Uniform Bound of the Entanglement for the Ground State of the Quantum Ising Model with Large Transverse Magnetic Field

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Abstract

We consider the ground state of the quantum Ising model with transverse field h in one dimension in a finite volume

$$\Lambda_m := \{-m, -m+1, \ldots, m+L\} \ .$$

For h sufficiently large we prove a bound for the entanglement of the interval $\Lambda_0 :=$

$\{0, .., L\}$ relative to its complement $\Lambda_m \backslash \Lambda_0$ which is uniform in m and L. The bound is established by means of a suitable cluster expansion.

Contents

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1 Introduction and results

A characteristic feature distinguishing quantum systems from the classical ones is that pure states of composite systems do not assign in general a definite pure state to their subsystems. In particular, in the framework of quantum statistical mechanics, the density matrix describing the state of a subsystem, which goes under the name of *reduced density matrix*, is obtained

taking the trace over the other system's components and in general corresponds to a mixed state. This property is called entanglement and the von Neumann entropy of the reduced density matrix of a subsystem can be considered as a measure of the entanglement of the subsystem itself. One can therefore address the problem of estimating the the von Neumann entropy (also known in the literature as *entanglement entropy*) of a subsystem w.r.t. the ground state of the entire system. There are few rigorous results in this direction, we refer the reader to [GOS] and reference therein for a more complete discussion on this topic.

The ground state of the quantum Ising model with a transverse magnetic field can be represented as a classical Ising model with one added continuous dimension [DLP]. In turn this classical Ising model can be represented via a suitable FK random cluster model [FK], [CKP], [AKN]. This last representation has been used for example in [GOS] to study the entanglement of the ground state in the supercritical regime.

In this paper we study the problem of entanglement for the supercritical quantum Ising model with transverse magnetic field by using the representation of [DLP]. We consider a Gibbs random field in $\mathbb{Z}^2$ in which the spins take values in the space of trajectories of a spin-flip process. For this model a cluster expansion was developed in [CG] and it was proved that it satisfies the conditions for convergence (see [KP]) when the parameter h corresponding to the strength of the transverse magnetic field is sufficiently large.

We consider the ground state of the quantum Ising model with transverse field h in one dimension in a finite volume

$$\Lambda_m := \{-m, -m+1, \dots, m+L\} \ . \tag{1}$$

By using this cluster expansion we prove that for h sufficiently large the entanglement of the interval $\Lambda_0 := \{0, .., L\}$ relative to its complement $\Lambda_m \backslash \Lambda_0$ is bounded by a constant uniformly in m and L.

It was proved in [GOS] that for h larger than some value corresponding to a percolation threshold the entanglement is bounded by a constant times $\log L$.

In section 1.2.1 we recall the definition of the quantum Ising model with transverse field on $\mathbb{Z}$.

In section 2 we recall and present in a more complete form the result about the cluster expansion for the one-dimensional interacting spin-flip process given in [CG]. We remark that, although in this paper all the computations are carried out for the one-dimensional model

with nearest-neighbour translation-invariant ferromagnetic couplings, the cluster expansion presented in section 2 can be performed for the model defined on $\mathbb{Z}^d, d \geq 1$, with bounded, finite-range, pairwise interactions.

In section 3.2 we recall the set up developed in [GOS] in order to estimate the entanglement entropy of the ground state of the system and prove the key estimates which will lead us to the uniform bound of this quantity.

If $\mathcal{H}_m := \mathcal{H}_{\Lambda_m}$ is the Hilbert space for the quantum system defined on Λ_m, considering the representation of $\mathcal{H}_m$ as $\mathcal{H}_{m,L} \otimes \mathcal{H}_L$, with $\mathcal{H}_L := \mathcal{H}_{\Lambda_0}$ and $\mathcal{H}_{m,L} := \mathcal{H}_{\Lambda_m \setminus \Lambda_0}$, let ρ_m^L be the trace over $\mathcal{H}_{m,L}$ of the density operator associated to the ground state of the system.

We will prove the following

Theorem 1 *Consider a one-dimensional quantum Ising model in a transverse magnetic field. There exists a positive value of the external magnetic field h^* such that, for any $h > h^*$, the* entanglement entropy *of the ground state $S\left(\rho_m^L\right) := -tr_{\mathcal{H}_L}\left(\rho_m^L \log \rho_m^L\right)$ is bounded by a constant uniformly in m, L with $m \geq 0, L \geq 1$.*

We stress that the cluster expansion can be carried out for one-dimensional quantum Ising models with transverse field with bounded, finite-range, translation-invariant, ferromagnetic interactions. Therefore our result can be generalised in a straightforward way to this case.

1.1 Notation

Given a set $A \subset \mathbb{R}^d, d \geq 1$, let us denote by A^c its complement. We also set $\mathcal{P}(A)$ to be the collection of all subsets of A, $\mathcal{P}_n(A) := \{B \in \mathcal{P}(A) : |B| = n\}$ and $\mathcal{P}_f(A) := \bigcup_{n \geq 1} \mathcal{P}_n(A)$, where $|B|$ is the cardinality of B. Given $B \subset A, \mathbf{1}_B$ denotes the indicator function of B. Hence, if B is a discrete set, for any $b \in B, \mathbf{1}_{\{b\}}(b') = \begin{cases} 1 & b = b' \\ 0 & b \neq b' \end{cases}$.

A sequence of $\{\Lambda_n\}_{n \in \mathbb{N}} \in \mathcal{P}_f\left(\mathbb{Z}^d\right), d \geq 1$, is called *cofinal* if for any $n \geq 1, \Lambda_n \subset \Lambda_{n+1}$ and $\{\Lambda_n\} \uparrow \mathbb{Z}^d$.

Let $\mathcal{B}\left(\mathbb{R}^d\right)$ be the Borel σ-algebra of $\mathbb{R}^d$ and λ^d be the Lebesgue measure on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$. A sequence of $\{I_n\}_{n \in \mathbb{N}} \in \mathcal{B}\left(\mathbb{R}^d\right)$ is called an *exhaustion* of $\mathbb{R}^d$ if, for any $n \geq 1, \lambda^d(I_n) < \infty, I_n \subset I_{n+1}$ and $\{I_n\} \uparrow \mathbb{R}^d$.

For any $x \in \mathbb{R}^d$, we set $|x| := \sum_{i=1}^d |x_i|$ and consequently, for any $A \subset \mathbb{R}^d, \operatorname{dist}(x, A) := \inf_{y \in A} |x - y|$.

Given a Hilbert space $\mathcal{H}$, let $\mathfrak{B}(\mathcal{H})$ be the Banach space of bounded linear operators on $\mathcal{H}$ with norm $\|\mathbf{A}\| := \sup_{\psi \in \mathcal{H} \,:\, \|\psi\|=1} \|\mathbf{A}\psi\|$ and $\mathfrak{T}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{H})$ the collection of trace class operators on $\mathcal{H}$ which is a Banach space if endowed with the norm $tr_{\mathcal{H}}(|\mathbf{A}|)$, where $tr_{\mathcal{H}}(\mathbf{A})$ denotes the trace of $\mathbf{A} \in \mathfrak{T}(\mathcal{H})$. Then, if $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, we denote by $tr_{\mathcal{H}_1} : \mathfrak{T}(\mathcal{H}_1 \otimes \mathcal{H}_2) \longmapsto \mathfrak{T}(\mathcal{H}_2)$ the partial trace w.r.t. $\mathcal{H}_1$.

1.1.1 Graphs

Let $G = (V, E)$ be a graph whose set of vertices and set of edges are given respectively by a finite or enumerable set V and $E \subset \mathcal{P}_2(V)$. $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq \mathcal{P}_2(V') \cap E$ is said to be a subgraph of G and this property is denoted by $G' \subseteq G$. If $G' \subseteq G$, we denote by $V(G')$ and $E(G')$ respectively the set of vertices and the collection of the edges of G'. $|V(G')|$ is called the *order* of G' while $|E(G')|$ is called its *size*. Given $G_1, G_2 \subseteq G$, we denote by $G_1 \cup G_2 := (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2)) \subset G$ the *graph union* of G_1 and G_2. Given $e \in E$, we denote by $V_e := \{v \in V : \mathbf{1}_e(v) = 1\}$ the set of *endpoints* of e, hence $e = \{v, v'\}$ iff $\{v, v'\} = V_e$. Moreover, given $E' \subseteq E$, we denote by $V(E') := \bigcup_{e \in E'} V_e$.

A *path* in G is a subgraph γ of G such that there is a bijection $\{0, .., |E(\gamma)|\} \ni i \longmapsto v(i) := x_i \in V(\gamma)$ with the property that any $e \in E(\gamma)$ can be represented as $\{x_{i-1}, x_i\}$ for $i = 1, .., |E(\gamma)|$. Two distinct vertices x, y of G are said to be *connected* if there exists a path $\gamma \subseteq G$ such that $x_0 = x,\ x_{|E(\gamma)|} = y$. Therefore, if γ is a path in G, we will denote by $|\gamma|$ its length $|E(\gamma)|$ and by $end(\gamma) := \left\{v \in V(\gamma) : \sum_{e \in E(\gamma)} \mathbf{1}_e(v) = 1\right\}$ the collection of its *endpoints*. Hence, for any $e \in E$, the graph $(V_e, e) \subset G$ is a path of length 1 and $end((V_e, e)) = V_e$. A graph G is said to be *connected* if any two distinct elements of $V(G)$ are connected. The maximal connected subgraphs of G are called *components* of G. Two connected subgraph $G_1, G_2 \subset G$ are connected by a path γ in G if $G_1 \cup G_2$ is not connected and $G_1 \cup G_2 \cup \gamma$ is a connected subgraph of G.

We denote by $\mathbb{L}^d$ the graph $\left(\mathbb{Z}^d, \mathbb{E}^d\right)$ with $\mathbb{E}^d := \left\{\{x, y\} \in \mathcal{P}_2\left(\mathbb{Z}^d\right) : |x - y| = 1\right\}$. If $\Lambda \subset \mathbb{Z}^d$, we also set $\partial\Lambda := \{y \in \Lambda^c : \operatorname{dist}(y, \Lambda) = 1\}$ and $\mathbb{L}^d_\Lambda := \left(\Lambda \cup \partial\Lambda, \mathbb{E}^d_\Lambda\right)$, where $\mathbb{E}^d_\Lambda := \left\{e \in \mathbb{E}^d : V_e \subset (\Lambda \cup \partial\Lambda)\right\}$.

1.2 The model

We consider the Hilbert space $\mathcal{H} := l^2(\{-1,1\},\mathbb{C})$ which is isomorphic to $\mathbb{C}^2$. The algebra $\mathcal{U} := M(2,\mathbb{C})$ of bounded linear operators on $\mathcal{H}$ is then generated by the Pauli matrices $\sigma^{(i)}, i = 1,2,3$ and by the identity I. In particular, unless differently specified, in the following we will always consider the representation of $\mathcal{U}$ with respect to which $\sigma^{(3)}$ is diagonal, i.e.

$$\sigma^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

and

$$\sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{3}$$

Let Λ be a finite connected subset of $\mathbb{Z}$ and set $\mathcal{H}_\Lambda := \bigotimes_{x\in\Lambda} \mathcal{H}_x$ where, for any $x \in \Lambda, \mathcal{H}_x$ is a copy of $\mathcal{H}$ at x. The finite volume Hamiltonian of the ferromagnetic quantum Ising model with transverse field is the linear operator on $\mathcal{H}_\Lambda$

$$\mathbf{H}_\Lambda(J,h) := -\frac{1}{2}J \sum_{x,y\in\Lambda\,:\,\{x,y\}\in\mathbb{E}} \sigma_x^{(3)}\sigma_y^{(3)} - h\sum_{x\in\Lambda}\sigma_x^{(1)} , \tag{4}$$

with $h > 0$ and $J \geq 0$ for any $x, y \in \Lambda$.

Given $\Lambda \subset\subset \mathbb{Z}$, it can be proven [CKP] that $\mathbf{H}_\Lambda(J,h)$ generates a positivity improving semigroup which by the Perron-Frobenius theorem has a unique ground state $\Psi_\Lambda \in \mathcal{H}_\Lambda$. The same argument applies to the operator $\mathbf{L}_\Lambda(h) : \mathcal{H}_\Lambda \circlearrowleft$ such that

$$\mathbf{L}_\Lambda(h) := h\sum_{x\in\Lambda}\left(\sigma_x^{(1)} - 1\right) , \tag{5}$$

whose ground state $\Psi_\Lambda^0 \in \mathcal{H}_\Lambda$ is such that $\langle\Psi_\Lambda, \Psi_\Lambda^0\rangle > 0$ and, for any element $\mathbf{A}$ of the Abelian subalgebra $\mathfrak{A}_\Lambda$, generated by $\left\{\sigma_x^{(3)}, x \in \Lambda\right\}$, of the algebra of linear operators on $\mathcal{H}_\Lambda$

$$\langle\Psi_\Lambda, \mathbf{A}\Psi_\Lambda\rangle = \lim_{\beta\to\infty} \frac{\left\langle \Psi_\Lambda^0, e^{-\frac{\beta}{2}\mathbf{H}_\Lambda(J,h)}\mathbf{A}e^{-\frac{\beta}{2}\mathbf{H}_\Lambda(J,h)}\Psi_\Lambda^0\right\rangle}{\langle\Psi_\Lambda^0, e^{-\beta\mathbf{H}_\Lambda(J,h)}\Psi_\Lambda^0\rangle} = \lim_{\beta\to\infty}\frac{tr_{\mathcal{H}_\Lambda}\left(e^{-\beta\mathbf{H}_\Lambda(J,h)}\mathbf{A}\right)}{tr_{\mathcal{H}_\Lambda}\left(e^{-\beta\mathbf{H}_\Lambda(J,h)}\right)} . \tag{6}$$

1.2.1 Spin-flip process description of the system

In [DLP] (Section 2.5) it has been shown that in the chosen representation for the Pauli matrices, for any $h > 0$, the linear operator $\mathbf{L}(h) := h\left(\sigma^{(1)} - 1\right)$ on $\mathcal{H}$ can be interpreted as

the generator of a continuous time Markov process with state space $\{-1,1\}$, the so called *spin-flip process*, with rate h. Namely, for any function f on $\{-1,1\}$

$$L(h)\,f(\xi) := h\left(f(-\xi) - f(\xi)\right)\ ,\ \xi \in \{-1,1\}\ . \tag{7}$$

Hence, given a Poisson point process $(N_h(t), t \in \mathbb{R})$ with intensity h, we can consider the random process which, with a little abuse of notation, we denote by

$$\mathbb{R} \ni t \longmapsto \sigma(t) := (-1)^{N_h(t)} \in \{-1,1\}\ , \tag{8}$$

that is the stationary measure μ defined by the semigroup generated by $L(h)$ on the measurable space $(\mathcal{D}, \mathcal{F})$, where $\mathcal{D}$ is the Skorokhod space $\mathbb{D}(\mathbb{R}, \{-1,1\})$ of piecewise $\{-1,1\}$-valued rcll (càdlàg) constant functions on $\mathbb{R}$ and $\mathcal{F}$ is the σ-algebra generated by the open sets in the associated Skorokhod topology.

Consequently, for any interval $I \subset \mathbb{R}$, let μ_I be the restriction of μ to the measurable space $(\mathcal{D}_I, \mathcal{F}_I)$, where

$$\mathcal{D}_I := \{\sigma \in \mathbb{D}(I, \{-1,1\}) : \sigma = \sigma' \restriction_I,\ \sigma' \in \mathcal{D}\} \tag{9}$$

and $\mathcal{F}_I$ is the σ-algebra generated by the open sets in the associated Skorokhod topology. Moreover, we denote by μ_I^p the probability distribution corresponding to periodic b.c.'s, that is conditional to

$$\begin{aligned} \mathcal{D}_I^p &:= \left\{\sigma \in \mathcal{D}_I : \sigma\left(-\frac{\beta}{2}\right) = \sigma\left(\frac{\beta}{2}\right)\right\} \\ &= \left\{\sigma \in \mathcal{D}_I : N_h\left(\frac{\beta}{2}\right) - N_h\left(-\frac{\beta}{2}\right) = 2k\ ,\ k \in \mathbb{Z}^+\right\}\ . \end{aligned} \tag{10}$$

Let $\mathfrak{D}_I := \mathcal{D}_I^{\mathbb{Z}}$ the configuration space of the random field $\mathbb{Z} \ni x \longmapsto \sigma_x \in \mathcal{D}_I, \mathfrak{F}_I := \mathcal{F}_I^{\otimes\mathbb{Z}}$ the σ-algebra generated by the cylinder events of $\mathfrak{D}_I$, and ν_I the product measure $\mu_I^{\otimes\mathbb{Z}}$.

The finite volume distribution Given $\beta > 0$, let us set $I := \left[-\frac{\beta}{2}, \frac{\beta}{2}\right]$. For any finite subset Λ of $\mathbb{Z}$ we denote by σ_Λ the restriction of the configuration $\sigma \in \mathfrak{D}_I$ to $\mathcal{D}_I^\Lambda$ and set $\sigma_\Lambda(t) := \{\sigma_x(t)\}_{x\in\Lambda}$, $\mathcal{F}_I^\Lambda := \mathcal{F}_I^{\otimes\Lambda}$ and $\mu_I^\Lambda := \mu_I^{\otimes\Lambda}$. We introduce the conditional Gibbs measure

$\nu_{I,\Lambda}^{\eta;\xi^+,\xi^-}$ on $\left(\mathcal{D}_I^\Lambda, \mathcal{F}_I^\Lambda\right)$ with density w.r.t. μ_I^Λ given by

$$Z_{\Lambda,I}^{-1}\left(\eta;\xi^+,\xi^-\right)\exp\left[J\sum_{x,y\in\Lambda\,:\,\{x,y\}\in\mathbb{E}}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}\mathrm{d}t\sigma_x\left(t\right)\sigma_y\left(t\right)+J\sum_{x\in\Lambda}\sum_{y\in\partial\Lambda}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}\mathrm{d}t\sigma_x\left(t\right)\eta_y\left(t\right)\right]\times$$
$$\times\prod_{x\in\Lambda}\mathbf{1}_{\{\xi_x^-\}}\left(\sigma_x\left(-\frac{\beta}{2}\right)\right)\mathbf{1}_{\{\xi_x^+\}}\left(\sigma_x\left(\frac{\beta}{2}\right)\right)\ , \tag{11}$$

where $\eta\in\mathcal{D}_I^{\Lambda^c}, \xi^+,\xi^-\in\Omega_\Lambda:=\{-1,1\}^\Lambda$ and $Z_{\Lambda,I}\left(\eta;\xi^+,\xi^-\right)$ is the normalizing constant.

In [DLP] it has been shown that the expected value of an observable $\mathbf{F}\in\mathfrak{A}_\Lambda$ in the equilibrium (KMS) state of the ferromagnetic quantum Ising model with transverse field at inverse temperature $\beta>0$ can be represented as the expected value w.r.t. the Gibbs distribution (11) with periodic b.c.'s at $t=\pm\frac{\beta}{2}$ of the function F on Ω_Λ corresponding to the spectral representation of $\mathbf{F}$ computed at $\sigma_\Lambda\left(0\right)\in\Omega_\Lambda$. Namely

$$\frac{tr_{\mathcal{H}_\Lambda}\left(e^{-\beta\mathbf{H}_\Lambda(h)}\mathbf{F}\right)}{tr_{\mathcal{H}_\Lambda}\left(e^{-\beta\mathbf{H}_\Lambda(h)}\right)}=\frac{tr_{\mathcal{H}_\Lambda}\left(e^{-\frac{\beta}{2}\mathbf{H}_\Lambda(h)}\mathbf{F}e^{-\frac{\beta}{2}\mathbf{H}_\Lambda(h)}\right)}{tr_{\mathcal{H}_\Lambda}\left(e^{-\beta\mathbf{H}_\Lambda(h)}\right)}=\nu_{I,\Lambda}^\eta\left(F\left(\sigma_\Lambda\left(0\right)\right)\right)\ , \tag{12}$$

where

$$\frac{\mathrm{d}\nu_{I,\Lambda}^\eta}{\mathrm{d}\mu_{I,\Lambda}^p}:=Z_{\Lambda,I}^{-1}\left(\eta\right)\exp\left[J\sum_{x,y\in\Lambda\,:\,\{x,y\}\in\mathbb{E}}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}\mathrm{d}t\sigma_x\left(t\right)\sigma_y\left(t\right)+J\sum_{x\in\Lambda}\sum_{y\in\partial\Lambda}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}\mathrm{d}t\sigma_x\left(t\right)\eta_y\left(t\right)\right] \tag{13}$$

is the density of the conditional Gibbs measure w.r.t. $\mu_{I,\Lambda}^p:=\left(\mu_I^p\right)^{\otimes\Lambda}$. Clearly the b.c. $\eta\in\mathcal{D}_I^{\Lambda^c}$ can be thought of as a time varying local external field in the direction of the spin field (see also [CKP] and [KL] for a general discussion).

Correlation inequalities imply that the expected value of local observables of the form $\prod_{x\in\Lambda}\sigma_x^{(z)}, \Lambda\subset\subset\mathbb{Z}$, in the ground state of the ferromagnetic quantum Ising model with transverse field can be computed from $\nu_{I_n,\Lambda_m}^\eta\left[\prod_{x\in\Lambda}\sigma_x\left(0\right)\right]$ by taking first the limit through an exhaustion $\{I_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}$ (i.e. $\beta\to\infty$) and then the limit $\{\Lambda_m\}_{m\in\mathbb{N}}\uparrow\mathbb{Z}$ (see also [CKP] Section 2). For sufficiently large values of the external field h, these limits can be shown to be independent of the b.c.'s by using the cluster expansion carried out in the next section.

Therefore, in the following, we will consider fixed b.c.'s at $\left\{(x,t)\in\Lambda\times I: t=\pm\frac{\beta}{2}\right\}$, free b.c.'s at $\{(x,t)\in\Lambda^c\times I\}$ and assume that β is a multiple of δ.

2 Cluster expansion

We perform a cluster expansion on the model and verify that, when h is sufficiently large, we can ensure that, for a suitable choice of the parameters, the condition of Kotecký and Preiss [KP] are satisfied and the cluster expansion is therefore convergent.

We stress that the following argument applies to a more general setup in which the model is defined on $\mathbb{Z}^d$, with $d \geq 1$, and the coupling between any pair of spins are bounded. We also remark that the requirement for the two-body interactions to be ferromagnetic is needed in order to guarantee, by means of correlation inequalities, the existence of the ground state, while translation-invariance and finite-rangeness are sufficient conditions for the existence of thermodynamics.

Given δ to be fixed later, we partition the trajectory of any spin-flip process $(\sigma_x(t), t \in I), x \in \Lambda$, into blocks of size δ (fig. 1). We will call the last coordinate of the vector in $\mathbb{R}^2$ corresponding to a point in $\mathbb{Z} \times \delta\mathbb{Z}$ the *vertical component*. Then, denoting an element x of $\mathbb{Z} \times \delta\mathbb{Z}$ by $x = (x_1, \delta x_2)$, we denote by $\mathbb{L}^2_\delta$ the graph whose set of vertices is $\mathbb{Z} \times \delta\mathbb{Z}$ and whose set of edges is $\mathbb{E}^2_\delta := \{\{x, y\} \in \mathcal{P}_2(\mathbb{Z} \times \delta\mathbb{Z}) : |x_1 - y_1| + |x_2 - y_2| = 1\}$.

Let us set $\mathbb{V} := \{\{x, y\} \in \mathbb{E}^2_\delta : x_1 = y_1\}$ the set of vertical edges in $\mathbb{L}^2_\delta$ and by $\mathbb{O} := \mathbb{E}^2_\delta \backslash \mathbb{V}$. Denoting by $\Delta := \Lambda \times (\delta\mathbb{Z} \cap I)$ we define $\mathbb{O}_\Delta := \left\{e \in \mathbb{O} : V_e \subset \left(\Delta \backslash \overline{\partial}\Delta\right)\right\}$ and $\mathbb{V}_\Delta := \{e \in \mathbb{V} : V_e \subset \Delta\}$. Moreover, we define

$$\partial^{\pm}\Delta := \left\{(x_1, \delta x_2) \in \Delta : x_1 \in \Lambda, \delta x_2 = \pm\frac{\beta}{2}\right\} \tag{14}$$

and set $\overline{\partial}\Delta := \partial^+\Delta \cup \partial^-\Delta$ and

$$\partial\Delta := \overline{\partial}\Delta \cup \{x \in \mathbb{Z} \times \delta\mathbb{Z} : x_1 \in \partial\Lambda, x_2 \in \delta\mathbb{Z} \cap I\} \ . \tag{15}$$

Then, denoting by $\Omega_D := \{-1, 1\}^D$, for any $D \subset \mathbb{Z} \times \delta\mathbb{Z}$, assuming b.c. $\xi = (\xi^+, \xi^-) \in \Omega_{\overline{\partial}\Delta} := \Omega_{\partial^+\Delta} \times \Omega_{\partial^-\Delta}$ at $\overline{\partial}\Delta$, with ξ^+, ξ^- appearing in (11), and free b.c.'s at $\partial\Delta \backslash \overline{\partial}\Delta$, we have

$$\begin{aligned} Z_\Delta(\xi) := Z_{\Lambda,I}\left(\xi^+, \xi^-\right) = \int \bigotimes_{z_1 \in \Lambda} \mu_I(\mathrm{d}\sigma_{z_1})\, e^{\sum_{x,y \in \Delta \,:\, \{x,y\} \in \mathbb{O}} W(\sigma_x, \sigma_y)} \times \\ \times \prod_{z_1 \in \Lambda} \mathbf{1}_{\{\xi^-_{z_1}\}}\left(\sigma_{z_1}\left(-\frac{\beta}{2}\right)\right) \mathbf{1}_{\{\xi^+_{z_1}\}}\left(\sigma_{z_1}\left(\frac{\beta}{2}\right)\right) \ , \end{aligned} \tag{16}$$

where, for any $x, y \in \Delta \backslash \overline{\partial}\Delta$,

$$W\left(\sigma_x, \sigma_y\right) = J \int_0^\delta dt \sigma_{x_1}\left(\delta x_2 + t\right) \sigma_{y_1}\left(\delta y_2 + t\right) \ . \tag{17}$$

Setting

$$e^{W(\sigma_x, \sigma_y)} = 1 + \left[e^{W(\sigma_x, \sigma_y)} - 1\right] \ , \tag{18}$$

$Z_\Delta\left(\xi\right)$ can be rewritten as

$$\begin{aligned} Z_\Delta\left(\xi\right) = & \sum_{\ell \in \mathcal{P}(\mathbb{O}_\Delta)} \int \bigotimes_{z_1 \in \Lambda} \mu_I\left(\mathrm{d}\sigma_{z_1}\right) \prod_{e \in \ell} \left[e^{\mathbf{1}_e(\{x,y\}) W(\sigma_x, \sigma_y)} - 1\right] \times \\ & \times \mathbf{1}_{\{\xi^-\}}\left(\sigma\left(-\frac{\beta}{2}\right)\right) \mathbf{1}_{\{\xi^+\}}\left(\sigma\left(\frac{\beta}{2}\right)\right) \\ = & \sum_{\ell \in \mathcal{P}(\mathbb{O}_\Delta)} \int \bigotimes_{z_1 \in \Lambda} \mu_I\left(\mathrm{d}\sigma_{z_1}\right) \prod_{e \in \ell} \mathbf{1}_e\left(\{x, y\}\right)\left(e^{W(\sigma_x, \sigma_y)} - 1\right) \times \\ & \times \mathbf{1}_{\{\xi^-\}}\left(\sigma\left(-\frac{\beta}{2}\right)\right) \mathbf{1}_{\{\xi^+\}}\left(\sigma\left(\frac{\beta}{2}\right)\right) \ . \end{aligned} \tag{19}$$

Given $\ell \in \mathcal{P}\left(\mathbb{O}_\Delta\right)$, for any $x_1 \in \Lambda$ we can integrate over the trajectories of the stationary process $\left(\sigma_{x_1}\left(t\right), t \in I\right)$ keeping fixed its values at δx_2 if $\left(x_1, \delta x_2\right) \in \ell$. This integral can be computed explicitely. Indeed, setting $V\left(\ell\right) := \left(\bigcup_{e \in \ell} V_e\right)$ and denoting by

$$V'\left(\ell\right) := \bigcup_{e \in \ell} \left\{z \in \Delta \cup \overline{\partial}\Delta : z_1 = x_1, z_2 = x_2 + 1, \left(x_1, \delta x_2\right) \in V_e\right\} \ , \tag{20}$$

we have

$$\begin{aligned} & \mu_{I,\Lambda}^\xi\left[\prod_{e \in \ell} \mathbf{1}_e\left(\{x, y\}\right)\left(e^{W(\sigma_x, \sigma_y)} - 1\right)\right] = \\ & \mu_{I,\Lambda}^\xi\left[\mu_{I,\Lambda}^\xi\left[\prod_{e \in \ell} \mathbf{1}_e\left(\{x, y\}\right)\left(e^{W(\sigma_x, \sigma_y)} - 1\right) \mid \left\{\sigma_{x_1}\left(\delta x_2\right)\right\}_{(x_1, \delta x_2) \in V(\ell) \cup V'(\ell)}\right]\right] \ , \end{aligned} \tag{21}$$

where we have set $\mu_{I,\Lambda}^\xi := \mu_{I,\Lambda}\left[\cdot | \sigma\left(\pm\frac{\beta}{2}\right) = \xi^\pm\right]$.

If $\left(x_1, \delta x_2\right), \left(x_1, \delta y_2\right) \in V\left(\ell\right)$ such that $y_2 \geq x_2 + 2$, and there is no other $\left(x_1, \delta z_2\right) \in V\left(\ell\right)$ such that $x_2 + 2 \leq z_2 \leq y_2 - 1$, we can integrate over the trajectories of $\left(\sigma_{x_1}\left(t\right), t \in I\right)$ with

given values at $t=\delta x_2+\delta,\delta y_2$. Let

$$x_2^{(1)}:=\min\left\{z_2\in\mathbb{Z}:(x_1,\delta z_2)\in V(\ell)\right\}\ , \tag{22}$$

$$x_2^{(i+1)}:=\min\left\{z_2\in\mathbb{Z}:(x_1,\delta z_2)\in V(\ell)\backslash\bigcup_{j=1}^{i}\left(x_1,\delta x_2^{(j)}\right)\right\}\ ,\ i\geq 1\ . \tag{23}$$

Then, $T_\ell(x_1):=\{z_2\in\mathbb{Z}:(x_1,\delta z_2)\in V(\ell)\}$ can be represented as the ordered set $T_\ell(x_1)=\left\{x_2^{(1)},..,x_2^{(|T_\ell(x_i)|)}\right\}$. For any $i=1,..,|T_\ell(x_1)|$, we denote by

$$y_2^{(i)}:=\left\{z_2\in\mathbb{Z}:(x_1,\delta z_2)\in\Delta\backslash V(\ell),z_2=x_2^{(i)}+1\right\} \tag{24}$$

and set

$$\overline{T}_\ell(x_1):=\left\{x_2^{(1)},..,x_2^{(|V_\ell(x_i)|)},x_2^{(|V_\ell(x_i)|+1)}:=\frac{\beta}{2\delta}\right\}\ , \tag{25}$$

$$\Gamma_\ell(x_1):=\left\{-\frac{\beta}{2\delta}=:y_2^{(0)},y_2^{(1)},..,y_2^{(|V_\ell(x_1)|)}\right\}\ . \tag{26}$$

Hence, denoting by

$$V_\ell(x_1):=\bigcup_{x_2\in T_\ell(x_1)}\{x\in\Delta:x=(x_1,\delta x_2)\}\ , \tag{27}$$

we get

$$\int\mu_I^{\xi_{x_1}}\left(\mathrm{d}\sigma_{x_1}|\left\{\sigma_{x_1}(\delta x_2)\right\}_{x_2\in T_{x_1}(\ell)\cup\Gamma_{x_1}(\ell)}\right)\prod_{e\in\ell\ :\ V_e\cap V_\ell(x_1)\neq\varnothing}\mathbf{1}_e(\{x,y\})\left(e^{W(\sigma_x,\sigma_y)}-1\right)= \tag{28}$$

$$\int\bigotimes_{i=1}^{|V_\ell(x_1)|}\mu_I^{\xi_{x_1}}\left(\mathrm{d}\sigma_{x_1}|\sigma_{x_1}\left(\delta x_2^{(i)}\right),\sigma_{x_1}\left(\delta y_2^{(i)}\right)\right)\prod_{e\in\ell\ :\ V_e\cap V_\ell(x_1)\neq\varnothing}\mathbf{1}_e(\{x,y\})\left(e^{W(\sigma_x,\sigma_y)}-1\right)\times$$

$$\times\prod_{i=0}^{|T_\ell(x_1)|}\frac{1+\sigma_{x_1}\left(\delta y_2^{(i)}\right)\sigma_{x_1}\left(\delta x_2^{(i+1)}\right)e^{-2h\delta\left(x_2^{(i+1)}-y_2^{(i)}\right)}}{2}\ ,$$

where we have used that, given $x_1\in\Lambda$, for any $t,s\in I$ with $t>s,\eta,\eta'\in\{-1,1\}$,

$$\mu_I\left[\mathbf{1}_{\{\eta'\}}(\sigma_x(t))|\sigma_x(s)=\eta\right]=\frac{1+\eta'\eta e^{-2h(t-s)}}{2}=\begin{cases}\frac{1+e^{-2h(y_2-x_2)}}{2} & \text{if } \eta'=\eta\\ \frac{1-e^{-2h(t-s)}}{2} & \text{if } \eta'=-\eta\end{cases}\ . \tag{29}$$

Therefore, setting

$$\Lambda(\ell):=\{x_1\in\Lambda:|V_\ell(x_1)|\geq 1\}\ , \tag{30}$$

since $\mu^{\xi}_{\Lambda,I} = \bigotimes_{x_1\in\Lambda(\ell)} \mu_I^{\xi_{x_1}}$, we obtain

$$Z_\Delta(\xi) = \sum_{\ell\in\mathcal{P}(\mathbb{O}_\Delta)} \int \bigotimes_{x_1\in\Lambda(\ell)} \mu_I^{\xi_{x_1}} \left(\mathrm{d}\sigma_{x_1} | \left\{\sigma_{x_1}(\delta x_2)\right\}_{x_2\in T_{x_1}(\ell)\cup\Gamma_{x_1}(\ell)} \right) \times \tag{31}$$
$$\times \prod_{x_1\in\Lambda\setminus\Lambda(\ell)} \frac{1+\xi^+_{x_1}\xi^-_{x_1}e^{-2h\beta}}{2} \prod_{e\in\ell} \mathbf{1}_e(\{x,y\}) \left(e^{W(\sigma_x,\sigma_y)}-1\right) \times$$
$$\times \prod_{i=0}^{|T_\ell(x_1)|} \frac{1+\sigma_{x_1}\left(\delta y_2^{(i)}\right)\sigma_{x_1}\left(\delta x_2^{(i+1)}\right)e^{-2h\delta\left(x_2^{(i+1)}-y_2^{(i)}\right)}}{2} .$$

It can be useful to represent $Z_\Delta(\xi)$ as the partition function of a classical spin system. Indeed, we can consider a classical spin system on $\mathbb{Z}\times\delta\mathbb{Z}$ by associating to any lattice point $(x_1,\delta x_2)\in\mathbb{Z}\times\delta\mathbb{Z}$ a random element, which we will still call *spin*, taking values in the space $\mathcal{D}_\delta$ of piecewise $\{-1,1\}$-valued functions on $[0,\delta]$ endowed with the Skorokhod topology, namely

$$\mathcal{D}_\delta := \{\sigma\in\mathbb{D}([0,\delta],\{-1,1\})\} . \tag{32}$$

Setting $\mathcal{S} := \mathcal{D}_\delta^{\mathbb{Z}}$, we denote by $\mathbf{S}$ the injection of $\mathcal{D}$ in $\mathcal{S}$ such that

$$\mathcal{D}\ni\sigma\longmapsto\mathbf{S}(\sigma) := \left\{\sigma^{(k)}\right\}_{k\in\mathbb{Z}}\in\mathcal{S} , \tag{33}$$

where $\forall k\in\mathbb{Z}, \sigma^{(k)}$ denotes the element of $\mathcal{D}_\delta$ representing the function $[0,\delta]\ni t\longmapsto\sigma^{(k)}(t) := \sigma(k\delta+t)\in\{-1,1\}$. Equipping $\mathcal{S}$ with the product topology, the push-forward of μ w.r.t. $\mathbf{S}$ on $(\mathcal{S},\mathfrak{S})$ with $\mathfrak{S}$ the product σ-algebra can be written as

$$\mu\circ\mathbf{S}^{-1}\left(\mathrm{d}\left\{\sigma^{(k)}\right\}_{k\in\mathbb{Z}}\right) = 2^{\frac{\beta}{\delta}-1}\bigotimes_{k\in\mathbb{Z}}\mu^\delta\left(\mathrm{d}\sigma^{(k)}\right)\prod_{k\in\mathbb{Z}}\delta_{\sigma^{(k)}(\delta),\sigma^{(k+1)}(0)} , \tag{34}$$

with

$$\mu^\delta\left(\mathrm{d}\sigma^{(k)}\right) := \mu_{[\delta k,\delta(k+1)]}(\mathrm{d}\sigma) , \ k\in\mathbb{Z} . \tag{35}$$

For any $x=(x_1,\delta x_2)\in\Delta$, with a little abuse of notation we denote by σ_x the element of $\mathcal{D}_\delta$ representing the function $[0,\delta]\ni t\longmapsto\sigma_{x_1}(\delta x_2+t)\in\{-1,1\}$. Hence, we denote by $\mathcal{S}_\Delta := \mathcal{D}_\delta^\Delta$ and by $\mathfrak{S}_\Delta := \{A\cap\mathcal{S}_\Delta : A\in\mathfrak{S}\}$. In particular, we can represent the Gibbs probability measure $\nu_{I,\Lambda}$ on $\left(\mathcal{D}_I^\Lambda,\mathcal{F}_I^\Lambda\right)$ specified by (11), with fixed b.c. $\xi\in\Omega_{\overline{\partial}\Delta}$ at $\overline{\partial}\Delta$ and

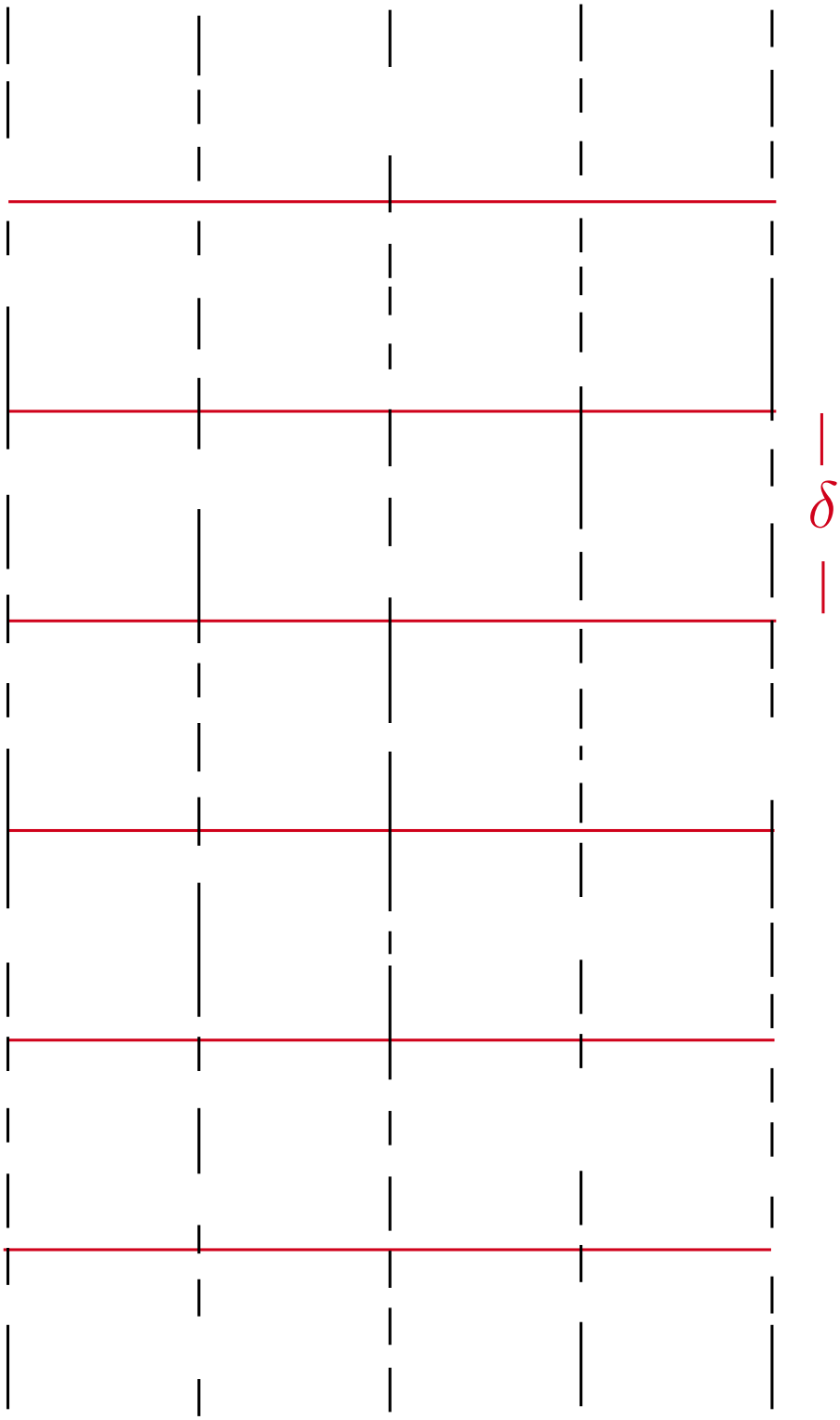

Figure 1: The construction of the *spins* in $\mathcal{D}_\delta$.

free b.c.'s at $\partial\Delta\backslash\overline{\partial}\Delta$, by the Gibbs probability measure $\nu_\delta^\xi\left(\mathrm{d}\sigma_\Delta\right)$ on $\left(\mathcal{S}_\Delta,\mathfrak{S}_\Delta\right)$ specified by the density

$$Z_\delta^{-1}\left(\xi\right)\exp\left[\sum_{x,y\in\Delta}\left(W_1\left(\sigma_x,\sigma_y\right)+\mathbf{1}_{\mathbb{O}_\Delta}\left(\{x,y\}\right)W\left(\sigma_x,\sigma_y\right)\right)\right]\mathbf{1}_{\{\xi^-\}}\left(\sigma_{\partial^-\Delta}\right)\mathbf{1}_{\{\xi^+\}}\left(\sigma_{\partial^+\Delta}\right) \tag{36}$$

w.r.t. the reference measure $\mu^\delta\left(\mathrm{d}\sigma_\Delta\right):=\bigotimes_{x\in\Delta}\mu^\delta\left(\mathrm{d}\sigma_x\right)$, associated to the interaction W_1+W, where, in view of the fact that, by (34), for any $x_1,x_2\in\mathbb{Z}$, the spins $\sigma_{(x_1,\delta x_2)},\sigma_{(x_1,\delta x_2+\delta)}\in\mathcal{D}_\delta$ must satisfy the compatibility condition $\sigma_{(x_1,\delta x_2)}\left(\delta\right)=\sigma_{(x_1,\delta x_2+\delta)}\left(0\right)$,

1. $W_1\left(\sigma_x,\sigma_y\right)=0$ if $\{x,y\}\in\mathbb{V}$ and if $x_2<y_2,\sigma_x\left(\delta\right)=\sigma_y\left(0\right)$ or if $y_2<x_2,\sigma_y\left(\delta\right)=\sigma_x\left(0\right)$;
2. $W_1\left(\sigma_x,\sigma_y\right)=-\infty$ if $\{x,y\}\in\mathbb{V}$ and if $x_2<y_2,\sigma_x\left(\delta\right)\neq\sigma_y\left(0\right)$ or if $y_2<x_2,\sigma_y\left(\delta\right)\neq\sigma_x\left(0\right)$.

Then, by the definition of the potential W_1,

$$e^{W_1(\sigma_x,\sigma_y)}=\left[\delta_{x_1,y_1}\left(\delta_{y_2,x_2+1}\delta_{\sigma_x(\delta),\sigma_y(0)}+\delta_{x_2,y_2+1}\delta_{\sigma_y(\delta),\sigma_x(0)}\right)+\left(1-\delta_{x_1,y_1}\right)\right]\ . \tag{37}$$

Hence,

$$\begin{aligned}Z_\Delta\left(\xi\right)=&\sum_{\ell\in\mathcal{P}(\mathbb{O}_\Delta)}\prod_{x_1\in\Lambda(\ell)}2^{|T_\ell(x_1)|}\int\bigotimes_{x_2\in T_\ell(x_1)}\mu^\delta\left(\mathrm{d}\sigma_{(x_1,\delta x_2)}\right)\prod_{x_2\in T_\ell(x_1)}e^{W_1\left(\sigma_{(x_1,\delta x_2)},\sigma_{(x_1,\delta x_2+\delta)}\right)}\times \\ &\times\prod_{e\in\ell}\mathbf{1}_e\left(\{x,y\}\right)\left(e^{W(\sigma_x,\sigma_y)}-1\right)\prod_{(x,y)\in\partial^+\Delta\times\partial^-\Delta\ :\ x_1=y_1\ ,\ x_1\in\Lambda\backslash\Lambda(\ell)}\frac{1+\xi_x\xi_y e^{-2h\beta}}{2}\times \\ &\times\prod_{i=0}^{|T_\ell(x_1)|}\frac{1+\sigma_{\left(x_1,\delta y_2^{(i)}\right)}\sigma_{\left(x_1,\delta x_2^{(i+1)}\right)}e^{-2h\delta\left(x_2^{(i+1)}-y_2^{(i)}\right)}}{2}\mathbf{1}_{\{\xi_{x_1}^-\}}\left(\sigma_{\left(x_1,\delta y_2^{(0)}\right)}\right)\mathbf{1}_{\{\xi_{x_1}^+\}}\left(\sigma_{\left(x_1,\delta y_2^{(|T_\ell(x_1)|+1)}\right)}\right)\ .\end{aligned} \tag{38}$$

Let us denote by $\Pi\left(\ell\right)$ the set of paths in $\left(\Delta,\mathbb{V}_\Delta\right)$ connecting any couple of points $\left(y,x\right)\in V\left(\ell\right)\cup\partial^-\Delta\times V\left(\ell\right)\cup\partial^+\Delta$ such that:

- if $y=\left(x_1,y_2\right)$ with $x_1\in\Lambda\left(\ell\right),x=\left(x_1,x_2\left(y\right)\right)$ with $x_2\left(y\right):=\min\left\{z_2\in\overline{T}_\ell\left(x_1\right):z_2\geq y_2+2\right\}$;
- if $y=\left(x_1,-\frac{\beta}{2}\right)$ with $x_1\in\Lambda\backslash\Lambda\left(\ell\right),x=\left(x_1,\frac{\beta}{2}\right)$.

Hence, we can write

$$
\begin{aligned}
Z_{\Delta}(\xi) = & \sum_{\ell \in \mathcal{P}(\mathbb{O}_{\Delta})} 2^{|V(\ell)|} \int \bigotimes_{x \in V(\ell)} \mu^{\delta}(\mathrm{d}\sigma_x) \prod_{\{x,y\} \in \mathbb{V}_{\Delta} \,:\, x \in V(\ell), y \in V'(\ell)} e^{W_1(\sigma_x, \sigma_y)} \times \\
& \times \prod_{e \in \ell} \mathbf{1}_e(\{x,y\}) \left(e^{W(\sigma_x,\sigma_y)} - 1\right) \times \\
& \times \prod_{\gamma \in \Pi(\ell)} \mathbf{1}_{end(\gamma)}(\{x,y\}) \frac{1 + \sigma_x \sigma_y e^{-2h\delta|x-y|}}{2} \times \\
& \times \left[(1 - \mathbf{1}_{\partial^+\Delta}(x)) + \mathbf{1}_{\partial^+\Delta}(x)\, \mathbf{1}_{\xi_x^+}(\sigma_x)\right] \left[(1 - \mathbf{1}_{\partial^-\Delta}(y)) + \mathbf{1}_{\partial^-\Delta}(y)\, \mathbf{1}_{\xi_y^-}(\sigma_y)\right] .
\end{aligned}
\tag{39}
$$

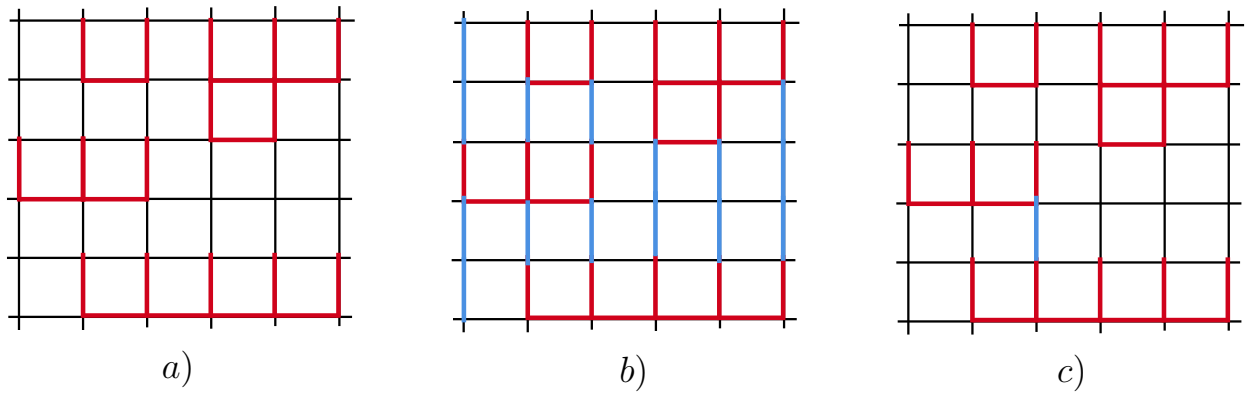

Figure 2: a) A subset ℓ of $\mathbb{O}_{\Delta}$ and the corresponding $G_e, e \in \ell$, b) the paths in $(\Delta, \mathbb{V}_{\Delta})$ connecting the graphs $G_e, e \in \ell$, c) three polymers associated to ℓ.

2.1 Reduction to a polymer gas model

Given $e \in \mathbb{O}$, let

$$
V'(e) := \{x \in \mathbb{Z} \times \delta\mathbb{Z} : (x_1, \delta(x_2 - 1)) \in V_e\} \tag{40}
$$

and

$$
E'(e) := \{e' \in \mathbb{V} : e' = \{x,y\}, x \in V_e, y \in V'(e)\} \ . \tag{41}
$$

We set $G_e := (V_e \cup V'(e), e \cup E'(e)) \subset \mathbb{L}^2_{\delta}$.

We call *polymer* a connected subgraph R of $\mathbb{L}^2_{\delta}$ which satisfies the following conditions:

1. for any $e \in E(R) \cap \mathbb{O}, G_e \subseteq R$;

2. if e and e' are two distinct edges in $E(R) \cap \mathbb{O}$, either $G_e \cup G_{e'}$ is a connected subgraph of $\mathbb{L}^2_\delta$ or, given a path γ connecting G_e and $G_{e'}$, for any $e'' \in E(\gamma) \cap \mathbb{O}, G_{e''} \subset R$.

Given a polymer R (an example is a connected subgraph of the graph in fig.2 c)) we set $\|R\| := |E(R)|$. Denoting by $\mathfrak{R}$ the set of polymers, $R, R' \in \mathfrak{R}$ are said to be *compatible*, and we write $R \sim R'$, if $V(R) \cap V(R') = \varnothing$, otherwise are said to be *incompatible* and we write $R \nsim R'$. Given $\mathcal{R} \subset \mathfrak{R}$, we denote by $\mathfrak{P}(\mathcal{R})$ the collection of the subsets of $\mathcal{R}$ consisting of mutually compatible polymers and by $\mathfrak{P}_0(\mathcal{R}) := \{\varrho \in \mathfrak{P}(\mathcal{R}) : \|\varrho\| < \infty\}$. We also set $\mathfrak{P} := \mathfrak{P}(\mathfrak{R}), \mathfrak{P}_0 := \mathfrak{P}_0(\mathfrak{R})$. Given $\mathcal{R} \in \mathcal{P}_f(\mathfrak{R})$ and $R \in \mathfrak{R}$ we write $\mathcal{R} \nsim R$ if there exists $R' \in \mathcal{R}$ such that $R' \nsim R$. Moreover, we call $\mathcal{R}$ a *polymer cluster* if it cannot be decomposed as a union of $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{P}_f(\mathfrak{R})$ such that every pair $R_1 \in \mathcal{R}_1, R_2 \in \mathcal{R}_2$ is compatible. We denote by $\mathcal{C}(\mathcal{R})$ the collection of polymer clusters in $\mathcal{R}$ and let $\mathcal{C}$ be the collection of polymer clusters in $\mathfrak{R}$.

Given a finite $\Delta := \Lambda \times I \subset \mathbb{Z} \times \delta\mathbb{Z}$ we denote by

$$V^+_\Delta := \bigcup_{e \in \mathbb{O} \,:\, V_e \subset \partial^+\Delta} V'(e) \tag{42}$$

and set $\mathfrak{R}_\Delta$ the collection of polymers $R \in \mathfrak{R}$ such that:

- $V(R) \subseteq \Delta \cup V^+_\Delta$;
- if $V(R) \cap \overline{\partial}\Delta \neq \varnothing$ then either $\partial^+\Delta$ or $\partial^-\Delta$ or $\overline{\partial}\Delta = \partial^+\Delta \cup \partial^-\Delta$ are contained in $V(R)$.

We also set $\mathfrak{P}_\Delta := \mathfrak{P}(\mathfrak{R}_\Delta)$. Then, for any $\mathcal{R} \subseteq \mathfrak{R}_\Delta$, we define

$$\mathcal{Z}\left(\mathcal{R}, \Phi^{h,\xi}\right) := \sum_{\varrho \in \mathfrak{P}(\mathcal{R})} \prod_{R \in \varrho} \Phi^{h,\xi}(R) \ , \tag{43}$$

where the function $\mathfrak{R} \ni R \longmapsto \Phi^{h,\xi}(R) \in \mathbb{R}^+$ is the *activity* of the polymer.

By (39),

$$
\begin{aligned}
Z_{\Delta}(\xi) = & \sum_{\ell \in \mathcal{P}(\mathbb{O}_{\Delta})} 2^{|V(\ell)|} \int \bigotimes_{x \in V(\ell)} \mu^{\delta}(\mathrm{d}\sigma_x) \prod_{\{x,y\} \in \mathbb{V}_{\Delta} \,:\, x \in V(\ell), y \in V'(\ell)} e^{W_1(\sigma_x,\sigma_y)} \times \\
& \times \prod_{e \in \ell} \mathbf{1}_e(\{x,y\}) \left(e^{W(\sigma_x,\sigma_y)} - 1\right) \times \\
& \times \sum_{\mathfrak{g} \in \mathcal{P}(\Pi(\ell))} \left(\frac{1}{2}\right)^{|\Pi(\ell) \setminus \mathfrak{g}|} \prod_{\gamma \in \mathfrak{g}} \mathbf{1}_{end(\gamma)}(\{x,y\}) \sigma_x \sigma_y \frac{e^{-2h\delta|E(\gamma)|}}{2} \times \\
& \times \left[(1 - \mathbf{1}_{\partial^+\Delta}(x)) + \mathbf{1}_{\partial^+\Delta}(x) \mathbf{1}_{\xi_x^+}(\sigma_x)\right] \left[(1 - \mathbf{1}_{\partial^-\Delta}(y)) + \mathbf{1}_{\partial^-\Delta}(y) \mathbf{1}_{\xi_y^-}(\sigma_y)\right] .
\end{aligned}
\tag{44}
$$

Then, given $\ell \in \mathcal{P}(\mathbb{O}_{\Delta})$ and $\mathfrak{g} \in \mathcal{P}(\Pi(\ell))$, the components of $\varrho(\ell) := \left(\bigcup_{e \in \overline{\ell}} G_e\right) \cup \left(\bigcup_{\gamma \in \mathfrak{g}} \gamma\right)$, with $\overline{\ell} := \ell \cup \left\{e \in \mathbb{O} \,:\, V_e \subset \overline{\partial}\Delta\right\}$, fit the definition of polymer, hence we can write

$$
\begin{aligned}
Z_{\Delta}(\xi) = & \sum_{\ell \in \mathcal{P}(\mathbb{O}_{\Delta})} \sum_{\varrho \in \mathfrak{P}_{\ell}} \prod_{R \in \varrho} 2^{|U_R|} \int \mu^{\delta}(\mathrm{d}\sigma_{U_R}) \prod_{\{x,y\} \in \mathbb{V}_{\Delta} \,:\, x \in U_R, y \in U'_R} e^{W_1(\sigma_x,\sigma_y)} \times \\
& \times \prod_{e \in E(R) \cap \mathbb{O}_{\Delta}} \mathbf{1}_e(\{x,y\}) \left(e^{W(\sigma_x,\sigma_y)} - 1\right) \times \\
& \times \prod_{\gamma \in \mathfrak{g}(R)} \mathbf{1}_{end(\gamma)}(\{x,y\}) \sigma_x \sigma_y \frac{e^{-2h\delta|E(\gamma)|}}{2} \prod_{\gamma \in \Pi(\ell) \setminus \mathfrak{g}(R)} \mathbf{1}_{end(\gamma)}(\{x,y\}) \frac{1}{2} \times \\
& \times \left[(1 - \mathbf{1}_{\partial^+\Delta}(x)) + \mathbf{1}_{\partial^+\Delta}(x) \mathbf{1}_{\xi_x^+}(\sigma_x)\right] \left[(1 - \mathbf{1}_{\partial^-\Delta}(y)) + \mathbf{1}_{\partial^-\Delta}(y) \mathbf{1}_{\xi_y^-}(\sigma_y)\right] ,
\end{aligned}
\tag{45}
$$

where, for any $\ell \in \mathcal{P}(\mathbb{O}_{\Delta})$, $\mathfrak{P}_{\ell}$ is the set of the collections of mutually compatible polymers which can be realised as union set of $\bigcup_{e \in \overline{\ell}} G_e$ with elements of $\Pi(\ell)$ and, for any polymer R in $\varrho \in \mathfrak{P}_{\ell}$, $U_R := V(\ell) \cap V(R)$, $U'_R := V'(\ell) \cap V(R)$ and $\mathfrak{g}(R) := \{\gamma \in \Pi(\ell) : \gamma \subset R\}$.

Given $\varrho \in \mathfrak{P}_{\Delta}$, setting $\ell(\varrho) := E(\varrho)$ and consequently $V(\varrho) := V(\ell(\rho))$ and $\Pi(\varrho) :=$

$\Pi\left(\ell\left(\rho\right)\right), Z_{\Delta}\left(\xi\right)$ can be rewritten as

$$
\begin{aligned}
Z_{\Delta}\left(\xi\right) = & \sum_{\varrho\in\mathfrak{P}_{\Delta}}\prod_{R\in\varrho}2^{|U_R|}\int\mu^{\delta}\left(\mathrm{d}\sigma_{U_R}\right)\prod_{\{x,y\}\in\mathbb{V}_{\Delta}\,:\,x\in U_R,y\in U'_R}e^{W_1(\sigma_x,\sigma_y)}\times \\
& \times\prod_{e\in E(R)\cap\mathbb{O}_{\Delta}}\mathbf{1}_e\left(\{x,y\}\right)\left(e^{W(\sigma_x,\sigma_y)}-1\right)\times \\
& \times\left(\frac{1}{2}\right)^{|\Pi(\varrho)\setminus\mathfrak{g}(R)|}\prod_{\gamma\in\mathfrak{g}(R)}\mathbf{1}_{end(\gamma)}\left(\{x,y\}\right)\sigma_x\sigma_y\frac{e^{-2h\delta|E(\gamma)|}}{2}\times \\
& \times\left[\left(1-\mathbf{1}_{\partial^+\Delta}\left(x\right)\right)+\mathbf{1}_{\partial^+\Delta}\left(x\right)\mathbf{1}_{\xi_x^+}\left(\sigma_x\right)\right]\left[\left(1-\mathbf{1}_{\partial^-\Delta}\left(y\right)\right)+\mathbf{1}_{\partial^-\Delta}\left(y\right)\mathbf{1}_{\xi_y^-}\left(\sigma_y\right)\right]\ .
\end{aligned}
\tag{46}
$$

Hence,

$$
Z_{\Delta}\left(\xi\right)=\mathcal{Z}\left(\mathfrak{R}_{\Delta},\Phi^{h,\xi}\right)=\sum_{\varrho\in\mathfrak{P}_{\Delta}}\prod_{R\in\varrho}\Phi^{h,\xi}\left(R\right)\ , \tag{47}
$$

with

$$
\begin{aligned}
\Phi^{h,\xi}\left(R\right) := & 2^{|U_R|}\int\mu^{\delta}\left(\mathrm{d}\sigma_{U_R}\right)\prod_{\{x,y\}\in\mathbb{V}_{\Delta}\,:\,x\in U_R,y\in U'_R}e^{W_1(\sigma_x,\sigma_y)}\times \\
& \times\prod_{e\in E(R)\cap\mathbb{O}}\mathbf{1}_e\left(\{x,y\}\right)\left(e^{W(\sigma_x,\sigma_y)}-1\right)\times \\
& \times\left(\frac{1}{2}\right)^{|\Pi(\varrho)\setminus\mathfrak{g}(R)|}\prod_{\gamma\in\mathfrak{g}(R)}\mathbf{1}_{end(\gamma)}\left(\{x,y\}\right)\sigma_x\sigma_y\frac{e^{-2h\delta|E(\gamma)|}}{2}\times \\
& \times\left[\left(1-\mathbf{1}_{\partial^+\Delta}\left(x\right)\right)+\mathbf{1}_{\partial^+\Delta}\left(x\right)\mathbf{1}_{\xi_x^+}\left(\sigma_x\right)\right]\left[\left(1-\mathbf{1}_{\partial^-\Delta}\left(y\right)\right)+\mathbf{1}_{\partial^-\Delta}\left(y\right)\mathbf{1}_{\xi_y^-}\left(\sigma_y\right)\right]\ .
\end{aligned}
\tag{48}
$$

Choosing $\delta=\frac{1}{\sqrt{h}}$, for any $\xi\in\Omega_{\overline{\partial}\Delta}, R\in\mathfrak{R}_{\Delta}$, we have

$$
\Phi^{h,\xi}\left(R\right)\leq\left(e^{\frac{J}{\sqrt{h}}}-1\right)^{|E(R)\cap\mathbb{O}|}e^{-2\sqrt{h}|E(R)\cap\mathbb{V}|}\leq e^{-a(h)\|R\|}\ , \tag{49}
$$

with

$$
e^{-a(h)}:=\max\left\{\left(e^{\frac{J}{\sqrt{h}}}-1\right),e^{-2\sqrt{h}}\right\}\ . \tag{50}
$$

We remark that to deal with b.c.'s that are free on the top and on the bottom of Δ, or are periodic in the vertical direction, the definition of the polymers activity must be changed slightly.

The previous bound implies that the cluster expansion is convergent when h is sufficiently large. Indeed we can choose a constant $c > 0$ such that

$$\sum_{R' \in \mathfrak{R}_\Delta \,:\, R' \nsim R} e^{c\|R'\|} e^{-a(h)\|R'\|} \leq \frac{c}{2} \|R\| \ , \tag{51}$$

which is a sufficient condition for the theorem in [KP] to hold. Therefore, for any $\mathcal{R} \subseteq \mathfrak{R}_\Delta$,

$$\log \mathcal{Z}\left(\mathfrak{R}_\Delta, \Phi^{h,\xi}\right) = \sum_{\mathcal{R}' \in \mathcal{C}(\mathcal{R})} \hat{\Phi}^{h,\xi}\left(\mathcal{R}'\right) \tag{52}$$

where, setting $\mathcal{C}_\Delta := \mathcal{C}\left(\mathfrak{R}_\Delta\right)$, in view of (51),

$$\mathcal{C}_\Delta \ni \mathcal{R} \longmapsto \hat{\Phi}^{h,\xi}\left(\mathcal{R}\right) := \sum_{\mathcal{R}' \in \mathcal{P}(\mathcal{R})} (-1)^{|\mathcal{R}|-|\mathcal{R}'|} \log \mathcal{Z}\left(\mathfrak{R}_\Delta, \Phi^{h,\xi}\right) \tag{53}$$

is such that, $\forall R \in \mathfrak{R}_\Delta$,

$$\sum_{\mathcal{R}' \in \mathcal{C}_\Delta \,:\, \mathcal{R}' \nsim R} \left|\hat{\Phi}^{h,\xi}\left(\mathcal{R}'\right)\right| e^{\frac{c}{2}\sum_{R' \in \mathcal{R}'} \|R'\|} \leq \frac{c}{2} \|R\| \ . \tag{54}$$

As already highlighted in the proposition in [KP], the last bound is the key ingredient to perform estimates of quantities which can be represented, in the setup of a polymer gas model, as ratios of partition functions of the form $\frac{\mathcal{Z}(\mathcal{R},\Phi)}{\mathcal{Z}(\mathcal{R},\Phi')}$. As will clearly appear in the next section the proof of Theorem 1 will indeed rely on estimates of this kind.

Moreover, the bound on the polymer activity also implies that two point correlation functions decay exponentially with the distance when h is large with an h-dependent decay constant.

3 Slit box variant of the model

From now on we set δ equal to $\frac{1}{\sqrt{h}}$. Let $\mathbb{H} := \{y \in \mathbb{Z} \times \delta\mathbb{Z} : y_2 = 0\}$ and $\mathbb{H}^+ := \{y \in \mathbb{Z} \times \delta\mathbb{Z} : y_2 > 0\}$, $\mathbb{H}^- := \{y \in \mathbb{Z} \times \delta\mathbb{Z} : y_2 < 0\}$.

Given a finite $\Lambda \subset \mathbb{Z}$, we denote by $\bar{\Lambda} := \{(x_1, 0) \in \mathbb{H} : x_1 \in \Lambda\}$ and set

$$\Lambda_\pm := \left\{ y \in \mathbb{R}^2 : y = \left(x_1, \pm\frac{1}{2}\right), x_1 \in \Lambda \right\} \ . \tag{55}$$

In order to discuss the asymptotic scaling of the entanglement entropy of the ground state of a block of spins, inspired by [GOS], we consider a modified model in which $\mathbb{L}^2_\delta$ is replaced by the graph $\bar{\mathbb{L}}^2_\delta$ in such a way that:

- each lattice point in $x = (x_1, 0) \in \bar{\Lambda}$ is replaced by two distinct vertices $x^+ := \left(x_1, \frac{1}{2}\right)$ and $x^- := \left(x_1, -\frac{1}{2}\right)$;
- each bond $e = \{x, y\} \in \mathbb{E}^2_\delta$ such that $x \in \bar{\Lambda}, y \in \mathbb{H}^+$ is replaced by $\{x^+, y\}$;
- each bond $e = \{x, y\} \in \mathbb{E}^2_\delta$ such that $x \in \bar{\Lambda}, y \in \mathbb{H}^-$ is replaced by $\{x^-, y\}$;
- each bond $e = \{x, y\} \in \mathbb{E}^2_\delta$ such that $x, y \in \bar{\Lambda}$ is replaced by the bonds $\{x^+, y^+\}, \{x^-, y^-\}$;
- each bond $e = \{x, y\} \in \mathbb{E}^2_\delta$ such that $y \in \bar{\Lambda}, x \in \partial\bar{\Lambda} \cap \mathbb{H}$ is replaced by the bonds $\{x, y^+\}, \{x, y^-\}$.

For any $\beta > 0$, let us set $I^+ := \left[0, \frac{\beta}{2}\right] \cap \delta\mathbb{Z}, I^- := \left[-\frac{\beta}{2}, 0\right] \cap \delta\mathbb{Z}$ and denote $I := I^+ \cup I^-$. Moreover, we set $\Delta := \Delta_+ \cup \Delta_-$, where $\Delta_\pm := \Lambda_\pm \times I^\pm$, and keep the definitions of $\partial^\pm\Delta, \overline{\partial}\Delta$ and $\partial\Delta$ given in (14) and (15). We also keep the definition of V^+_Δ given in (42) and define $\bar{\mathbb{V}}_\Delta$ and $\bar{\mathbb{O}}_\Delta$ according to the definitions of $\mathbb{V}_\Delta$ and $\mathbb{O}_\Delta$ given at the beginning of Section 2.

Then, the generic matrix element $\rho^\beta_\Lambda(\epsilon^+, \epsilon^-)$, with $\epsilon^\pm \in \Omega_\Lambda$, of the density operator ρ^β_Λ on $\mathcal{H}_\Lambda$ associated to the Hamiltonian (4) can be rewritten in terms of a Gibbsian specification $\frac{\nu^p_\delta(\mathrm{d}\sigma_\Delta)}{\mu^\delta(\mathrm{d}\sigma_\Delta)}$ for a spin model defined on $\bar{\mathbb{L}}^2_\delta$ by a two-body potential $W_1 + W$ analogous to that given in (36) with periodic b.c.'s at $\overline{\partial}\Delta$.

Let us set $R_+ := \bigcup_{e \in \bar{\mathbb{O}}_\Delta \,:\, V_e \subset \Lambda_+} G_e$ and define R_- to be the graph such that $V(R_-) := \Lambda_- \cup \bar{\Lambda}$ and $E(R_-) := \left\{e \in \bar{\mathbb{O}}_\Delta \,:\, V_e \subset \Lambda_-\right\} \cup \left\{\{x, y\} \in \mathbb{R}^2 : x \in \Lambda_-, y \in \bar{\Lambda}\right\}$. We denote by $\mathfrak{R}$ the union set of $\{R_+, R_-\}$ with the collection of polymers R in $\bar{\mathbb{L}}^2_\delta$ such that $V(R) \subset \Lambda^c_+$. Hence, assuming periodic b.c.'s at $\overline{\partial}\Delta$, fixed b.c. $(\epsilon^+, \epsilon^-) \in \Omega_{\Lambda_+} \times \Omega_{\Lambda_-}$ and free b.c.'s at $\partial\Delta$, denoting by $\Phi^h_{\sigma_{\Lambda_+} = \epsilon^+, \sigma_{\Lambda_-} = \epsilon^-}$ the activity of the polymers in

$$\mathfrak{R}_\Delta := \left\{R \in \mathfrak{R} : V(R) \subseteq \Delta \cup \bar{\Lambda}\right\} \ , \tag{56}$$

by (47), we have

$$\rho^\beta_\Lambda\left(\epsilon^+, \epsilon^-\right) = \frac{\nu^p_\delta\left(\mathbf{1}_{\left\{\sigma_{\Lambda_+} = \epsilon^+, \sigma_{\Lambda_-} = \epsilon^-\right\}}\right)}{\nu^p_\delta\left(\mathbf{1}_{\left\{\sigma_{\Lambda_+} = \sigma_{\Lambda_-}\right\}}\right)} = \frac{\mathcal{Z}\left(\mathfrak{R}_\Delta, \Phi^h_{\sigma_{\Lambda_+} = \epsilon^+, \sigma_{\Lambda_-} = \epsilon^-}\right)}{\mathcal{Z}\left(\mathfrak{R}_\Delta, \Phi^h_{\sigma_{\Lambda_+} = \sigma_{\Lambda_-}}\right)} \tag{57}$$

with

$$\mathcal{Z}\left(\mathfrak{R}_\Delta, \Phi^h_{\sigma_{\Lambda_+} = \sigma_{\Lambda_-}}\right) = \sum_{\epsilon \in \Omega_\Lambda} \mathcal{Z}\left(\mathfrak{R}_\Delta, \Phi^h_{\sigma_{\Lambda_+} = \sigma_{\Lambda_-} = \epsilon}\right) \ . \tag{58}$$

3.1 The reduced density operator

From now on we will keep our notation as close as possible to that introduced in [GOS]. Let us set for $m, L \in \mathbb{N}$,

$$\Lambda_m := \{-m, -m+1, \ldots, m+L\} \ , \ \Lambda_0 := \{0, .., L\} \tag{59}$$

and $\Lambda_m^{\pm} := (\Lambda_m)_{\pm}, \Lambda_{\pm} := (\Lambda_0)_{\pm}, \Delta_m := \Delta_m^+ \cup \Delta_m^-$, where $\Delta_m^{\pm} := \Lambda_m^{\pm} \times I^{\pm}$. Considering the representation of $\mathcal{H}_m := \mathcal{H}_{\Lambda_m}$ as $\mathcal{H}_{m,L} \otimes \mathcal{H}_L$, where $\mathcal{H}_{m,L} := \mathcal{H}_{\Lambda_m \setminus \Lambda_0}, \mathcal{H}_L := \mathcal{H}_{\Lambda_0}$, we denote by $\bar{\rho}_m^{L,\beta}$ the partial trace of $\rho_m^{\beta} := \rho_{\Lambda_m}^{\beta}$ w.r.t. $\mathcal{H}_{m,L}$. Then, the generic matrix element $\rho_m^{L,\beta}(\epsilon^+, \epsilon^-) := \rho_{\Lambda_m}^{\Lambda_0,\beta}(\epsilon^+, \epsilon^-)$, with $\epsilon^{\pm} \in \Omega_L := \Omega_{\Lambda_0}$, of the *reduced density operator* $\rho_m^{L,\beta} := \frac{\bar{\rho}_m^{L,\beta}}{tr_{\mathcal{H}_{\Lambda_0}} \bar{\rho}_m^{L,\beta}}$ on $\mathcal{H}_L$ writes

$$\begin{aligned}
\rho_m^{L,\beta}\left(\epsilon^+, \epsilon^-\right) &= \frac{\nu_\delta^p\left(\mathbf{1}_{\{\sigma_{L^+}=\epsilon^+, \sigma_{L^-}=\epsilon^-\}}\right)}{\nu_\delta^p\left(\mathbf{1}_{\{\sigma_{L^+}=\sigma_{L^-}\}}\right)} \\
&= \frac{\sum_{\epsilon' \in \Omega_{m,L}} \mathcal{Z}\left(\mathfrak{R}_m, \Phi^h_{\sigma_{m,L}=\epsilon', \sigma_{\Lambda_+}=\epsilon^+, \sigma_{\Lambda_-}=\epsilon^-}\right)}{\sum_{\epsilon \in \Omega_L} \sum_{\epsilon' \in \Omega_{m,L}} \mathcal{Z}\left(\mathfrak{R}_m, \Phi^h_{\sigma_{m,L}=\epsilon', \sigma_{\Lambda_+}=\sigma_{\Lambda_-}=\epsilon}\right)} \\
&= \frac{\mathcal{Z}\left(\mathfrak{R}_m, \Phi^h_{\sigma_{\Lambda_+}=\epsilon^+, \sigma_{\Lambda_-}=\epsilon^-}\right)}{\mathcal{Z}\left(\mathfrak{R}_m, \Phi^h_{\sigma_{\Lambda_+}=\sigma_{\Lambda_-}}\right)} \ ,
\end{aligned} \tag{60}$$

where, $\sigma_{L^{\pm}} := \sigma_{\Lambda_{\pm}}, \sigma_{m,L} := \sigma_{\Lambda_m \setminus \Lambda_0}, \Omega_{m,L} := \Omega_{\Lambda_m \setminus \Lambda_0}$ and $\mathfrak{R}_m := \mathfrak{R}_{\Delta_m}$ with $\mathfrak{R}_{\Delta_m}$ defined as in (56).

Since in the limit of $\beta \to \infty$ and then of $\{\Lambda_m\} \uparrow \mathbb{Z}$ the Gibbs measures defined in (11) for different b.c.'s converge weakly to the same limit, the same conclusion holds for the Gibbs measures defined in (36). Therefore, assuming b.c. $\xi \in \Omega_{\partial^+\Delta_m \cup \partial^-\Delta_m}$ at $\overline{\partial}\Delta_m$ and free b.c.'s at $\partial\Delta_m \setminus \overline{\partial}\Delta_m$ for any event $\{\cdot\} \in \mathfrak{S}_{\Lambda_+ \cup \Lambda_-}$, we set

$$\phi_{m,\beta}\{\cdot\} := \frac{\nu_\delta^\xi\left(\mathbf{1}_{\{\cdot\}}\right)}{\nu_\delta^\xi\left(\mathbf{1}_{\Omega_{\Lambda_+ \cup \Lambda_-}}\right)} \tag{61}$$

and define

$$\tilde{\rho}_m^{L,\beta}\left(\epsilon^+,\epsilon^-\right) := \frac{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+}=\sigma_{L^-}\right\}} = \frac{\mathcal{Z}\left(\mathfrak{R}_m,\Phi^{h,\xi}_{\sigma_{\Lambda_+}=\epsilon^+,\sigma_{\Lambda_-}=\epsilon^-}\right)}{\mathcal{Z}\left(\mathfrak{R}_m,\Phi^{h,\xi}_{\sigma_{\Lambda_+}=\sigma_{\Lambda_-}}\right)} . \tag{62}$$

where here $\mathfrak{R}_m$ is the set of polymers R in $\mathfrak{R}$ with $V(R)\subseteq\Delta_m\cup\bar{\Lambda}\cup V^+_{\Delta_m}$ such that, if $V(R)\cap\overline{\partial}\Delta\neq\varnothing$, then $V(R)$ contains either $\partial^+\Delta$ or $\partial^-\Delta$ or both. To simplify the notation, in the following we will also set $\mathfrak{P}_m:=\mathfrak{P}_{\Delta_m}$ and $\mathcal{C}_m:=\mathcal{C}_{\Delta_m}$.

Lemma 2 *There exists a positive value of the external magnetic field h^* such that, for any $h>h^*$ and $L,m\in\mathbb{N}$, uniformly in $\beta>0,\epsilon^+,\epsilon^-\in\Omega_L$ and in the b.c. $\xi\in\Omega_{\overline{\partial}\Delta_m}$,*

$$e^{-\psi(c)}\leq\frac{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+\right\}\phi_{m,\beta}\left\{\sigma_{L^-}=\epsilon^-\right\}}\leq e^{\psi(c)} , \tag{63}$$

where $\psi(c):=\frac{8}{1-e^{-\frac{c}{2}}}$ with c the constant appearing in (54).

Proof. We proceed as in the proof of the proposition in [KP]. Since by (48) for any $R\in\mathfrak{R}_m$ compatible with R_+ and R_-, $\Phi^{h,\xi}_{\sigma_{\Lambda_+}=\epsilon^+,\sigma_{\Lambda_-}=\epsilon^-}(R)=\Phi^{h,\xi}(R)$, by (52) we have

$$\begin{aligned}\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\} &= \frac{\mathcal{Z}\left(\mathfrak{R}_m,\Phi^{h,\xi}_{\sigma_{\Lambda_+}=\epsilon^+,\sigma_{\Lambda_-}=\epsilon^-}\right)}{\mathcal{Z}\left(\mathfrak{R}_m,\Phi^{h,\xi}\right)} \\ &= \exp\left\{\sum_{\mathcal{R}\in\mathcal{C}_m\,:\,\mathcal{R}\nsim R_+\cup R_-}\hat{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})-\hat{\Phi}^{h,\xi}(\mathcal{R})\right\} .\end{aligned} \tag{64}$$

Moreover, because if $R\in\mathfrak{R}_m$ is compatible with R_+, $\Phi^{h,\xi}_{\sigma_{L^+}=\epsilon^+}(R)=\Phi^{h,\xi}(R)$ and analogously if $R\in\mathfrak{R}_m$ is compatible with R_-, $\Phi^{h,\xi}_{\sigma_{L^-}=\epsilon^-}(R)=\Phi^{h,\xi}(R)$,

$$\begin{aligned}\phi_{m,\beta}\left\{\sigma_{L^\pm}=\epsilon^\pm\right\} &= \frac{\sum_{\epsilon^\mp\in\Omega_L}\mathcal{Z}\left(\mathfrak{R}_m,\Phi^{h,\xi}_{\sigma_{L^\pm}=\epsilon^\pm,\sigma_{L^\mp}=\epsilon^\mp}\right)}{\mathcal{Z}\left(\mathfrak{R}_m,\Phi^{h,\xi}\right)} \\ &= \frac{\mathcal{Z}\left(\mathfrak{R}_m,\Phi^{h,\xi}_{\sigma_{L^\pm}=\epsilon^\pm}\right)}{\mathcal{Z}\left(\mathfrak{R}_m,\Phi^{h,\xi}\right)} \\ &= \exp\left\{\sum_{\mathcal{R}\in\mathcal{C}_m\,:\,\mathcal{R}\nsim R_\pm}\hat{\Phi}^{h,\xi}_{\sigma_{L^\pm}=\epsilon^\pm}(\mathcal{R})-\hat{\Phi}^{h,\xi}(\mathcal{R})\right\} .\end{aligned} \tag{65}$$

Therefore, setting, for any $\mathcal{R} \in \mathcal{C}_m, V(\mathcal{R}) := \bigcup_{R \in \mathcal{R}} V(R)$ and

$$\bar{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R}) := \max\left\{\left|\hat{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})\right|, \left|\hat{\Phi}^{h,\xi}(\mathcal{R})\right|, \left|\hat{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})\right|, \left|\hat{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})\right|\right\} , \tag{66}$$

by (54),

$$\frac{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+\right\}\phi_{m,\beta}\left\{\sigma_{L^-}=\epsilon^-\right\}} = \tag{67}$$

$$\exp\left\{\sum_{\mathcal{R}\in\mathcal{C}_m\,:\,\mathcal{R}\nsim R_+,\mathcal{R}\nsim R_-} \hat{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R}) - \hat{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+}(\mathcal{R}) - \hat{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^-}(\mathcal{R}) + \hat{\Phi}^{h,\xi}(\mathcal{R})\right\}$$

$$\leq \exp\left\{4\sum_{\mathcal{R}\in\mathcal{C}_m\,:\,\mathcal{R}\nsim R_+,\mathcal{R}\nsim R_-} \bar{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})\right\}$$

$$\leq \exp\left\{4\sum_{x\in\Lambda_+} e^{-\frac{c}{2}\inf_{\mathcal{R}\in\mathcal{C}_m\,:\,\mathcal{R}\nsim R_-,V(\mathcal{R})\ni x}\sum_{R\in\mathcal{R}}\|R\|}\times\right.$$

$$\left.\times\sum_{\mathcal{R}\in\mathcal{C}_m\,:\,\mathcal{R}\nsim R_-,V(\mathcal{R})\ni x} \bar{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})\, e^{\frac{c}{2}\sum_{R\in\mathcal{R}}\|R\|}\right\}$$

$$\leq \exp\left\{2c\,\|R_-\|\,e^{-\frac{c}{2}\|R_-\|}\sum_{k=0}^{L} e^{-\frac{c}{2}k\wedge(L-k)}\right\} \leq e^{\frac{8}{1-e^{-\frac{c}{2}}}} .$$

The lower bound in (63) follows from the estimate

$$\frac{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+\right\}\phi_{m,\beta}\left\{\sigma_{L^-}=\epsilon^-\right\}} \geq \exp -4\left\{\sum_{\mathcal{R}\in\mathcal{C}_{\Delta_m}\,:\,\mathcal{R}\nsim R_+,\mathcal{R}\nsim R_-} \bar{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})\right\} . \tag{68}$$

■

Lemma 3 *There exists a positive value of the external magnetic field h^* such that, for any $h > h^*$ and any $L, m \in \mathbb{N}$, uniformly in $\beta > 0, \epsilon^+, \epsilon^- \in \Omega_L$ and in the b.c.'s $\xi \in \Omega_{\overline{\partial}\Delta_n}$ and $\eta_n \in \mathcal{S}_{\Delta_n^c\setminus\overline{\partial}\Delta_n}$,*

$$e^{-\frac{\psi(c)}{2}e^{-cm}} \leq \frac{\phi_{n,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}} \leq e^{\frac{\psi(c)}{2}e^{-cm}} . \tag{69}$$

Proof. Let us assume b.c. $\xi \in \Omega_{\partial^+\Delta_n \cup \partial^-\Delta_n}$ at $\overline{\partial}\Delta_n$ and free b.c.'s at $\partial\Delta_n \backslash \overline{\partial}\Delta_n$. The proof of (69) for more general b.c.'s will follow directly from the one carried out for this case since a change in the b.c.'s affects only the definition of the polymers activity in the cluster expansion. As in the proof of the preceding Lemma we have

$$\frac{\phi_{n,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}} = \tag{70}$$
$$\frac{\exp\left\{\sum_{\mathcal{R}\in\mathcal{C}_n\,:\,\mathcal{R}\nsim(R_+\cup R_-)}\hat{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})-\hat{\Phi}^{h,\xi}(\mathcal{R})\right\}}{\exp\left\{\sum_{\mathcal{R}\in\mathcal{C}_m\,:\,\mathcal{R}\nsim(R_+\cup R_-)}\hat{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})-\hat{\Phi}^{h,\xi}(\mathcal{R})\right\}}\,.$$

Since $\mathfrak{R}_n \supset \mathfrak{R}_m$, setting

$$\bar{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R}) := \max\left\{\left|\hat{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})\right|,\left|\hat{\Phi}^{h,\xi}(\mathcal{R})\right|\right\}\,, \tag{71}$$

we have

$$\frac{\phi_{n,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}} \le \exp\left\{2\sum_{\mathcal{R}\in\mathcal{C}_n\,:\,V(\mathcal{R})\cap(\Delta_n\triangle\Delta_m)\neq\varnothing,\mathcal{R}\nsim R_+,\mathcal{R}\nsim R_-}\bar{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})\right\} \tag{72}$$
$$\le \exp\left\{2\sum_{x\in\Lambda_+} e^{-\frac{c}{2}\inf_{\mathcal{R}\in\mathcal{C}_n\,:\,\mathcal{R}\nsim R_-,V(\mathcal{R})\cap(\Delta_n\triangle\Delta_m)\neq\varnothing,V(\mathcal{R})\ni x}\sum_{R\in\mathcal{R}}\|R\|}\times\right.$$
$$\left.\times\sum_{\mathcal{R}\in\mathcal{C}_n\,:\,\mathcal{R}\nsim R_-,V(\mathcal{R})\cap(\Delta_n\triangle\Delta_m)\neq\varnothing,V(\mathcal{R})\ni x}\bar{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})\,e^{\frac{c}{2}\sum_{R\in\mathcal{R}}\|R\|}\right\}$$
$$\le \exp\left\{c\,\|R_-\|\,e^{-c\left(\frac{\|R_-\|}{2}+m\right)}\sum_{k=0}^{L}e^{-\frac{c}{2}k\wedge(L-k)}\right\} \le e^{\frac{4}{1-e^{-\frac{c}{2}}}e^{-cm}}\,.$$

The lower bound in (69) follows from the estimate

$$\frac{\phi_{n,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}} \ge \exp\left\{-2\sum_{\mathcal{R}\in\mathcal{C}_n\,:\,V(\mathcal{R})\cap(\Delta_n\triangle\Delta_m)\neq\varnothing,\mathcal{R}\nsim R_+,\mathcal{R}\nsim R_-}\bar{\Phi}^{h,\xi}_{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-}(\mathcal{R})\right\}\,. \tag{73}$$

■

Lemma 4 *There exists a positive value of the external magnetic field h^* such that, for any $h > h^*$ and any $L, m \in \mathbb{N}$, uniformly in $\beta > 0, \epsilon^+, \epsilon^- \in \Omega_L$ and in the b.c.'s $\xi \in \Omega_{\overline{\partial}\Delta_n}$ and $\eta_n \in \mathcal{S}_{\Delta_n^c \setminus \overline{\partial}\Delta_n}$,*

$$\left| \frac{\phi_{n,\beta}\left\{\sigma_{L^+} = \sigma_{L^-}\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+} = \sigma_{L^-}\right\}} - \frac{\phi_{n,\beta}\left\{\sigma_{L^+} = \epsilon^+, \sigma_{L^-} = \epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+} = \epsilon^+, \sigma_{L^-} = \epsilon^-\right\}} \right| \leq e^{\frac{\psi(c)}{2}} e^{-cm} \frac{\psi(c)}{2} e^{-cm} \ . \tag{74}$$

Proof. Proceeding as in the proof of the previous result, as well as in the proof of the statement (iii) in the thesis of the Proposition in [KP],

$$\begin{aligned}
&\left| \frac{\phi_{n,\beta}\left\{\sigma_{L^+} = \sigma_{L^-}\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+} = \sigma_{L^-}\right\}} - \frac{\phi_{n,\beta}\left\{\sigma_{L^+} = \epsilon^+, \sigma_{L^-} = \epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+} = \epsilon^+, \sigma_{L^-} = \epsilon^-\right\}} \right| \\
&= \Bigg| \exp \sum_{\mathcal{R} \in \mathcal{C}_n \,:\, V(\mathcal{R}) \cap (\Delta_n \triangle \Delta_m) \neq \varnothing, \mathcal{R} \nsim R_+, \mathcal{R} \nsim R_-} \hat{\Phi}^{h,\xi}_{\sigma_{L^+} = \sigma_{L^-}}(\mathcal{R}) - \hat{\Phi}^{h,\xi}(\mathcal{R}) \qquad (75) \\
&- \exp \sum_{\mathcal{R} \in \mathcal{C}_n \,:\, V(\mathcal{R}) \cap (\Delta_n \triangle \Delta_m) \neq \varnothing, \mathcal{R} \nsim R_+, \mathcal{R} \nsim R_-} \hat{\Phi}^{h,\xi}_{\sigma_{L^+} = \epsilon^+, \sigma_{L^-} = \epsilon^-}(\mathcal{R}) - \hat{\Phi}^{h,\xi}(\mathcal{R}) \Bigg| \qquad (76) \\
&\leq e^{\frac{\psi(c)}{2}} e^{-cm} \left| \sum_{\mathcal{R} \in \mathcal{C}_n \,:\, V(\mathcal{R}) \cap (\Delta_n \triangle \Delta_m) \neq \varnothing, \mathcal{R} \nsim R_+, \mathcal{R} \nsim R_-} \hat{\Phi}^{h,\xi}_{\sigma_{L^+} = \sigma_{L^-}}(\mathcal{R}) - \hat{\Phi}^{h,\xi}_{\sigma_{L^+} = \epsilon^+, \sigma_{L^-} = \epsilon^-}(\mathcal{R}) \right| \\
&\leq e^{\frac{\psi(c)}{2}} e^{-cm} \sum_{\mathcal{R} \in \mathcal{C}_n \,:\, V(\mathcal{R}) \cap (\Delta_n \triangle \Delta_m) \neq \varnothing, \mathcal{R} \nsim R_+, \mathcal{R} \nsim R_-} \left(\left| \hat{\Phi}^{h,\xi}_{\sigma_{L^+} = \sigma_{L^-}}(\mathcal{R}) \right| + \left| \hat{\Phi}^{h,\xi}_{\sigma_{L^+} = \epsilon^+, \sigma_{L^-} = \epsilon^-}(\mathcal{R}) \right| \right) \\
&\leq e^{\frac{\psi(c)}{2}} e^{-cm} \frac{\psi(c)}{2} e^{-cm} \ .
\end{aligned}$$

■

3.2 Entanglement entropy

In order to prove Theorem 1 we follow the same strategy of the proof of Theorem 2.8 in [GOS] to which we refer the reader for the details of the computations.

As a matter of fact, it follows from the estimate (77) given below that there exist $C := C(c), C' := C'(c) > 0$, such that, for any $k \geq K := \lceil C^{-1} \ln C' \rceil$, the norm of $\rho^L_{k+1} - \rho^L_k$ is bounded by $C' e^{-C(k-K)}$. Therefore, denoting by $\mathbf{d}$ the smallest value between the dimension of $\mathcal{H}_L$ and that of $\mathcal{H}_{m,L}$, since $S\left(\rho^L_m\right) = -\sum_{i=1}^{\mathbf{d}} \alpha_i\left(\rho^L_m\right) \log \alpha_i\left(\rho^L_m\right)$, where $\left\{\alpha_i\left(\rho^L_m\right)\right\}_{i=1}^{\mathbf{d}}$ is the vector of the eigenvalues of ρ^L_m arranged in decreasing order, if $2 \leq m \leq K$, we

get that $S\left(\rho_m^L\right)$ is smaller than $2K$. On the other hand, if $K \geq m$, iterating the bound of $\max_{i\geq 1}\left|\alpha_i\left(\rho_{K+r+1}^L\right)-\alpha_i\left(\rho_{K+r}^L\right)\right| \leq C'e^{-Cr}, r \geq 0$, one can prove that there exist $C_0 := C_0(c) > 0, \iota := \iota(c) > 2$ such that $\alpha_i\left(\rho_m^L\right) \leq C_0 + \alpha_i\left(\rho_K^L\right)$, for $i \leq 2^{2K}$, and $\alpha_i\left(\rho_m^L\right) \leq \frac{C_0}{i^\iota}$ for $i > 2^K$, which leads to the bound $S\left(\rho_m^L\right) \leq C_1 K, C_1 := C_1(c) > 0$.

Proposition 5 *There exists a positive value of the external magnetic field h^* such that, for any $h > h^*$ and for any $L, m, n \in \mathbb{N}$ such that $m < n$,*

$$\left\|\tilde{\rho}_m^L - \tilde{\rho}_n^L\right\| \leq e^{\frac{\psi(c)}{2}\left(e^{-cm}+6\right)}\frac{\psi(c)}{2}e^{-cm} \ . \tag{77}$$

Proof. Proceeding as in the proof of Theorem 2.2 in [GOS], we are reduced to estimate the following quantity

$$\sum_{\epsilon^+,\epsilon^-\in\Omega_L} b\left(\epsilon^+\right) b\left(\epsilon^-\right)\left|\tilde{\rho}_m^{L,\beta}\left(\epsilon^+,\epsilon^-\right)-\tilde{\rho}_n^{L,\beta}\left(\epsilon^+,\epsilon^-\right)\right| \ , \tag{78}$$

for any real-valued positive function b on Ω_L such that $\sum_{\epsilon\in\Omega_L} b^2(\epsilon) = 1$. But

$$\begin{aligned}\left|\tilde{\rho}_m^{L,\beta}\left(\epsilon^+,\epsilon^-\right)-\tilde{\rho}_n^{L,\beta}\left(\epsilon^+,\epsilon^-\right)\right| &\leq \left|\frac{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+}=\sigma_{L^-}\right\}}-\frac{\phi_{n,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{n,\beta}\left\{\sigma_{L^+}=\sigma_{L^-}\right\}}\right| \\ &= \frac{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{n,\beta}\left\{\sigma_{L^+}=\sigma_{L^-}\right\}}\times \\ &\times\left|\frac{\phi_{n,\beta}\left\{\sigma_{L^+}=\sigma_{L^-}\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+}=\sigma_{L^-}\right\}}-\frac{\phi_{n,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}\right| \ .\end{aligned} \tag{79}$$

Moreover, by (63),

$$\begin{aligned}\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\} &= \frac{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+,\sigma_{L^-}=\epsilon^-\right\}}{\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+\right\}\phi_{m,\beta}\left\{\sigma_{L^-}=\epsilon^-\right\}}\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+\right\}\phi_{m,\beta}\left\{\sigma_{L^-}=\epsilon^-\right\} \\ &\leq e^{\psi(c)}\phi_{m,\beta}\left\{\sigma_{L^+}=\epsilon^+\right\}\phi_{m,\beta}\left\{\sigma_{L^-}=\epsilon^-\right\}\end{aligned} \tag{80}$$

and, by the symmetry under the reflection w.r.t. the horizontal axis,

$$\begin{aligned}\phi_{n,\beta}\left\{\sigma_{L^+}=\sigma_{L^-}\right\} &= \sum_{\epsilon\in\Omega_L}\frac{\phi_{n,\beta}\left\{\sigma_{L^+}=\sigma_{L^-}=\epsilon\right\}}{\phi_{n,\beta}\left\{\sigma_{L^+}=\epsilon\right\}\phi_{n,\beta}\left\{\sigma_{L^-}=\epsilon\right\}}\phi_{n,\beta}\left\{\sigma_{L^+}=\epsilon\right\}\phi_{n,\beta}\left\{\sigma_{L^-}=\epsilon\right\} \\ &\geq e^{-\psi(c)}\sum_{\epsilon\in\Omega_L}\phi_{n,\beta}^2\left\{\sigma_{L^+}=\epsilon\right\} \ .\end{aligned} \tag{81}$$

Therefore, by (74), (78) is bounded by

$$e^{\frac{\psi(c)}{2}\left(e^{-cm}+4\right)}\frac{\psi\left(c\right)}{2}e^{-cm}\sum_{\epsilon^{+},\epsilon^{-}\in\Omega_{L}}b\left(\epsilon^{+}\right)b\left(\epsilon^{-}\right)\frac{\phi_{m,\beta}\left\{\sigma_{L^{+}}=\epsilon^{+}\right\}\phi_{m,\beta}\left\{\sigma_{L^{-}}=\epsilon^{-}\right\}}{\sqrt{\sum_{\epsilon^{+}\in\Omega_{L}}\phi_{m,\beta}^{2}\left\{\sigma_{L^{+}}=\epsilon^{+}\right\}}\sqrt{\sum_{\epsilon^{-}\in\Omega_{L}}\phi_{m,\beta}^{2}\left\{\sigma_{L^{-}}=\epsilon^{-}\right\}}}\times \tag{82}$$

$$\times\frac{\sum_{\epsilon\in\Omega_{L}}\phi_{m,\beta}^{2}\left\{\sigma_{L^{+}}=\epsilon\right\}}{\sum_{\epsilon\in\Omega_{L}}\phi_{n,\beta}^{2}\left\{\sigma_{L^{+}}=\epsilon\right\}}\ .$$

Proceeding as in the proof of (69), for any $\epsilon\in\Omega_{L}$, we get the bound

$$\frac{\phi_{n,\beta}\left\{\sigma_{L^{+}}=\epsilon\right\}}{\phi_{m,\beta}\left\{\sigma_{L^{+}}=\epsilon\right\}}\geq e^{-\psi(c)}\ . \tag{83}$$

Hence, (78) is smaller than

$$e^{\frac{\psi(c)}{2}\left(e^{-cm}+6\right)}\frac{\psi\left(c\right)}{2}e^{-cm}\left|\sum_{\epsilon\in\Omega_{L}}b\left(\epsilon\right)\frac{\phi_{m,\beta}\left\{\sigma_{L^{+}}=\epsilon\right\}}{\sqrt{\sum_{\epsilon'\in\Omega_{L}}\phi_{m,\beta}^{2}\left\{\sigma_{L^{+}}=\epsilon'\right\}}}\right|^{2} \tag{84}$$

and by the Schwarz inequality we get (77). ■

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