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Intersection numbers from companion tensor algebra

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ABSTRACT: Twisted period integrals are ubiquitous in theoretical physics and mathematics, where they inhabit a finite-dimensional vector space governed by an inner product known as the intersection number. In this work, we uncover the associated tensor structures of intersection numbers and integrate them with the fibration method to develop a novel evaluation scheme. Companion matrices allow us to cast the computation of the intersection numbers in terms of a matrix operator calculus within the ambient tensor space. For illustrative purposes, our algorithm has been successfully applied to the numerical decomposition of a sample of two-loop integrals, coming from planar five-point massless functions, representing a significant advancement for the direct projection of Feynman integrals to master integrals via intersection numbers.

KEYWORDS: Differential and Algebraic Geometry, Higher-Order Perturbative Calculations, Scattering Amplitudes

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1 Introduction

Understanding the properties of *twisted period integrals* is of fundamental importance in theoretical physics and mathematics, as they appear in a plethora of applications, ranging from Correlation Functions in Classical and Quantum Field Theory, Feynman Integral and Scattering Amplitudes computation, Cosmological Wavefunctions of the Universe, String Theory, and Gravitational Waves Physics, as well as in the formal studies of special functions like the Aomoto-Gauß Hypergeometric functions, Euler-Mellin integrals, and Gelfand-Kapranov-Zelevinsky systems, and their applications in the realm of differential geometry, combinatorics, and statistics.

Defined as integrals of a differential n -form weighted by a regulated multivalued function, known as the *twist*, that vanishes at the boundary of the integration domain, twisted period integrals admit a finite-dimensional vector space structure, whose characteristics can be explored using the framework of intersection theory for twisted cohomology [1–13]. Differential n -forms live in a vector space defined as the quotient space of the closed forms

modulo the exact forms, known as twisted de Rham cohomology group [14–17]. The dimension, as well as the algebraic and analytic properties of this group, depend on the geometry of the twist, namely on its zeroes and its critical points. Once a basis of forms spanning this space has been introduced, it is possible to derive linear and quadratic relations among the elements, as well as differential and difference equations. The elementary quantities that define such relations, are given by *intersection numbers* of meromorphic differential n -forms, which act as an inner product in the cohomology space. Developing optimal algorithms for calculating intersection numbers for meromorphic twisted n -forms remains a challenge of significant interest to both mathematicians and physicists, and different methods have been developed, leveraging the twisted version of the Stokes’ theorem [18]. The most efficient approach to evaluate n -form intersection numbers nowadays, as detailed in [16, 17, 19], is based on the concept of *fibration* [10], an approach that amounts to recursive evaluation of intersection numbers for vector-valued 1-forms, for which a systematic algorithm is known [14]. Improving the evaluation of 1-form intersection numbers is then of fundamental importance in order to make the fibration algorithm more efficient. One simplification arises from exploiting the invariance of the representative of the cohomology classes and by avoiding the use of algebraic extensions [20]. Singularities not regulated by the twist were originally addressed by adding extra analytic regulators [16, 17], which made computations more cumbersome. Remarkably, such singularities are naturally incorporated in the framework of *relative twisted cohomology* [21], and are managed using multivariate Leray coboundaries, eliminating the need for additional parameters [22, 23]. A noteworthy enhancement in this evaluation process, which avoids the appearance of algebraic extension, has been introduced in [24] and applied in [25], exploiting techniques from computational algebraic geometry such as *polynomial ideals* and the global residue theorem, and employing the technology of the finite fields reconstruction of rational functions [26, 27]. As an alternative to the fibration technique, a direct computation method has been implemented in the logarithmic case [2, 8, 28, 29], while for generic meromorphic n -forms only partial results are known in literature [2, 18, 30].

It has been recently proposed another technique [31], based on the solution of the *secondary equation* built from the *Pfaffian system* of differential equations for the generators of the cohomology group [9, 32], obtained via an efficient algorithm for construction of such systems by means of the Macaulay matrix (see also [33] for another application). In mathematics, intersection theory has been widely used to study the vector space structure of Aomoto-Gelfand and Gauß hypergeometric integrals, as well as Gelfand-Kapranov-Zelevinsky hypergeometric systems [31, 34, 35]. Our main focus in developing intersection theory is to apply it to Feynman integrals [14], which are among the key elements in perturbative Quantum Field Theory (pQFT), and can be seen as singular limit of the former mathematical structures [36] (see also [37, 38]). After a systematic change to a parametric representation [14, 39–41], the twisted period integral structure becomes evident, and n -forms of dimensionally regulated Feynman integrals become elements of a relative cohomology group, with relative surfaces corresponding to the physical propagators. In this framework, linear relations among differential n -forms are known as *integration-by-parts* (IBP) identities [42, 43], in which integrals are decomposed in terms of a basis of master integrals (MIs). These integrals obey differential equations, whose canonical form [44–48] (in presence of generalised polylogarithms

as well as of elliptic functions), properties [49], and their symmetries [50, 51] can be investigated in terms of intersection numbers. Moreover, the symbol letters, which are related to the singularities appearing in the DEs, have been studied using intersection theory alongside a novel understanding of the recursive structure of the Baikov representation [52–54]. Besides that, quadratic relations among FIs are known as twisted Riemann bilinear relations, and they can also be investigated by means of intersection numbers [55–57]. In the previous work [25], these techniques were applied to the decomposition of two-loop four-point Feynman Integrals, planar and non-planar, massless and with one external mass, requiring the evaluation of 9-form intersection numbers. Beyond FIs, twisted cohomology found many new applications in physics: it can be applied to Correlation Functions in quantum mechanics [58], and lattice gauge theory [59–62]. Recent studies showed the applicability also to Fourier transform [63], seen as confluent hypergeometric function [64, 65] such as those appearing in gravitational waves waveforms computations [66].¹ Cosmological wavefunctions of the Universe [68], that for years have been computed using standard pQFT techniques, are now being understood to have a twisted period integral structure [69–72], where the intersection theory can be applied.

In this work, we propose an improved framework to evaluate n -form intersection numbers that reveals the underlying tensor structures.

Evaluation of 1-form intersection numbers requires local holomorphic solutions of differential equations near the singularities of the connection. Usually, these solutions are found by constructing ansätze at each singularity. However, as shown in [24], this process can be bypassed using polynomial division method and the global residue theorem. By introducing an appropriate polynomial ideal [25], whose vanishing set encompasses the singularities of the connection, and choosing an element of the quotient ring as the new unique ansatz, we simplify the computation.

We advance this idea by treating the ansatz as an element of a tensor space composed of three factors: the vector-valued cohomology of the fibration layer, the quotient ring, and the space of Laurent expansions around the vanishing set of the polynomial ideal. Adopting this approach, we reformulate the computation of intersection numbers in the language of matrix calculus. Companion matrices, which naturally encode the polynomial division operation in terms of matrix multiplication, are central to this reformulation. Additionally, the Weyl algebra of differential operators gets represented by infinite-dimensional operators acting on the space of Laurent expansions.

By combining these two representations, the differential equation is reinterpreted through tensor product companion matrices with four indices acting on the ansatz, while the global residue is translated into a covector on the tensor space. This approach not only illuminates the mathematical properties and patterns of intersection numbers, but also greatly enhances computational performance. The primary advantage of the proposed algorithm lies in its reliance on algebraic and matrix operations, which can be efficiently coupled with finite field reconstruction techniques.

To demonstrate the effectiveness of our method, we apply it to the numerical decomposition of two-loop five-point functions, which require the computation of 11-form intersection

¹See also [67] for another application to post-Minkowskian integrals.

numbers. This application marks a significant milestone in the full decomposition of two-loop Feynman integrals using projection through intersection numbers.

The work is organised as follows: in section 2 we give an overview of intersection numbers for relative twisted de Rham cohomology, and of the twisted period integral structure of Feynman integrals. In section 3 we introduce the tensor structure of intersection numbers and the companion matrix representation. In section 4 we show the effectiveness of our novel method by applying it to the reduction of massless two-loop five-point functions. We provide concluding remarks in section 5.

The manuscript contains three appendices: in appendix A we show that polynomial division operation can be read as a change of coordinates. In appendix B we show a pedagogical example in which we carry out a step-by-step evaluation of intersection numbers in a univariate example, and in appendix C we give details on the computer implementation of our framework.

For our research, the following software has been used: LITERED [73, 74], FINITEFLOW [27], FERMAT [75], FERMATICA [76], MATHEMATICA, SINGULAR [77] and its MATHEMATICA interface SINGULAR.M [78], and JAXODRAW [79, 80].

2 Relative twisted cohomology and intersection theory

In this section we review in a pragmatic way the main concepts of twisted cohomology and intersection theory, focusing on the computational aspects of 1- and n -form intersection numbers, in the framework [25] combining the *global residue theorem* [20], the *polynomial division* method [24], and *relative twisted cohomology* [22, 23].

2.1 Twisted period integrals

Objects of our study are *twisted period integrals* and their duals, defined as bilinear products between differential n -forms: $\varphi = \widehat{\varphi}(z) d^n z \equiv \widehat{\varphi}(z) dz_1 \wedge \dots \wedge dz_n$ (similarly for the dual form $\varphi^\vee$), and n -dimensional integration domains $\mathcal{C}$ and $\mathcal{C}^\vee$:

$$I = \int_{\mathcal{C}} u \varphi := \langle \varphi | \mathcal{C} \rangle, \quad I^\vee = \int_{\mathcal{C}^\vee} u^{-1} \varphi^\vee := [\mathcal{C}^\vee | \varphi^\vee], \quad u = \prod_i B_i^{\gamma_i}, \quad (2.1)$$

where u is a multivalued function called the *twist*, with polynomial factors $B_i = B_i(z)$ and generic exponents γ_i .

The integration domains are defined in such a way that the twist vanishes on their boundaries: $\prod_i B_i(\partial\mathcal{C}) = \prod_i B_i(\partial\mathcal{C}^\vee) = 0$, ensuring that the integrals (2.1) obey integration-by-parts (IBPs) identities:

$$\langle \nabla_\omega \phi | \mathcal{C} \rangle = 0, \quad [\mathcal{C}^\vee | \nabla_{-\omega} \phi^\vee] = 0, \quad (2.2)$$

where ϕ and $\phi^\vee$ are generic $(n-1)$ -forms, and $\nabla_{\pm\omega}$ is a covariant derivative with connection ω defined as

$$\nabla_{\pm\omega} := d \pm \omega, \quad \text{with} \quad \omega := d \log(u) = \sum_{i=1}^n \widehat{\omega}_i dz_i, \quad \text{and} \quad \widehat{\omega}_i := \partial_{z_i} \log(u). \quad (2.3)$$

As a result, $\langle \varphi |$ and $|\varphi^\vee\rangle$ are elements of the n^{th} relative cohomology group H^n and its dual $H^{\vee n}$, which we write, following the notation of [21–23], as

$$\langle \varphi | \in H^n := H^n(T, D, \nabla_\omega), \quad |\varphi^\vee\rangle \in H^{\vee n} := H^n(T^\vee, D^\vee, \nabla_{-\omega}). \quad (2.4)$$

The ambient spaces are the Zariski-open subsets of the complex n -dimensional space

$$T = \mathbb{C}^n \setminus \mathcal{P}_\omega \setminus D^\vee, \quad T^\vee = \mathbb{C}^n \setminus \mathcal{P}_\omega \setminus D, \quad (2.5)$$

where the polar set $\mathcal{P}_\omega = V(\prod_i B_i)$, is the vanishing locus V of the polynomial factors (2.1), and $D, D^\vee$ are relative boundaries, which are sets of singularities not regulated by the twist.

Analogously, the integration domains $\mathcal{C}$ and $\mathcal{C}^\vee$ are elements of the relative n^{th} homology group H_n and its dual $H_n^\vee$. De Rham's theorem ensures that the four vector spaces are isomorphic, hence they have the same finite dimension, which can be evaluated as:

$$\begin{aligned} \nu &= \dim(H^n) = \dim(H^{\vee n}) = \dim(H_n) = \dim(H_n^\vee) \\ &= \text{number of zeros of } d \log(u_\rho). \end{aligned} \quad (2.6)$$

Assuming $D = V(D_1 \cdot \dots \cdot D_{n_D})$ and $D^\vee = V(D_1^\vee \cdot \dots \cdot D_{n_D^\vee}^\vee)$, with n_D and $n_D^\vee$ number of relative surfaces, u_ρ is the fully regulated twist, defined as:

$$u_\rho = u(z) \prod_{i=1}^{n_D} (D_i)^{\rho_i} \prod_{j=1}^{n_D^\vee} (D_j^\vee)^{\tau_j}, \quad (2.7)$$

where ρ_i and τ_j are analytic regulators. The dimension (2.6) is also related to the number of critical points of the Morse height function $\log(|u_\rho|)$, see [40].

Besides the two pairing that define the integrals $\langle \varphi | \mathcal{C} \rangle$ and the dual $[\mathcal{C}^\vee | \varphi^\vee]$, it is possible to define the other two quantities: *intersection numbers* for twisted cycles $[\mathcal{C}^\vee | \mathcal{C}]$ and twisted cocycles $\langle \varphi | \varphi^\vee \rangle$. The latter is going to be the main subject of this work.

2.2 Linear relations

The finite dimensionality of H^n and $H^{\vee n}$, ensures the existence of bases $\{\langle e_i | \}_{i=1}^\nu$ and $\{| h_i \rangle \}_{i=1}^\nu$ belonging respectively to H^n and $H^{\vee n}$, from which it is possible to construct square matrix of intersection numbers called the **C**-matrix or *metric*

$$\mathbf{C}_{ij} := \langle e_i | h_j \rangle. \quad (2.8)$$

The fact that the rank of this matrix is ν means that the intersection numbers define a non-degenerate scalar product. Moreover, any arbitrary cocycle $\langle \varphi |$ can be decomposed in terms of the basis elements [14, 15] via the *master decomposition formula*

$$\langle \varphi | = \sum_{i=1}^\nu c_i \langle e_i |, \quad \text{with} \quad c_i = \sum_{j=1}^\nu \langle \varphi | h_j \rangle (\mathbf{C}^{-1})_{ji}. \quad (2.9)$$

Consequently, any twisted-period integral I can be decomposed in terms of a basis of *master integrals* (MIs) $\mathcal{I}_i$ as

$$I = \langle \varphi | \mathcal{C} \rangle = \sum_{i=1}^\nu c_i \langle e_i | \mathcal{C} \rangle = \sum_{i=1}^\nu c_i \mathcal{I}_i, \quad (2.10)$$

with the coefficients of the decomposition c_i given by eq. (2.9) and $\mathcal{I}_i = \langle e_i | \mathcal{C} \rangle$.

2.3 1-form intersection numbers

In the case of 1-forms, intersection numbers between twisted cocycles of the type:

$$\varphi \in H^1(T, D; \nabla_\omega) \quad \text{and} \quad \varphi^\vee \in H^1(T^\vee, D^\vee; \nabla_{-\omega}) \quad (2.11)$$

are defined as integrals of the product of a representative form and its dual, after an appropriate regularisation procedure:

$$\langle \varphi | \varphi^\vee \rangle := \frac{1}{2\pi i} \int_{\mathcal{X}} \iota(\varphi) \wedge \iota^\vee(\varphi^\vee). \quad (2.12)$$

In this case $\mathcal{X} = T \cap T^\vee$, and the regulators act on the forms schematically² as:

$$\begin{aligned} \iota(\varphi) &:= \varphi - \sum_{p \in \mathcal{P}_\omega \cup D} \nabla_\omega[(1 - \theta_{z,p})\psi], \\ \iota^\vee(\varphi^\vee) &:= \varphi^\vee - \sum_{p \in \mathcal{P}_\omega \cup D^\vee} \nabla_{-\omega}[(1 - \theta_{z,p})\psi^\vee], \end{aligned} \quad (2.13)$$

with the Heaviside functions defined as:

$$\theta_{z,p} := \theta(|z-p| - \epsilon), \quad (2.14)$$

where we introduced a small real parameter $0 < \epsilon \ll 1$. Above the ψ and $\psi^\vee$ are the local solutions of the differential equations:

$$\nabla_\omega \psi = \varphi, \quad \nabla_{-\omega} \psi^\vee = \varphi^\vee \quad (2.15)$$

on T and $T^\vee$ respectively. As originally shown in [21], intersection numbers of twisted 1-forms with integration variable z can be evaluated as:

$$\langle \varphi | \varphi^\vee \rangle = - \sum_{p \in \mathcal{P}_\omega} \text{Res}_{z=p}(\psi^\vee \varphi) - \sum_{p \in D^\vee} \text{Res}_{z=p}(\psi^\vee \varphi) + \sum_{p \in D} \text{Res}_{z=p}(\psi \varphi^\vee). \quad (2.16)$$

For the computation of the intersection numbers it is sufficient to find the *local* solutions of eq. (2.15) around the poles $\mathcal{P}_\omega$ (as well as the corresponding relative set D or $D^\vee$) contributing to the residue formula (2.16). As discussed³ in [20, 24], the sum over the poles $\mathcal{P}_\omega$ can be computed as the *global residue*

$$\sum_{p \in \mathcal{P}_\omega} \text{Res}_{z=p}(\psi^\vee \varphi) = \text{Res}_{(\mathcal{B})}(\psi^\vee \varphi), \quad (2.17)$$

where $\mathcal{B}(z)$ is a univariate polynomial vanishing on the polar set (see appendix C for more details on its choice)

$$\mathcal{P}_\omega \subseteq V(\mathcal{B}). \quad (2.18)$$

²Strictly speaking, the local solutions ψ appearing in eq. (2.13) should bear an additional index p corresponding to the points they are localized at. In the rest of this review we will avoid it to simplify the notation, however see appendix A for more details.

³See also [81] for a nice review.

The vector of local solutions (2.15) can be viewed⁴ as an element of the quotient ring [25] $\mathcal{Q} := \mathbb{K}[z]/\langle \mathcal{B} \rangle$, where $\mathbb{K}[z]$ is the space of polynomials in the variable z with coefficients in the field $\mathbb{K}$, $\langle \mathcal{B} \rangle$ is the ideal generated by the polynomial $\mathcal{B}(z) - \beta$ deformed by the symbolic parameter β .

In our applications, $D = \emptyset$ and only simple poles in $D^\vee$ will appear, hence the last term on the r.h.s. of eq. (2.16) will not contribute to the intersection number. This is always the case for FIs in Baikov representation, since higher order poles can be mapped into derivatives of the twist via integration by parts.

Moreover, we will distinguish the case in which $\varphi^\vee$ is a boundary-supported form, meaning a form living on $D^\vee$ as defined below, or not.

Regulated case. If $\varphi^\vee$ is not a boundary-supported form, only the first term on the r.h.s. of eq. (2.16) will contribute to the intersection number, so it becomes:

$$\langle \varphi | \varphi^\vee \rangle = -\text{Res}_{\langle \mathcal{B} \rangle} \left(\varphi \psi \right), \quad (2.19)$$

where from now on we omit the $\vee$ sign from the solution. Therefore, working in $\mathcal{Q}$, the intersection number can be evaluated by solving the system of equations:

$$\langle \varphi | \varphi^\vee \rangle + \text{Res}_{\langle \mathcal{B} \rangle} \left(\varphi \psi \right) = 0, \quad (2.20)$$

$$\left[\hat{\nabla}_{-\omega} \psi - \hat{\varphi}^\vee \right]_{\langle \mathcal{B} \rangle} = 0, \quad (2.21)$$

where the bracket $\left[\right]_{\langle \mathcal{B} \rangle}$ stands for the remainder of the polynomial division w.r.t. $\mathcal{B}$, and

$$\hat{\nabla}_{-\omega} \equiv (\partial_z \mathcal{B}) \partial_\beta - \hat{\omega} + \partial_z. \quad (2.22)$$

The solution can be obtained by the following ansatz [24]:

$$\psi(\beta, z) = \sum_{a=0}^{\kappa-1} \sum_{n \in \mathbb{Z}} z^a \beta^n \psi_{an}, \quad \text{with } \kappa := \deg(\mathcal{B}), \quad (2.23)$$

where the Laurent series expansion coefficients ψ_{an} vanish for small enough n . The global residue can then be evaluated as:

$$\langle \varphi | \varphi^\vee \rangle = -\frac{1}{\ell_c} \left[\hat{\varphi} \psi \right]_{\langle \mathcal{B} \rangle} \Big|_{z^{\kappa-1} \beta^{-1}}, \quad (2.24)$$

that is the coefficient of the term $z^{\kappa-1} \beta^{-1}$ divided by the leading coefficient ℓ_c of the $\mathcal{B}$ polynomial.

⁴See appendix A for details of this transformation.

Relative case. Assuming $z \in D^\vee$ to be a non-regulated boundary, we allow the existence of boundary-supported forms in $H^{\vee 1}$ defined via the Leray coboundary operation [21–23, 25]

$$\delta_z(\phi^\vee(z)) := \frac{u(z)}{u(0)} d\theta_{z,0} \phi^\vee(z), \quad (2.25)$$

where $\phi^\vee(z)$ is a generic univariate function regular at $z = 0$ and the Heaviside function is defined as in eq. (2.14). In this case, only the second term in eq. (2.16) will contribute, and intersection numbers localise on the non-regulated pole as:

$$\langle \varphi | \delta_z(\phi^\vee(z)) \rangle := \text{Res}_{z=0} \left(\frac{u(z)}{u(0)} \varphi \phi^\vee(z) \right). \quad (2.26)$$

Let us remark, that an analogous left form can be introduced for $z \in D$, however it will not be used in the applications considered in this work.

2.4 n -form intersection numbers

Intersection numbers for n -forms can be evaluated using the *fibration*-based approach introduced in [10, 16, 17, 19] and discussed more recently in [22–25], in which the integration variables are considered one at a time. Once an order of variables has been fixed,⁵ for example $\{z_n, \dots, z_1\}$, where on the right appears the innermost variable, multivariate intersection numbers can be evaluated recursively.

Let us focus on the m^{th} fiber, assuming that all the $(m-1)$ -variate building blocks are known. Specifically, given the bases $\{e_i^{(\mathbf{m}-1)}\}_{i=1}^{\nu_{m-1}}$ and $\{h_i^{(\mathbf{m}-1)}\}_{i=1}^{\nu_{m-1}}$ of H^{m-1} and its dual $H^{\vee m-1}$, respectively, we define the $\mathbf{C}$ -matrix at the $(m-1)^{\text{th}}$ layer as:

$$\mathbf{C}_{ij}^{(m-1)} := \langle e_i^{(\mathbf{m}-1)} | h_j^{(\mathbf{m}-1)} \rangle. \quad (2.27)$$

We introduce the covariant derivatives on the m^{th} fibration layer

$$(\nabla_{\Omega^{(m)}})_{ij} := \delta_{ij} dz_m + \Omega_{ji}^{(m)}, \quad (\nabla_{\Omega^{\vee(m)}})_{ij} := \delta_{ij} dz_m + \Omega_{ij}^{\vee(m)}, \quad (2.28)$$

where δ_{ij} denotes the Kronecker delta function. The matrix-valued connection operators $\Omega^{(m)}$ and $\Omega^{\vee(m)}$ are defined by the following systems of differential equations, fulfilled by the bases and dual bases elements

$$dz_m \langle e_i^{(\mathbf{m}-1)} | = \sum_{j=1}^{\nu_{m-1}} \Omega_{ij}^{(m)} \langle e_j^{(\mathbf{m}-1)} |, \quad dz_m | h_i^{(\mathbf{m}-1)} \rangle = - \sum_{j=1}^{\nu_{m-1}} \Omega_{ji}^{\vee(m)} | h_j^{(\mathbf{m}-1)} \rangle, \quad (2.29)$$

and computed in terms of intersection numbers, from the master decomposition formulae

$$\begin{aligned} \hat{\Omega}_{ij}^{(m)} &= \sum_{k=1}^{\nu_{m-1}} \langle (\partial_{z_m} + \hat{\omega}_m) e_i^{(\mathbf{m}-1)} | h_k^{(\mathbf{m}-1)} \rangle (\mathbf{C}_{(m-1)}^{-1})_{kj}, \\ \hat{\Omega}_{ji}^{\vee(m)} &= - \sum_{k=1}^{\nu_{m-1}} (\mathbf{C}_{(m-1)}^{-1})_{ik} \langle e_k^{(\mathbf{m}-1)} | (\partial_{z_m} - \hat{\omega}_m) h_j^{(\mathbf{m}-1)} \rangle. \end{aligned} \quad (2.30)$$

⁵In principle, any order of variables is valid here. In practice, it is chosen to minimize the number of independent basis elements in each layer of the fibration.

Differential m -forms can be rewritten as:

$$\varphi^{(\mathbf{m})} = \sum_{i=1}^{\nu_{m-1}} \varphi_i^{(m)} \wedge e_i^{(\mathbf{m}-1)}, \quad \varphi^{\vee(\mathbf{m})} = \sum_{i=1}^{\nu_{m-1}} \varphi_i^{\vee(m)} \wedge h_i^{(\mathbf{m}-1)}, \quad (2.31)$$

where the projection coefficients $\varphi_i^{(m)}$ and $\varphi_i^{\vee(m)}$ are indeed 1-forms and are computed through the master decomposition formulae

$$\begin{aligned} \varphi_i^{(m)} &= \sum_{j=1}^{\nu_{m-1}} \langle \varphi^{(\mathbf{m})} | h_j^{(\mathbf{m}-1)} \rangle (\mathbf{C}_{(m-1)}^{-1})_{ji}, \\ \varphi_i^{\vee(m)} &= \sum_{j=1}^{\nu_{m-1}} (\mathbf{C}_{(m-1)}^{-1})_{ij} \langle e_j^{(\mathbf{m}-1)} | \varphi^{\vee(\mathbf{m})} \rangle. \end{aligned} \quad (2.32)$$

As a starting point for the discussion of the novel ideas presented in this work, we conveniently cast them [10, 19] in vector-valued 1-forms of length ν_{m-1}

$$\begin{aligned} \varphi^{(m)} &= [\varphi_1^{(m)} \dots \varphi_{\nu_{m-1}}^{(m)}]^T \in H^1(T^{(m)}, D^{(m)}; \nabla_{\Omega^{(m)}}), \\ \varphi^{\vee(m)} &= [\varphi_1^{\vee(m)} \dots \varphi_{\nu_{m-1}}^{\vee(m)}]^T \in H^1(T^{\vee(m)}, D^{\vee(m)}; \nabla_{\Omega^{\vee(m)}}), \end{aligned} \quad (2.33)$$

where:

$$T^{(m)} = \mathbb{C} \setminus \mathcal{P}_{\Omega^{(m)}} \setminus D^{\vee(m)}, \quad T^{\vee} = \mathbb{C} \setminus \mathcal{P}_{\Omega^{(m)}} \setminus D^{(m)}, \quad (2.34)$$

and the polar set $\mathcal{P}_{\Omega^{(m)}}$ is the set of singularities of $\Omega^{(m)}$, while $D^{(m)}$ and $D^{\vee(m)}$ are the relative boundaries at layer m . Hence, intersection numbers for differential m -forms can be evaluated as:

$$\begin{aligned} \langle \varphi^{(\mathbf{m})} | \varphi^{\vee(\mathbf{m})} \rangle &= - \sum_{p \in \mathcal{P}_{\Omega^{(m)}}} \text{Res}_{z_m=p} \left(\sum_{i=1}^{\nu_{m-1}} \langle \varphi^{(\mathbf{m})} | h_i^{(\mathbf{m}-1)} \rangle \psi_i^{\vee(m)} \right) \\ &\quad - \sum_{p \in D^{\vee(m)}} \text{Res}_{z_m=p} \left(\sum_{i=1}^{\nu_{m-1}} \langle \varphi^{(\mathbf{m})} | h_i^{(\mathbf{m}-1)} \rangle \psi_i^{\vee(m)} \right) \\ &\quad + \sum_{p \in D^{(m)}} \text{Res}_{z_m=p} \left(\sum_{i=1}^{\nu_{m-1}} \psi_i^{(m)} \langle e_i^{(\mathbf{m}-1)} | \varphi^{\vee(\mathbf{m})} \rangle \right), \end{aligned} \quad (2.35)$$

where ψ and $\psi^{\vee}$ are solutions to the systems of differential equations:

$$\nabla_{\Omega^{(m)}} \psi^{(m)} = \varphi^{(m)}, \quad \nabla_{\Omega^{\vee(m)}} \psi^{\vee(m)} = \varphi^{\vee(m)}. \quad (2.36)$$

As in the case of 1-forms, it is sufficient to solve the system (2.36) around the set of singularities where the residues are computed. Being $\mathcal{B}(z_m)$ an univariate polynomial vanishing on the set of points $\mathcal{P}_{\Omega^{(m)}}$, the sum over the poles $\mathcal{P}_{\Omega^{(m)}}$ can be computed as the global residue:

$$\sum_{p \in \mathcal{P}_{\Omega^{(m)}}} \text{Res}_{z_m=p} \left(\langle \varphi^{(\mathbf{m})} | h^{(\mathbf{m}-1)} \rangle \cdot \psi^{\vee(m)} \right) = \text{Res}_{\langle \mathcal{B}^{(m)} \rangle} \left(\langle \varphi^{(\mathbf{m})} | h^{(\mathbf{m}-1)} \rangle \cdot \psi^{\vee(m)} \right). \quad (2.37)$$

In our applications, $D = \emptyset$ and only simple poles in $D^{\vee}$ will appear, hence the last term in the sum in r.h.s. of eq. (2.35) will not contribute to the intersection number.

Regulated case. Assuming that $\varphi^{\vee(\mathbf{m})}$ is not a boundary-supported form on $D^\vee$, only the first term on the r.h.s. of eq. (2.35) will contribute to the intersection number, and eq. (2.35) becomes:

$$\langle \varphi^{(\mathbf{m})} | \varphi^{\vee(\mathbf{m})} \rangle = -\text{Res}_{\langle \mathcal{B}^{(m)} \rangle} \left(\langle \varphi^{(\mathbf{m})} | h^{(\mathbf{m}-1)} \rangle \cdot \psi^{(m)} \right). \quad (2.38)$$

Working in the quotient ring $\mathcal{Q} := \mathbb{K}[z_m] / \langle \mathcal{B}^{(m)} \rangle$, where $\langle \mathcal{B}^{(m)} \rangle$ denotes the ideal generated by the $\mathcal{B}(z_m) - \beta$ polynomial in the z_m variable, the system of equations can be rewritten as:

$$\langle \varphi^{(\mathbf{m})} | \varphi^{\vee(\mathbf{m})} \rangle + \text{Res}_{\langle \mathcal{B}^{(m)} \rangle} \left(\langle \varphi^{(\mathbf{m})} | h^{(\mathbf{m}-1)} \rangle \cdot \psi^{(m)} \right) = 0, \quad (2.39)$$

$$\left[\widehat{\nabla}_{\Omega^{\vee(m)}} \psi^{(m)} - \widehat{\varphi}^{\vee(m)} \right]_{\langle \mathcal{B}^{(m)} \rangle} = 0, \quad (2.40)$$

with

$$\widehat{\nabla}_{\Omega^{\vee(m)}} \equiv (\partial_{z_m} \mathcal{B}^{(m)}) \partial_\beta + \widehat{\Omega}^{\vee(m)} + \partial_{z_m}, \quad (2.41)$$

where we omitted the cohomology group indices for brevity, and it can be solved using the ansatz

$$\psi_i^{(m)}(\beta, z_m) = \sum_{a=0}^{\kappa-1} \sum_{n \in \mathbb{Z}} z_m^a \beta^n \psi_{ian}^{(m)} \quad \text{for } i = 1, \dots, \nu_{m-1}, \quad (2.42)$$

similar to eq. (2.23).

Relative case. Assuming $D^\vee = V(z_1 \cdot \dots \cdot z_m)$ to be non-regulated boundaries, we allow the existence of multivariate boundary-supported forms defined via the Leray coboundary operation:

$$\delta_{z_1, \dots, z_m}(\phi^{\vee(\mathbf{n}-\mathbf{m})}) := \frac{u}{u(0)} \bigwedge_{i=1}^m d\theta_{z_i, 0} \wedge \phi^{\vee(\mathbf{n}-\mathbf{m})}, \quad (2.43)$$

where $\phi^{\vee(\mathbf{n}-\mathbf{m})}$ is a generic $(n-m)$ -form, the Heaviside functions are defined in eq. (2.14), and, in shorthand notation, $u(0) := u|_{z_1, \dots, z_m=0}$. In this case, intersection numbers of n -forms can be evaluated as:

$$\langle \varphi^{(\mathbf{n})} | \delta_{z_1, \dots, z_m}(\phi^{\vee(\mathbf{n}-\mathbf{m})}) \rangle = \left\langle \text{Res}_{z_1=0, \dots, z_m=0} \left(\frac{u}{u(0)} \varphi^{(\mathbf{n})} \right) | \phi^{\vee(\mathbf{n}-\mathbf{m})} \right\rangle, \quad (2.44)$$

where the $(n-m)$ -form intersection numbers on the r.h.s. is evaluated using eq. (2.40).

2.5 Feynman integrals as twisted period integrals

Feynman Integrals (FIs) can be considered as twisted period integrals. In the *Baikov representation* [39, 41], a FI with E independent external momenta, and L loops can be written in terms of $N = L \cdot E + L(L+1)/2$ integration variables z_i , corresponding to the generalised set of denominators

$$I = \kappa \int_{\mathcal{C}} u \varphi = \langle \varphi | \mathcal{C} \rangle, \quad u = \mathcal{B}^\gamma, \quad \varphi = \frac{d^N z}{\prod_{i=1}^N z_i^{a_i}}, \quad (2.45)$$

where $\mathcal{B}$ is the Baikov polynomial, which vanishes on the boundary of the integration domain $\mathcal{B}(\partial\mathcal{C}) = 0$ guaranteeing that the integrals (2.45) obey IBPs.

Let us assume that the first n of the generalised denominators appear as propagators, $\{z_1, \dots, z_n\}$, whereas the remaining integration variables $\{z_{n+1}, \dots, z_N\}$ can enter only in the numerator as irreducible scalar products. The differential n -forms we will be dealing with are of the type:

$$\varphi = \hat{\varphi}(z) d^N z, \quad \hat{\varphi}(z) = \frac{\prod_{j=n+1}^N z_j^{-a_j}}{\prod_{i=1}^n z_i^{a_i}}. \quad (2.46)$$

In this case, the definitions for the cohomology group and its dual, given in eq. (2.4), specify as:

$$\mathcal{P}_\omega = V(\mathcal{B}), \quad D = \emptyset, \quad D^\vee = V(z_1 \cdot z_2 \cdot \dots \cdot z_n), \quad (2.47)$$

and the dimension of the cohomology group ν , hence the number of master integrals, are given by eq. (2.6), with the regulated twist (2.7). Any FI $I = \langle \varphi | \mathcal{C} \rangle$ can be decomposed in terms of a basis of master integrals (MIs) $\{\mathcal{I}_i\}_{i=1}^\nu$, where $\mathcal{I}_i = \langle e_i | \mathcal{C} \rangle$, as in eq. (2.10), and the coefficients of the decomposition c_i are given by eq. (2.9), and the N -form intersection numbers are computed using the fibration approach described in section 2.4.

We define an integral sector $\mathcal{S} = (\sigma_1, \dots, \sigma_N)$ as the set of points $(a_1, \dots, a_N) \in \mathbb{Z}^N$, such that $\sigma_i = \theta(a_i - 1/2)$. We notice that for each sector a subset of relative surfaces $D_\mathcal{S}^\vee = D^\vee \cap \prod_{i=1}^N z_i^{\sigma_i}$ appears. The dimension of the cohomology group ν can be evaluated as sum of the dimensions of each sector $\nu_\mathcal{S}$

$$\nu = \sum_{\mathcal{S}} \nu_\mathcal{S}, \quad (2.48)$$

where the dimension $\nu_\mathcal{S}$ can be evaluated as:

$$\nu_\mathcal{S} = \text{number of solutions of } \omega_\mathcal{S} = 0, \quad \omega_\mathcal{S} = d \log(u_\mathcal{S}), \quad (2.49)$$

with $u_\mathcal{S}$ being the twist of the sector $\mathcal{S}$, obtained from u by cutting the relative surfaces $D_\mathcal{S}^\vee$. In the application discussed in section 4, we successfully used this algorithm to generate bases in each layer of the fibration by choosing $\nu_\mathcal{S}$ independent elements in each sector.⁶

With this we conclude our overview of the relative twisted cohomology framework and advance to the tensorial reformulation of the intersection number computation process.

3 Tensor structure of intersection numbers

The fibration method for evaluation of intersection numbers involves solving system of equations, such as those found in eqs. (2.20), (2.21), for univariate forms, and in eqs. (2.39), (2.40), for the multivariate cases. Matrix calculus provides a powerful approach to reformulating these computations, offering deeper insight into the mathematical properties and patterns of intersection numbers while significantly boosting computational efficiency.

⁶A proof-of-concept implementation of this algorithm, together with examples of its usage, can be found in the following GitHub repository https://github.com/GiacomoBrunello/pentabox_decomposition.

In computational algebraic geometry, *companion matrices* [82, 83]⁷ allow for the reinterpretation of polynomial division as matrix multiplication. In this section, we investigate how companion matrices can be used to develop a novel scheme for computing intersection numbers, ultimately determining them through matrix operations.

3.1 The three vector spaces

Let us fix the fibration layer m and denote by z the corresponding Baikov variable. In the following we will omit the layer index whenever it does not cause confusion. Recall, that the ansatz (2.42) is parameterized by 3 indices

$$\psi_i^{(m)} = \sum_{a,n} z^a \beta^n \psi_{ian}, \quad (3.1)$$

which motivates us to distinguish the following 3 vector spaces:

1. Vector space of ν -dimensional vectors labeled by the first index $i = 1, \dots, \nu$ in eq. (3.1), namely

$$\mathbb{K}^\nu, \quad (3.2)$$

closely related to the linear space of vector-valued 1-forms introduced in eqs. (2.33).

2. The second index $a = 0, \dots, \kappa - 1$ parameterizes the set of irreducible monomials of the ideal $\langle \mathcal{B}(z) - \beta \rangle$, that is a basis of the quotient ring $\mathcal{Q}$ viewed as a vector space

$$\mathcal{Q} = \text{Span}_{\mathbb{K}}(1, \dots, z^{\kappa-1}), \quad \kappa := \deg(\mathcal{B}(z)). \quad (3.3)$$

It is then natural to interpret the $\mathcal{B}(z) - \beta = 0$ equation as a many-to-one coordinate change $z \mapsto \beta$, and think of $\mathcal{Q}$ as a vector bundle of rank κ with β parameterizing the base and irreducible monomials forming the basis in fibres (see appendix A for more details).

3. Finally, the last index $n \in \mathbb{Z}$ runs over the powers of the variable β that appear in the Laurent series expansion

$$\mathcal{L} = \text{Span}_{\mathbb{K}}(\dots, \beta^{-1}, \beta^0, \beta^1, \dots). \quad (3.4)$$

The $z \mapsto \beta$ coordinate change introduced above effectively glues all the roots of the $\mathcal{B}(z)$ polynomial in the z -plane to the origin of the β -plane. The space (3.4) captures the local behavior of functions at this point.

The solution ψ to the differential equation system (3.1) belongs to the tensor product of these three spaces:

$$\psi^{(m)} \in \mathbb{K}^\nu \otimes \mathcal{Q} \otimes \mathcal{L}. \quad (3.5)$$

In the following we will represent the differential equation system (2.21), (2.40) as a linear operator $\mathcal{T}_{\nabla}$ acting on the space (3.5)

$$\mathcal{T}_{\nabla_{\Omega^\vee}} \cdot \psi^{(m)} - \hat{\varphi}^{\vee(m)} = 0. \quad (3.6)$$

This formulation allows the solution to be obtained with linear algebra methods.

⁷See also [84] for a nice review.

3.1.1 Polynomial companion matrices

For the ideal generated by a monic⁸ degree κ polynomial

$$\langle \mathcal{B} \rangle \equiv \langle \mathcal{B}(z) - \beta \rangle = \langle b_0 - \beta + z b_1 + \dots + z^{\kappa-1} b_{\kappa-1} + z^\kappa \rangle, \quad (3.7)$$

we choose as a basis of the quotient ring the list of monomials shown in eq. (3.3). In the monomial basis of eq. (3.3), the companion matrix representation for the basic multiplication and differentiation operators read

$$Q_z := \begin{bmatrix} 0 & & & -b_0 + \beta \\ 1 & 0 & & -b_1 \\ & 1 & 0 & -b_2 \\ & & \ddots & \vdots \\ & & & 1 & 0 & -b_{\kappa-2} \\ & & & & 1 & -b_{\kappa-1} \end{bmatrix} \quad \begin{matrix} \uparrow \\ \vdots \\ \downarrow \end{matrix} \quad \begin{matrix} \leftarrow \kappa \rightarrow \\ \leftarrow \kappa \rightarrow \end{matrix}$$

$$Q_{\partial_z} := \begin{bmatrix} 0 & 1 & & \\ & 0 & 2 & \\ & & 0 & 3 \\ & & & \ddots & \ddots \\ & & & & \kappa-1 & \\ & & & & & 0 \end{bmatrix} \quad \begin{matrix} \uparrow \\ \vdots \\ \downarrow \end{matrix} \quad \begin{matrix} \leftarrow \kappa \rightarrow \\ \leftarrow \kappa \rightarrow \end{matrix} \quad (3.8)$$

where we explicitly showed the 0 entries on the main diagonal for the reader's convenience.

3.1.2 Series companion matrices

To operate on Laurent series expansions in the β variable, we employ the infinite matrix representation of the Weyl algebra composed of the two operators

$$L_\beta := \begin{matrix} \vdots \\ \vdots \\ -1 \rightarrow \\ 0 \rightarrow \\ 1 \rightarrow \\ 2 \rightarrow \\ \vdots \end{matrix} \begin{bmatrix} \ddots & & & \\ & \ddots & & \\ & & 1 & 0 \\ & & & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 & 0 \\ & & & & & & \ddots \end{bmatrix}$$

$$L_{\partial_\beta} := \begin{matrix} \vdots \\ \vdots \\ -1 \rightarrow \\ 0 \rightarrow \\ 1 \rightarrow \\ 2 \rightarrow \\ \vdots \end{matrix} \begin{bmatrix} \ddots & & & \\ & \ddots & & \\ & & 0 & 0 \\ & & & 0 & 1 \\ & & & & 0 & 2 \\ & & & & & 0 & 3 \\ & & & & & & \ddots \end{bmatrix} \quad (3.9)$$

In practice, however, only a finite number of the ansätze coefficients (2.23), (2.42) contribute to a given intersection number, suggesting that the infinite matrix representation (3.9) should be somehow restricted so that the matrices become finite.

Let us denote the leading exponents of the Laurent series expansions in β of the building blocks of the main linear system (2.21), (2.40) as

$$\min_\beta(\widehat{\Omega}^{\vee(m)}) = -1, \quad \min_\beta(\langle \varphi^{(\mathbf{m})} | h^{(\mathbf{m}-1)} \rangle) = \mu, \quad \min_\beta(\widehat{\varphi}^{\vee(m)}) = \mu^\vee, \quad (3.10)$$

where we would like to emphasize the simple pole condition for the connection matrix, which we always can satisfy with an appropriate choice of the $\mathcal{B}(z)$ polynomial (3.7) (see ap-

⁸Here we normalized the B_κ coefficient to 1 to avoid clutter in the subsequent formulae.

pendix C for more details). Now we may introduce the restricted analog⁹ of the infinite representation (3.9) reading

$$L_\beta := \begin{array}{c} \mu+1 \rightarrow \\ \vdots \\ -1 \rightarrow \\ 0 \rightarrow \\ 1 \rightarrow \\ \vdots \\ -\mu^\vee \rightarrow \end{array} \left[\begin{array}{cccc} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & 1 & 0 & \\ & & 1 & 0 \\ & & & \ddots & \ddots \\ & & & 1 & 0 \\ & & & & 1 & 0 \end{array} \right] \begin{array}{c} \uparrow \\ 1+\mu \\ \downarrow \end{array} \begin{array}{c} \leftarrow -\mu^\vee - \mu + 1 \rightarrow \end{array}$$

$$L_{\partial_\beta} := \begin{array}{c} \mu+1 \rightarrow \\ \vdots \\ -1 \rightarrow \\ 0 \rightarrow \\ 1 \rightarrow \\ \vdots \\ -\mu^\vee \rightarrow \end{array} \left[\begin{array}{cccc} 0 & \mu & & \\ & 0 & \ddots & \\ & & 0 & 0 \\ & & 0 & 1 \\ & & & 0 & 2 \\ & & & \ddots & \ddots \\ & & & 0 & -\mu^\vee \\ & & & & 0 \end{array} \right] \begin{array}{c} \uparrow \\ 1+\mu \\ \downarrow \end{array} \begin{array}{c} \leftarrow -\mu^\vee - \mu + 1 \rightarrow \end{array} \quad (3.11)$$

Here for each matrix we show the row labels on the left (which may start with negative integers when $\mu < 0$), and overall number of rows and columns on the right and at the bottom respectively. We also highlight with grey the row corresponding to the β^{-1} term of the Laurent expansion as it plays an important role for the global residue used in following.

3.1.3 Companion tensor representation

We are now in position to combine the two matrix representations introduced above into a single tensor representation that is going to be used for intersection number computation later. For simplicity, we will focus on the univariate case (2.23), as the generalization to multivariate (2.42) is straightforward.

Step 1: polynomial reduction. The key idea is to rewrite the ansatz (2.23) for an element of the quotient ring (3.3) in a vector form

$$\psi = \sum_{a=0}^{\kappa-1} z^a \psi_a(\beta) \equiv \begin{bmatrix} 1 & z & \dots & z^{\kappa-1} \end{bmatrix} \cdot \begin{bmatrix} \psi_0(\beta) \\ \psi_1(\beta) \\ \vdots \\ \psi_{\kappa-1}(\beta) \end{bmatrix}, \quad (3.12)$$

where we regard the row vector of irreducible monomials z^a on the left as a basis (of $\mathcal{Q}$ viewed as a vector bundle) and collect the coefficients $\psi_a(\beta)$ on the right.

Inside of the quotient ring (3.3), multiplication by some polynomial

$$p(z, \beta) = \sum_a z^a p_a(\beta) \rightsquigarrow Q_p = \sum_a (Q_z)^a p_a(\beta) \quad (3.13)$$

⁹Strictly speaking, finite matrices do not form a representation of a Weyl algebra. For example, the finite matrix L_β shown in eq. (3.11) does not have a well-defined inverse: the possible candidate $L_{1/\beta}$ does not give a pure identity matrix in the products $L_\beta \cdot L_{1/\beta} = \text{diag}(0, 1, \dots, 1)$ or $L_{1/\beta} \cdot L_\beta = \text{diag}(1, \dots, 1, 0)$. Nevertheless, in practice we may use the rules (3.11) after a (symbolic) series expansion in the $\beta \rightarrow 0$ limit.

is captured by the companion matrix Q_p , constructed using the basic monomial matrix (3.8) (see appendix C for an alternative method). Similarly, multiplication by some rational function consisting of polynomial numerator $n(z, \beta)$ and denominator $d(z, \beta)$ is expressed by

$$f(z, \beta) = \frac{n(z, \beta)}{d(z, \beta)} \rightsquigarrow Q_f = Q_n \cdot (Q_d)^{-1}, \quad (3.14)$$

i.e. a product of the numerator and the inverse of the denominator matrices.¹⁰ Companion matrix Q_f encodes the result of the polynomial reduction of the product of the ansatz (3.12) and the rational function (3.14) as matrix-vector multiplication

$$\left[f \psi \right]_{\langle \mathcal{B} \rangle} = \sum_{a_1, a_2=0}^{\kappa-1} z^{a_1} (Q_f)_{a_1 a_2} \psi_{a_2}(\beta) \equiv \begin{bmatrix} 1 & z & \dots & z^{\kappa-1} \end{bmatrix} \cdot Q_f \cdot \begin{bmatrix} \psi_0(\beta) \\ \psi_1(\beta) \\ \vdots \\ \psi_{\kappa-1}(\beta) \end{bmatrix}. \quad (3.15)$$

Upon this substitution, the companion matrix representation of eq. (2.21) reads:

$$Q_{\widehat{\nabla}_{-\omega}} \cdot \psi - \widehat{\varphi}^\vee = 0, \quad (3.16)$$

where

$$Q_{\widehat{\nabla}_{-\omega}} \equiv Q_{\partial_z \mathcal{B}} \partial_\beta - Q_{\widehat{\omega}} + Q_{\partial_z}. \quad (3.17)$$

Step 2: series expansion. Now let us include the Laurent series expansion in $\beta \rightarrow 0$ of the ansatz (3.12) encapsulated in an infinite matrix of coefficients

$$\psi \Big|_{\beta \rightarrow 0} = \sum_{a=0}^{\kappa-1} \sum_n z^a \beta^n \psi_{an} \quad (3.18)$$

$$\equiv \begin{bmatrix} 1 & z & \dots & z^{\kappa-1} \end{bmatrix} \cdot \left(\dots + \begin{bmatrix} \psi_{0n} \\ \psi_{1n} \\ \vdots \\ \psi_{\kappa-1n} \end{bmatrix} \beta^n + \begin{bmatrix} \psi_{0n+1} \\ \psi_{1n+1} \\ \vdots \\ \psi_{\kappa-1n+1} \end{bmatrix} \beta^{n+1} + \dots \right) \quad (3.19)$$

$$\equiv \begin{bmatrix} 1 & z & \dots & z^{\kappa-1} \end{bmatrix} \cdot \begin{bmatrix} \dots & \psi_{0n} & \psi_{0n+1} & \dots \\ \dots & \psi_{1n} & \psi_{1n+1} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \dots & \psi_{\kappa-1n} & \psi_{\kappa-1n+1} & \dots \end{bmatrix} \cdot \begin{bmatrix} \beta^n \\ \beta^{n+1} \\ \vdots \end{bmatrix}. \quad (3.20)$$

Almost all the basic operators get represented in terms of factorized tensors. The exception is the z -monomial multiplication operator: to derive its representation we first expand the matrix Q_z shown in eq. (3.8) as a linear polynomial in β with matrix coefficients $Q_{z,0}$ and $Q_{z,1}$, namely

$$z \rightsquigarrow Q_z = Q_{z,0} + Q_{z,1} \beta, \quad (3.21)$$

and only then substitute β with the corresponding matrix L_β from eq. (3.9) or (3.11).

¹⁰The order of operations here does not matter, as companion matrices Q_f for multiplication operators (i.e. excluding Q_{∂_z}) form a commutative algebra.

Step 3: global residue and tensor algebra. The global residue (2.39) acts as a covector on the space of ψ_{an} , and in the basis of (3.20), it simply extracts the coefficient of the $\psi_{\kappa-1,-1}$ component. This extraction can be carried out by multiplying with the row vector $R \equiv E_{\kappa-1} \otimes E_{-1}$, where E_j is a row vector with a single 1 in the j^{th} position. The intersection numbers can then be evaluated by applying the following list of *substitution rules*:

$$\begin{aligned}
 z & \rightsquigarrow \mathcal{T}_z = \mathbb{1} \otimes Q_{z,0} + L_\beta \otimes Q_{z,1}, \\
 \partial_z & \rightsquigarrow \mathcal{T}_{\partial_z} = \mathbb{1} \otimes Q_{\partial_z}, \\
 \beta & \rightsquigarrow \mathcal{T}_\beta = L_\beta \otimes \mathbb{1}, \\
 \partial_\beta & \rightsquigarrow \mathcal{T}_{\partial_\beta} = L_{\partial_\beta} \otimes \mathbb{1}, \\
 \text{Res}_{\langle \mathcal{B} \rangle} & \rightsquigarrow R = E_{\kappa-1} \otimes E_{-1},
 \end{aligned} \tag{3.22}$$

where, in the case of restricted representation (3.11), the covector R is just the unit row vector with the element 1 in the $|\mu|\kappa$ position (and all the other elements are vanishing)

$$R := \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.23}$$

$\uparrow$
 $|\mu|\kappa$

which effectively encodes the outcome of the global residue in eqs. (2.20), (2.39). Multiplication by the series expansion of function (3.14)

$$f(z, \beta) \Big|_{\beta \rightarrow 0} = \sum_{an} z^a \beta^n f_{an} \rightsquigarrow \mathcal{T}_f = \sum_{an} (\mathcal{T}_z)^a \cdot (\mathcal{T}_\beta)^n f_{an} \tag{3.24}$$

is then repackaged into a rank-4 companion tensor $\mathcal{T}_f$ as

$$\left[f \psi \right]_{\langle \mathcal{B} \rangle} \Big|_{\beta \rightarrow 0} = \sum_{a_1, a_2=0}^{\kappa-1} \sum_{n_1, n_2} z^{a_1} \beta^{n_1} (\mathcal{T}_f)_{n_1 n_2 a_1 a_2} \psi_{a_2 n_2}(\beta), \tag{3.25}$$

where the $\mathcal{T}_f$ tensor is defined as,

$$\mathcal{T}_f = \sum_{an} \mathbb{1} \otimes (Q_{z,0} + L_\beta \otimes Q_{z,1})^a \cdot (L_\beta \otimes \mathbb{1})^n f_{an}. \tag{3.26}$$

The substitution rules collected in this subsection complete the *dictionary* of rules required for the tensor representation of the systems of differential equations (3.6), which lie at the core of the evaluation of *intersection numbers by tensor algebra* method discussed next.

3.2 Companion tensors for intersection numbers

Restoring the notation of section 2 and using the replacements (3.22), the *companion tensor representation* of the univariate system (2.20), (2.21), required in the evaluation of the intersection numbers for differential 1-forms, reads

$$\langle \varphi | \varphi^\vee \rangle + R \cdot \mathcal{T}_\varphi \cdot \psi = 0, \tag{3.27}$$

$$\mathcal{T}_{\widehat{\nabla} - \omega} \cdot \psi - \widehat{\varphi}^\vee = 0, \tag{3.28}$$

where

$$\mathcal{T}_{\widehat{\nabla}_{-\omega}} \equiv \mathcal{T}_{\partial_z \mathcal{B}} \cdot \mathcal{T}_{\partial_\beta} - \mathcal{T}_{\widehat{\omega}} + \mathcal{T}_{\partial_z} . \quad (3.29)$$

A simple application of generation and solution of such a system can be found in appendix B.

Similarly, the *companion tensor representation* of the multivariate system (2.39), (2.40), required in the evaluation of the intersection numbers for differential m -forms, becomes

$$\langle \varphi^{(\mathbf{m})} | \varphi^{\vee(\mathbf{m})} \rangle + R \cdot \mathcal{T}_{\langle \varphi^{(\mathbf{m})} | h^{(\mathbf{m}-1)} \rangle} \cdot \psi^{(m)} = 0 , \quad (3.30)$$

$$\mathcal{T}_{\widehat{\nabla}_{\Omega^\vee}} \cdot \psi^{(m)} - \widehat{\varphi}^{\vee(m)} = 0 , \quad (3.31)$$

where

$$\mathcal{T}_{\widehat{\nabla}_{\Omega^\vee}} \equiv \mathcal{T}_{\partial_z \mathcal{B}} \cdot \mathcal{T}_{\partial_\beta} + \mathcal{T}_{\widehat{\Omega}^\vee} + \mathcal{T}_{\partial_z} . \quad (3.32)$$

The overall structure of the system (3.30), (3.31) can be consolidated into the following inhomogeneous matrix system

$$\left[\begin{array}{c|c} 1 & R \cdot \mathcal{T}_{\langle \varphi^{(\mathbf{m})} | h^{(\mathbf{m}-1)} \rangle} \\ \hline 0 & \mathcal{T}_{\widehat{\nabla}_{\Omega^\vee}} \\ \vdots & \\ 0 & \end{array} \right] \cdot \left[\begin{array}{c} \langle \varphi^{(\mathbf{m})} | \varphi^{\vee(\mathbf{m})} \rangle \\ \hline \psi^{(m)} \end{array} \right] = \left[\begin{array}{c} 0 \\ \hline \widehat{\varphi}^{\vee(m)} \end{array} \right] \quad (3.33)$$

for the augmented column of unknowns that unites the intersection number and the ansatz together. This system has to be solved *only* for the first unknown — the intersection number [85] (see also [86]). We notice, that in practice, this formulation proves to be highly robust, enabling solutions even in cases when the system is ill-defined, such as when the connection matrix has a resonant spectrum (i.e., eigenvalues that differ by integers).

The tensor systems of equations presented in this section constitute the first major result of this communication. They offer a new perspective on evaluation of intersection numbers of differential n -forms, applicable in both physics and mathematics. The key novel feature of the proposed algorithm is its reliance on a companion tensor representation of differential operators that effectively replaces arithmetic operations of polynomial division with matrix multiplication operating *directly* on the remainders of this division.

Next we will show a non-trivial application of our formalism.

4 Decomposition of two-loop five-point massless planar integrals

The goal of this section, is to apply the algorithm described in section 3 for the numerical decomposition of massless two-loop five-point functions in terms of master integrals. A complete analytic decomposition is currently infeasible, as the most computationally intensive parts of the problem require runtimes on the order of a day or more. As such, pursuing a full analytic result is beyond the scope of this work. Further improvements in the efficiency

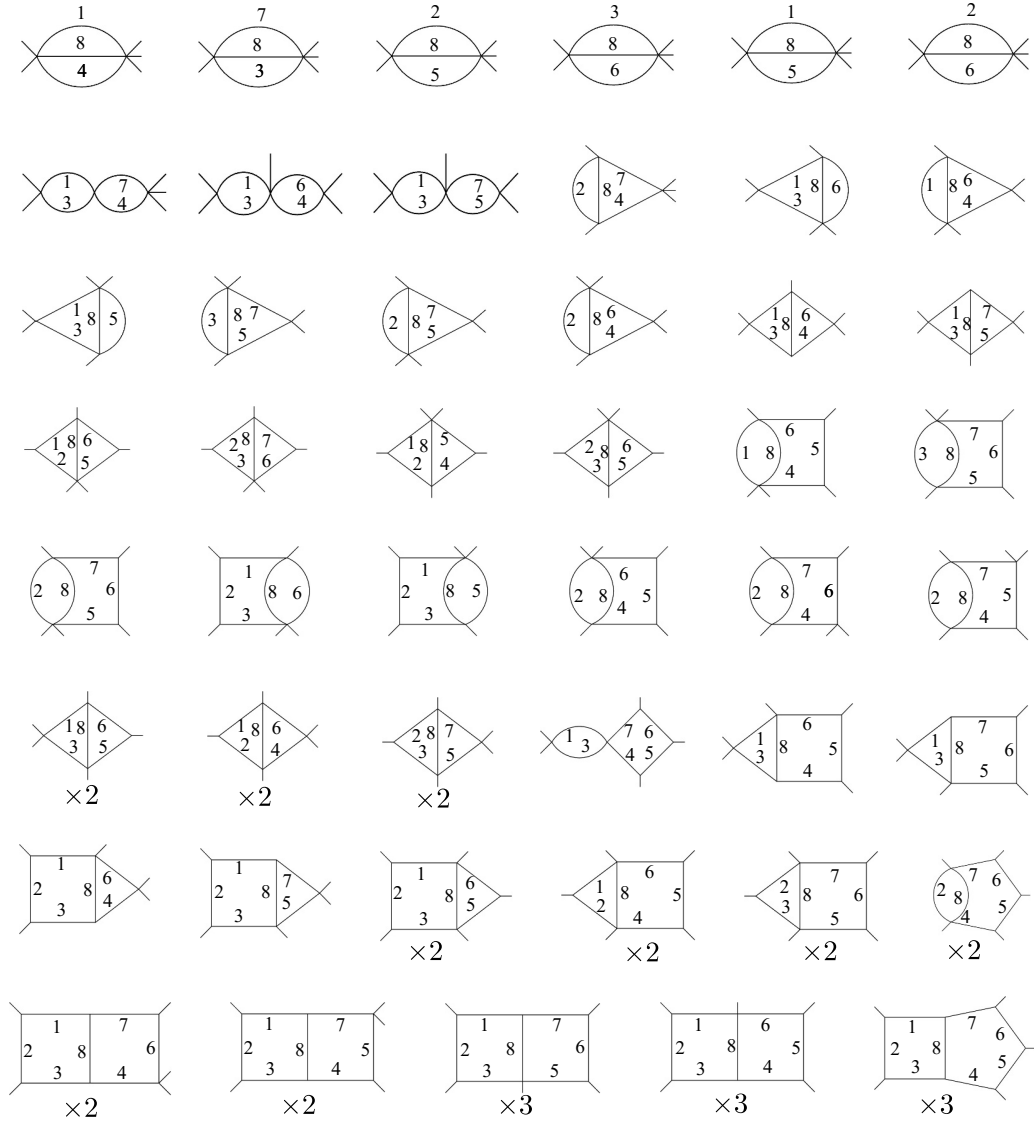


Figure 1. The 47 sectors of the 62 master integrals defined in eq. (4.6), corresponding to the massless two-loop five-points integral family (symmetry relations have not been applied).

of our implementation are left for future investigation. The integral family is defined in terms of 11 generalised denominators:

$$\begin{aligned}
 z_1 &= k_1^2, & z_2 &= (k_1 + p_1)^2, & z_3 &= (k_1 + p_1 + p_2)^2, & z_4 &= (k_2 + p_1 + p_2)^2, \\
 z_5 &= (k_2 + p_1 + p_2 + p_3)^2, & z_6 &= (k_2 - p_5)^2, & z_7 &= k_2^2, & z_8 &= (k_1 - k_2)^2, \\
 z_9 &= (k_2 + p_1)^2, & z_{10} &= (k_1 + p_1 + p_2 + p_3)^2, & z_{11} &= (k_1 - p_5)^2,
 \end{aligned} \tag{4.1}$$

as:

$$I_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11}} = \int d^{11}z \, u(\mathbf{z}) \frac{z_9^{-a_9} z_{10}^{-a_{10}} z_{11}^{-a_{11}}}{z_1^{a_1} z_2^{a_2} z_3^{a_3} z_4^{a_4} z_5^{a_5} z_6^{a_6} z_7^{a_7} z_8^{a_8}} \tag{4.2}$$

z_9, z_{10}, z_{11} are irreducible scalar products, and hence the set of relative boundaries is given by:

$$D^\vee = V\left(\prod_{i=1}^8 z_i\right). \quad (4.3)$$

The kinematics is such that:

$$\begin{aligned} p_i^2 &= 0, & s_{12} &= (p_1+p_2)^2, & s_{23} &= (p_2+p_3)^2, \\ s_{34} &= (p_3+p_4)^2, & s_{45} &= (p_4+p_5)^2, & s_{51} &= (p_5+p_1)^2. \end{aligned} \quad (4.4)$$

Therefore, our computation involves six external parameters

$$\vec{x} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, d\}. \quad (4.5)$$

We evaluate the intersection numbers numerically, restricting $\vec{x}$ to generic numerical values,¹¹ for example $\vec{x}_0 = \{1/3, -1/5, -1/7, 3/7, 5/13, 79/11\}$.

This integral family has (before application of the symmetry relations) $\nu = 62$ master integrals, which we may pick¹² as depicted in figure 1, as:

$$\begin{aligned} J_1 &= I_{10010001000}, & J_2 &= I_{00100011000}, & J_3 &= I_{01001001000}, & J_4 &= I_{00100101000}, \\ J_5 &= I_{10001001000}, & J_6 &= I_{01000101000}, & J_7 &= I_{10110010000}, & J_8 &= I_{10110100000}, \\ J_9 &= I_{10101010000}, & J_{10} &= I_{01010011000}, & J_{11} &= I_{10100101000}, & J_{12} &= I_{10010101000}, \\ J_{13} &= I_{10101001000}, & J_{14} &= I_{00101011000}, & J_{15} &= I_{01001011000}, & J_{16} &= I_{01010101000}, \\ J_{17} &= I_{10110101000}, & J_{18} &= I_{10101011000}, & J_{19} &= I_{11001101000}, & J_{20} &= I_{01100111000}, \\ J_{21} &= I_{11011001000}, & J_{22} &= I_{01101101000}, & J_{23} &= I_{10011101000}, & J_{24} &= I_{00101111000}, \\ J_{25} &= I_{01001111000}, & J_{26} &= I_{11100101000}, & J_{27} &= I_{11101001000}, & J_{28} &= I_{01011101000}, \\ J_{29} &= I_{01010111000}, & J_{30} &= I_{01011011000}, & J_{31} &= I_{10101101000}, & J_{32} &= I_{101011-11000}, \\ J_{33} &= I_{11010101000}, & J_{34} &= I_{110101-11000}, & J_{35} &= I_{01101011000}, & J_{36} &= I_{01101011-100}, \\ J_{37} &= I_{10111110000}, & J_{38} &= I_{10111101000}, & J_{39} &= I_{10101111000}, & J_{40} &= I_{11110101000}, \\ J_{41} &= I_{11101011000}, & J_{42} &= I_{11101101000}, & J_{43} &= I_{111011-11000}, & J_{44} &= I_{11011101000}, \\ J_{45} &= I_{11011101-100}, & J_{46} &= I_{01101111000}, & J_{47} &= I_{01101111-100}, & J_{48} &= I_{01011111000}, \\ J_{49} &= I_{01011111-100}, & J_{50} &= I_{11110111000}, & J_{51} &= I_{11110111-100}, & J_{52} &= I_{11111011000}, \\ J_{53} &= I_{11111011-100}, & J_{54} &= I_{11101111000}, & J_{55} &= I_{111-11111000}, & J_{56} &= I_{11101111-100}, \\ J_{57} &= I_{11111101000}, & J_{58} &= I_{111111-11000}, & J_{59} &= I_{11111101-100}, & J_{60} &= I_{11111111000}, \\ J_{61} &= I_{11111111-100}, & J_{62} &= I_{111111110-10}. \end{aligned} \quad (4.6)$$

Our MATHEMATICA and FINITEFLOW implementation can take as input any integral of the type of eq. (4.2). We identify two types of applications, to check and gauge the potential of our algorithm

¹¹The choice of kinematic region does not affect the performance of our implementation. We select generic numerical values that are away from the singularities of the process. For the proof-of-concept experiments, the integers appearing in the rational numbers are chosen to be relatively small to simplify the finite field reconstruction.

¹²We may use any other basis of independent integrals as well.

- **Higher-rank integrals.** We analyze the performance of our method when applied to the decomposition of high-rank integrals of the form:

$$I_n = \int d^{11}z \, u(\mathbf{z}) \frac{z_9^n}{z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8} = I_{111111111-n00}, \quad (4.7)$$

for the cases where the exponent n takes the values $n = 2, 5, 10, 15, 20$.

- **Higher-dimensions integrals.** We investigate the decomposition of shifted-dimension integrals, generated by the insertion of integer powers of the Baikov polynomial in the numerator. For the considered case, we decompose single and double power of $\mathcal{B}$, corresponding to a pentabox integral in $d + 2$ and $d + 4$ [87, 88] dimensions, formed by a linear combination of 412 monomials up to rank 4, and 169744 monomials up to rank 8, respectively:

$$I_{\mathcal{B}^n} = \int d^{11}z \, u(\mathbf{z}) \frac{\mathcal{B}^n}{z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8}, \quad \text{for } n = 1, 2. \quad (4.8)$$

The results are expressed in terms of master integrals of eq. (4.6) via a complete set of spanning cuts, as:

$$I = \sum_{i=1}^{62} c_i J_i. \quad (4.9)$$

The explicit expressions for the twist $u(\mathbf{z})$, the master integrals J_i , the bases of the (dual) co-homology groups, and the numerical values of the coefficients c_i , computed by through the master decomposition formula (2.9) in terms of intersection numbers, can be found in the supplementary material file `pentabox_massless.m`, and in the corresponding GitHub repository https://github.com/GiacomoBrunello/pentabox_decomposition.¹³

The set of spanning cuts is given by the maximal cuts of the first ten master integrals $\{J_1, \dots, J_{10}\}$ of eq. (4.6). For each cut, after a choice of fibration has been chosen, the bases for each layer have been automatically generated using the algorithm described in section 2.5. The cuts over J_3, J_4 , and their symmetric partners J_6, J_5 , represent the major bottleneck of this calculation. For J_3 , choosing as fibration order $\{z_4, z_9, z_7, z_6, z_1, z_3, z_{10}, z_{11}\}$, the dimensions of the internal bases read (from outer to inner): $\{31, 28, 12, 4, 2, 2, 1, 1\}$. For J_4 , choosing as fibration order $\{z_4, z_9, z_7, z_5, z_2, z_1, z_{10}, z_{11}\}$, the dimensions of the internal bases read: $\{27, 26, 18, 6, 3, 2, 1, 1\}$. This complexity required the development of an advanced computational implementation in FINITEFLOW [26, 27] to construct companion matrices and perform tensor algebra. The details are described in appendix C. The successful decomposition¹⁴ of a special monomial of rank $n = 20$ and the complete decomposition of the pentabox integrals in $d + 2$ and $d + 4$ dimensions stands as the second major result of this communication. It completes and extends earlier studies exploring the decomposition of 2-to-2

¹³Computations were performed on a machine with an AMD EPYC 7282 16-Core Processor (x86_64 architecture), featuring 32 logical CPUs at 2.79 GHz and 125 GiB of system memory.

¹⁴The numerical decomposition coefficients have been successfully compared against those obtained via standard IBP techniques, employing NEATIBP [89] to generate systems and reducing them via FINITEFLOW, up to $n = 10$.

and 2-to-3 2-loop integrals [25, 90], dealing with the decomposition of an involved combination of integrals. It marks a milestone in the decomposition of two-loop Feynman integrals into master integrals through intersection numbers projection, indicating that this novel method could become a valid option in presence of integrals with higher numerators or denominator powers. Notably, our tests indicate that the complexity of intersection number computations scales linearly with the rank of the integrals' numerators (at least for the integrals shown in eq. (4.7)), therefore motivating further applications and developments of intersection numbers-based methods to decompose multi-scale integrals in higher-multiplicity cases within gauge theories or effective field theories, in tandem with the standard IBP-based approaches.

5 Conclusions

Twisted period integrals appear in many applications of theoretical physics and mathematics, making it pivotal to understand their nature. These integrals belong to a finite-dimensional vector space whose geometry is governed by an inner product, known as the intersection number.

In this work, we identified and explored the underlying tensor structures of this inner product, empowering a novel computational procedure for evaluating intersection numbers of multivariate differential forms.

Within the fibration-based approach, n -form intersection numbers are calculated iteratively, one variable at a time, on separate layers. The architecture of the cohomology group on each layer is shaped by the connection and its geometric properties: the zeroes determine the dimension of the cohomology group as a vector space, while the singularities characterize the localization points of intersection numbers. The connection also defines the differential equation for parallel transport, with the intersection number arising from the sum of residues of the local vector-valued holomorphic solutions at the localization points.

The singularities of the connection can be naturally collected in the vanishing set of a suitably chosen interpolating polynomial ideal. A linear deformation of the ideal generator with a new symbolic parameter then constitutes a many-to-one change of variables. In this new coordinate system, intersection numbers localize at a single point: the origin.

This perspective allowed us to introduce the ambient tensor space that organically hosts the problem of evaluating intersection numbers. The space consists of three components. The family of local vector-valued solutions at localization points is glued together and gauged to become an element of the quotient ring, yielding the first two components of the tensor space. Since only the local behavior of the solution around the origin matters, it makes sense to work directly with the series expansions in the new variable, forming the third factor of the tensor space.

Through this representation, the differential equation turns into a matrix operator acting on the tensor space, while the residue becomes a simple covector, with companion matrices serving as the building blocks. As a result, intersection numbers can be computed using simple algebraic and matrix operations, with finite field reconstruction significantly enhancing performance.

Our novel companion-matrix-based framework has been applied to the numerical reduction of massless two-loop five-point functions up to rank 20, requiring the computation of 11-form

intersection numbers. This result represents a milestone in the decomposition of Feynman integrals and Euler-Mellin integrals into master integrals using intersection numbers.

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A Polynomial division as gauge transformation

Local solutions of (2.15) are labeled by the points of $V(\mathcal{B})$. Let us assume for simplicity that there are no multiple poles, i.e.

$$V(\mathcal{B}) = \{p_1(\beta), \dots, p_\kappa(\beta)\} \quad (\text{A.1})$$

and all the points $p_i(0)$ are different. The generating equation for the ideal (3.7)

$$\mathcal{B}(z) - \beta = 0 \quad (\text{A.2})$$

is a many-to-one polynomial map, such that $\mathcal{B}(p_i(\beta)) \equiv \beta$ for each i , see [81] for further discussion.

We may thus perform the coordinate change $z \mapsto \beta$ of the connection (2.3) near each point $p_i(0)$ and organize the local solutions to the differential equation (2.15) into a column vector

$$\psi = \left[\psi_{p_1}(\beta) \ \psi_{p_2}(\beta) \ \dots \ \psi_{p_\kappa}(\beta) \right]^T, \quad (\text{A.3})$$

which thus becomes a vector bundle of rank κ . This vector of local solutions is related to the ansatz (3.12) via the Vandermonde matrix (see also [91] for more applications)

$$\begin{bmatrix} \psi_{p_1}(\beta) \\ \psi_{p_2}(\beta) \\ \vdots \\ \psi_{p_\kappa}(\beta) \end{bmatrix} = \begin{bmatrix} 1 & p_1 & p_1^2 & \dots & p_1^{\kappa-1} \\ 1 & p_2 & p_2^2 & \dots & p_2^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p_\kappa & p_\kappa^2 & \dots & p_\kappa^{\kappa-1} \end{bmatrix} \cdot \begin{bmatrix} \psi_0(\beta) \\ \psi_1(\beta) \\ \vdots \\ \psi_{\kappa-1}(\beta) \end{bmatrix}, \quad (\text{A.4})$$

establishing a simple relation between the two descriptions via a gauge transformation.

B A pedagogical example

Consider a family of univariate twisted period integrals of the form

$$I_a = \int_{\mathcal{C}} u \varphi, \quad u = z^\rho \mathcal{B}(z)^\gamma, \quad \varphi = \frac{dz}{z^a}, \quad (\text{B.1})$$

where exponents $a \in \mathbb{Z}$ and $\rho, \gamma \in \mathbb{C}$, and $\mathcal{B}(z)$ the following quadratic monic polynomial:

$$\mathcal{B}(z) = b_0 + b_1 z + z^2, \quad (\text{B.2})$$

with some generic coefficients $b_0, b_1 \in \mathbb{K}$. The connection (2.3) is

$$\hat{\omega}(z) = \frac{\rho}{z} + \gamma \frac{\partial_z \mathcal{B}}{\mathcal{B}}, \quad (\text{B.3})$$

and its polar set is

$$\mathcal{P}_\omega = \{0\} \cup V(\mathcal{B}). \quad (\text{B.4})$$

To illustrate the computational scheme introduced in this communication, let us focus on the intersection number:

$$\langle \varphi \mid \varphi^\vee \rangle = \langle dz/\mathcal{B}^2 \mid dz/\mathcal{B} \rangle. \quad (\text{B.5})$$

It can be easily seen, that the pole at $z = 0$ from the set (B.4) does not contribute to this intersection number, so below we will only consider the vanishing locus $V(\mathcal{B})$ for brevity.

B.1 Step 1: polynomial reduction

The ideal generated by the polynomial (B.2) induces a 2-dimensional quotient space (3.3)

$$\mathcal{Q} = \text{Span}_{\mathbb{K}(\beta)}(1, z). \quad (\text{B.6})$$

This implies that the ansatz (2.23) is a 2-dimensional column vector

$$\psi(z, \beta) = \psi_0(\beta) + z \psi_1(\beta) \equiv \begin{bmatrix} 1 & z \end{bmatrix} \cdot \begin{bmatrix} \psi_0(\beta) \\ \psi_1(\beta) \end{bmatrix}, \quad (\text{B.7})$$

and the companion matrices (3.8) are explicitly

$$Q_z = \begin{bmatrix} 0 & -b_0 + \beta \\ 1 & -b_1 \end{bmatrix}, \quad Q_{\partial_z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (\text{B.8})$$

The differential equation (2.21) turns into a rank-2 linear system as in eq. (3.16)

$$Q_{\hat{\nabla}_{-\omega}} \cdot \begin{bmatrix} \psi_0(\beta) \\ \psi_1(\beta) \end{bmatrix} - \begin{bmatrix} \beta^{-1} \\ 0 \end{bmatrix} = 0, \quad (\text{B.9})$$

where:

$$Q_{\hat{\nabla}_{-\omega}} \equiv Q_{\partial_z \mathcal{B}} \partial_\beta - Q_{\hat{\omega}} + Q_{\partial_z}. \quad (\text{B.10})$$

Here the companion matrix representation of the $\hat{\omega}$ -term follows directly from eq. (B.3) using the method of section 3.1.3, giving

$$Q_{\hat{\omega}} = \frac{\gamma}{\beta} Q_{\partial_z \mathcal{B}} + \rho Q_{1/z}, \quad (\text{B.11})$$

and the two appearing matrices are

$$Q_{\partial_z \mathcal{B}} \equiv b_1 \mathbb{1} + 2 Q_z = \begin{bmatrix} b_1 & 2(-b_0 + \beta) \\ 2 & -b_1 \end{bmatrix}, \quad (\text{B.12})$$

$$Q_{1/z} \equiv Q_z^{-1} = \frac{1}{-b_0 + \beta} \begin{bmatrix} b_1 & -b_0 + \beta \\ 1 & 0 \end{bmatrix}. \quad (\text{B.13})$$

Finally, to simplify the subsequent formulae we find it convenient to multiply the system (B.9) from the left with the inverse matrix $Q_{\partial_z \mathcal{B}}^{-1}$ that takes the following form:

$$Q_{\partial_z \mathcal{B}}^{-1} = \frac{1}{\Delta + 4\beta} \begin{bmatrix} b_1 & 2(-b_0 + \beta) \\ 2 & -b_1 \end{bmatrix}, \quad \Delta := b_1^2 - 4b_0, \quad (\text{B.14})$$

where we recognize $\Delta = \text{Disc}_z(\mathcal{B}(z))$ as the discriminant of the polynomial (B.2). The system thus becomes

$$\left(\partial_\beta - \frac{\gamma}{\beta} - \rho Q_{\partial_z \mathcal{B}}^{-1} \cdot Q_{1/z} + Q_{\partial_z \mathcal{B}}^{-1} \cdot Q_{\partial_z} \right) \cdot \begin{bmatrix} \psi_0(\beta) \\ \psi_1(\beta) \end{bmatrix} - Q_{\partial_z \mathcal{B}}^{-1} \cdot \begin{bmatrix} \beta^{-1} \\ 0 \end{bmatrix} = 0. \quad (\text{B.15})$$

To compute the intersection number (B.5) we are only interested in the $\beta \rightarrow 0$ series expansion of the solution (B.7) to the system (B.9), which we turn to now.

B.2 Step 2: series expansion

The general $\beta \rightarrow 0$ series expansion of the ansatz (3.20) specifies in the case of eq. (B.7) to

$$\psi(z, \beta) \Big|_{\beta \rightarrow 0} = \sum_{a=0}^1 \sum_{n \geq 0} z^a \psi_{an} \beta^n \equiv \begin{bmatrix} 1 & z \end{bmatrix} \cdot \begin{bmatrix} \cdots & \psi_{0-1} & \psi_{00} & \psi_{01} & \cdots \\ \cdots & \psi_{1-1} & \psi_{10} & \psi_{11} & \cdots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ \beta^{-1} \\ 1 \\ \beta \\ \vdots \end{bmatrix}. \quad (\text{B.16})$$

For the specific case of the intersection number (B.5), the values of the exponents (3.10) are $\mu = -2$ and $\mu^\vee = -1$, so it is enough to bound the ansatz above to the interval $\{\beta^{-1}, 1, \beta\}$. It can be easily seen, that the $\psi_{a,-1}$ are going to be fixed to 0, so we will drop them in the following for simplicity. The series expansion of the system (B.15) reads

$$\left(\partial_\beta - \frac{\gamma}{\beta} + \frac{-\rho}{\Delta} \begin{bmatrix} \frac{2b_0 - b_1^2}{b_0} & b_1 \\ -\frac{b_1}{b_0} & 2 \end{bmatrix} + \frac{1}{\Delta} \begin{bmatrix} 0 & b_1 \\ 0 & 2 \end{bmatrix} + \dots \right) \cdot \left(\begin{bmatrix} \psi_{00} \\ \psi_{10} \end{bmatrix} + \begin{bmatrix} \psi_{01} \\ \psi_{11} \end{bmatrix} \beta \right) - \left(\frac{1}{\beta \Delta} + \frac{-4}{\Delta^2} + \dots \right) \begin{bmatrix} b_1 \\ 2 \end{bmatrix} = 0, \quad (\text{B.17})$$

where the opacity of the colors represent the degree in β of the collected terms.

B.3 Step 3: global residue and tensor algebra

Now we substitute the individual β -dependent operators with their matrix representatives (3.11), for example

$$\partial_\beta \rightsquigarrow L_{\partial_\beta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \beta^{-1} \rightsquigarrow L_{\beta^{-1}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad (\text{B.18})$$

leading to the tensor representation of the differential operator in eq. (B.17) of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} -\gamma & 0 \\ 0 & -\gamma \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \frac{-\rho}{\Delta} \frac{2b_0 - b_1^2}{b_0} & \frac{1-\rho}{\Delta} b_1 \\ \frac{\rho}{\Delta} \frac{b_1}{b_0} & \frac{2(1-\rho)}{\Delta} \end{bmatrix} + \dots \quad (\text{B.19})$$

The full system (B.17) thus turns into

$$\begin{bmatrix} * & * & -\gamma & 0 & 0 & 0 \\ * & * & 0 & -\gamma & 0 & 0 \\ * & * & \frac{-\rho}{\Delta} \frac{2b_0 - b_1^2}{b_0} & \frac{1-\rho}{\Delta} b_1 & -\gamma + 1 & 0 \\ * & * & \frac{\rho}{\Delta} \frac{b_1}{b_0} & \frac{2(1-\rho)}{\Delta} & 0 & -\gamma + 1 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \psi_{00} \\ \psi_{10} \\ \psi_{01} \\ \psi_{11} \end{bmatrix} - \begin{bmatrix} \frac{b_1}{\Delta} \\ \frac{2}{\Delta} \\ \frac{-4b_1}{\Delta^2} \\ \frac{-8}{\Delta^2} \\ * \\ * \end{bmatrix} = 0, \quad (\text{B.20})$$

we hid the irrelevant entries of the system under the $*$ signs. Next, to obtain the matrix form (3.33), we augment eq. (B.20) with the additional equation (3.30). This equation represents the global residue by a dot product with the row vector (3.23)

$$\begin{aligned} \langle \varphi | \varphi^\vee \rangle &= -\text{Res}_{\langle \mathcal{B} \rangle}(\varphi | \psi) \\ &= - \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \psi_{00} \\ \psi_{10} \\ \psi_{01} \\ \psi_{11} \end{bmatrix} = - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \psi_{00} \\ \psi_{10} \\ \psi_{01} \\ \psi_{11} \end{bmatrix}. \quad (\text{B.21}) \end{aligned}$$

The linear system that involves the non-zero elements of the ansatz [85, 86] is obtained by removing the first column and the last row from eq. (B.20)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & -\gamma & 0 & 0 & 0 \\ 0 & 0 & -\gamma & 0 & 0 \\ 0 & \frac{-\rho}{\Delta} \frac{2b_0 - b_1^2}{b_0} & \frac{1-\rho}{\Delta} b_1 & -\gamma + 1 & 0 \\ 0 & \frac{\rho}{\Delta} \frac{b_1}{b_0} & \frac{2(1-\rho)}{\Delta} & 0 & -\gamma + 1 \end{bmatrix} \cdot \begin{bmatrix} \langle \varphi | \varphi^\vee \rangle \\ \psi_{00} \\ \psi_{10} \\ \psi_{01} \\ \psi_{11} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{b_1}{\Delta} \\ \frac{2}{\Delta} \\ \frac{-4b_1}{\Delta^2} \\ \frac{-8}{\Delta^2} \end{bmatrix}, \quad (\text{B.22})$$

which we solve for the value of the intersection number $\langle \varphi | \varphi^\vee \rangle$ only. The final result reads

$$\langle \varphi | \varphi^\vee \rangle = -\psi_{11} = \frac{8}{\Delta^2(\gamma-1)} + \frac{-4b_0 + \Delta\rho}{b_0\Delta^2(\gamma-1)\gamma}, \quad (\text{B.23})$$

in agreement with [92]. This computation follows closely our computer implementation based on FINITEFLOW [27], whose general structure is shown in figure 2 and some details are summarized in the following appendix.

C Implementation details

In this appendix we collect the details of our computer implementation of the framework presented in section 3.

Intersection matrices. The systems shown in section 3.2 are not limited to computation of a single given intersection number $\langle \varphi | \varphi^\vee \rangle$. Indeed, in our implementation we construct and solve the same *single* linear system for a collection of $\{\varphi_1, \dots, \varphi_A\}$ and $\{\varphi_1^\vee, \dots, \varphi_B^\vee\}$ (subscripts here temporarily denote different cocycles in a list and not projection components) to determine the corresponding $A \times B$ intersection matrix in one go. In the context of the example discussed in appendix B above, that corresponds to the **column** on the r.h.s. becoming a wider matrix with B columns, and the top **row** on the l.h.s. growing into a taller matrix with A rows.

Choice of the polynomial ideal. There are several methods to construct the polynomial (3.7). Importantly it must contain (a subset of) the singularities $\mathcal{P}_{\Omega(m)}$ of the connection matrix $\Omega^{(m)}$ or its dual $\Omega^{\vee(m)}$ at a given layer m . To satisfy the crucial condition (3.10) we may choose the $\mathcal{B}(z)$ as the least common multiple (LCM) of all the denominators appearing in the $\Omega^\vee$ connection matrix, as proposed in [25]. Alternatively, we may process separately each irreducible factor in the LCM with its highest multiplicity, and add up the individual contributions in the end. In this way only simple poles in the β -variable may appear in the connection matrix, which dramatically simplifies the construction of the tensor representation (3.32) in practice.

The infinity. One possible prescription for computing the residue contribution at $z = \infty$ is to take $\mathcal{B}(z) := z^n$, where n is the highest order pole of $\Omega^\vee$, and use exactly the same solving procedure as before. Alternatively, we may bring the $z = \infty$ to some finite position z_∞ via the Möbius coordinate change

$$z \mapsto \frac{z}{z - z_\infty}. \quad (\text{C.1})$$

However, in practice, we have not observed significant performance improvements with this strategy, primarily due to the growth of polynomials in intermediate expressions. Therefore, a careful consideration will be necessary in future implementations of the Möbius transformation approach.

Polynomial reduction via the Sylvester matrix. To build the companion matrices presented in section 3.1.1 we may also just directly polynomial reduce numerator and denominator of individual entries of $\Omega^\vee$, $\varphi^{(m)}$, and $\varphi^{\vee(m)}$, as an alternative to eq. (3.13). To do that we build the augmented Sylvester¹⁵ matrix [83, 94] of some polynomial $p(z) = p_0 + z p_1 + \dots + z^n p_n$ we wish to reduce modulo the chosen polynomial (3.7)

$$S_{p,B} = \left[\begin{array}{c|cc} \begin{matrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{matrix} & \begin{matrix} p_n & p_{n-1} & \dots & p_1 \\ & p_n & p_{n-1} & \dots \\ & & \ddots & \ddots \\ & & & p_n & p_{n-1} \end{matrix} & \begin{matrix} p_0 & & & & \\ & p_0 & & & \\ & & p_0 & & \\ & & & p_0 & \\ & & & & p_0 \end{matrix} \\ \hline \begin{matrix} 1 & b_{\kappa-1} & \dots & b_1 & b_0 \\ & 1 & b_{\kappa-1} & \dots & b_1 \\ & & \ddots & \ddots & \ddots \\ & & & 1 & b_{\kappa-1} \\ & & & & 1 \end{matrix} & \begin{matrix} b_0 & & & & \\ & b_0 & & & \\ & & b_0 & & \\ & & & b_0 & \\ & & & & b_0 \end{matrix} \end{array} \right] \quad (C.2)$$

In the top right corner we highlighted the $\kappa \times \kappa$ block, which, upon row reduction, will contain the companion matrix Q_p up to transposition and reversing the order of rows and columns:

$$\text{RowReduce}[S_{p,B}] = \left[\begin{array}{c|c} \mathbb{1} & \tilde{Q}_p \\ \hline & * \end{array} \right] \implies Q_p = (\tilde{Q}_p)^T // \text{Reverse} // \text{Map}[\text{Reverse}] . \quad (C.3)$$

It is also possible to find the companion matrix representation of a rational function with this technique, see [95] for details.

FiniteFlow data graph. We find FINITEFLOW [26, 27] to be an indispensable tool for our implementation due to flexibility of its user interface and support for tensor constructions that proved to be fundamental for the successful application presented in section 4. Indeed, many tensor operations like the outer product and contractions can naturally be done within FINITEFLOW via the nodes `FFAlgTakeAndAdd` and `FFAlgTakeAndAddBL` (with some automatization of the pattern creation <https://github.com/vchestnov/utis>). To compute intersection numbers using this technology, we convert the input data $\{\Omega^\vee, \varphi, \varphi^\vee\}$ first into companion matrix representation (3.14) utilizing the Sylvester matrix technique (C.2), then series expand the produced matrices in $\beta \rightarrow 0$ limit using the `FFAlgLaurent` node, to finally construct the companion tensor representation (3.26). We then solve the system (2.21), (2.40) with the `FFAlgNodeSparseSolver` node specifying only the intersection number $\langle \varphi | \varphi^\vee \rangle$ itself as the needed unknown variable to solve for. We repeat the same procedure for the pole at infinity, add all the contributions with the `FFAlgTakeAndAdd` node, and reconstruct only the final sum of contributions to the given intersection number. The schematic form of the data graph that we construct within FINITEFLOW at each layer is shown in figure 2.

Data Availability Statement. This article has no associated data or the data will not be deposited.

¹⁵The same idea applies to multivariate polynomial ideals, in which case the corresponding matrix bears the name of F.S. Macaulay [93].

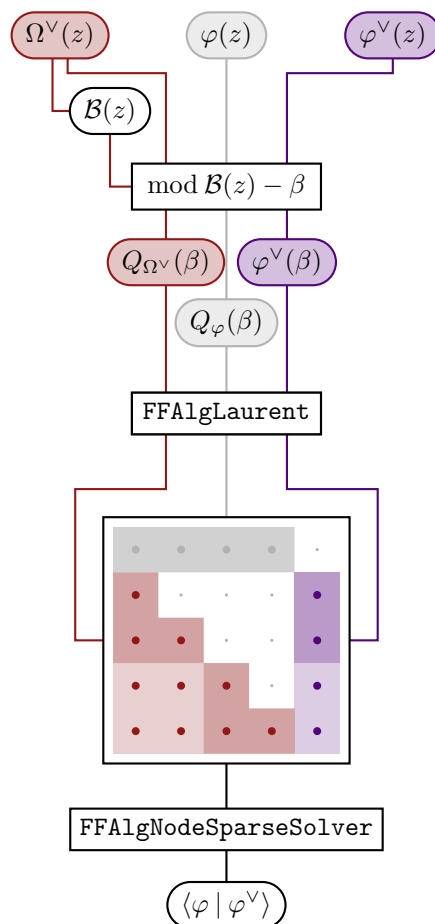


Figure 2. Flowchart of the data graph constructed in FINTEFLOW for computing intersection numbers and described in appendix C. Square in the middle represents the linear system of equations (3.27), (3.28) and (3.30), (3.31), see eq. (B.22) for an example.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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