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Substitution Effects in Intertemporal Problems[†]

By DAVIDE DRAGONE AND PAOLO VANIN*

We consider a broad class of intertemporal economic problems and characterize the short-run and long-run responses of the demand for a good to a permanent increase in its market price. Depending on the interplay between self-productivity and time discounting, we show that dynamic substitution effects can generate price elasticities of opposite signs in the short run and in the long run. (JEL D11, D15, H20, J22, J24)

Price effects and intertemporal choices are at the heart of economic analysis. In static consumer problems with two goods, it is well known that substitution effects are always negative so that consumers respond to an increase in the relative price of a good by reducing its consumption.

In dynamic extensions of such problems in which present choices affect future conditions through some relevant state variable such as habit, human capital, or health, consumers' responses may be substantially different. While this has long been informally known, no general characterization is available, perhaps surprisingly.

In this paper, we characterize substitution effects for a broad class of intertemporal consumer and worker problems. We identify time discounting and self-productivity as key determinants. A state variable is self-productive or self-depleting depending on whether it contributes positively or negatively to its own accumulation. Under self-productivity we show that dynamic substitution effects have opposite signs in the short run and in the long run, with the discount rate determining whether consumption initially falls and eventually increases or the other way around. Under self-depletion, short-run and long-run responses have the same negative sign.

As an example, think of an agent who chooses between schooling and working over time. If human capital is self-productive (i.e., if the net production of new knowledge increases in the agent's stock of existing knowledge), we show that a wage increase induces patient agents to work less and study more in the short run in order to work more and reap the benefits of the larger stock of human capital in the

*Dragone: University of Bologna, Department of Economics (email: davide.dragone@unibo.it); Vanin: University of Bologna, Department of Economics (email: paolo.vanin@unibo.it). John Asker was coeditor for this article. We thank Antonio Cabrales, Giacomo Calzolari, Chris Cronin, Vincenzo Denicolò, Matthew Ellman, James Heckman, Paolo Manasse, Antonio Minniti, Tito Pietra, Holger Strulik, Giulio Zanella, and participants in the 2015 ASSET Conference in Granada, the 2015 EEA Congress in Mannheim, the 2015 UPF-GPEFM Alumni Meeting in Barcelona, the 2016 PET Conference in Rio de Janeiro, the 2016 SIE Conference in Milan, the 2016 Brucchi Luchino Workshop in Bologna, the 2017 Workshop on Stochastic Optimal Control in Gothenburg, the 2017 Health and Labor Conference in Essen, and seminars at the Department of Applied Economics in Palma and the Department of Economics in Bologna for useful comments and suggestions. The usual disclaimer applies.

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long run; impatient agents respond in the opposite way. If human capital is instead self-depleting (for instance, due to depreciation), short-run and long-run responses have the same sign: after a wage increase, agents work more both in the short run and in the long run and invest less in human capital.

Our results contribute to different strands of literature. First, we characterize dynamic substitution effects in a general setup. This sheds light on the key assumptions about self-productivity and discounting that produce differences and commonalities among the apparently widely different models of health behavior, human capital, endogenous preferences, addiction, and habit formation that originate from Grossman (1972); Blinder and Weiss (1976); Stigler and Becker (1977); Becker and Murphy (1988); and Carroll, Overland, and Weil (2000).

Second, our paper contributes to the literature on the comparative statics and comparative dynamics properties of intertemporal optimization models (see, among others, Oniki 1973; Epstein 1978; Otani 1982; and Caputo 1990, 1997). We show under which conditions dynamic substitution effects are positive or negative. An empirical implication is that heterogeneity in individual discount rates can produce an attenuation bias in estimated price responses. This result is particularly relevant when writing and solving computational models that estimate the dynamic responses to tax changes or income subsidies (see Hall 2010 and Ljungqvist and Sargent 2012 for an overview). An implication for policy making is that there are conditions under which policies that seem effective in the short run backfire in the long run, and vice versa.

Third, our paper relates to the literature on taxation in models with endogenous human capital, which has devoted substantial attention to the roles of asymmetric information, age-dependent taxation, and stochastic returns, but less to the aspects that are the focus of the present paper.¹ We take a complementary perspective: we abstract from all these elements to show in the cleanest way how the short-run and long-run responses to labor income taxes depend on the interplay between self-productivity and time discounting. More broadly, we contribute to the large literature on taxation in dynamic models investigating, e.g., health behavior, addiction, or obesity (see Cawley and Ruhm 2012 for an overview).

The paper is structured as follows: Section I presents the general intertemporal problem and the main results. Sections II and III specialize the results to consumer and labor supply problems. Section IV concludes.

I. An Intertemporal Consumer/Worker Problem

Consider the following intertemporal model, which nests consumer and labor supply problems as special cases:

$$(1) \quad \max_{\{x,y\}} \int_0^{\infty} e^{-\rho t} U(x(t), y(t), Z(t)) dt$$

¹ See Makris and Pavan (2018) for a discussion and in particular, among others, the works of Kapička (2006, 2015); Krause (2009); Best and Kleven (2013); Stantcheva (2017); and Kapička and Neira (2019).

subject to

$$(2) \quad \dot{Z}(t) = f(x(t), Z(t)),$$

$$(3) \quad \dot{A}(t) = rA(t) + M(t) + g(x(t), Z(t), p) - y(t),$$

$$(4) \quad A(0) = A^0, \quad Z(0) = Z^0,$$

where $x \geq 0$ and $y \geq 0$ are consumption goods, Z is any form of capital (such as human capital, habit, health, or addiction), and A represents assets.² Equation (2) describes the law of motion of state variable Z , and equation (3) describes the dynamic budget constraint. Parameter $\rho > 0$ is the intertemporal discount rate. To maintain generality, we make no functional form assumptions on utility or the law of motion of Z .³

Based on equation (2), we introduce the following classification (subscripts denote partial derivatives).

DEFINITION 1: *State variable Z is self-depleting if $f_Z < 0$ and self-productive if $f_Z > 0$.*

The case of self-depleting Z covers all intertemporal problems where the state variable contributes to its dynamics only through depreciation. This is the case, for instance, in Grossman's (1972) health accumulation model, Becker and Murphy's (1988) model of rational addiction, and the models of habit formation considered in Carroll, Overland, and Weil (2000) and Chetty and Szeidl (2016), which share the common assumption $\dot{Z} = \bar{f}(x) - \delta Z$ with $\bar{f}, \bar{f}' > 0$. The case of self-productive Z includes the models of human capital considered in, among others, Weiss (1972) and Cunha and Heckman (2007), or Dalgaard and Strulik's (2014) model of health deficit accumulation, in which the state variable contributes to its own accumulation according to $\dot{Z} = \bar{f}(Z) - x$. In richer models where, e.g., $\dot{Z} = \bar{f}(x, Z) - \delta Z$, the state variable can be self-depleting under certain conditions and self-productive under others, as shown in the labor supply models treated in Section III.

Equation (3) represents the dynamics of asset accumulation where r is the interest rate, $M \geq 0$ is an exogenous instantaneous income flow, and y is the numeraire good. The formulation of (3) is flexible enough to encompass consumption as well as labor supply choices. In consumer problems, $g(x, Z, p) = m(Z) - px$, where $m(Z)$ is the endogenous component of income (if any) and px is expenditure on x at the relative price p . Hence in a consumer problem, the budget constraint equation becomes

$$(5) \quad \dot{A} = rA + M + m(Z) - px - y.$$

In labor supply problems, $g(x, Z, p) = pL(x, Z)$, and the budget constraint is

$$(6) \quad \dot{A} = rA + M + pL(x, Z) - y,$$

²Henceforth the time arguments are omitted to simplify notation.

³The case where the evolution of Z also depends on y —i.e., $\dot{Z} = f(x, y, Z)$ —is studied in the Appendix.

where x represents leisure, $L(x, Z)$ is effective labor (which may depend on Z ; e.g., if Z is human capital), $L_x < 0$, and p is the market wage of effective labor.

Problem (1)–(4) is solved applying Pontryagin's maximum principle, as detailed in the Appendix. We assume concavity of the corresponding Hamiltonian function, that the market interest rate r is equal to the discount rate ρ , and that price changes are compensated to keep the marginal utility of assets constant (for a similar approach see, e.g., Heckman 1974 and Becker and Murphy 1988). This form of compensation is convenient because it neutralizes income effects so that dynamic price effects are entirely driven by dynamic substitution effects.⁴ The corresponding system of optimal trajectories for x and y is

$$(7) \quad \dot{x} = \frac{\mathcal{H}_{yy}}{\mathcal{H}_{xy}^2 - \mathcal{H}_{xx}\mathcal{H}_{yy}} [(f_Z - \rho)(U_x + g_x U_y) - f_x(U_Z + g_Z U_y)],$$

$$(8) \quad \dot{y} = -\frac{\mathcal{H}_{xy}}{\mathcal{H}_{yy}} \dot{x},$$

where \mathcal{H} represents the Hamiltonian function associated with the maximization problem.

Equating to zero expressions (2), (3), (7), and (8) yields the steady state (x^L, y^L, Z^L, A^L) . Our goal is to establish how an unexpected permanent change in price affects consumption when the system is at a steady state with saddle-point stability. For expositional simplicity, in the proceeding we focus on x and Z , and we report the corresponding analysis for y and A in the Appendix.

We investigate dynamic substitution effects over two different time horizons. The short-run effect describes the consumption response on impact, when the price change is announced and implemented. The short-run effect is computed considering the policy function $\hat{x}(Z, A)$ leading to the steady state and computing $x_p^S \equiv \partial \hat{x}(Z^L, A^L) / \partial p$.⁵ The long-run effect describes the change in the steady-state values $x_p^L \equiv \partial x^L / \partial p$ and $Z_p^L \equiv \partial Z^L / \partial p$.

PROPOSITION 1: *Consider an unexpected, permanent change in price p , occurring when all variables are at their saddle-point-stable steady-state level and compensated to maintain the marginal utility of wealth constant. The short-run and long-run responses of x and Z are*

$$(9) \quad x_p^S = a[f_x g_{pZ} - (f_Z - \rho) g_{xp}] \varepsilon,$$

$$(10) \quad x_p^L = a[f_x g_{pZ} - (f_Z - \rho) g_{xp}] f_Z,$$

$$(11) \quad Z_p^L = -a[f_x g_{pZ} - (f_Z - \rho) g_{xp}] f_x,$$

⁴The main insights generalize to the case in which the interest rate depends on the level of assets, as it is common in macroeconomic models, and price changes are not compensated to maintain the marginal-utility-of-wealth constant (see Dragone and Vanin 2015).

⁵Under specific assumptions one can write the closed-form expression of $\hat{x}(Z, A)$ as, e.g., in Becker and Murphy's (1988) rational addiction model. For our results, however, this is not needed.

where concavity and saddle-point stability imply $a, \varepsilon, f_x/f_Z < 0$.

Term ε is the negative eigenvalue associated to the Jacobian matrix computed at the steady state, whereas a is a function of the primitives of the model (see equations A41 and A50 in the Appendix). Given that the expression in square brackets is the same in all equations, the relation between the short-run and long-run price responses is particularly simple.

PROPOSITION 2: *The short-run and long-run price responses are related as follows:*

$$(12) \quad x_p^L = \frac{f_Z}{\varepsilon} x_p^S,$$

$$(13) \quad Z_p^L = -\frac{f_x}{f_Z} x_p^L.$$

Hence,

- If Z is self-depleting,
 - x_p^S and x_p^L have the same sign,
 - demand for x is more elastic in the short run than in the long run if $f_Z > \varepsilon$ and more rigid otherwise, and
- If Z is self-productive, x_p^S and x_p^L have opposite signs, and
- Z_p^L has the same sign as x_p^L .

These results are general and they highlight the importance of self-productivity and self-depletion for the sign and magnitude of price responses over time. The following sections, focusing on consumer and labor supply problems, show how the interplay between self-productivity and time discounting determines the specific signs of the short-run and long-run price responses.

II. Consumer Problem

In a consumer problem, the dynamic budget constraint is (5) because $g_{pZ} = 0$ and $g_{xp} = -1$. Hence the price responses (9)–(11) of Proposition 1 simplify as follows:

$$(14) \quad x_p^S = a(f_Z - \rho)\varepsilon,$$

$$(15) \quad x_p^L = a(f_Z - \rho)f_Z,$$

$$(16) \quad Z_p^L = -a(f_Z - \rho)f_x.$$

This allows us to be more specific about the direction of the price responses.

PROPOSITION 3: *In a dynamic consumer problem,*

- (i) *If Z is self-depleting, price responses of x are always negative, and the state variable Z decreases ($x_p^S, x_p^L, Z_p^L < 0$).*
- (ii) *If Z is self-productive, the sign of price responses depends on ρ relative to f_Z . Specifically,*
- *If the consumer is sufficiently impatient ($\rho > f_Z$), consumption of x decreases on impact and increases in the long run ($x_p^S < 0, x_p^L > 0$); the state variable Z increases ($Z_p^L > 0$).*
 - *If the consumer is sufficiently patient ($\rho < f_Z$), consumption of x increases on impact and decreases in the long run ($x_p^S > 0, x_p^L < 0$); the state variable Z decreases ($Z_p^L < 0$).*

The above proposition shows how the specific sign of the short-run and long-run price responses depends on the interplay between f_Z and ρ . When Z is self-depleting, a forward-looking agent substitutes consumption away from the more expensive good and, relative to the initial steady state, she reduces the quantity demanded at any point in time, as shown in Figure 1.⁶ This is similar to what happens in a static model.

Common examples of this case are consumer problems in which the law of motion is of the following kind:

$$(17) \quad \dot{Z}(t) = x^\alpha - \delta Z,$$

with $\alpha \in (0, 1]$ and $\delta \in (0, 1)$. The interpretation of x and Z depends on the specific application. In habit models and in rational addiction models, x is consumption and Z is the addiction stock or the habit stock (Becker and Murphy 1988; Chetty and Szeidl 2016). In health accumulation models, x is health investment and Z is the health condition (Grossman 1972; Galama and Kapteyn 2011). While the role of x and Z in the utility function depends on the specific model, these differences are irrelevant for our results.⁷ In fact, inspection of the dynamics of Z immediately reveals that Z is self-depleting because $f_Z = -\delta < 0$. Hence, using Proposition 3 we can immediately predict that price effects are negative, both in the short run and in the long run.

If Z is self-productive instead, whether the short-run response is positive or negative (remembering that the long-run response has the opposite sign) depends on the time discount rate ρ . The intuition behind this result lies in the tension between

⁶Whether there is overshooting in the short run relative to the long run depends on whether the policy function is upward sloping or, as in Figure 1, downward sloping.

⁷For example, in addiction and taste-formation models, consumption capital raises the marginal utility of x ($U_{xZ} > 0$): the more I listen to music, the more I appreciate it; the more I consume drugs, the more I crave them. In habit-formation models, instead, $U_{xZ} < 0$: the more I get used to a good, the less I value it. See Dragone and Ziebarth (2017) for an analysis of the two cases in the context of consumption of novel versus familiar goods.

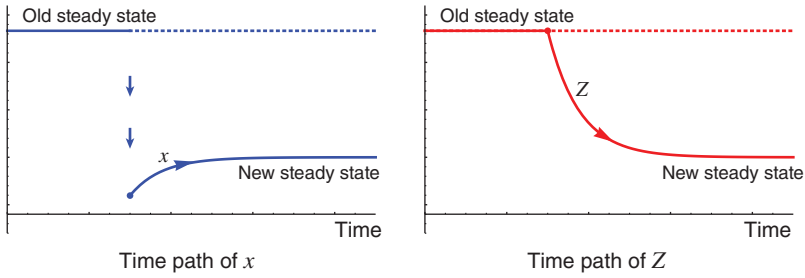


FIGURE 1. SELF-DEPLETING ($f_Z < 0$)

Notes: The figure shows short-run and long-run responses to a price increase. Consumption of x decreases both on impact and in the long run, with overshooting in the short run, if $\varepsilon < f_Z$. The left panel shows the time path for x , and the right panel shows the time path for Z .

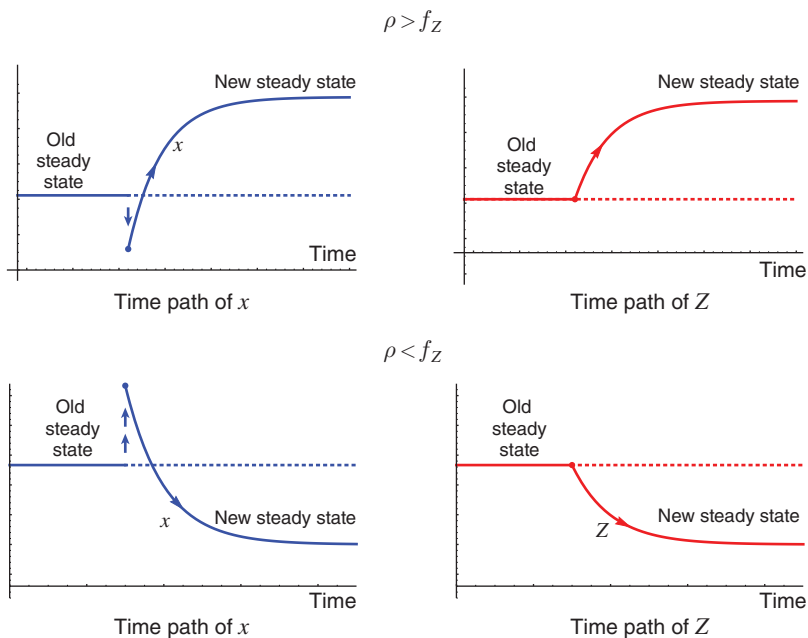
self-productivity and impatience, because self-productivity amplifies the consequences of current behavior over time while time discounting reduces the relevance of such future consequences. If impatience dominates ($\rho > f_Z$), present outcomes matter relatively more than future ones, and the agent immediately reduces consumption of good x when it becomes more expensive. The emphasis on short-run outcomes, however, produces a persistent increase in the state variable that will eventually lead the agent to reverse her initial choice and, in the long run, increase consumption of x (Figure 2, upper panels).⁸ If, instead, the agent is sufficiently patient ($f_Z > \rho$), in the long run she aims at consuming less of the relatively more expensive good (as in a static model). In order to do so, she is willing to bear the short-run cost of higher consumption, which persistently decreases Z and, over time, allows for lower consumption (Figure 2, lower panels).

As an example of self-productivity in consumer problems, consider the case in which the law of motion is

$$(18) \quad \dot{Z} = \delta Z - x^\alpha.$$

For instance, this is the case of Dalgaard and Strulik's (2014) and Dragone and Strulik's (2017) models of health deficit accumulation, where Z is the stock of health deficits and x is health care investment. Since Z is self-productive ($f_Z = \delta > 0$), health care has opposite short-run and long-run price effects (Proposition 2). Accordingly, if the agent is sufficiently patient, after an increase in the price of health care she will demand more health care in the short run and less in the long run. If, instead, she is sufficiently impatient, the opposite holds (Proposition 3).

⁸With self-productivity it must be the case that in steady state, $f_x < 0$. Hence, a short-run reduction in consumption triggers an increase in the state variable, followed by further increases due to self-productivity. Since the policy function is upward sloping, consumption will track the evolution of the state variable that was originated by the initial change in consumption.

FIGURE 2. SELF-PRODUCTIVE ($f_Z > 0$)

Notes: The figure shows that short-run and long-run responses of x to a price increase have opposite signs. After a price increase, a relatively impatient agent ($\rho > f_Z$) reduces consumption on impact (as in a static model) and increases it in the long run. A relatively patient agent ($\rho < f_Z$) does the opposite: she increases consumption in the short run and decreases it (as in a static model) in the long run. The left panels show the time path for x , and the right panels show the time path for Z .

III. Labor Supply and Human Capital Accumulation

Results are slightly more complex but qualitatively similar for labor supply problems, in which budget constraint (6) applies. The price responses (9)–(11) of Proposition 1 are

$$(19) \quad x_p^S = a[f_x L_Z - (f_Z - \rho)L_x]\varepsilon,$$

$$(20) \quad x_p^L = a[f_x L_Z - (f_Z - \rho)L_x]f_Z,$$

$$(21) \quad Z_p^L = -a[f_x L_Z - (f_Z - \rho)L_x]f_x.$$

Comparing these expressions to the analog expressions (14)–(16) of consumer problems, the only difference is the content in square brackets. Accordingly, one can state the exact equivalent of Proposition 3 (which we do not report here, for brevity), with the only difference here being that the critical value of ρ is $f_Z - f_x L_Z/L_x$ rather than just f_Z .

Further insights can be obtained considering an example where, unlike equations (17) and (18), the sign of f_Z in steady state depends on the parameters of the model. Consider a schooling model where Z is human capital and x is time devoted to schooling.⁹ The agent's problem can be written as

$$(22) \quad \max_{\{x,y\}} \int_0^\infty e^{-\rho t} u(x,y) dt$$

subject to

$$(23) \quad \dot{Z} = x^\alpha Z^\gamma - \delta Z,$$

$$(24) \quad \dot{A} = rA + M + pL(x,Z) - y,$$

where $\alpha, \gamma, \delta > 0$, effective labor $L(x,Z)$ is increasing in human capital and decreasing in nonlabor (schooling) time ($L_Z > 0, L_x < 0$), and p is the wage rate of effective labor.

Human capital is produced with schooling and existing human capital, and it depreciates over time. The law of motion (23) thus entails a self-productive component (the first term) and a self-depleting component (the second term) whose relative value in steady state only depends on the value of γ : human capital Z is self-productive in steady state if $\gamma > 1$ (i.e., when there are increasing marginal returns to knowledge in the production of new knowledge) and self-depleting if $\gamma < 1$ (i.e., when marginal returns are decreasing).¹⁰

Determining which case is empirically relevant is outside the scope of this paper.¹¹ For our purposes, it suffices to observe that due to Proposition 2, whether γ is larger or smaller than one is enough to predict whether in this model the short-run and long-run responses of labor supply to a wage increase have the same sign ($\gamma < 1$) or opposite signs ($\gamma > 1$). In the latter case, under more specific functional assumptions, Proposition 3 allows us to determine the conditions under which labor supply first increases and then decreases, or the other way around.

IV. Discussion and Conclusion

This paper studies short-run and long-run price effects for a broad class of dynamic consumer/worker problems. We provide a theoretical answer to the following simple but fundamental questions: Are dynamic substitution effects always negative? Can they change sign over different time horizons? Can demand be more

⁹Labor supply models fall into two broad classes: training models, which focus on the direct investment in human capital through either schooling or on-the-job training (as pioneered in Ben-Porath 1967; Heckman 1976; and Blinder and Weiss 1976), and learning-by-doing (or experience) models, which consider human capital accumulation as a byproduct of work activity (see Arrow 1962 and Weiss 1972 for seminal contributions). Both classes of models have been very influential in the literature, and they are both described by problem (1)–(4) and by (6).

¹⁰To see it, observe that in steady state the requirement $\dot{Z} = 0$ implies $x^\alpha Z^{\gamma-1} = \delta$. Since $f_Z = \gamma x^\alpha Z^{\gamma-1} - \delta$, we can replace and get $f_Z = \delta(\gamma - 1)$, which implies that $f_Z > 0$ if and only if $\gamma > 1$.

¹¹Heckman, Lochner, and Taber (1998) and Kapička (2006, 2015), among others, assume decreasing returns ($\gamma < 1$), but Trostel (2004) presents evidence that marginal returns are increasing ($\gamma > 1$) for low levels of human capital and decreasing ($\gamma < 1$) for high levels.

elastic in the short run than in the long run? On what primitives of the model does it depend?

We first characterize short-run and long-run price responses in a general model that encompasses as special cases models of endogenous preferences, health, and human capital. We find that short-run and long-run substitution effects may have opposite signs. Under assumptions that are commonly used in the literature, this possibility materializes when the state variable is self-productive, a condition that can be satisfied, for instance, in models of human capital accumulation and endogenous preferences, in which capital is an input for its own accumulation. With self-productive capital, after a price increase, impatient agents reduce consumption today and increase it tomorrow while patient ones do the opposite. If instead the state variable is self-depleting, as easily happens in models with capital depreciation (including those of rational addiction and health capital), both short-run and long-run price responses are negative.

In most models used in the literature, our results allow for an immediate assessment of the sign of price responses over time by simple inspection of the intertemporal problem. We illustrate this possibility considering workhorse models of consumption and labor supply that have been used extensively in the literature. For example, in consumer problems with health capital (Grossman 1972; Galama and Van Kippersluis 2018) the law of motion features self-depletion; hence, health investment is predicted to decrease both in the short run and in the long run when its price increases. On the contrary, health deficit accumulation models feature self-productivity (Dalgaard and Strulik 2014); hence, the price responses in the short run and in the long run will have opposite signs. Models of human capital often involve a self-productive and a self-depleting element (see, e.g., Ben-Porath 1967; Heckman 1976; Blinder and Weiss 1976; and Magnac, Pistoiesi, and Roux 2018). In such a case, the determination of the short-run and long-run substitution effects depends on which of the two effects dominates in steady state.

As intertemporal consumer problems are a building block of dynamic macroeconomic models, our results are also relevant for business cycle and growth theories, although an explicit analysis of general equilibrium models, possibly with a stochastic component, is outside the scope of the present paper. While an infinite time horizon is relevant in many applications and may represent a finite but uncertain life duration, it would be interesting to extend the analysis to a finite (and certain) time horizon, as well as to the effects of temporary price changes. This is left for future research.

MATHEMATICAL APPENDIX

A. Solving the General Intertemporal Problem

In this Appendix we derive the solution for the general intertemporal problem in which the law of motion depends on both consumption goods and on the state variable Z ; i.e.,

$$(A1) \quad \max_{\{x,y\}} \int_0^{\infty} e^{-\rho t} U(x(t), y(t), Z(t)) dt$$

subject to

$$(A2) \quad \dot{Z}(t) = f(x(t), y(t), Z(t)),$$

$$(A3) \quad \dot{A}(t) = rA(t) + M(t) + g(x(t), Z(t), p) - y(t),$$

$$(A4) \quad A(0) = A_0, \quad Z(0) = Z_0,$$

where $r = \rho > 0$. The case presented in the main text amounts to assuming that $f_y = 0$. The associated current-value Hamiltonian function is

$$(A5) \quad \mathcal{H}(x, y, Z, A, \mu, \lambda; p, M) = U(x, y, Z) + \lambda[rA + M + g(x, Z, p) - y] \\ + \mu f(x, y, Z),$$

where λ and μ are the costate variables associated with states A and Z , respectively. The following conditions are necessary for an internal solution:

$$(A6) \quad \mathcal{H}_x = U_x(x, y, Z) + \lambda g_x(x, Z, p) + \mu f_x(x, y, Z) = 0,$$

$$(A7) \quad \mathcal{H}_y = U_y(x, y, Z) - \lambda + \mu f_y(x, y, Z) = 0,$$

$$(A8) \quad \dot{\mu} = \rho\mu - \mathcal{H}_Z(x, y, Z, A, \mu, \lambda; p, M),$$

$$(A9) \quad \dot{\lambda} = \lambda(\rho - r),$$

together with (A2), (A3), (A4), and with the transversality conditions $\lim_{t \rightarrow \infty} e^{-\rho t} \mu(t) Z(t) = 0$ and $\lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) A(t) = 0$. The above conditions are also sufficient for a maximum if $\mathcal{H}(x, y, Z, A, \mu, \lambda; p, M)$ is concave in state and control variables (Mangasarian 1966; Seierstad and Sydsaeter 1977). We assume that this is indeed the case and that \mathcal{H}_{xx} and \mathcal{H}_{yy} are strictly negative.

The first-order conditions (A6)–(A7) determine the optimal value of x and y as functions of the state and costate variables, of the market price, and of the exogenous component of income:

$$(A10) \quad x^* = x^*(Z, A, \mu, \lambda; p, M),$$

$$(A11) \quad y^* = y^*(Z, A, \mu, \lambda; p, M).$$

Replacing (x^*, y^*) in (A2), (A3), (A8), and (A9) yields the optimal dynamics of state and costate variables:¹²

$$(A12) \quad \dot{Z} = f(x^*, y^*, Z),$$

$$(A13) \quad \dot{A} = rA + M + g(x^*, Z, p) - y^*,$$

$$(A14) \quad \dot{\mu} = \rho\mu - \mathcal{H}_Z(x^*, y^*, Z, A, \mu, \lambda; p, M),$$

$$(A15) \quad \dot{\lambda} = \lambda(\rho - r).$$

The solution $\{Z(t), A(t), \mu(t), \lambda(t)\}$ of the above system of differential equations represents the optimal trajectory of the state and costate variables given the initial values of the state variables, the transversality conditions, p , and M (among the other variables). This solution, once plugged into (A10) and (A11), determines the time path of x and y .

Note that since in this model $r = \rho$, equation (A15) implies that the shadow price of assets is constant over time: $\lambda(t) = \bar{\lambda}$ for all t . In general, $\bar{\lambda}$ depends on the parameters of the model, possibly including market prices and the exogenous component of income, i.e., $\bar{\lambda} = \bar{\lambda}(p, M)$. Its specific value is obtained by imposing additional constraints. For example, one can reasonably assume that assets go to zero (or some positive value) in steady state. With $\lambda = \bar{\lambda}$, the dimensionality of the system of differential equations effectively reduces to three equations only ((A12), (A13), and (A14)). Given the initial values of the state variables and the transversality conditions, the solution $\{Z(t), A(t), \mu(t)\}$ of the reduced system of differential equations represents the optimal trajectory of the state and costate variables, which depends on p and M , and that determines the optimal path of x and y .

We focus on solutions leading to a saddle-point-stable steady state,

$$(A16) \quad x^L = x^*(Z^L, A^L, \mu^L, \bar{\lambda}; p, M),$$

$$(A17) \quad y^L = y^*(Z^L, A^L, \mu^L, \bar{\lambda}; p, M),$$

where

$$(A18) \quad Z^L = Z^L(p, M), \quad A^L = A^L(p, M), \quad \mu^L = \mu^L(p, M)$$

are the steady-state values of state and costate variables that satisfy (A12), (A13), and (A14) with equality.

¹² Alternatively, one could derive the solution in terms of state and control variables. We opt for the current procedure to highlight the dynamics of the states and the corresponding shadow prices, and to show how our method can be applied to problems with any number of control variables.

To assess the saddle-point stability of the steady state, consider the determinant of the three-dimensional Jacobian matrix \mathbf{J} associated to (A12), (A13), and (A14), with $\lambda = \bar{\lambda}$:

$$(A19) \quad |\mathbf{J}| = \rho f_Z(\rho - f_Z) + \frac{\rho}{\mathcal{H}_{xx}\mathcal{H}_{yy} - \mathcal{H}_{xy}^2} \\ \times \left\{ (f_y\mathcal{H}_{xz} - f_x\mathcal{H}_{yz})^2 - (f_y^2\mathcal{H}_{xx} - 2f_xf_y\mathcal{H}_{xy} + f_x^2\mathcal{H}_{yy})\mathcal{H}_{zz} \right. \\ \left. - (2f_Z - \rho) [(f_y\mathcal{H}_{xy} - f_x\mathcal{H}_{yy})\mathcal{H}_{xz} + (f_x\mathcal{H}_{xy} - f_y\mathcal{H}_{xx})\mathcal{H}_{yz}] \right\}.$$

The steady state has saddle-point stability when the Jacobian matrix \mathbf{J} admits one negative eigenvalue ε (there is at most one), implying $|\mathbf{J}| < 0$.

The response of the steady state to price changes determines the long-run price response. To determine the short-run response, we consider how the saddle path leading to the steady state changes after a price shock. Notice that the saddle path can either be expressed as a function of the state variables (the feedback representation of the solution) or as a function of time (open-loop solution; Dockner et al. 2000; Barro and Sala-i Martin 2004). Here we focus on the feedback representation. As λ is fixed, it will be enough to consider how the saddle path of μ ,

$$(A20) \quad \hat{\mu}(Z, A; p, M),$$

changes when p or M change. This information will be used to obtain the short-run response of x and y . Plugging (A20) into (A10) and (A11) yields the saddle paths of x and y in feedback form, i.e., the policy functions of x and y as functions of the state variables, given p and M :

$$(A21) \quad \hat{x} = \hat{x}(Z, A; p, M) = x^*(Z, A, \hat{\mu}(Z, A; p, M), \bar{\lambda}(p, M); p, M),$$

$$(A22) \quad \hat{y} = \hat{y}(Z, A; p, M) = y^*(Z, A, \hat{\mu}(Z, A; p, M), \bar{\lambda}(p, M); p, M).$$

B. Long-Run Substitution Effects

Consider (A16) and (A17). The change of steady-state consumption of x after an increase in price p is computed as follows:

$$(A23) \quad x_p^L = \frac{\partial x^*}{\partial p} + \frac{\partial x^*}{\partial Z} Z_p^L + \frac{\partial x^*}{\partial A} A_p^L + \frac{\partial x^*}{\partial \mu} \mu_p^L + \frac{\partial x^*}{\partial \lambda} \frac{\partial \bar{\lambda}}{\partial p},$$

$$(A24) \quad y_p^L = \frac{\partial y^*}{\partial p} + \frac{\partial y^*}{\partial Z} Z_p^L + \frac{\partial y^*}{\partial A} A_p^L + \frac{\partial y^*}{\partial \mu} \mu_p^L + \frac{\partial y^*}{\partial \lambda} \frac{\partial \bar{\lambda}}{\partial p},$$

where $\partial \bar{\lambda} / \partial p = 0$ if the price change is compensated so as to leave the marginal value of assets $\bar{\lambda}$ constant.

To compute the above equations, apply Cramer's rule to the first-order conditions (A6) and (A7). This yields

$$(A25) \quad \frac{\partial x^*}{\partial p} = -\frac{g_{xp}\bar{\lambda}}{\Omega} \mathcal{H}_{yy}, \quad \frac{\partial y^*}{\partial p} = \frac{g_{xp}\bar{\lambda}}{\Omega} \mathcal{H}_{xy},$$

$$(A26) \quad \frac{\partial x^*}{\partial Z} = \frac{\mathcal{H}_{xy}\mathcal{H}_{yZ} - \mathcal{H}_{xZ}\mathcal{H}_{yy}}{\Omega}, \quad \frac{\partial y^*}{\partial Z} = \frac{\mathcal{H}_{xy}\mathcal{H}_{xZ} - \mathcal{H}_{yZ}\mathcal{H}_{xx}}{\Omega},$$

$$(A27) \quad \frac{\partial x^*}{\partial A} = \frac{\partial x^*}{\partial M} = 0, \quad \frac{\partial y^*}{\partial A} = \frac{\partial y^*}{\partial M} = 0,$$

$$(A28) \quad \frac{\partial x^*}{\partial \mu} = \frac{f_y \mathcal{H}_{xy} - f_x \mathcal{H}_{yy}}{\Omega}, \quad \frac{\partial y^*}{\partial \mu} = \frac{f_x \mathcal{H}_{xy} - f_y \mathcal{H}_{xx}}{\Omega},$$

$$(A29) \quad \frac{\partial x^*}{\partial \lambda} = -\frac{\mathcal{H}_{xy} + g_x \mathcal{H}_{yy}}{\Omega}, \quad \frac{\partial y^*}{\partial \lambda} = \frac{\mathcal{H}_{xx} + g_x \mathcal{H}_{xy}}{\Omega},$$

where

$$(A30) \quad \Omega = \mathcal{H}_{xx}\mathcal{H}_{yy} - \mathcal{H}_{xy}^2,$$

which is positive by strict concavity. Then consider equations (A12), (A13), and (A14), which equal zero in steady state, and apply Cramer's rule again to obtain

$$(A31) \quad Z_p^L = \frac{\rho\bar{\lambda}}{\Omega|\mathbf{J}|} \left[g_{xp}(f_Z - \rho)(f_x \mathcal{H}_{yy} - f_y \mathcal{H}_{xy}) + f_x^2 g_{pZ} \mathcal{H}_{yy} \right. \\ \left. + f_y^2 (g_{pZ} \mathcal{H}_{xx} - g_{xp} \mathcal{H}_{xZ}) + (g_{xp} \mathcal{H}_{yZ} - 2g_{pZ} \mathcal{H}_{xy}) f_x f_y \right],$$

(A32)

$$A_p^L = \frac{\bar{\lambda}}{\Omega|\mathbf{J}|} \left\{ [(\mathcal{H}_{yZ} + g_Z \mathcal{H}_{yy})f_x - (\mathcal{H}_{xy} + g_x \mathcal{H}_{yy})f_Z] [(f_Z - \rho)g_{xp} - f_x g_{pZ}] \right. \\ \left. + [g_x(g_{xp} \mathcal{H}_{ZZ} - g_{pZ} \mathcal{H}_{xZ}) + g_{pZ}(g_Z \mathcal{H}_{xx} - g_x \mathcal{H}_{xZ}) + g_Z(g_x - g_{xp}) \mathcal{H}_{xZ}] f_y^2 \right. \\ \left. - [g_{xp}(\mathcal{H}_{ZZ} + g_Z \mathcal{H}_{yZ}) - g_{pZ}(\mathcal{H}_{xZ} + 2g_Z \mathcal{H}_{xy} - g_x \mathcal{H}_{yZ})] f_x f_y \right. \\ \left. - [(\mathcal{H}_{xx} + g_x \mathcal{H}_{xy})g_{pZ} - g_{xp}(\mathcal{H}_{xZ} + g_x \mathcal{H}_{yZ})] f_Z f_y \right. \\ \left. + (g_x \mathcal{H}_{yZ} - g_Z \mathcal{H}_{xy})(f_Z - \rho)f_y g_{xp} \right\} + g_p A_M^L,$$

$$(A33) \quad \mu_p^L = \frac{\rho\bar{\lambda}}{\Omega|\mathbf{J}|} \left[(g_{xp} \mathcal{H}_{yZ} - g_{pZ} \mathcal{H}_{xy})(f_Z \mathcal{H}_{xy} - f_x \mathcal{H}_{yZ}) \right. \\ \left. - (g_{xp} \mathcal{H}_{ZZ} - g_{pZ} \mathcal{H}_{xZ})(f_y \mathcal{H}_{xy} - f_x \mathcal{H}_{yy}) \right. \\ \left. + (g_{xp} \mathcal{H}_{xZ} - g_{pZ} \mathcal{H}_{xx})(f_y \mathcal{H}_{yZ} - f_Z \mathcal{H}_{yy}) \right].$$

Expressions (A31) and (A32) describe the long-run price effect on Z and A . The long-run price responses for x and y are found by replacing the above expressions in (A23) and (A24), which for the general case yields

$$(A34) \quad x_p^L = \theta [f_x g_{pZ} - (f_Z - \rho) g_{xp}] f_Z \mathcal{H}_{yy} + Df_y,$$

$$(A35) \quad y_p^L = \theta [f_x g_{pZ} - (f_Z - \rho) g_{xp}] (f_x \mathcal{H}_{yZ} - f_Z \mathcal{H}_{xy}) + Ef_y,$$

where

$$(A36) \quad \theta = -\frac{\bar{\lambda}\rho}{\Omega|\mathbf{J}|},$$

$$(A37) \quad D = \theta \{ g_{pZ} (f_y \mathcal{H}_{xZ} - f_Z \mathcal{H}_{xy} - f_x \mathcal{H}_{yZ}) + g_{xp} [\mathcal{H}_{yZ} (2f_Z - \rho) - f_y \mathcal{H}_{ZZ}] \},$$

$$(A38) \quad E = \theta [g_{pZ} (f_Z \mathcal{H}_{xx} - f_x \mathcal{H}_{xZ}) + g_{xp} (f_x \mathcal{H}_{ZZ} - f_Z \mathcal{H}_{xZ})].$$

These general expressions are not particularly insightful per se. Notably, their formulation simplifies considerably in specific economic applications of interest.

For instance, when $f_y = 0$, as in the class of models considered in Section I, the long-run responses of x and Z boil down to equations (10) and (11) in Proposition 1,

$$(A39) \quad x_p^L = a [f_x g_{pZ} - (f_Z - \rho) g_{xp}] f_Z,$$

$$(A40) \quad Z_p^L = -a [f_x g_{pZ} - (f_Z - \rho) g_{xp}] f_x,$$

where

$$(A41) \quad a = \theta \mathcal{H}_{yy} = -\frac{\bar{\lambda}\rho}{\Omega|\mathbf{J}|} \mathcal{H}_{yy} < 0.$$

Moreover, in a consumer problem, $g_{pZ} = 0$ and $g_{xp} = -1$; hence (A39) and (A40) further simplify to equations (15) and (16):

$$(A42) \quad x_p^L = a (f_Z - \rho) f_Z,$$

$$(A43) \quad Z_p^L = -a (f_Z - \rho) f_x.$$

As a side result, note that the same procedure can be used to compute the response to a change in the exogenous component of income. Consider first the long-run response:

$$(A44) \quad x_M^L = \frac{\partial x^*}{\partial M} + \frac{\partial x^*}{\partial Z} Z_M^L + \frac{\partial x^*}{\partial A} A_M^L + \frac{\partial x^*}{\partial \lambda} \frac{\partial \bar{\lambda}}{\partial M} + \frac{\partial x^*}{\partial \mu} \mu_M^L,$$

$$(A45) \quad y_M^L = \frac{\partial y^*}{\partial M} + \frac{\partial y^*}{\partial Z} Z_M^L + \frac{\partial y^*}{\partial A} A_M^L + \frac{\partial y^*}{\partial \lambda} \frac{\partial \bar{\lambda}}{\partial M} + \frac{\partial y^*}{\partial \mu} \mu_M^L.$$

Since $\partial x^*/\partial A = \partial x^*/\partial M = \partial y^*/\partial A = \partial y^*/\partial M = \partial Z^L/\partial M = \partial \bar{\lambda}/\partial M = \partial \mu^L/\partial M = 0$, we obtain that, when the marginal utility of wealth is maintained constant, the long-run dynamic income effects at the steady state are nil.

C. Short-Run Substitution Effects

To compute short-run price effects, consider (A10) and (A11) and recall that in the short run state variables A and Z are given. Given that λ is fixed, the short-run responses of x and y , when all variables are at their steady-state level, only depend on the direct effect of price and on its indirect effect through the costate variable μ :

$$(A46) \quad x_p^S = \frac{\partial x^*}{\partial p} + \frac{\partial x^*}{\partial \mu} \mu_p^S,$$

$$(A47) \quad y_p^S = \frac{\partial y^*}{\partial p} + \frac{\partial y^*}{\partial \mu} \mu_p^S.$$

The terms $\partial x^*/\partial p, \partial x^*/\partial \mu, \partial y^*/\partial p, \partial y^*/\partial \mu$ are described in equations (A25) and (A28). To obtain μ_p^S , we need to assess how the saddle path of μ responds to a price change. Even without knowing its specific functional form, we can approximate it around the steady state. Take a first-order linear expansion of (A12), (A13), and (A14):

$$(A48) \quad \begin{pmatrix} \dot{Z} \\ \dot{A} \\ \dot{\mu} \end{pmatrix} = \mathbf{J} \cdot \begin{pmatrix} Z - Z^L \\ A - A^L \\ \mu - \mu^L \end{pmatrix}.$$

Consider the eigenvector $(1, \xi_2, \xi_3)$ associated to the negative eigenvalue ε of the Jacobian matrix \mathbf{J} , with

$$(A49) \quad \xi_3 = \frac{1}{\Phi} [(\varepsilon - f_Z)\Omega + f_x(\mathcal{H}_{xZ}\mathcal{H}_{yy} - \mathcal{H}_{xy}\mathcal{H}_{yZ}) + f_y(\mathcal{H}_{xx}\mathcal{H}_{yZ} - \mathcal{H}_{xZ}\mathcal{H}_{xy})],$$

$$(A50) \quad \varepsilon = \frac{1}{2} \left(\rho - \sqrt{\rho^2 - 4 \frac{|\mathbf{J}|}{\rho}} \right) < 0,$$

$$(A51) \quad \Phi = -f_y^2 \mathcal{H}_{xx} + 2f_x f_y \mathcal{H}_{xy} - f_x^2 \mathcal{H}_{yy} > 0.$$

The particular solution of the system of ordinary linear differential equations (A48) is

$$(A52) \quad Z(t) = Z^L + \alpha e^{\varepsilon t},$$

$$(A53) \quad A(t) = A^L + \alpha e^{\varepsilon t} \xi_2,$$

$$(A54) \quad \mu(t) = \mu^L + \alpha e^{\varepsilon t} \xi_3,$$

where α is a constant that depends on initial values. Replacing $\alpha e^{\epsilon t} = Z(t) - Z^L$ in (A54) yields the feedback solution of μ as a function of the state variable Z only,

$$(A55) \quad \hat{\mu}(Z) = \mu^L + (Z - Z^L) \xi_3.$$

Taking the derivative of (A55) with respect to p yields, for any A , the short-run effect of a price change

$$(A56) \quad \mu_p^S \equiv \frac{\partial \hat{\mu}(Z)}{\partial p} = \mu_p^L - \xi_3 Z_p^L + (Z - Z^L) \frac{\partial \xi_3}{\partial p}.$$

In proximity of Z^L , the last term is negligible and the above expression simplifies to

$$(A57) \quad \mu_p^S = \mu_p^L - \xi_3 Z_p^L.$$

Replacing (A57) in (A46)–(A47) and rearranging yields the short-run price response of x and y ,

$$(A58) \quad x_p^S = C(f_x \mathcal{H}_{yy} - f_y \mathcal{H}_{xy}) + \frac{\bar{\lambda}}{\Phi} f_y^2 g_{xp},$$

$$(A59) \quad y_p^S = C(f_y \mathcal{H}_{xx} - f_x \mathcal{H}_{xy}) - \frac{\bar{\lambda}}{\Phi} f_x f_y g_{xp},$$

where

$$(A60) \quad C = \epsilon \theta \left\{ g_{pZ} + \frac{g_{xp}}{\Phi} [(f_x \mathcal{H}_{yy} - f_y \mathcal{H}_{xy})(f_Z - \rho) + (f_y \mathcal{H}_{xZ} - f_x \mathcal{H}_{yZ}) f_y] \right\}.$$

When $f_y = 0$, the above expressions simplify considerably. For example, the short-run response of x becomes

$$(A61) \quad x_p^S = a\epsilon [f_x g_{pZ} - (f_Z - \rho) g_{xp}],$$

as shown in equation (9) of Proposition 1.

In a consumer problem $g_{pZ} = 0$ and $g_{xp} = -1$; hence (A61) further simplifies to

$$(A62) \quad x_p^S = a\epsilon (f_Z - \rho),$$

as shown in equation (14).

Note that neither expression depends on assets or income because income effects are also nil in the short run. To see it, consider

$$(A63) \quad x_M^S = \frac{\partial x^*}{\partial M} + \frac{\partial x^*}{\partial \mu} \mu_M^S,$$

$$(A64) \quad y_M^S = \frac{\partial y^*}{\partial M} + \frac{\partial y^*}{\partial \mu} \mu_M^S.$$

Taking the derivative of (A55) with respect to M yields

$$(A65) \quad \mu_M^S = \mu_M^L - \xi_3 Z_M^L = 0.$$

Together with the fact that $\partial x^*/\partial M = \partial y^*/\partial M = 0$, we conclude that $x_M^S = y_M^S = 0$.

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