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The cauchy-leray integral: Counterexamples to the Lp-theory

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THE CAUCHY-LERAY INTEGRAL:
COUNTER-EXAMPLES TO THE L^p-THEORY

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ABSTRACT. We prove the optimality of the hypotheses guaranteeing the L^p-boundedness for the Cauchy-Leray integral in $\mathbb{C}^n$, $n \geq 2$, obtained in [LS-4].

Two domains, both elementary in nature, show that the geometric requirement of strong $\mathbb{C}$-linear convexity, and the regularity of order 2, are both necessary.

1. INTRODUCTION

The purpose of this paper is to present counter-examples that show that the L^p results for the Cauchy-Leray integral obtained in [LS-4] are essentially optimal. A second paper in this series will deal with counter-examples for the Cauchy-Szegő projection, relevant to [LS-5].

Recall that in the case of the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, the Cauchy-Leray integral and the Cauchy-Szegő projection agree, and the same holds for the unbounded realization U_0 of $\mathbb{B}$. However when $n \geq 2$, for more general domains, these two operators are quite different. For the former, the result obtained in [LS-4] states:

Suppose D is a bounded domain in $\mathbb{C}^n$ whose boundary is of class $C^{1,1}$ and which is strongly $\mathbb{C}$-linearly convex. Then the induced Cauchy-Leray transform is bounded on $L^p(bD, d\sigma)$ for $1 < p < \infty$.

Our counter-examples show that these two restrictions on the boundary of D, the geometric condition of strong $\mathbb{C}$-linear convexity, and the regularity of degree 2, are in fact optimal.

We consider the following simple bounded domains in $\mathbb{C}^2$. Here $z_j = x_j + i\,y_j$, $j = 1, 2$. The first domain is defined by

$$(1) \qquad\qquad |z_2|^2 + x_1^2 + y_1^4 \; < \; 1 \,.$$

This is the domain given up to a translation by (9) in Section 3. It has a C^∞ (in fact real-analytic) boundary, is strongly pseudo-convex, but not strongly $\mathbb{C}$-linearly convex. We should point out that the Cauchy-Szegő projection of this domain is bounded in L^p for any $1 < p < \infty$ by earlier results in [KS] and [MS].

The second example is

$$(2) \qquad |z_2|^2 + |x_1|^m + y_1^2 \; < \; 1 \,, \quad \text{where} \quad 1 < m < 2.$$

This domain is given (up to a translation) by (20) in Section 5. The domain has a boundary of class $C^{2-\epsilon}$, ($\epsilon = 2 - m$), is strongly $\mathbb{C}$-linearly convex (and hence strongly pseudoconvex in the sense discussed in Section 5).

We show that for each domain the induced Cauchy-Leray transform is not bounded on L^p for any p, $1 \le p \le \infty$, in the sense that there is a function $f \in L^p(bD, d\sigma)$ and a subset $\mathbf{S}' \subset bD$ disjoint from the support of f for which the inequality

$$\|\mathbf{C}(f)\|_{L^p(\mathbf{S}', d\sigma)} \le A_p \|f\|_{L^p(bD, d\sigma)}$$

fails. See Theorem 2 for the precise statement.

The idea of the proof is as follows: we assume that L^p-boundedness holds for one of these domains. Then an appropriate scaling and limiting argument shows that this positive result implies a corresponding conclusion in a limiting model domain, where it is much easier to supply an explicit counter-example.

For the first example the limiting domain is the unbounded half-space $\{z \, : \, 2\,\mathrm{Im}\,z_2 > x_1^2\}$ (called D_0 in (5)), which is holomorphically equivalent to the more familiar half-space U_0, $\{z \, : \, 2\,\mathrm{Im}\,z_2 > |z_1|^2\}$, which itself is holomorphically equivalent to the unit ball. Note that the last two are strongly $\mathbb{C}$-linearly convex, but D_0 is not. The hint that one might be led to a counter-example for D_0, and then for the domain (1) is that its Cauchy-Leray operator is not "pseudo-local"; (see (6) which shows that the kernel is singular away from the diagonal).

The analysis of the second domain, represented by (2), is parallel to that of the first domain. For example, the corresponding limiting domain is $\{z \, : \, 2\,\mathrm{Im}\,z_2 > |x_1|^m\}$, see (21).

A remark is in order about the family of boundary measures $\{d\mu_a\}_a$ given in (29) that define the L^p spaces, $L^p = L^p(bD, d\mu_a)$ in the above results. Three examples of $d\mu_a$ are significant in many circumstances.

First, the induced Lebesgue measure $d\sigma$; second, the Leray-Levi measure $d\lambda$, see (4); and third, the Fefferman measure [B2], [F-1], [G]. When the domain is smooth and strongly pseudo-convex the three measures give the same L^p spaces, so that the counter-example holds for the domain (1) for all such measures. In example (2) these measures are essentially different, yet the counter-example still holds in all cases.

We should also call the reader's attention to the earlier relevant work in [BaLa] where counter-examples are given for Cauchy-Leray integrals. However the less explicit and more complex nature of their construction and proof limit the results to the case $p = 2$. It should be stressed that when $n \geq 2$ in general the Cauchy-Leray transform is far from "self-adjoint" and so failure of L^2-boundedness does not imply failure for any p, $p \neq 2$.

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2. The Cauchy-Leray integral; the model domain D_0

For a bounded domain (say of class C^2) D in $\mathbb{C}^n$, with a defining function ρ, the corresponding Cauchy-Leray integral is

$$(3) \qquad \mathbf{C}(f)(z) = \int_{bD} \frac{1}{\Delta(w, z)^n} \, f(w) \, d\lambda(w) \quad z \in D.$$

Here f is (say) a bounded function on bD,

$$\Delta(w, z) = \langle \partial\rho(w), w - z \rangle = \sum_{j=1}^{n} \frac{\partial\rho(w)}{\partial w_j}(w_j - z_j),$$

and

$$(4) \qquad d\lambda(w) = \frac{j^*}{(2\pi i)^n} \left(\partial\rho \wedge (\bar{\partial}\partial\rho)^{n-1} \right)$$

is the Leray-Levi measure (with j^* the pull-back under the inclusion: $bD \hookrightarrow \mathbb{C}^n$). We have $d\lambda(w) = \Gamma(w) \, d\sigma(w)$, with $d\sigma$ the induced Lebesgue measure on bD and

$$\Gamma(w) = \frac{(n-1)!}{4\pi^n} \mathcal{L}(w) \, |\nabla\rho(w)|,$$

where $\mathcal{L}(w)$ is the determinant of the Levi-form acting on the maximal complex subspace of the tangent space $T_w(bD)$ at $w \in bD$ (See e.g., [Ra, Ch. 7].)

It is worth recalling two intrinsic properties of (3):

- Its invariance under changes of coordinates that are given either by translations or unitary mappings of the space $\mathbb{C}^n$, see [Bo].
- The independence of the Cauchy-Leray integral (3) of the particular defining function ρ of D.

We consider first the unbounded "model domain" D_0 in $\mathbb{C}^2$. With $z = (z_1, z_2)$, $z_j = x_j + i\, y_j$, it is defined by

$$(5) \qquad D_0 = \{z \; : \; 2\, \mathrm{Im} z_2 > x_1^2\},$$

which is to be compared with the more familiar form of D_0 given by $\{z \; : \; 2\, \mathrm{Im} z_2 > |z_1|^2\}$. These two domains are biholomorphically equivalent via the mappings $z_1 \leftrightarrow z_1$, $z_2 \leftrightarrow z_2 \pm i\, z_1^2$. Now the complex tangent space of these domains at the origin is the subspace $\{(z_1, 0)\}$. So since $|z_1|^2 = x_1^2 + y_1^2$, and on $\mathbb{R}^2$ this is positive definite, this implies that the second domain is strongly $\mathbb{C}$-linearly convex. However because the form x_1^2 is degenerate along the direction y_1, it follows that strong $\mathbb{C}$-linear convexity fails for the domain (4). (For more about these convexities see [APS]; [Hö]; [LS-3, Sect. 3.3].)

With $\rho_0(z) = x_1^2 - 2y_2$, and $\Delta_0(w, z) = \langle \partial \rho_0(w), w - z \rangle$, and if we write $w = (w_1, w_2)$, $w_j = u_j + i\, v_j$, $j = 1, 2$, a simple calculation gives that with w and $z \in bD_0$,

$$(6) \qquad \Delta_0(w, z) = \frac{1}{2}\big((u_1 - x_1)^2 + 2i\,(u_1(v_1 - y_1) + u_2 - x_2)\big).$$

Now for fixed $z \in bD_0$, observe that $\Delta_0(w, z)$ vanishes on the one-dimensional variety given by $u_1 = x_1$, $u_1(v_1 - y_1) + u_2 = x_2$. Also the Leray-Levi measure $d\lambda_0$ corresponding to ρ_0 is $du_1\, dv_1\, du_2/(4\pi^2) \approx d\sigma_0$, the induced Lebesgue measure on bD_0, if we are near the origin.

To construct a counterexample for the model domain D_0 we first choose a small constant a, which we will keep fixed throughout ($a = 1/12$ will do). Then for any δ, $0 < \delta < 1$, we define the following two sets in the parameter space:

$$(7) \qquad \begin{cases} \mathbf{U} & = \; \big\{|u_1| \le a\delta^2\,, \; |v_1| \le \frac{1}{2}\,, \; |u_2| \le a\delta^2\big\} \\[2mm] \mathbf{U}' & = \; \big\{\delta \le |x_1| \le 2\delta\,, \; |y_1| \le \frac{1}{2}\,, \; |x_2| \le a\delta^2\big\} \end{cases}$$

with $w \in bD_0$ written as $(u_1 + i\, v_1, u_2 + i\, u_1^2/2)$ and $z \in bD_0$ written as $(x_1 + i\, y_1, x_2 + i\, x_1^2/2)$. If $\mathbf{S}_0$ and $\mathbf{S}_0'$ are the corresponding sets in bD_0, then it follows that $\Delta_0(w, z) \ne 0$ whenever $w \in \mathbf{S}_0$ and $z \in \mathbf{S}_0'$. Also

$$\sigma_0(\mathbf{S}_0) \approx \lambda_0(\mathbf{S}_0) \approx m(\mathbf{U}) \approx \delta^4 \quad \text{while} \quad \sigma_0(\mathbf{S}_0') \approx \lambda_0(\mathbf{S}_0') \approx m(\mathbf{U}') \approx \delta^3$$

where σ_0 denotes the induced Lebesgue measure on bD_0 and m is the standard Euclidean measure on the three-dimensional parameter space.

We shall now test the presumed inequality

$$(8) \qquad \|\mathbf{C}_0(f_0)\|_{L^p(\mathbf{S}_0', \, d\sigma_0)} \leq A_p \, \|f_0\|_{L^p(bD_0, \, d\sigma_0)} \, ,$$

when f_0 is assumed to have support in $\mathbf{S}_0$. Under these circumstances $\mathbf{C}_0(f_0)(z)$ is well-defined as an absolutely convergent integral

$$\int_{\mathbf{S}_0} \frac{1}{|\Delta_0(w, z)|^2} \, f_0(w) \, d\lambda_0(w) \, ,$$

when $z \in \mathbf{S}_0'$, in view of the non-vanishing of $\Delta_0(w, z)$ for these w and z. The constant A_p is of course assumed to be independent of f_0.

Proposition 1. *The inequality* (8) *fails for* $f_0 = \chi_{\mathbf{S}_0}$ *(the characteristic function of* $\mathbf{S}_0$*) for every* p, $1 \leq p \leq \infty$.

Proof. Throughout this proof we will use the notation C to denote a constant which may not be the same in different occurrences.

It is clear by (6) and (7) that we have:

$$\operatorname{Re} \Delta_0(w, z) \geq \frac{\delta^2}{4} \quad \text{and} \quad |\operatorname{Im} \Delta_0(w, z)| \leq a\delta^2 + 2a\delta^2 = 3a\delta^2.$$

Then $(\operatorname{Re} \Delta_0(w, z))^2 \geq 2 \, |\operatorname{Im} \Delta_0(w, z)|^2$, (which holds if $a \leq 1/12$). But $|\Delta_0(w, z)| \leq C\delta^2$, thus

$$\operatorname{Re} \frac{1}{(\Delta_0(w, z))^2} \geq C\delta^4 \, .$$

Now take $f_0 = \chi_{\mathbf{S}_0}$, the characteristic function of $\mathbf{S}_0$. Therefore

$$\operatorname{Re}\big(\mathbf{C}_0(f_0)\big)(z) \geq C\delta^{-4}\lambda_0(\mathbf{S}_0) \geq C > 0 \, , \quad \text{for} \quad z \in \mathbf{S}_0' \, .$$

So

$$\|\mathbf{C}_0(f_0)\|^p_{L^p(\mathbf{S}_0')} \geq C\lambda_0(\mathbf{S}_0') \geq C\delta^3 \, ,$$

while

$$\|f_0\|^p_{L^p(bD_0)} \leq \lambda_0(\mathbf{S}_0) \leq C\delta^4 \, .$$

Since δ^3 is not $O(\delta^4)$ for small δ, (8) cannot hold, when $p < \infty$. The case $p = \infty$ requires a separate but simpler argument which we omit. $\qquad \square$

3. The first counter-example

We turn to the domain (1) in $\mathbb{C}^2$ and it is useful to consider a translate of it, given by

$$(9) \qquad\qquad D = \{|z_2 - i|^2 + x_1^2 + y_1^4 < 1\}\,.$$

We will show after rescaling and a passage to the limit, that we can reduce consideration of D to D_0. From (9) it is clear that

$$\rho(z) = x_1^2 + y_1^4 + x_2^2 + y_2^2 - 2y_2$$

is a defining function for D, and that ρ is strongly pluri-subharmonic, so D is strongly pseudo-convex. Moreover, since each of the four one-variable functions, x_1^2, y_1^4, x_2^2, $y_2^2 - 2y_2$ are strictly convex, the domain D is itself strictly convex. We note that

$$\mathrm{Re}\,\Delta(w,z) = \mathrm{Re}\langle\partial\rho(w), w - z\rangle = \frac{1}{2}\big(\nabla\rho(w), w - z\big)_{\mathbb{R}}$$

where $(\cdot,\cdot)_{\mathbb{R}}$ is the real inner product induced on $\mathbb{R}^4 = \mathbb{C}^2$ from $\langle\cdot,\cdot\rangle$. With $w = (w_1, w_2)$, $w_j = u_j + i\,v_j$, $j = 1, 2$, we claim that

$$(10) \qquad\qquad \big(\nabla\rho(w), w - z\big)_{\mathbb{R}} \geq$$

$$\geq (x_1 - u_1)^2 + (x_2 - u_2)^2 + (y_2 - v_2)^2 + (v_1^2 + y_1^2)(v_1 - y_1)^2,$$

when $w, z \in bD$.

To prove (10) we use the identity

$$f(\beta) - f(\alpha) = (\beta - \alpha)f'(\alpha) + \int_\alpha^\beta (\beta - \alpha)f''(t)\,dt$$

for the functions $f_1 = u_1^2$, $f_2 = u_2^2$, $f_3 = v_2^2 - 2v_2$, and $f_4 = v_1^4$. Similarly, we replace $w = (u_1 + i\,v_1, u_2 + i\,v_2)$ by $z = (x_1 + i\,y_1, x_2 + i\,y_2)$ and add the corresponding identities. Taking into account that $\rho(w) = f_1 + f_2 + f_3 + f_4$ wih $\rho(w) = 0$ and the similar fact for $\rho(z)$, together with the observations that

$$\int_\alpha^\beta (\beta - \alpha)f''(t)\,dt = (\beta - \alpha)^2 \quad \text{if} \quad f''(t) = 2\,,$$

and

$$\int_\alpha^\beta (\beta - \alpha)f''(t)\,dt \geq (\beta^2 + \alpha^2)(\beta - \alpha)^2, \quad \text{if} \quad f''(t) = 12\,t^2\,,$$

then yields (10).

Now (10) shows that if w and $z \in bD$, $w \neq z$, then z lies on one side of the (real) tangent plane to bD at w. By convexity of D, the same holds for $z \in \overline{D} \setminus \{w\}$.

Turning to the Cauchy-Leray integral of D we recall two preliminary facts. First $\mathbf{C}(f)(z)$ is holomorphic in $z \in D$, if f is say an integrable function on bD. Second, whenever F is a holomorphic function in D which is continuous in $\overline{D}$, and $f = F\big|_{bD}$, then $\mathbf{C}(f)(z) = F(z)$, $z \in D$. The latter fact follows from the Cauchy-Fantappiè formalism (see [Ra] and [LS-3]).

Note that when f is a bounded function supported in a closed set $\mathbf{S}$ in bD, then $\mathbf{C}(f)(z)$ is well-defined as a convergent integral whenever z is outside the support $\mathbf{S}$. So certainly the extendability of $\mathbf{C}$ to a bounded operator on L^p would imply

$$(11) \qquad \|\mathbf{C}(f)\|_{L^p(\mathbf{S}')} \leq A_p \|f\|_{L^p}$$

whenever $\mathbf{S}'$ is disjoint from $\mathbf{S}$, with A_p independent of f, $\mathbf{S}$, and $\mathbf{S}'$.

Theorem 2. *For any p, $1 \leq p \leq \infty$, the presumed inequality* (11) *fails when tested for a bound A_p independent of f and its support $\mathbf{S}$, and with $\mathbf{S}'$ disjoint from $\mathbf{S}$.*

A further statement along these lines is made possible by the following fact (whose proof is given in [LS-2]): whenever f is of class C^1 on the boundary bD, $\mathbf{C}(f)$ is extendable to a continuous function on $\overline{D}$. With this we can define the Cauchy-Leray transform $\mathcal{C}(f)$ for such f, by $\mathcal{C}(f) = \mathbf{C}(f)\big|_{bD}$.

Corollary 3. *The mapping $f \mapsto \mathcal{C}(f)$, initially defined for C^1 functions f is not extendable to a bounded operator on $L^p(bD)$.*

4. Proof of Theorem 2

We shall obtain a contradiction to (11) by a scaling argument that leads us back to Proposition 1.

We define the maps τ_ϵ, $\epsilon > 0$, on $\mathbb{C}^2$ by $\tau_\epsilon(z_1, z_2) = (\epsilon z_1, \epsilon^2 z_2)$ and set $D_\epsilon = \tau_{\epsilon^{-1}}(D)$, with D as in (9). Then $\rho(z) = x_1^2 + y_1^4 + x_2^2 + y_2^2 - 2y_2$ is a defining function for D and $\rho_\epsilon(z) = \epsilon^{-2}\rho(\tau_\epsilon(z))$ is a defining function for D_ϵ. Note that

$$\rho_\epsilon(z) = x_1^2 - 2y_2 + \epsilon^2(y_1^4 + x_2^2 + y_2^2),$$

from which it is clear that the domains D_ϵ increase as ϵ decreases, with limit our model domain $D_0 = \{\rho_0(z) < 0\}$ and $\rho_0(z) = x_1^2 - 2y_2$.

Observe also that

$$\rho_\epsilon(z) \to \rho_0(z), \quad \text{and} \quad \nabla\rho_\epsilon(z) \to \nabla\rho_0(z)$$

uniformly on compact subsets of $\mathbb{C}^2$.

We next define the "transported" measures $d\lambda_\epsilon$ on bD_ϵ by

$$(12) \qquad \int\limits_{w \in bD_\epsilon} F(w)\, d\lambda_\epsilon(w) \;=\; \epsilon^{-4} \int\limits_{w \in bD} F(\tau_{\epsilon^{-1}}(w))\, d\lambda(w)$$

for any continuous functions F defined on bD_ϵ.

Here $d\lambda$ is the Leray-Levi measure on bD, but note that $d\lambda_\epsilon$ is not the Leray-Levi measure on bD_ϵ: as a result the operator $\mathbf{C}_\epsilon$ defined below is *not* the Cauchy-Leray integral of bD_ϵ. We can also define the corresponding action of τ_ϵ on functions f on bD by

$$\tau_\epsilon(f) = f \circ \tau_\epsilon\,.$$

Then by what has been said above, τ_ϵ maps $L^p(bD, d\lambda)$ to $L^p(bD_\epsilon, d\lambda_\epsilon)$ and we have the "isometry":

$$(13) \qquad\qquad \|f\|_{L^p(bD, d\lambda)} \;=\; \epsilon^{4/p}\, \|\tau_\epsilon(f)\|_{L^p(bD_\epsilon, d\lambda_\epsilon)}\,.$$

Now let $\Delta_\epsilon(w, z) = \langle \partial\rho_\epsilon(w), w - z\rangle$. Then

$$\Delta(\tau_\epsilon(w), \tau_\epsilon(z)) = \langle \partial\rho(\tau_\epsilon w), \tau_\epsilon(w-z)\rangle = \epsilon^2 \langle \partial\rho_\epsilon(w), w-z\rangle = \epsilon^2 \Delta_\epsilon(w, z)\,.$$

Hence

$$(14) \qquad\qquad \Delta_\epsilon(w, z) \;=\; \epsilon^{-2}\, \Delta(\tau_\epsilon(w), \tau_\epsilon(z)).$$

We now define the operator $\mathbf{C}_\epsilon$, by setting

$$(15) \qquad\qquad \mathbf{C}_\epsilon(F)(z) \;=\; \int\limits_{bD_\epsilon} \frac{1}{\Delta_\epsilon(w, z)^2}\, F(w)\, d\lambda_\epsilon(w)$$

which is well-defined as the integral above for any bounded function F, as long as z is outside the support of F. Our next claim is that

$$(16) \qquad\qquad \mathbf{C}_\epsilon(F)(z) \;=\; \mathcal{C}(F \circ \tau_{\epsilon^{-1}})(\tau_\epsilon(z)),$$

for bounded F on bD_ϵ, if $z \in bD_\epsilon$ lies outside the support of F. In fact going back to the definition (3), and using (12) and (14), we see that

$$\mathbf{C}(F \circ \tau_{\epsilon^{-1}})(\tau_\epsilon(z)) \;=\; \int\limits_{bD} \frac{1}{\Delta(w, \tau_\epsilon(z))^2}\, F(\tau_{\epsilon^{-1}}(w))\, d\lambda(w) \;=$$

$$\epsilon^4 \int\limits_{bD_\epsilon} \frac{1}{\Delta(\tau_\epsilon(w), \tau_\epsilon(z))^2}\, F(w)\, d\lambda_\epsilon(w) \;=\; \int\limits_{bD_\epsilon} \frac{1}{\Delta_\epsilon(w, z)^2}\, F(w)\, d\lambda_\epsilon(w)$$

showing (16).

Next, if (11) held, then by (13) we would also have

$$(17) \qquad\qquad \|\mathbf{C}_\epsilon(F)\|_{L^p(\mathbf{S}'_\epsilon, d\lambda_\epsilon)} \leq A_p \|F\|_{L^p(bD_\epsilon, d\lambda_\epsilon)}$$

whenever F is a bounded function on bD_ϵ and $\mathbf{S}'_\epsilon$ is a closed subset of bD_ϵ, disjoint from the support of F.

At this point we restrict the attention to the unit ball $\mathbb{B}$ in $\mathbb{C}^2$, and we exploit the common coordinate system for $bD_0 \cap \mathbb{B}$, and $bD_\epsilon \cap \mathbb{B}$, when ϵ is small. That is in this ball, bD_0 is the graph over $(x_1 + i\, y_1, x_2)$ given by $(x_1 + i\, y_1, x_2 + i\, x_1^2/2)$, while bD_ϵ is the graph given by $(x_1 + i\, y_1, x_2 + i\, (x_1^2/2 + \Phi_\epsilon(x_1, y_1, x_2)))$, with $\Phi_\epsilon(x_1, y_1, x_2) = O(\epsilon^2)$.

Now if in these coordinates we write $d\lambda = \Lambda(x_1, y_1, x_2)\, dx_1\, dy_1\, dx_2$ and $d\lambda_\epsilon = \Lambda_\epsilon(x_1, y_1, x_2)\, dx_1\, dy_1\, dx_2$, then by (12) we have

$$(18) \qquad \Lambda_\epsilon(x_1, y_1, x_2) \;=\; \Lambda(\epsilon\, x_1, \epsilon\, y_1, \epsilon^2 x_2)\,.$$

Finally we take $\mathbf{S}$ to be the set in bD corresponding to $\mathbf{U}$ in (7), and f to be the function $\chi_\mathbf{S}$, the characteristic function of $\mathbf{S}$. We lift f to functions on bD_ϵ and bD_0 respectively, by setting $F(z_\epsilon) = f(x_1, y_1, x_2)$ when $z_\epsilon = (x_1 + i\, y_1, x_2 + i\, (x_1^2/2 + \Phi_\epsilon(x_1, y_1, x_2)))$, and $f_0(z_0) = f(x_1, y_1, x_2)$ when $z_0 = (x_1 + i\, y_1, x_2 + i\, x_1^2/2)$.

We also lift the sets $\mathbf{U}$ and $\mathbf{U}'$ in (7), to $\mathbf{S}_\epsilon$ and $\mathbf{S}'_\epsilon$ (subsets of bD_ϵ) in the same way. Our claim is with that notation

$$(19) \qquad \mathbf{C}_\epsilon(F)(z_\epsilon) \;\to\; \mathbf{C}_0(f_0)(z_0)\,, \quad \text{if } z_0 \in \mathbf{S}'_0\,.$$

In fact,

$$\mathbf{C}_\epsilon(F)(z_\epsilon) \;=\; \int\limits_\mathbf{S} \frac{1}{\Delta_\epsilon(w, z_\epsilon)^2}\, f(u_1, v_1, u_2)\, \Lambda_\epsilon(u_1, v_1, u_2)\, du_1\, dv_1\, du_2\,.$$

However by (18) $\Lambda_\epsilon(u_1, v_1, u_2) \to \Lambda(0,0,0)$, and moreover $\Delta_\epsilon(w, z_\epsilon) \to \Delta_0(w, z_0)$ because $\nabla\rho_\epsilon \to \nabla\rho_0$, while $\Delta_0(w, z_0) \neq 0$ if $w \in \mathbf{S}_0$ and $z_0 \in \mathbf{S}'_0$. This gives (19).

As a result (17) leads to

$$\|\mathbf{C}_0(f_0)\|_{L^p(\mathbf{S}'_0, d\lambda_0)} \leq A_p \|f_0\|_{L^p(bD_0, d\lambda_0)}$$

which contradicts Proposition 1, proving the theorem.

5. The second counter-example

Here the domain will be taken to be

$$(20) \qquad D = \{|z_2 - i|^2 + |x_1|^m + y_2^2 < 1\}, \quad \text{with} \quad 1 < m < 2\,.$$

Its model domain is

$$(21) \qquad D_0 \;=\; \{2\, \mathrm{Im}\, z_2 > |x_1|^m\}.$$

For any f that is bounded on bD, the Cauchy-Leray integral $\mathbf{C}(f)(z)$ is well-defined for z that lies in the complement of the support of f.

As in the previous sections we will show that the mapping $f \mapsto \mathbf{C}(f)$ fails to be bounded in L^p in the sense that the proposed inequality

$$(22) \qquad \|\mathbf{C}(f)\|_{L^p(\mathbf{S}',d\sigma)} \leq A_p \|f\|_{L^p(bD,d\sigma)}$$

cannot hold. Here $\mathbf{S}'$ is any set disjoint from the support of f, and the bound A_p is assumed independent of f and $\mathbf{S}'$.

The proof of this assertion follows the same lines as in Sections 2-4 for the domain (9), and so we will only discuss the minor differences that occur.

The defining function of D is $\rho(z) = |x_1|^m + y_1^2 + x_2^2 + y_2^2 - 2y_2$, and that of D_0 is $\rho_0(z) = |x_1|^m - 2y_2$. Note that both domains are of class $C^{2-\alpha}$, with $\alpha = 2 - m$. Also since $|x_1|^m$, y_1^2, x_2^2, $y_2^2 - 2y_2$ are strongly convex functions of one variable, the domain D is strongly convex, hence strongly $\mathbb{C}$-linearly convex, and in fact strongly pseudoconvex in the following sense: the domain D is exhausted by an increasing family of smooth domains $\{D_\gamma\}_\gamma$ which are uniformly strongly pseudo-convex, with defining function: $\rho_\gamma(z) = (x_1^2 + \gamma)^{m/2} + y_1^2 + x_2^2 + y_2^2 - 2y_2$.

This convexity implies that $\operatorname{Re}\Delta(w, z) > 0$ for $w \in bD$ and $z \in \overline{D}$, except when $z = w$.

Returning to the model domain, if $\Delta_0(w, z) = \langle \partial\rho_0(w), w - z \rangle$ a calculation gives

$$(23) \qquad \Delta_0(w, z) =$$

$$= |x_1|^m - |u_1|^m + m\,[u_1]^{m-1}(u_1 - x_1) + i\left(m\,[u_1]^{m-1}(v_1 - y_1) + 2(x_2 - y_2)\right).$$

Here we have used the notation

$$[u_1]^{m-1} \;=\; \frac{1}{m}\frac{d}{du_1}|u_1|^m \;=\; |u_1|^{m-1}\operatorname{sign} u_1\,.$$

Now we set

$$(24) \qquad \begin{cases} \mathbf{U} \;=\; \{|u_1| \leq a\,\delta^2\,,\ |v_1| \leq \delta^{2-m}\,,\ |u_2| \leq \delta^m\} \\[2mm] \mathbf{U}' \;=\; \{\delta \leq |x_1| \leq 2\delta\,,\ |y_1| \leq \delta^{2-m}\,,\ |x_2| \leq \delta^m\} \end{cases}$$

and let $\mathbf{S}_0$ and $\mathbf{S}_0'$ be the corresponding induced sets on bD_0.

We have that near the origin

$$d\lambda_0 \approx |u_1|^{m-2}\,du_1\,dv_1\,du_2, \quad \text{and} \quad d\sigma_0 \approx du_1\,dv_1\,du_2\,,$$

where $d\lambda_0$ and $d\sigma_0$ are the Leray-Levi measure and the induced Lebesgue measure on bD_0. Thus

$$(25) \qquad\qquad \lambda_0(\mathbf{S}_0) \;\approx\; \delta^{2m}$$

$$(26) \qquad\qquad \sigma_0(\mathbf{S}_0) \;\approx\; \delta^4\,, \quad \text{and} \quad \sigma_0(\mathbf{S}_0') \;\approx\; \delta^3\,.$$

By (23) it follows that

$$\operatorname{Re}\Delta_0(w, z) \gtrsim \delta^m, \quad \text{while} \quad |\operatorname{Im}\Delta_0(w, z)| \lesssim \delta^m,$$

and if we choose a sufficiently small, then

$$\operatorname{Re}\frac{1}{\left(\Delta_0(w, z)\right)^2} \ge c\,\delta^{-2m} \quad \text{if } w \in \mathbf{S}_0, \quad \text{while } z \in \mathbf{S}_0'.$$

Taking f_0 to be the characteristic function of $\mathbf{S}_0$, we therefore get $\operatorname{Re}\mathbf{C}_0(f_0)(z) \ge c > 0$, for $z \in \mathbf{S}_0'$. Hence this gives a contradiction to (22) in the case when $\mathbf{C}_0$ is the Cauchy-Leray integral of the model domain D_0.

To pass to the domain D we carry out the scaling via

$$(27) \qquad\qquad \tau_\epsilon(z_1, z_2) = (\epsilon z_1, \epsilon^m z_2)$$

The domain $D_\epsilon = \tau_{\epsilon^{-1}}(D)$ has a defining function

$$\rho_\epsilon(z) = \epsilon^{-m}\rho(\tau_\epsilon(z)) = \rho_0(z) + \epsilon^m(x_2^2 + y_2^2) + \epsilon^{2-m}y_1^2,$$

which converges to $\rho_0(z) = |x_1|^m - 2y_2$.

We also set $\Delta_\epsilon(w, z) = \langle\partial\rho_\epsilon(w), w - z\rangle$. The transported measure $d\lambda_\epsilon$ on D_ϵ is now defined by the identity

$$(28) \qquad \int\limits_{w \in bD_\epsilon} F(w)\,d\lambda_\epsilon(w) = \epsilon^{-2m}\int\limits_{w \in bD} F(\tau_{\epsilon^{-1}}(w))\,d\lambda(w),$$

(compare with (12)).

Finally, if we assumed that (22) held for the domain D we can then see by the reasoning in Section 4 that the corresponding result would hold for the model domain D_0 achieving our desired contradiction.

We point out in closing that (22) fails not only for the L^p norms taken with the induced Lebesgue measure but others as well. Consider the measures $d\mu_a$ on bD given by

$$(29) \qquad\qquad d\mu_a = \mathcal{L}^a\,d\sigma,$$

with $\mathcal{L}$ the determinant of the Levi-form, and $d\sigma$ the induced Lebesgue measure.

Then the argument above shows that (22) fails when

$$-\infty < a < \frac{1}{2 - m},$$

(because $\mathcal{L}^a \approx |x_1|^{(m-2)a}$). Here the factor ϵ^{-2m} in (28) is replaced with $\epsilon^{a(2-m)-2-m}$ to reflect the new transported measures $d\mu_{a,\epsilon}$.

The case $a = 0$ corresponds to induced Lebesgue measure; the case $a = 1$, to the Leray-Levi measure $d\lambda$, and the case $a = 1/3$ corresponds

to the Fefferman measure [B2], [F-1], [G]. (Here the expression "A corresponds to B" may take the meaning that $A \approx B$.)

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