = (-1)^{\binom{n}{2}} \times \\
& \quad \sum_{\sigma \in \mathbf{S}_n} (-1)^{|\sigma|} (e_{\sigma(n)1} + (n-1)\delta_{\sigma(n)1}) (e_{\sigma(n-1)2} + (n-2)\delta_{\sigma(n-1)2}) \cdots e_{\sigma(1)n} \\
& = \sum_{\tau \in \mathbf{S}_n} (-1)^{|\tau|} (e_{\tau(1)1} + (n-1)\delta_{\tau(1)1}) (e_{\tau(2)2} + (n-2)\delta_{\tau(2)2}) \cdots e_{\tau(n)n} \\
& = \mathbf{cdet} \begin{pmatrix} e_{11} + (n-1) & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} + (n-2) & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{pmatrix} \in \mathbf{U}(\mathfrak{gl}(n)),
\end{aligned}$$

the Capelli column determinant in $\mathbf{U}(\mathfrak{gl}(n))$.

Then

$$\mathcal{K} \left(\mathbf{cdet} \begin{pmatrix} e_{11} + (n-1) & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} + (n-2) & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{pmatrix} \right) = \mathcal{K}([n \cdots 21|12 \cdots n]),$$

that equals

$$(n \cdots 21|12 \cdots n) = \mathbf{det} \begin{pmatrix} (1|1) & \cdots & (1|n) \\ \vdots & & \vdots \\ (n|1) & \cdots & (n|n) \end{pmatrix} \in \mathbb{C}[M_{n,n}],$$

by Corollary 7.5. \square

In the enveloping algebra $\mathbf{U}(\mathfrak{gl}(n))$, given any integer $k = 1, 2, \dots, n$, consider the k -th *Capelli element*:

$$\mathbf{H}_k(n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k]. \quad (21)$$

By the same argument of Proposition 8.6,

$$\mathbf{H}_k(n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbf{cdet} \begin{pmatrix} e_{i_1, i_1} + (k-1) & e_{i_1, i_2} & \cdots & e_{i_1, i_k} \\ e_{i_2, i_1} & e_{i_2, i_2} + (k-2) & \cdots & e_{i_2, i_k} \\ \vdots & \vdots & \ddots & \vdots \\ e_{i_k, i_1} & e_{i_k, i_2} & \cdots & e_{i_k, i_k} \end{pmatrix},$$

and the operator \mathcal{K} maps $\mathbf{H}_k(n)$ to the polynomial

$$\begin{aligned} \mathbf{h}_k(n) &= \sum_{1 \leq i_1 < \dots < i_k \leq n} (i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k) \\ &= \sum_{1 \leq i_1 < \dots < i_k \leq n} \det \begin{pmatrix} (i_1 | i_1) & \dots & (i_1 | i_k) \\ \vdots & & \vdots \\ (i_k | i_1) & \dots & (i_k | i_k) \end{pmatrix} \in \mathbb{C}[M_{n,n}]. \end{aligned}$$

Notice that the polynomials $\mathbf{h}_k(n)$'s appear as coefficients (in $\mathbb{C}[M_{n,n}]$) of the characteristic polynomial:

$$P_{M_{n,n}}(t) = \det(tI - M_{n,n}) = t^n + \sum_{i=1}^n (-1)^i \mathbf{h}_i(n) t^{n-i}.$$

Clearly, $\mathbf{h}_k(n)$ is $ad_{gl(n)}$ -invariant in $\mathbb{C}[M_{n,n}]$ and, therefore, $\mathbf{H}_k(n)$ is a *central element* of the enveloping algebra $\mathbf{U}(gl(n))$.

In passing we recall Capelli's Theorem ([9] and [10], see also [6]):

Proposition 8.7.

$$\zeta(n) = \mathbb{C}[\mathbf{H}_1(n), \mathbf{H}_2(n), \dots, \mathbf{H}_n(n)].$$

Moreover, the $\mathbf{H}_k(n)$'s are algebraically independent.

In general, given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$, $\lambda_1 \leq n$, consider the sum of Capelli bitableaux

$$\mathbf{K}_\lambda(n) = \sum_S [S|S],$$

where the sum is extended to all row-increasing tableaux S , $sh(S) = \lambda$ (the $\mathbf{K}_\lambda(n)$'s are called *shaped* Capelli elements in [7]). Notice that the elements $\mathbf{K}_\lambda(n)$ are *radically different* from the elements $\mathbf{H}_\lambda(n) = \mathbf{H}_{\lambda_1}(n) \cdots \mathbf{H}_{\lambda_p}(n)$.

From Corollary 7.5, Eqs. (3) and (4) and *row* skew-symmetry of bitableaux, we infer

Proposition 8.8. *We have*

$$\mathcal{K}(\mathbf{K}_\lambda(n)) = (-1)^{\binom{|\lambda|}{2}} \mathbf{h}_{\lambda_1}(n) \mathbf{h}_{\lambda_2}(n) \cdots \mathbf{h}_{\lambda_p}(n), \quad |\lambda| = \sum_i \lambda_i.$$

Hence, the elements $\mathbf{K}_\lambda(n)$ are central. By Corollary 7.9, the following statements are equivalent:

- The $\mathbf{K}_\lambda(n)$ -basis theorem for $\zeta(n)$ [7]:

Proposition 8.9. *The set*

$$\{\mathbf{K}_\lambda(n); \lambda_1 \leq n\}$$

is a linear basis of $\zeta(n)$.

Notice that the elements $\mathbf{K}_\lambda(n)$ are *radically different* from the *quantum immanants* of [21], [22] and [8].

- The well-known theorem for the algebra of invariants $\mathbb{C}[M_{n,n}]^{ad_{gl(n)}}$:

Proposition 8.10.

$$\mathbb{C}[M_{n,n}]^{ad_{gl(n)}} = \mathbb{C}[\mathbf{h}_1(n), \mathbf{h}_2(n), \dots, \mathbf{h}_n(n)].$$

Moreover, the $\mathbf{h}_k(n)$'s are algebraically independent.

Proposition 8.10 is usually stated in terms of the algebra $\mathbb{C}[M_{n,n}]^{GL(n)} = \mathbb{C}[M_{n,n}]^{ad_{gl(n)}}$, where $\mathbb{C}[M_{n,n}]^{GL(n)}$ is the subalgebra of invariants with respect to the *conjugation action* of the general linear group $GL(n)$ on $\mathbb{C}[M_{n,n}]$ (see, e.g. [20], [14], [24]).

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