Second gradient Green-Naghdi type thermo-elasticity and viscoelasticity

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Mauro Fabrizio[a], Franca Franchi[a,*], Roberta Nibbi[a]

[a]*Department of Mathematics, University of Bologna, 5 Piazza di Porta S. Donato, 40126 Bologna, Italy*

Abstract

In this paper we perform a thermodynamically consistent theory for second gradient Green-Naghdi revisited type thermo-elasticity and viscoelasticity by developing a non-standard version of the virtual powers method. We emphasize the incontrovertible effectiveness of variational arguments to approach non local continuum structures. Notably, non local Kelvin-Voigt type viscoelastic models and non local/local revised Green-Naghdi rigid heat conduction theories are recovered as special constitutive settings, also providing a fruitful comparison with pre-existent modelings.

Keywords: Virtual powers, second gradient thermo-viscoelasticity, Green-Naghdi and Kelvin-Voigt theories.

Devoted to Prof. Brian Straughan on the occasion of his 75th birthday: a great researcher, but above all a great lifelong friend!

1. Introduction

We face some issues related to the thermodynamic description of non local thermo-elastic materials, within the small deformations approach, with the aim of enhancing the strategy of the virtual powers format. Specifically, we focus on second gradient linear elasticity in the presence of analogous non local thermal properties, within a revisited Green-Naghdi (shortly G-N) type heat theory. We observe that G-N heat theories (see e.g. [13, 14, 15]) have aroused the great attention of many researchers, also for constructive criticisms, see e.g. [4, 17]. There is also awareness that, due to their employment in many areas of continuum mechanics with interesting applications to micro- and nano-thermal devices, a reexamination of their local/non local constitutive features is worthwhile, see [16, 17, 18]

The introduction of suitable retardation parameters, responsible for memory effects, leads to a non local thermo-viscoelastic theory of Kelvin-Voigt (shortly K-V) type, which might be easily generalized by incorporating the dependence on past histories. Nowadays, K-V type viscoelasticity has been increasingly occupying attention in literature, also within generalized theories (see e.g [20, 10] and references therein). Our constitutive theories are going to allow for memory effects at micro- and nano-scales interacting phenomena, where different size effects on heat conduction and elastic deformations become essential and standard local theories are proven to fail [22].

Moreover, non local theories have acquired a fundamental importance in very different research fields, ranging from physics and engineering to biology and medicine, even mixing the phenomena together and hence assessing multidisciplinary scale effects.

As well known, the so called Rational Thermodynamics approach (see [8] and references therein) addresses only simple (local) materials, therefore the two Laws must be re-adapted for describing non local structures. In our opinion the virtual powers method represents the most efficient strategy to face such complex materials: besides weak formulations, which need less smoothness requirements, it guarantees a universal and a priori tool since it is based on the rheological aspects under study and furnishes all the (balance and imbalance) local equations together with the appropriate constitutive boundary conditions (see e.g. [12, 5] and references therein). In this paper we just follow the guidelines of this method, within a non-standard version.

The outline of the paper is as follows. In Section 2, besides some preliminary notations and definitions, we develop a revised version of the virtual powers method, based on the notion of the coldness displacement, and perform novel non isothermal second order gradients elastic/viscoelastic settings, under analogous non local thermal properties through a revisitation of G-N heat

*Corresponding author
Email address: franca.franchi@unibo.it (Franca Franchi)

theories. In Section 3 non local K-V type viscoelastic models and non local/local revisited G-N type rigid heat theory are recovered as special constitutive settings. generalizable to further hereditary effects, also furnishing some comparisons with pre-existent literature modelings.

2. Second gradient thermo-elastic and viscoelastic models: preliminaries, the guidelines of the general virtual powers scheme and thermodynamic restrictions

As a brief review of kinematical aspects, let a typical simple/non simple body be pointwise identified with the initial configuration $\mathcal{B}(0)$ of the ordinary 3D Euclidean space. Let $\mathcal{P}(0)$ be an arbitrary body part (sub-body) of $\mathcal{B}(0)$, with outward unit normal $\mathbf{n}$ on its boundary. So we work on the space-time cylinder $\mathcal{P}(0) \times (0, T)$. In what follows, we address the Lagrangian description of Continuum Mechanics, within the small deformations approach, using the standard notations but granting us some slight nomenclature changes to avoid "writing distractions".

A simple dot stands for the scalar product between vectors or tensors of any order, independently of contracted index pair. As usual, the material position $\mathbf{X}$ is related to the current position $\mathbf{x}(t)$, via the material displacement vector $\mathbf{u}$, as $\mathbf{u} = \mathbf{x}(t) - \mathbf{X}$, and, for brevity, henceforth the space dependence of the fields is generally omitted. Further an upper dot over a symbol denotes the time derivative, so that the velocity field is $\mathbf{v} = \dot{\mathbf{u}}$. The tensor $\mathbf{E} = sym(\nabla\mathbf{u})$ represents the linearized form of the Green-St-Venant strain tensor to denote the infinitesimal strain tensor, and ρ_0 stands for the (positive and constant) reference mass density. Without any misunderstanding the first Piola-Kirchhoff stress tensor will be still denoted by $\mathbf{T}$, likewise we use the notation θ for the material temperature, $\mathbf{q}$ for the heat flux vector in its material form and so on for the internal energy density e and the entropy density η, the body force density $\mathbf{b}$ and the heat supply density r.

In analogy with the displacement vector $\mathbf{u}$, we introduce the coldness displacement, denoted by $\hat{k}$, defined as a time primitive of the coldness $k = 1/\theta$, so that $\dot{\hat{k}} = k$.

The "kinetics" of our thermo-elastic behavior is characterized by the pair $(\mathbf{u}, \hat{k})$, so that the pair $(\dot{\mathbf{u}}, \dot{\hat{k}})$ represents the real "velocities", whereas the virtual (test) "velocities" will be given by $(\delta\mathbf{u}, \delta\hat{k})$. As usual, the virtual velocities are supposed sufficiently smooth with compact support on the space-time cylinder in study.

Let us also define the modified Helmholtz free energy per unit mass

$$\tilde{\psi} = k\psi = ke - \eta\,, \tag{1}$$

replacing the standard free energy $\psi = e - \theta\eta$. Therefore, dealing with thermal properties, the natural variable becomes the coldness rather than the temperature.

We conclude the first part of this section, with some useful considerations over the strategic notion of the state σ and the associated process P within a local/non local constitutive theory. On the Lagrangian point of view, for simple isothermal linearly elastic materials one has $\sigma = \nabla\mathbf{u}$ with process $P = \nabla\dot{\mathbf{u}}$. The first Piola-Kirchhoff stress tensor $\mathbf{T}$ depends on the present value of $\nabla\mathbf{u}$, so that the internal mechanical power density has form $\mathbf{T} \cdot \nabla\dot{\mathbf{u}}$.

More generally, the deformation can be better approximated considering higher order strain gradients. Herein we focus on second strain gradient materials, so the effective stress tensor splits as follows:

$$\mathbf{T} = \mathbf{T}_2 - \nabla \cdot \mathbf{T}_3\,, \tag{2}$$

where the hyper-stress $\mathbf{T}_3$ is a third order tensor, which is supposed to be symmetric with respect to the last two indices. In a simplified constitutive setting, $\mathbf{T}_3$ may be chosen proportional to the gradient of $\mathbf{T}_2$, via a parameter representing an internal length modulo able to capture different size-dependent effects [9, 1]. Meanwhile one may propose a non isothermal simple/non simple elastic theory, within the classical (Fourier, Cattaneo, Green-Naghdi) simple heat theories: the state is then re-defined to incorporate thermal variables, like, by way of our idea, the coldness and the coldness displacement gradient, whereas the process expands to include their time derivatives. Nevertheless, it is not unrealistic to argue on a complex heat theory, for which it is possible to give a suitable definition of higher order thermal effects. Henceforth, for our aims, we concentrate on second gradient thermo-elastic materials.

To better understand, in the case of a rigid heat conductor of grade 2, the effective heat flux vector splits as

$$\mathbf{q} = \mathbf{q}_1 - \nabla \cdot \mathbf{Q}\,, \tag{3}$$

where the second order symmetric tensor $\mathbf{Q}$ plays the role of a hyper-heat flux tensor, so that its divergence accounts for the flux exchanging between bulk and surface points. Again, as before, in a simplified frame, it may be chosen proportional to the gradient of the classical heat flux vector $\mathbf{q}_1$. So, only one additional parameter, with the role of an internal length, enters to address micro- to nano-devices and the simple stationary/rate constitutive

setting for $\mathbf{q}_1$ affects the non local thermal features. As a consequence, the state σ and the process P broaden due to non local behaviors.

We now propose a non standard version of the virtual powers method, based on the notion of the coldness displacement, to derive all the balance and imbalance laws, together with the appropriate boundary conditions, for the non local thermo-mechanical structures we have in mind.

Let us first begin to work with the weak formulation of the momentum equation

$$\rho_0 \ddot{\mathbf{u}} = \nabla \cdot \mathbf{T} + \rho_0 \mathbf{b} \tag{4}$$

for all reference sub-bodies $\mathcal{P}(0) \subset \mathcal{B}(0)$ and for any sufficiently smooth virtual velocity field $\delta\mathbf{u}$.

Henceforth, following [11], the virtual and the real internal mechanical powers are expressed as

$$\mathcal{P}_m^{(i)}(\mathcal{P}(0); \delta\mathbf{u}) = \int_{\mathcal{P}(0)} \mathbf{T} \cdot \nabla\delta\mathbf{u} \, dV = \int_{\mathcal{P}(0)} \tilde{p}_m^{(i)} \, dV \tag{5}$$

$$\mathcal{P}_m^{(i)}(\mathcal{P}(0)) = \int_{\mathcal{P}(0)} \mathbf{T} \cdot \nabla\dot{\mathbf{u}} \, dV = \int_{\mathcal{P}(0)} p_m^{(i)} \, dV \tag{6}$$

where $\delta\mathbf{u}$ is replaced by the the real velocity $\dot{\mathbf{u}}$ and $p_m^{(i)}$ represents the density of the global internal mechanical power. By denoting the external virtual mechanical power with $\mathcal{P}_m^{(e)}(\mathcal{P}(0); \delta\mathbf{u})$, the weak momentum equation becomes a physical balance in the d'Alembert form, as follows

$$\int_{\mathcal{P}(0)} \rho_0 \ddot{\mathbf{u}} \cdot \delta\mathbf{u} \, dV = \mathcal{P}_m^{(e)}(\mathcal{P}(0); \delta\mathbf{u}) - \mathcal{P}_m^{(i)}(\mathcal{P}(0); \delta\mathbf{u}) \tag{7}$$

where the left-hand side represents the virtual power of acceleration forces and

$$\begin{aligned} \mathcal{P}_m^{(e)}(\mathcal{P}(0); \delta\mathbf{u}) &= \int_{\mathcal{P}(0)} (\nabla \cdot (\mathbf{T}\delta\mathbf{u}) + \rho_0 \mathbf{b} \cdot \delta\mathbf{u}) \, dV \\ &= \int_{\partial\mathcal{P}(0)} \mathbf{T}\mathbf{n} \cdot \delta\mathbf{u} \, da + \int_{\mathcal{P}(0)} \rho_0 \mathbf{b} \cdot \delta\mathbf{u} \, dV . \end{aligned}$$

Hence the Kinetic Energy Theorem can be written as

$$\frac{d}{dt} \int_{\mathcal{P}(0)} \rho_0 \frac{\dot{\mathbf{u}}^2}{2} \, dV = \mathcal{P}_m^{(e)}(\mathcal{P}(0)) - \mathcal{P}_m^{(i)}(\mathcal{P}(0)) , \tag{8}$$

where the external mechanical power is

$$\mathcal{P}_m^{(e)}(\mathcal{P}(0)) = \int_{\partial\mathcal{P}(0)} \mathbf{T}\mathbf{n} \cdot \dot{\mathbf{u}} \, da + \int_{\mathcal{P}(0)} \rho_0 \mathbf{b} \cdot \dot{\mathbf{u}} \, dV . \tag{9}$$

Within this approach, the local form of the First Law of Thermodynamics becomes

$$\rho_0 \dot{e} - p_m^{(i)} = \rho_0 h \tag{10}$$

where, from the heat balance equation, the right-hand side satisfies

$$\rho_0 h = -\nabla \cdot \mathbf{q} + \rho_0 r , \tag{11}$$

$(\mathbf{q}, \rho_0 r)$ being the internal energy thermal inflow.

This last equation, multiplied by $\delta\hat{k}$ and integrated over $\mathcal{P}(0)$, leads to

$$\mathcal{P}_{en}^{(i)}(\mathcal{P}(0); \delta\hat{k}) = \mathcal{P}_{en}^{(e)}(\mathcal{P}(0); \delta\hat{k}) \tag{12}$$

where the virtual internal and external entropy powers (actions) write

$$\begin{aligned} \mathcal{P}_{en}^{(i)}(\mathcal{P}(0); \delta\hat{k}) &= \int_{\mathcal{P}(0)} \left(\rho_0 h \delta\hat{k} - \mathbf{q} \cdot \nabla\delta\hat{k} \right) \, dV \\ \mathcal{P}_{en}^{(e)}(\mathcal{P}(0); \delta\hat{k}) &= \int_{\mathcal{P}(0)} \left(-\nabla \cdot \left(\mathbf{q} \, \delta\hat{k} \right) + \rho_0 r \delta\hat{k} \right) \, dV \end{aligned} \tag{13}$$

which, in a complete analogy with the mechanical background, lead to the definitions

$$\begin{aligned} \mathcal{P}_{en}^{(i)}(\mathcal{P}(0)) &= \int_{\mathcal{P}(0)} (\rho_0 h k - \mathbf{q} \cdot \nabla k) \, dV = \int_{\mathcal{P}(0)} p_{en}^{(i)} \, dV \\ \mathcal{P}_{en}^{(e)}(\mathcal{P}(0)) &= \int_{\mathcal{P}(0)} (-\nabla \cdot (\mathbf{q} \, k) + \rho_0 r k) \, dV , \end{aligned}$$

$p_{en}^{(i)}$ being the density of the real internal entropy power. So, under suitable hypotheses of smoothness, the Second Law of Thermodynamics can be locally written as

$$\rho_0 \dot{\eta} \geq \rho_0 h k - \nabla k \cdot \mathbf{q} = p_{en}^{(i)} . \tag{14}$$

The interlacement between (10), multiplied by k, and (14) yields

$$\rho_0 \dot{\eta} \geq \rho_0 \dot{e} k - p_m^{(i)} k - \nabla k \cdot \mathbf{q} \tag{15}$$

which, via the definition of the modified free energy $\tilde{\psi}$, defined in (1), leads to the Clausius-Duhem inequality

$$-\rho_0 \dot{\tilde{\psi}} + \rho_0 e \dot{k} + p_m^{(i)} k + \nabla k \cdot \mathbf{q} \geq 0 . \tag{16}$$

It is worth to observe that this inequality holds also for non local thermo-mechanical structures by suitably updating the internal mechanical and entropic densities. The universality of this thermodynamic approach just stands on different expressions of the internal virtual/real powers according to the non local properties of the material under study.

For a second-gradient elastic structure, the effective stress tensor has the form (2) so the internal mechanical powers writes

$$\begin{aligned} \mathcal{P}_m^{(i)}(\mathcal{P}(0)) &= \int_{\mathcal{P}(0)} (\mathbf{T_2} \cdot \nabla\dot{\mathbf{u}} + \mathbf{T_3} \cdot \nabla\nabla\dot{\mathbf{u}}) \, dV \\ &= \int_{\mathcal{P}(0)} p_m^{(i)} \, dV , \end{aligned} \tag{17}$$

exhibiting a linear dependence on the mechanical process $P_m = (\nabla\dot{\mathbf{u}}, \nabla\nabla\dot{\mathbf{u}})$. Correspondently, in view of standard identities, the external mechanical power becomes

$$\begin{aligned}\mathcal{P}_m^{(e)}(\mathcal{P}(0)) = &\int_{\partial\mathcal{P}(0)} (\mathbf{T}\mathbf{n}\cdot\dot{\mathbf{u}} + \mathbf{T_3}\mathbf{n}\cdot\nabla\dot{\mathbf{u}})\ d\,a \\ &+ \int_{\mathcal{P}(0)} \rho_0 \mathbf{b}\cdot\dot{\mathbf{u}}\ d\,V\,,\end{aligned} \tag{18}$$

in which the classical traction problem is to be implemented by higher order boundary conditions due to the presence of the hyper-stress tensor, yielding the so called constitutive boundary conditions (see e.g. [11]), responsible for those surface effects typical of micro- and nano-scales phenomena.

Likewise, for a non local thermal behavior of grade 2, where the effective heat flux vector has the form (3), the expression for the internal entropic power becomes

$$\begin{aligned}\mathcal{P}_{en}^{(i)}(\mathcal{P}(0)) &= \int_{\mathcal{P}(0)} p_{en}^{(i)}\ d\,V \\ &= \int_{\mathcal{P}(0)} \left(\rho_0 hk - \mathbf{q_1}\cdot\nabla\dot{\hat{k}} - \mathbf{Q}\cdot\nabla\nabla\dot{\hat{k}}\right)\ d\,V\,,\end{aligned} \tag{19}$$

just showing a linear dependence on the thermal process $P_{en} = (\nabla\dot{\hat{k}}, \nabla\nabla\dot{\hat{k}})$. Obviously, for the fully coupled thermo-elastic theory of grade 2, the process is given by $P = (\nabla\dot{\mathbf{u}}, \nabla\dot{\hat{k}}, \nabla\nabla\dot{\mathbf{u}}, \nabla\nabla\dot{\hat{k}})$ and the internal coupled power is just the sum of (17) and (19).

Inequality (15) generalizes as follows

$$\rho_0\dot{\eta} \geq \rho_0\dot{e}k - p_m^{(i)}k - \mathbf{q_1}\cdot\nabla\dot{\hat{k}} - \mathbf{Q}\cdot\nabla\nabla\dot{\hat{k}}\,,$$

whereas, from (16), we recover the general Clausius-Duhem inequality for our theory

$$\begin{aligned}&-\rho_0\dot{\tilde{\psi}} + \rho_0 e\dot{k} + (\mathbf{T_2}\cdot\nabla\dot{\mathbf{u}} + \mathbf{T_3}\cdot\nabla\nabla\dot{\mathbf{u}})\,k \\ &+\mathbf{q_1}\cdot\nabla k + \mathbf{Q}\cdot\nabla\nabla k \geq 0\,.\end{aligned} \tag{20}$$

It is interesting to note that (20) is trivially generalizable to incorporate higher-order gradients.

The next step of this Section is the application of the classical Coleman-Noll arguments towards the thermodynamic restrictions on our constitutive setting. In order to describe non local anisotropic G-N thermo-elastic and viscoelastic effects, we let $\tilde{\psi}$ be continuously differentiable with respect to all the independent variables at the current time t, representing the state $\sigma = (k, \nabla\mathbf{u}, \nabla\nabla\mathbf{u}, \nabla\hat{k}, \nabla\nabla\hat{k})$. Upon evaluation of $\dot{\tilde{\psi}}$ and substitution in (20), also generalizing the standard thermodynamics arguments, we immediately find

$$e = \frac{\partial\tilde{\psi}}{\partial k} \quad , \qquad \eta = k^2\frac{\partial\psi}{\partial k} \quad .$$

From the split residual dissipation inequality, we recover the further restrictions

$$k\left(\mathbf{T_2} - \rho_0\frac{\partial\psi}{\partial\nabla\mathbf{u}}\right)\cdot\nabla\dot{\mathbf{u}} \geq 0\,,\ k\left(\mathbf{T_3} - \rho_0\frac{\partial\psi}{\partial\nabla\nabla\mathbf{u}}\right)\cdot\nabla\nabla\dot{\mathbf{u}} \geq 0$$
$$\left(\mathbf{q_1} - \rho_0\frac{\partial\tilde{\psi}}{\partial\nabla\hat{k}}\right)\cdot\nabla k \geq 0\,, \quad \left(\mathbf{Q} - \rho_0\frac{\partial\tilde{\psi}}{\partial\nabla\nabla\hat{k}}\right)\cdot\nabla\nabla k \geq 0\,.$$

Therefore we propose the constitutive relations

$$\mathbf{T_2} = \rho_0\frac{\partial\psi}{\partial\nabla\mathbf{u}} + \mathbf{D_1}\nabla\dot{\mathbf{u}}\,,\ \mathbf{T_3} = \rho_0\frac{\partial\psi}{\partial\nabla\nabla\mathbf{u}} + \mathbf{D_2}\nabla\nabla\dot{\mathbf{u}}\,, \tag{21}$$

$$\mathbf{q_1} = \rho_0\frac{\partial\tilde{\psi}}{\partial\nabla\hat{k}} + \mathbf{K_1}\nabla k\,, \quad \mathbf{Q} = \rho_0\frac{\partial\tilde{\psi}}{\partial\nabla\nabla\hat{k}} + \mathbf{K_2}\nabla\nabla k\,, \tag{22}$$

where $\mathbf{D_1}, \mathbf{D_2}, \mathbf{K_1}, \mathbf{K_2}$ are non-negative tensorial valued functions, possibly dependent on the coldness. The coefficients $\mathbf{D_1}, \mathbf{D_2}$ may be interpreted as viscoelastic and hyper-viscoelastic tensorial moduli; likewise the additional $\mathbf{K_1}$ and $\mathbf{K_2}$ play the role of conductivity and hyper-conductivity tensors, respectively.

By summarizing, different non local GN thermo-viscoelastic theories depend on different expressions for the free energy potentials. Notably, a significant number of additional constitutive parameters enter due to the complex structure under study.

To better clarify, in linearly isotropic frameworks, it is known that fourth-order viscoelastic or hyper-conductivity tensors can be written in terms of two non negative phenomenological parameters. On the other hand, the isotropic decomposition of the sixth-order isotropic hyper-viscoelastic (or elastic) tensor accounts for five additional constitutive moduli, playing the role of hyper-viscoelasticities (elasticities) ([7, 12]). Indeed, on a mechanical point of view, a further simplified case comes out under the constraint $\nabla\cdot\mathbf{u} = 0$; following [11], the hyper-viscoelasticities (elasticities) reduce to three. Hence the part of the effective stress tensor, independent on the choice the free energy ψ, has the following form

$$(\mathbf{D_1}\nabla\dot{\mathbf{u}})_{ij} - (\mathbf{D_2}\nabla\nabla\dot{\mathbf{u}})_{ijk,k} = \hat{\mu}(\dot{\mathbf{u}}_{i,j} + \dot{\mathbf{u}}_{j,i}) - (\hat{\eta}_1 + \hat{\eta}_3)(\Delta\dot{\mathbf{u}}_{i,j})\ ,$$

where the two positive coefficients $(\hat{\lambda}, \hat{\mu})$ play the role of the viscoelastic constants, while $\hat{\eta}_1$ and $\hat{\eta}_3$ represent the hyper-viscoelasticities. As a consequence, its divergence becomes

$$\nabla\cdot[\mathbf{D_1}\nabla\dot{\mathbf{u}} - \nabla\cdot(\mathbf{D_2}\nabla\nabla\dot{\mathbf{u}})] = \hat{\mu}\Delta\dot{\mathbf{u}} - (\hat{\eta}_1 + \hat{\eta}_3)\Delta^2\dot{\mathbf{u}} \quad .$$

Now, in this isotropic frame, let us choose a quadratic form for ψ with respect to its dependence both on the gradient and the double gradient of $\mathbf{u}$, accounting for the standard Hookean law, (λ, μ) being the Lamé constants.

So, in view of (21), the effective stress tensor for this second gradient viscoelasticity reads

$$\mathbf{T} = 2\mu\left(1 + \frac{\hat{\mu}}{\mu}\frac{\partial}{\partial t}\right)\mathbf{E} - (\eta_1 + \eta_3)\left(1 + \frac{(\hat{\eta}_1 + \hat{\eta}_3)}{(\eta_1 + \eta_3)}\frac{\partial}{\partial t}\right)\Delta\nabla\mathbf{u} \ ,$$

where η_1 and η_3 represent the hyper-elastic Lamé constants. In this way, usual dimensional arguments lead to the definition of two natural parameters, referred to as gradient- gradient rate-lengths

$$L_1^2 = (\eta_1 + \eta_3)/\mu \ , \qquad L_2^2 = (\hat{\eta}_1 + \hat{\eta}_3)/\hat{\mu} \ ,$$

which, in turn, play a relevant role for future applications, since these non local theories are generally employed in predicting the size-dependent behavior of micro- and nano-structures, at both mechanical and thermal scales, see e.g. [21].

It is worth to infer that, besides the standard conditions $\mu > 0$ and $\hat{\mu} > 0$, the thermodynamics restrictions guarantee the non negativeness of both $(\eta_1 + \eta_3)$ and $(\hat{\eta}_1 + \hat{\eta}_3)$ and that the right-hand side of the momentum equation (4), in the absence of body forces and thermal interaction terms, would become

$$\nabla \cdot \mathbf{T} = \mu(1 - L_1^2\Delta)\Delta\mathbf{u} + \hat{\mu}(1 - L_2^2\Delta)\Delta\dot{\mathbf{u}} \ .$$

3. Special settings

In this section some special quadratic type Helmholtz free energies, already introduced for simple thermo-viscoelasticity under G-N type heat properties, are suitably adapted to address simplified non local thermo-mechanical structures of grade 2.

3.1. A non local viscoelastic theory of K-V type

As a special example of a purely mechanical setting we address a non local isotropic linear elastic material.

Our thermodynamic development yields a second-order strain and strain rate K-V type linear viscoelasticity, where, following the simplified proposal by [2], the third order hyper-stress $\mathbf{T_3}$ is supposed to be proportional to the gradients of the Hookean and viscoelastic parts of $\mathbf{T_2}$ via the material parameters L_1^2 and L_2^2, with the role of deformation and deformation rate induced internal lengths, respectively. In this simplified context the elastic/viscoelastic Lamé constants are just the standard Lamé elastic/viscoelastic ones multiplied by L_1^2/L_2^2, which corresponds to the relations between sixth- and fourth-order tensors

$$\mathbf{D}_{ijklmn} = L^2\mathbf{C}_{ijlm}\delta_{kn} \ ,$$

within the isotropic formulation, δ_{kn} being the Kronecker symbol.

In this way only two internal length parameters are needed, in addition to the four elastic/viscoelastic Lamé constants, and the effective stress tensor has form

$$\begin{aligned}\mathbf{T} =& \mathbf{T_2} - \nabla \cdot \mathbf{T_3} \\ =& 2\mu(1 - L_1^2\Delta)\mathbf{E} + \lambda(1 - L_1^2\Delta)(\mathrm{tr}\mathbf{E})\mathbb{I} \\ &+ 2\hat{\mu}(1 - L_2^2\Delta)\dot{\mathbf{E}} + \hat{\lambda}(1 - L_2^2\Delta)(\mathrm{tr}\dot{\mathbf{E}})\mathbb{I} \ .\end{aligned}$$

Let us now propose a quick comparison with a dual-phase-lag elasticity theory by Tzou (see in [19]), in its reduction to a single-phase-lag one via a temporal shifting, leading to a small difference delay time $\tau > 0$ with the role of a retardation time. By performing the usual first-order Taylor series expansion, which is responsible of short memory effects, we point out that due to the following identifications

$$\hat{\mu} = \tau\mu \ , \quad \hat{\lambda} = \tau\lambda \ , \quad L_2^2 = L_1^2 \ ,$$

within the above constitutive relation, our non-local K-V type viscoelasticity becomes a dual-phase-lag non-local elasticity.

Hence, in the simplest case only one gradient coefficient appears, in addition to the standard Lamé constants and τ. Therefore, for a non local second-order strain and strain rate K-V viscoelasticity, the momentum equation (4), in terms of the displacement vector $\mathbf{u}$, reads

$$\begin{aligned}\rho_0\ddot{\mathbf{u}} =& \mu(1 - L_1^2\Delta)\Delta\mathbf{u} \\ &+ (\lambda + \mu)(1 - L_1^2\Delta)\nabla\nabla \cdot \mathbf{u} + \mu\tau(1 - L_1^2\Delta)\Delta\dot{\mathbf{u}} \\ &+ \tau(\lambda + \mu)(1 - L_1^2\Delta)\nabla\nabla \cdot \dot{\mathbf{u}} + \rho_0\mathbf{b} \ .\end{aligned} \tag{23}$$

It is worth to observe that an interaction term, accounting for temperature driven motions, might be added on the right-hand side of (23) [3].

3.2. Non local/local revisited G-N rigid heat theories

In this subsection we focus on a non local rigid heat conduction theory by re-examining the G-N type thermal properties, within an isotropic frame. In view of (22), by choosing for the modified free energy $\tilde{\psi}$ a quadratic dependence on the coldness displacement gradient and its double gradient, let $m = m(k) > 0$ be the G-N coefficient, whereas g_1 and g_2, depending on k too, may be referred to as the hyper-G-N coefficients; likewise the scalar valued function $K_1 = K_1(k) > 0$ defines the classical conductivity, whereas χ_1 and χ_2 represent the hyper-conductivity coefficients depending on at most k, such that

$$\mathbf{K_2}\nabla\nabla k = 2\chi_1\nabla\nabla k + \chi_2\Delta k\mathbb{I} \ .$$

Hence the constitutive relation for the total heat flux vector reads

$$\mathbf{q} = m\nabla\hat{k} + K_1\nabla k - \nabla\cdot\left(2g_1\nabla\nabla\hat{k} + g_2\Delta\hat{k}\mathbb{I} + 2\chi_1\nabla\nabla k + \chi_2\Delta k\mathbb{I}\right) .$$

So, limiting ourselves for easiness to the constant coefficients case, at the right-hand side of (11) we would have

$$-\nabla\cdot\mathbf{q} = -m\left[1 - L_3^2\Delta\right]\Delta\hat{k} - K_1\left[1 - L_4^2\Delta\right]\Delta k ,$$

where, by using the same dimensional arguments as in the mechanical structure, $L_3^2 = (2g_1 + g_2)/m$ and $L_4^2 = (2\chi_1 + \chi_2)/K_1$ may be referred to as the coldness displacement and the coldness displacement rate gradient lengths, respectively. Besides the simple G-N type and conductivity coefficients, the non locality needs therefore the additional presence of only two internal length parameters, which will be relevant for the description of thermal devices at small lengths [1, 11].

Also, restricting now our attention to a local thermal structure of a revisited G-N type but assuming that the coefficients depend on the coldness, we propose the modified free energy

$$\tilde{\psi}_2(k, \nabla\hat{k}) = \tilde{\psi}_E(k) + \frac{1}{2\rho_0}m(k)\nabla\hat{k}\cdot\nabla\hat{k} \tag{24}$$

with $m = m(k)$ being a positive valued scalar function. Thus, from (22), the heat flux satisfies the constitutive relation

$$\mathbf{q} = \mathbf{q_1} = m(k)\nabla\hat{k} + K_1(k)\nabla k ,$$

which, in turn, yields the classical Fourier Law, with possibly constant/non constant thermal conductivity, in the limit case $m(k) \to 0$.

By adopting the G-N form for $\tilde{\psi}_E$, i.e. $\tilde{\psi}_E = c(1 + \log k)$ with $c > 0$ constant, the internal energy density $e = \dfrac{\partial\tilde{\psi}_2}{\partial k}$ becomes

$$e = \frac{c}{k} + \frac{1}{2\rho_0}m'(k)\nabla\hat{k}\cdot\nabla\hat{k} ,$$

namely $e = e(k, \nabla\hat{k})$. So, the thermal equation (10)-(11) for this simple theory reads

$$\left(-\rho_0\frac{c}{k^2} + \frac{1}{2}m''(k)\nabla\hat{k}\cdot\nabla\hat{k}\right)\dot{k} = -m(k)\Delta\hat{k} - K_1(k)\Delta k - 2m'(k)\nabla\hat{k}\cdot\nabla k - K_1'(k)\nabla k\cdot\nabla\dot{k} + \rho_0 r .$$

However, in order to perform a fruitful comparison with other simple G-N type heat theories, we furnish a brief reformulation of our thermodynamics developments in terms of the temperature θ and the standard temperature displacement α, recalling that its time derivative is just the temperature $\theta = 1/k$ ([4, 17]).

As a first step we avoid the last term in (20) due to the non local behavior and rewrite the fourth addendum as $-\dfrac{1}{\theta^2}\nabla\theta\cdot\mathbf{q}$. Finally, we neglect the terms due to the mechanical structure and we focus on a modified free energy potential, now depending on θ and $\nabla\alpha$.

Taking into account the link between the free energy ψ and the modified free energy $\tilde{\psi}$, inequality (20) reads as follows

$$\frac{\rho_0}{\theta}\left[\psi - \theta\frac{\partial\psi}{\partial\theta} - e\right]\dot{\theta} - \rho_0\frac{\partial\psi}{\partial\nabla\alpha}\cdot\nabla\theta - \frac{1}{\theta}\nabla\theta\cdot\mathbf{q} \geq 0 .$$

So, standard thermodynamics arguments yield the classical relations $e = \psi - \theta\dfrac{\partial\psi}{\partial\theta}$, $\eta = -\dfrac{\partial\psi}{\partial\theta}$, together with the residual inequality

$$\left(\frac{\mathbf{q}}{\theta} + \rho_0\frac{\partial\psi}{\partial\nabla\alpha}\right)\cdot\nabla\theta \leq 0 .$$

Therefore, for the entropy influx [4] we have

$$\frac{\mathbf{q}}{\theta} = -\rho_0\frac{\partial\psi}{\partial\nabla\alpha} - \hat{K}(\theta)\nabla\theta ,$$

$\hat{K}$ being a positive valued scalar function depending on the temperature. If we choose the following free energy

$$\psi_3 = c(\theta - \theta\log\theta) + \frac{\hat{m}(\theta)}{2\rho_0}\nabla\alpha\cdot\nabla\alpha ,$$

where $\hat{m}$ is a positive scalar valued function and, as before, the positive constant c stands for the specific heat, our constitutive assumption for the entropy influx becomes

$$\frac{\mathbf{q}}{\theta} = -\hat{m}(\theta)\nabla\alpha - \hat{K}(\theta)\nabla\theta .$$

As a consequence, we have that the heat flux and the internal energy density have form

$$\mathbf{q} = -\theta\hat{m}(\theta)\nabla\alpha - \theta\hat{K}(\theta)\nabla\theta ,$$
$$e = \frac{1}{2\rho_0}\left(\hat{m}(\theta) - \theta\hat{m}'(\theta)\right)\nabla\alpha\cdot\nabla\alpha + c\theta .$$

Finally, by setting

$$c_\alpha = c_\alpha(\theta, \nabla\alpha) = c - \frac{\theta}{2\rho_0}\hat{m}''(\theta)\nabla\alpha\cdot\nabla\alpha$$

with the role of the modified G-N type specific heat and $m(\theta) = \theta\hat{m}(\theta)$, $K(\theta) = \theta\hat{K}(\theta)$, for this constitutive the-

ory, we recover the general quasi-linear parabolic thermal equation

$$\rho_0 c_\alpha \dot{\theta} = -2\left(\frac{m(\theta)}{\theta} - m'(\theta)\right)\nabla\alpha\cdot\nabla\theta + m(\theta)\Delta\alpha + K(\theta)\Delta\theta + K'(\theta)\nabla\theta\cdot\nabla\theta + \rho_0 r\,. \tag{25}$$

The expression of the modified specific heat in terms of the new function m is

$$c_\alpha = c - \frac{1}{2\rho_0}\left(m''(\theta) - \frac{2}{\theta}m'(\theta) + \frac{2}{\theta^2}m(\theta)\right)\nabla\alpha\cdot\nabla\alpha\,.$$

The local energy equation (25) may be easily compared with the classical heat equation under the Fourier's Law, when the new function m tends to zero. As a special case we have a constant conductivity whenever $\hat{K} = K_0/\theta$ (K_0 being constant), as in Section 2 of [17], whereas a porous medium equation (shortly PME) with exponent 2 is recoverable for $\hat{K} = K_0/\theta_0$, θ_0 being a reference temperature, as in the pioneering constitutive assumption by Green and Naghdi. In fact, in absence of external sources and by means of the temporal scaling $t' = t\hat{K}/2\rho_0 c$, (25) reduces to the non linear degenerate parabolic equation $\frac{d}{dt'}\theta = \Delta\theta^2$, which, as well known (see e.g. [10]), exhibits the mathematical property of finite wave propagation, in the framework of characteristic surfaces. Moreover, when m and K are both constants, namely $\hat{m} = m_0/\theta$ and $\hat{K} = K_0/\theta$, (25) becomes the heat conduction equation performed in Section 4 of [17] when constrained to static rigid conduction, according to their alternative strategy. In this case, the m'- and K'-terms disappear and hence

$$c_\alpha = c - \frac{1}{\rho_0\theta^2}m_0\nabla\alpha\cdot\nabla\alpha\,.$$

In fact, our heat equation appears more complicated to deal with from a purely mathematical point of view, but the generalization to non constant functions $m(\theta)$ and $K(\theta)$ allows applications in physical/astrophysical settings, in the presence of very low-/very high-temperature ranges.

As a final comment, just thinking of the thermoelastic description of ice at cryogenic temperatures, see e.g. references in [17], one may include an interaction term on the right-hand side of (25), due to the thermoelastic coupling.

4. Concluding remarks

The primary idea of this paper consists in stating the universal validity of the virtual powers format to derive thermodynamically consistent nonlocal continuum theories. In our opinion, the key support points are

- a variational topic towards the weak formulation of initial-boundary value problems with less smoothness requirements;
- higher-order constitutive boundary conditions justified by higher-order gradient continuum theories;
- the a priori role played by the concepts of state and process to assess non local behaviors;
- internal mechanical and entropic powers formulated through a linear dependence on the quantities within the process and hence easily generalizable to assess different non local behaviors.

We achieve these conclusions by developing an extended virtual powers scheme based on the notion of the coldness displacement, in order to perform novel non isothermal second order gradients elastic/viscoelastic settings, under analogous non local thermal properties through a revisitation of G-N heat theories.

Notably, non local K-V type viscoelastic models and nonlocal/local revised G-N type rigid heat conduction theories are recovered as special constitutive settings and fruitful comparisons with pre-existent literature modelings are provided.

Also, the need of taking a minimum number of constitutive/internal length scale parameters in triggering size-dependent effects is underlined.

Finally we think that, to improve memory aspects, our frameworks might be implemented by involving the Caputo-Fabrizio fractional calculus [6].

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