THE RAMSEY PROPERTIES FOR GRASSMANNIANS OVER $\mathbb{R}$, $\mathbb{C}$

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Abstract. In this note we study and obtain factorization theorems for colorings of matrices and Grassmannians over $\mathbb{R}$ and $\mathbb{C}$, which can be considered metric versions of the Dual Ramsey Theorem for Boolean matrices and of the Graham-Leeb-Rothschild Theorem for Grassmannians over a finite field.

Introduction

One of the most powerful principles in Ramsey theory is the dual Ramsey theorem of R. L. Graham and B. L. Rothschild [9]. It trivially implies the classical Ramsey theorem or the much more involved Hales-Jewett Theorem. The Dual Ramsey theorem is the particular instance of the Rota's conjecture for Grassmannians over the boolean field $\mathbb{F}_2$, and it indeed implies the Rota's conjecture for an arbitrary finite field, proved by Graham, Leeb and Rothschild (GLR) in [8]. These statements can be categorized as a structural Ramsey theorem, the Dual Ramsey theorem as the result for finite Boolean algebras or for finite dimensional vector spaces over the boolean field $\mathbb{F}_2$, and the (GLR) Theorem as its natural generalization to finite dimensional vector spaces over an arbitrary finite field $\mathbb{F}_p$. In this paper we study the case of the infinite fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$ in its metric form: Suppose that we endow the n-dimensional vector space $\mathbb{F}^n$ with a norm $\mathtt{m}$. We can naturally identify each k-dimensional subspace V of $\mathbb{F}^n$ with its unit ball $\mathrm{Ball}(V, \mathtt{m}) = \{v \in V : \mathtt{m}(v) \leqslant 1\}$. Thus, we can measure the distance between V and W by computing the Hausdorff distance $\Lambda_{\mathtt{m}}$ between the compact and convex sets $\mathrm{Ball}(V, \mathtt{m})$ and $\mathrm{Ball}(W, \mathtt{m})$. Instead of trying to understand only discrete colorings $c : \mathrm{Gr}(k, \mathbb{F}^n) \to r := \{0, 1, \cdots, r-1\}$ we can now work with 1-Lipschitz mappings, called here compact colorings, $c : (\mathrm{Gr}(k, \mathbb{F}^n), \Lambda_{\mathtt{m}}) \to (K, d_K)$ into a compact metric space (K, d_K) and ask how the restrictions of c to Grassmannians $\mathrm{Gr}(k, V)$ that are *congruent* to $\mathrm{Gr}(k, \mathbb{F}^m)$ look like. In this context, a reasonable notion of congruence $\mathrm{Gr}(k, V) \sim_{\mathtt{m}} \mathrm{Gr}(k, W)$ is that $(V, \mathtt{m})$ and $(W, \mathtt{m})$ are linearly isometric, or equivalently when there is an affine and symmetric bijection sending the dual unit ball $\mathrm{Ball}(V^*, \mathtt{m}^*)$ onto the dual unit ball $\mathrm{Ball}(W^*, \mathtt{m}^*)$ (see the introduction in §2.1.3 for basic definitions, and [20, Chapter 4] for a complete exposition). Notice that the set-mapping associated to a linear isometry from V onto W defines a $\Lambda_{\mathtt{m}}$-isometry from $\mathrm{Gr}(k, V)$ onto $\mathrm{Gr}(k, W)$. The corresponding quotient $\mathrm{Gr}(k, \mathbb{F}^n)/\sim_{\mathtt{m}}$ is canonically identified with the class $\mathcal{B}_k(\mathbb{F}^n, \mathtt{m})$ of isometric types of k-dimensional subspaces of $(\mathbb{F}^n, \mathtt{m})$, a closed subset of the *Banach-Mazur compactum* $\mathcal{B}_k$. In this paper we show that for the p-norms $\|(a_j)_j\|_p := (\sum_j |a_j|^p)^{1/p}$, if

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$p \in [1,\infty[\setminus(2\mathbb{N}+4)$, and for the sup norm $\|(a_j)_j\|_\infty := \max_j |a_j|$, we have that on each quotient $\mathcal{B}_k(\mathbb{F}^n, \|\cdot\|_p)$ there is a compatible "Gromov-Hausdorff"-metric γ_p, called here *extrinsic metric*, such that for every $k, m \in \mathbb{N}$, every compact metric space (K, d_K) and every $\varepsilon > 0$ there is a dimension n such that for every compact coloring $c : (\mathrm{Gr}(k, \mathbb{F}^n), \Lambda_{\|\cdot\|_p}) \to (K, d_K)$ there is some $V \in \mathrm{Gr}(m, \mathbb{F}^n)$ that is $\|\cdot\|_p$-congruent to $\mathbb{F}^m$ and there is a compact coloring $\widehat{c} : (\mathcal{B}_k(\mathbb{F}^n, \|\cdot\|_p), \gamma_p) \to (K, d_K)$ such that $d_K(\widehat{c}([W]_{\sim_{\mathtt{m}}}), c(W)) \leqslant \varepsilon$ for every $W \in \mathrm{Gr}(k, V)$.

In a similar way, we study factorizations of compact colorings of matrices of two kinds: $n \times k$-full rank matrices and n-square matrices of rank k, denoted by $M^k_{n,k}$ and by M^k_n, respectively. When the field $\mathbb{F}$ is finite, we show that for large enough n, for every coloring $c : M^k_{n,k} \to r$ there is some matrix $R \in M^m_{n,m}$ in *reduced column echelon form* and a unique $\widehat{c} : \mathrm{GL}(\mathbb{F}^k) \to r$ such that $c(R \cdot A) = \widehat{c}(\mathrm{red}(A))$ for every $A \in M^k_{m,k}$, where $\mathrm{red}(A)$ is the k-square invertible matrix such that $A \cdot \mathrm{red}(A)$ is in reduced column echelon form. We prove that colorings of M^k_n are factorized in a similar way by, in addition, using the full rank factorization of matrices. We then analyze the colorings of these matrices over the fields $\mathbb{R}, \mathbb{C}$, and we compute the corresponding Ramsey factors in the metric context for the p-norms.

The proofs for the infinite fields are based on the crucial fact that when $\mathtt{m}$ is a norm on the vector space $\mathbb{F}^\infty$, the space of sequences $(a_n)_n$ with finitely many non-zero entries, have an approximate Ramsey property called *steady approximate Ramsey property*, then there is a unique Banach space $\widehat{E}$ such that $E := (\mathbb{F}^\infty, \mathtt{m})$ can be linearly isometrically embedded into $\widehat{E}$, $\mathcal{B}_k(E)$ is dense in $\mathcal{B}_k(\widehat{E})$, and such that the group $\mathrm{Iso}(\widehat{E})$ of linear isometries of $\widehat{E}$, with its strong operator topology, is *extremely amenable*, that is, every continuous action of $\mathrm{Iso}(\widehat{E})$ on a compact space has a fixed point. The corresponding spaces to the p-norms are the Lebesgue spaces $L_p[0,1]$ if $p < \infty$, and the Gurarij space for the sup-norm.

The use of tools from topological dynamics on a pure approximate Ramsey problem is not accidental. The recent *Kechris-Pestov-Todorcevic correspondence* in its discrete and metric versions characterizes the extreme amenability of automorphism groups of Fraïssé (discrete/metric) structures in terms of the (approximate) Ramsey property of the collection of finitely generated substructures (see [6, 14, 15]).

The paper is organized as follows. We first study Ramsey properties of matrices over $\mathbb{F}_2$ and then over an arbitrary finite field $\mathbb{F}$. In particular, we provide in Theorem 1.7 another proof of the Rota's conjecture as a straightforward consequence of the Dual Ramsey theorem. To do this, we use basic tools from linear algebra, mainly the reduced column echelon form, that interestingly corresponds to some surjection being *rigid* with respect to the antilexicographical ordering, and that determines the Ramsey property (Proposition 1.9). We finish this section by introducing in Proposition 1.14 a uniqueness principle for these Ramsey factorizations. The rest of the sections are devoted to the study of Ramsey factorizations of matrices and Grassmannians over the fields $\mathbb{R}, \mathbb{C}$. Different principles and known facts from Banach space theory play a fundamental role, so we expose them with enough details. We start the second section by introducing the main concepts, namely ε-factors, and the Ramsey factors, including the extrinsic metrics, for full rank $n \times k$-matrices, Grassmannians, and $n \times n$-matrices of rank k, and we present our main results in Theorem 2.8, Theorem 2.16 and Theorem 2.25, respectively. The third section is devoted to the proofs of the factorization results exposed in section two. We recall the steady approximate Ramsey property (SARP$^+$) of a family of finite dimensional normed spaces and the extreme amenability of a topological group. We explain in Corollary 3.11 when a normed space of the form $E = (\mathbb{F}^\infty, \mathtt{m})$ is associated to a unique Fraïssé Banach space $\widehat{E}$ with an extremely amenable group of isometries, and how this gives Ramsey factors related to E. In Subsection 3.1 we analyze these factors and we prove that they are the ones presented in Section two (Theorem 3.12). We finish with an appendix where we analyze the special case of the sup-norm, and we give explicit definitions of extrinsic metrics.

1. The Dual Ramsey Theorem and matrices over finite fields

To keep the notation unified, let $\mathbb{F}^\infty$ be the vector space over $\mathbb{F}$ consisting of all eventually zero sequences $(a_j)_{j\in\mathbb{N}}$. Let $(u_j)_{j\in\mathbb{N}}$ be the *unit basis* of $\mathbb{F}^\infty$, that is, each u_j is the sequence whose only non-zero entry is 1 at the j^{th}-coordinate. In this way we identify $\mathbb{F}^n$ with the subspace $\langle u_j\rangle_{j<n}$ of $\mathbb{F}^\infty$, and then $\mathbb{F}^\infty$ with the increasing union of all $\mathbb{F}^n$.

Given $\alpha,\beta\in\mathbb{N}\cup\{\infty\}$, let $M_{\alpha,\beta}(\mathbb{F})$ be the collection of $\alpha\times\beta$-matrices with finitely many non-zero entries. In a similar manner as before, given $n\leqslant\alpha$ and $m\leqslant\beta$, a $m\times n$-matrix $A=(a_{i,j})_{i<n,j<m}$ is identified, if needed, with the $\alpha\times\beta$-matrix $B=(b_{i,j})_{i\leqslant\alpha,j\leqslant\beta}$ by keeping the old ones unchanged $b_{i,j}=a_{i,j}$ for $i<n$ and $j<m$, and by declaring the new entries as zero $b_{i,j}=0$ for $n\leqslant i<\alpha$ and $m\leqslant j<\beta$. So, we write $M_{\alpha,\beta}(\mathbb{F})=\bigcup_{n\leqslant\alpha,m\leqslant\beta}M_{n,m}(\mathbb{F})$, increasing union. Let $M^k_{\alpha,\beta}(\mathbb{F})$ be the set of all $\alpha\times\beta$-matrices of rank k with entries in $\mathbb{F}$. To lighten the notation, when there is no possible confusion, we will write $M_{\alpha,\beta}$, $M^k_{\alpha,\beta}$,... to denote $M_{\alpha,\beta}(\mathbb{F})$, $M^k_{\alpha,\beta}(\mathbb{F})$,...

There are several equivalent ways to present the dual Ramsey theorem (DRT) of Graham and Rothschild [9]. Among these, there is a factorization result for *Boolean matrices* stated below as Theorem 1.4. Motivated by this, we study Ramsey-theoretical factorization results for colorings of other classes of matrices. We begin with matrices with entries in a finite field, and then conclude, in the next section, with matrices over $\mathbb{R}$ or $\mathbb{C}$.

It is well known, for example using the Gauss-Jordan elimination method, that an $n\times m$-matrix A has a decomposition $A=\mathrm{red}(A)\cdot\tau(A)$ where $\mathrm{red}(A)$ is in reduced column echelon form and $\tau(A)$ is an invertible $m\times m$-matrix, that is unique when A has rank m. We prove that when the field is finite any finite coloring of matrices over a finite field is determined, in a precise way, by τ. This can be seen as an extension of the well known result of Graham, Leeb, and Rothschild on Grassmannians over a finite field [8].

Definition 1.1 (Factors)**.** Let X be a set and $r\in\mathbb{N}$. An *r-coloring* of X is a mapping $c:X\to r=\{0,1\dots,r-1\}$. A subset Y of X is *c-monochromatic* if c is constant on Y. We say that a mapping $\pi:X\to K$ is a *factor* of $c:X\to r$ if there is some $\widetilde{c}:K\to r$ such that $c=\widetilde{c}\circ\pi$. Finally, π is a *factor of c in $Y\subseteq X$* if $\pi\restriction_Y$ is a factor of $c\restriction_Y$. So, Y is c-monochromatic when the trivial constant map $\pi:X\to\{0\}=1$ is a factor of c in Y.

We now recall the *Dual Ramsey Theorem* (DRT) of Graham and Rothschild [9] (see also [17], [23]). For convenience, we present its formulation in terms of rigid surjections between finite linear orderings. Given two linear orderings $(R,<_R)$ and $(S,<_S)$, a surjective map $f:R\to S$ is called a *rigid surjection* when $\min f^{-1}(s_0)<_R\min f^{-1}(s_1)$ for every $s_0,s_1\in S$ such that $s_0<_S s_1$. We let $\mathrm{Epi}(R,S)$ be the collection of rigid surjections from R to S.

Theorem 1.2 (Graham–Rothschild)**.** *For every finite linear orderings R and S such that $|R|<|S|$ and every $r\in\mathbb{N}$ there exists an integer n such that, considering n naturally ordered, every r-coloring of $\mathrm{Epi}(n,R)$ has a monochromatic set of the form $\mathrm{Epi}(S,R)\circ\gamma=\{\sigma\circ\gamma:\sigma\in\mathrm{Epi}(S,R)\}$ for some $\gamma\in\mathrm{Epi}(n,S)$.*

1.1. **Ramsey properties of colorings of Boolean matrices.**

Perhaps the most common formulation of the *dual Ramsey Theorem* of Graham and Rothschild is done in terms of partitions. Given $k,m,n\in\mathbb{N}$, let $\mathcal{E}_m(n)$ be the set of all partitions of n into m pieces. Given $\mathcal{P}\in\mathcal{E}_m(n)$, let $\langle\mathcal{P}\rangle_k$ be the set of all partitions $\mathcal{Q}$ of n with k pieces that are coarser than $\mathcal{P}$, i.e., such that each piece of $\mathcal{Q}$ is a union of pieces of $\mathcal{P}$.

Theorem (DRT, partitions version)**.** *For every $k,m\in\mathbb{N}$ and $r\in\mathbb{N}$ there is $n\in\mathbb{N}$ such that every r-coloring of $\mathcal{E}_k(n)$ has a monochromatic set of the form $\langle\mathcal{P}\rangle_k$ for some $\mathcal{P}\in\mathcal{E}_m(n)$.*

The following three reformulations of the Dual Ramsey Theorem are *structural Ramsey results* for finite Boolean algebras.

Theorem (DRT, Boolean algebras)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be finite Boolean algebras, and let $r \in \mathbb{N}$. Then there exists a finite Boolean algebra $\mathcal{C}$ such that every r-coloring of the set $\binom{\mathcal{C}}{\mathcal{A}}$ of isomorphic copies of $\mathcal{A}$ inside $\mathcal{C}$ admits a monochromatic set of the form $\binom{\mathcal{B}_0}{\mathcal{A}}$ for some $\mathcal{B}_0 \in \binom{\mathcal{C}}{\mathcal{B}}$.*

Let $\mathcal{A}$ be a finite Boolean algebra. Any $a \in \mathcal{A}$ is represented as

$$a = \bigvee_{x \in \Gamma_a} x,$$

for a unique set of atoms Γ_a. So, any linear ordering $<$ on the sets of atoms $\mathrm{At}(\mathcal{A})$ of $\mathcal{A}$ *extends* to $\mathcal{A}$ by defining $a < b$ iff $\min_<(\Gamma_a \triangle \Gamma_b) \in \Gamma_a$. Following [14], we will say that $(\mathcal{A}, <)$ is a *canonically ordered (c.o.)* Boolean algebra. Given c.o. Boolean algebras $\mathcal{A}$ and $\mathcal{B}$, let $\mathrm{Emb}_<(\mathcal{A}, \mathcal{B})$ be the collection of ordering-preserving embeddings from $\mathcal{A}$ into $\mathcal{B}$, respectively.

Theorem 1.3 (DRT, canonically ordered Boolean algebras)**.** *Given c.o. Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ and $r \in \mathbb{N}$, there is a c.o. Boolean algebra $\mathcal{C}$ such that each r-coloring of $\mathrm{Emb}_<(\mathcal{A}, \mathcal{C})$ has a monochromatic set of the form $\varrho \circ \mathrm{Emb}_<(\mathcal{A}, \mathcal{B})$ for some $\varrho \in \mathrm{Emb}_<(\mathcal{B}, \mathcal{C})$.*

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are finite Boolean algebras with k and n atoms, respectively. Any embedding from $\mathcal{A}$ to $\mathcal{B}$ has a corresponding *representing* $n \times k$ matrix with entries in $\{0, 1\}$. We call the matrices arising in this fashion *Boolean matrices.* The set of $n \times k$ Boolean matrices will be denoted by $M^{\mathrm{ba}}_{n,k}$, i.e., the set of $n \times k$ matrix with entries in $\{0, 1\}$ whose columns (which can be identified with subsets of n) form a k-partition of n. We let $M^{\mathrm{oba}}_{n,k}$ be the set of Boolean $n \times k$-matrices that correspond to order-preserving embeddings between c.o. Boolean algebras. These are precisely the set of Boolean matrices whose columns $(P_i)_{i \in k}$ furthermore satisfy that the position of the first non-zero value of P_i is strictly smaller than the position of the first non-zero value of P_{i+1} for every $i < k - 1$.

In the following, we identify a permutation σ of k with the associated $k \times k$ permutation matrix. This allows one to identify the group $\mathcal{S}_k$ of permutations of k with a group of unitary matrices. Let $\pi : M^{\mathrm{ba}}_{n,k} \to \mathcal{S}_k$ be the function assigning to a matrix A the unique element $\pi(A)$ of $\mathcal{S}_k$ such that $A = A_< \cdot \pi(A)$ for some (uniquely determined) matrix $A_< \in M^{\mathrm{oba}}_{n,k}$. Given an $n \times m$-matrix A, we let $A \cdot M^{\mathrm{ba}}_{m,k} = \{A \cdot B \, : \, B \in M^{\mathrm{ba}}_{m,k}\}$.

Theorem 1.4 (DRT, Boolean matrices)**.** *For every $k, m \in \mathbb{N}$ and $r \in \mathbb{N}$ there is n such that for every $c : M^{\mathrm{ba}}_{n,k} \to r$ there is $R \in M^{\mathrm{oba}}_{n,m}$ such that π is a factor of c in $R \cdot M^{\mathrm{ba}}_{m,k}$. That is, the color of $R \cdot B$ depends only on $\pi(B) = \pi(R \cdot B)$ for every $B \in M^{\mathrm{ba}}_{m,k}$.*

Proof. Let $\mathcal{C}$ be a c.o. Boolean algebra obtained by applying the Dual Ramsey Theorem for c.o. Boolean algebras—Theorem 1.3—to the power sets $\mathcal{P}(k)$, $\mathcal{P}(m)$ canonically ordered as above by $s < t$ if and only if $\min(s \triangle t) \in s$, and to the number of colors $r^{\mathcal{S}_k}$. Without loss of generality we can assume that $\mathcal{C}$ is equal to $\mathcal{P}(n)$ for some $n \in \omega$, since any c.o. Boolean algebra is of this form. We claim that such an n satisfies the desired conclusions. Indeed, fix a coloring $c : M^{\mathrm{ba}}_{n,k} \to r$. This induces a coloring $f : \mathrm{Emb}_<(\mathcal{P}(k), \mathcal{P}(n)) \to r^{\mathcal{S}_k}$ as follows. Let γ be an element of $\mathrm{Emb}_<(\mathcal{P}(k), \mathcal{P}(n))$, and let $A_\gamma \in M^{\mathrm{ba}}_{n,k}$ be the corresponding representing matrix. Define then $f(\gamma)$ to be the element $(c(A_\gamma \cdot \sigma))_{\sigma \in \mathcal{S}_k}$ of $r^{\mathcal{S}_k}$. By the choice of $\mathcal{C} = \mathcal{P}(n)$ there exists $\varrho \in \mathrm{Emb}_<(\mathcal{P}(m), \mathcal{P}(n))$ such that f is constant on $\varrho \circ \mathrm{Emb}_<(\mathcal{P}(k), \mathcal{P}(m))$. Let now $\widetilde{c} \in r^{\mathcal{S}_k}$ be the constant value of f. It is now easy to see that $R := A_\varrho$ satisfies what we want $c(R \cdot B) = \widetilde{c}(\pi(B))$ for every $B \in M^{\mathrm{ba}}_{m,k}$. $\square$

1.2. **Ramsey properties of colorings of matrices over a finite field.** It is natural to consider Ramsey properties of other classes of matrices over a field $\mathbb{F}$. We are going to see that for $\mathbb{F}$ finite there is a factorization result similar to the DRT for Boolean matrices, that extends the well known theorem, the vector space Ramsey theorem, by Graham, Leeb and Rothschild on Grassmannians $\mathrm{Gr}(k,V)$, the family of all k-dimensional subspaces of a vector space V over $\mathbb{F}$. In the following, given a sequence (x_i) in a vector space E, we let $\langle x_i\rangle$ be its linear span inside E. Notice that the linear span of the empty sequence $(\,)$ is the trivial subspace $\{0\}$.

Theorem 1.5 (Graham-Leeb-Rothschild [8]; see also [10])**.** *Given $k,m,r\in\mathbb{N}$ there is $n\in\mathbb{N}$ such that every r-coloring of $\mathrm{Gr}(k,\mathbb{F}^n)$ has a monochromatic set of the form $\mathrm{Gr}(k,R)$ for some $R\in\mathrm{Gr}(m,\mathbb{F}^n)$.*

This result is a particular case of the factorization theorem for injective matrices. Recall that a $p\times q$-matrix $A=(a_{ij})$ is in *reduced row echelon form* (RREF) when there is $p_0\leqslant p$ and (a unique) strictly increasing sequence $(j_i)_{i<p_0}$ of integers $<q$ such that

i) $A\cdot u_{j_i}=u_i$ for every $i<p_0$ and
ii) $\langle A\cdot u_j\rangle_{j<j_i}=\langle u_l\rangle_{l<i}$ for every $i<p_0$.

When A is in RREF and it has rank p, we define I_A as the $q\times p$-matrix with entries in $\{0,1\}$, and whose nonzero entries are in the positions (j_i,i) $(i<p)$. For example for the field $\mathbb{F}_5$ and

$$A=\begin{pmatrix}1&2&0&3&0&1\\0&0&1&4&0&2\\0&0&0&0&1&3\end{pmatrix}\text{ we have } I_A=\begin{pmatrix}1&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&1\\0&0&0\end{pmatrix} \tag{1}$$

It follows that I_A is a right inverse to A, i.e., $A\cdot I_A=\mathrm{Id}_p$. A matrix A is in *reduced column echelon form (RCEF)* when its transpose A^{t} is in RREF. Let $\mathcal{E}_{n,m}(\mathbb{F})$, $\mathcal{E}(\mathbb{F})$ be the collection of $n\times m$-matrices of rank m in RCEF and of full rank matrices in RCEF, respectively. Notice that $\mathcal{E}_{n,m}(\mathbb{F})$ is non-empty exactly when $n\geqslant m$.

Definition 1.6. Let $\tau: M^k_{\infty,k}\to\mathrm{GL}(\mathbb{F}^k)$ be the mapping that assigns to each $A\in M^k_{\infty,k}(\mathbb{F})$ the unique $k\times k$-invertible matrix $\tau(A)$ such that $A\cdot\tau(A)$ is in RCEF. Let also $\mathrm{red}_c(A):=A\cdot\tau(A)$.

Theorem 1.7 (Factorization of colorings of full rank matrices over a finite field)**.** *Given $k,m,r\in\mathbb{N}$ there is $n\in\mathbb{N}$ such that for every $c: M^k_{n,k}(\mathbb{F})\to r$ there is $R\in\mathcal{E}_{n,m}(\mathbb{F})$ such that τ is a factor of c in $R\cdot M^k_{m,k}(\mathbb{F})$.*

This gives immediately the Graham-Leeb-Rothschild Theorem—Theorem 1.5—as every k-dimensional subspace of $\mathbb{F}^n$ can be represented as the linear span of the columns of a matrix in RCEF. The proof of Theorem 1.7 is a direct consequence of the DRT and the next propositions. In the following, we fix an ordering $<$ on the finite field $\mathbb{F}$ such that $0<1$ are the first two elements of $\mathbb{F}$. We let $\mathbb{F}^k$ be endowed with the corresponding antilexicographic order $<_{\mathrm{alex}}$ and we define $\Phi_{n,k}:\mathrm{Epi}(n,\mathbb{F}^k)\to M^k_{n,k}$ as the function assigning to each rigid surjection f the matrix whose rows are $f(j)$ for every $j<n$. A key feature of the antilexicographic order in this context is that given $x\in\mathbb{F}^k$ and $j<k$ we have that $x\in\langle u_l\rangle_{l<j}$ if and only if $x<_{\mathrm{alex}}u_j$.

Lemma 1.8. *$\Phi_{n,k}(f)$ is a full rank matrix in RCEF.*

Proof. It is clear that $\Phi_{n,k}(f)$ is a full rank matrix. We prove that it is in RCEF. Let A be the transpose of $\Phi_{n,k}(f)$. For each $i\in k$, let $j_i:=\min\{j<n\,:\,A\cdot u_j=u_i\}$. Then $(j_i)_{i<k}$ is strictly increasing, since f is a rigid surjection, and if $j<j_i$, then $A\cdot u_j<_{\mathrm{alex}}u_i$, by the definition of j_i, and the rigidity of f. Therefore $A\cdot u_j\in\langle u_l\rangle_{l<i}$. Consequently, A is in RREF. $\square$

The next is the key relation between matrices in RREF and rigid surjections that will allow us to use the dual Ramsey Theorem and prove Theorem 1.7.

Proposition 1.9. *For $A \in M_{k,n}^k(\mathbb{F})$ the following are equivalent.*

i) A is in RREF.

ii) The linear map $T_A : \mathbb{F}^n \to \mathbb{F}^k$ represented by A in the corresponding unit bases is a rigid surjection and for every $i < k$ there is a column of A equal to u_i.

In particular we have the following.

Corollary 1.10. *Suppose that $A \in M_{n,m}^m(\mathbb{F})$ and $B \in M_{m,k}^k(\mathbb{F})$.*

a) If A and B are in RCEF (resp. RREF) then $A \cdot B$ is also in RCEF (resp. RREF).

b) If A is in RCEF then $\tau(A \cdot B) = \tau(B)$. □

Proof of Proposition 1.9. i)⇒ii) Suppose that A is in RREF. We will prove that the canonical linear operator $T_A : \mathbb{F}^n \to \mathbb{F}^k$, $T_A(u_i) := A \cdot u_i$ for $i < n$ is a rigid surjection from $\mathbb{F}^n$ to $\mathbb{F}^k$ endowed with the antilexicographical order $<_{\mathrm{alex}}$ described before. Let $(j_i)_{i<k}$ be the strictly increasing sequence in n witnessing that A is in RREF. By linearity, $T_A(0) = 0$. Fix now $w \in \mathbb{F}^k$.

Claim 1.10.1. $\min_{<_{\mathrm{alex}}} (T_A)^{-1}(w) = I_A \cdot w$.

From this, since $I_A : \mathbb{F}^k \to \mathbb{F}^n$ is $<_{\mathrm{alex}}$-increasing, we obtain that T_A is a rigid surjection.

Proof of Claim: Applied to the example in (1) and to $w = (1, 2, 3)$, it should be clear that the spread $I_A \cdot (1, 2, 3) = (1, 0, 2, 0, 3, 0)$ of $(1, 2, 3)$ is the $<_{\mathrm{alex}}$-least element of the preimage of $(1, 2, 3)$ under T_A. We give a detailed proof. Suppose that $(v_j)_{j<n} = \bar{v} = \min_{<_{\mathrm{alex}}}\{v \in \mathbb{F}^n : A \cdot v = w\}$. Set $z = (z_j)_j := I_A(w)$. We prove by induction on $i < k$ that $v_j = z_j$ for every $j \geqslant j_{k-i-1}$. Suppose that $i = 0$. Since for every $j > j_{k-1}$ one has that $z_j = 0$, we obtain that $v_j = 0$, by $<_{\mathrm{alex}}$-minimality of $\bar{v}$. Let $(A)_{k-1}$ be the $(k-1)^{\mathrm{th}}$-row of A. It follows that $(A)_{k-1} = u_{j_{k-1}} + y$, where $y \in \langle u_j \rangle_{j>j_{k-1}}$. Hence,

$$z_{j_{k-1}} = w_{k-1} = (A)_{k-1} \cdot \bar{v} = v_{j_{k-1}}.$$

Suppose that the conclusion holds for i, that is, $v_j = z_j$ for every $j \geqslant j_{k-i-1}$. We will prove that it also holds for $i + 1$. Since $v \leqslant_{\mathrm{alex}} z$, and $z_j = 0$ for every $j_{k-i'-2} < j < j_{k-i'-1}$ and $0 \leqslant i' \leqslant i$, we obtain that $v_j = 0$ for such j's. Then the $(k-i-2)^{\mathrm{nd}}$ row of A is of the form $(A)_{k-i-2} = u_{j_{k-i-2}} + y$ with y in the span of $\{u_j : j > j_{k-i-2}, j \neq j_p \text{ for all } p\}$. It follows that

$$z_{j_{k-i-2}} = w_{k-i-2} = (A)_{k-i-2} \cdot \bar{v} = v_{j_{k-i-2}}.$$ □

ii)⇒i) Now suppose that T_A is a rigid surjection from $\mathbb{F}^n$ to $\mathbb{F}^k$ with respect to the antilexicographical orderings, and that for every $i < k$ a column of A is u_i. For each $i < k$, let j_i be the first such column of A. We prove that $(j_i)_{i<k}$ witnesses that A is in RREF, that is:

Claim 1.10.2. $T_A \langle u_j \rangle_{j<j_i} = \langle u_l \rangle_{l<i}$ for every $i < k$.

Proof of Claim: The proof is by induction on i. If $i = 0$, then $T_A \langle u_j \rangle_{j<j_0} = \{0\}$ because u_0 is the second element of $\mathbb{F}^n$ in the antilex ordering, while the first element is the zero vector. Suppose the result is true for i, and let us extend it to $i + 1$. In particular, we know that $j_{i+1} > j_i$, and it is clear that $\langle u_l \rangle_{l \leqslant i} \subseteq T_A \langle u_j \rangle_{j \leqslant j_i} \subseteq T_A \langle u_j \rangle_{j<j_{i+1}}$. Suppose towards a contradiction that there exists j such that $j_i < j < j_{i+1}$ and $T_A(u_j) \notin \langle u_l \rangle_{l \leqslant i}$. Denote by ξ the least such j, and set $y := T_A(u_\xi)$. Since $y \notin \langle u_l \rangle_{l \leqslant i}$, it follows that $u_{i+1} \leqslant_{\mathrm{alex}} y$, and since j_{i+1} is the minimal j such that $T_A(u_j) = u_{i+1}$, it follows that in fact $u_{i+1} <_{\mathrm{alex}} y$. We are assuming that T_A is a rigid surjection, so

$$\min(T_A)^{-1}(u_{i+1}) <_{\mathrm{alex}} \min(T_A)^{-1}(y) \leqslant_{\mathrm{alex}} u_\xi.$$

This means that there is some $x <_{\mathrm{alex}} u_\xi$ such that $T_A(x) = u_{i+1}$; that is, there is some $x \in \langle u_j \rangle_{j<\xi}$ with $T_A(x) = u_{i+1}$, and this is impossible because by the minimality of ξ we know that $T_A\langle u_j\rangle_{j<\xi} = \langle u_l\rangle_{l\leqslant i}$. □ □

Proof of Theorem 1.7. Fix all parameters. We consider $\mathbb{F}^k$ and $\mathbb{F}^m$ antilexicographically ordered by $<_{\mathrm{alex}}$ (as explained before). Let n be obtained from the linear orderings $(\mathbb{F}^k, <_{\mathrm{alex}})$, $(\mathbb{F}^m, <_{\mathrm{alex}}$) and the number of colors r^λ, where $\lambda = \prod_{i=0}^{k-1}(p^k - p^i)$ is the order of the group $\mathrm{GL}(\mathbb{F}^k)$, by applying the Dual Ramsey Theorem for rigid surjections (Theorem 1.2). We claim that n satisfies the desired conclusions. Fix a coloring $c : M^k_{n,k}(\mathbb{F}) \to r$. Let $c_0 : \mathrm{Epi}(n, \mathbb{F}^k) \to r^{\mathrm{GL}(\mathbb{F}^k)}$ be the coloring defined by $c_0(\sigma) := (c(\Phi_{k,n}(\sigma)\cdot\Gamma^{-1}))_{\Gamma\in\mathrm{GL}(\mathbb{F}^k)}$ for $\sigma \in \mathrm{Epi}(n, \mathbb{F}^k)$. By the choice of n, there exists $\varrho \in \mathrm{Epi}(n, \mathbb{F}^m)$ such that c_0 is constant on $\mathrm{Epi}(\mathbb{F}^m, \mathbb{F}^k)\circ\varrho$ with constant value $\widetilde{c} \in r^{\mathrm{GL}(\mathbb{F}^k)}$. Let $R := \Phi_{n,m}(\varrho)$. We claim that R and $\widetilde{c}$ satisfy the conclusion of the statement in the theorem. It follows from Proposition 1.8 that $R \in \mathcal{E}_{n,m}(\mathbb{F})$. Now let $A \in M^k_{m,k}(\mathbb{F})$. We have to prove that $c(R\cdot A) = \widetilde{c}(\tau(R\cdot A))$. First, note that $\tau(R\cdot A) = \tau(A)$, because R is in RCEF. Let B be the transpose of $\mathrm{red}_c(A)$ (i.e., B is the RREF of the transpose of A), and let $T_B : \mathbb{F}^m \to \mathbb{F}^k$ be the linear operator defined by B in the corresponding canonical bases. We know by Proposition 1.9 that $T_B \in \mathrm{Epi}(\mathbb{F}^m, \mathbb{F}^k)$.

Claim 1.10.3. $\Phi_{n,k}(T_B \circ \varrho) = R \cdot \mathrm{red}_c(A)$.

Proof of Claim: Fix $j < m$. Then the j^{th}-row $(\Phi_{n,k}(T_B\circ\varrho))_j$ of $\Phi_{n,k}(T_B\circ\varrho)$ is the row vector $T_B(\varrho(j))$. Hence,

$$(\Phi_{n,k}(T_B\circ\varrho))_j = T_B(\varrho(j)) = ((\mathrm{red}_c(A))^{\mathrm{t}}\cdot((R)_j)^{\mathrm{t}})^{\mathrm{t}} = (R)_j\cdot\mathrm{red}_c(A) = (R\cdot\mathrm{red}_c(A))_j. \quad \square$$

So, given $\Gamma \in \mathrm{GL}_k(\mathbb{F})$ we have that

$$c(R\cdot A) = c(R\cdot\mathrm{red}_c A\cdot\tau(A)^{-1}) = (c_0(R\cdot\mathrm{red}_c A))(\tau(A)) = \widetilde{c}(\tau(A)) = \widetilde{c}(\tau(R\cdot A)). \quad \square$$

1.2.1. *Square matrices of rank k.* We present the Ramsey factorization for finite colorings of square matrices. Recall that every $n\times m$-matrix A of rank k has a *full rank decomposition* $A = B\cdot C$ where $B \in M^k_{n,k}$ and $C \in M^k_{k,m}$.

Definition 1.11. Given k and n, let $\tau^{(2)} : M^k_{n,n} \to \mathrm{GL}(\mathbb{F}^k)$ be the mapping uniquely defined by the relation $A = A_0\cdot\tau^{(2)}(A)\cdot A_1^{\mathrm{t}}$ for some $A_0, A_1 \in \mathcal{E}_{n,k}(\mathbb{F})$.

It is routine to see that $\tau^{(2)}$ is well defined.

Theorem 1.12 (Factorization of colorings of square matrices over a finite field)**.** *For every $k, m, r \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that for every $c : M^k_{n,n}(\mathbb{F}) \to r$ there are $R_0, R_1 \in \mathcal{E}_{n,m}(\mathbb{F})$ such that $\tau^{(2)}$ is a factor of c in $R_0\cdot M^k_{m,m}(\mathbb{F})\cdot R_1^{\mathrm{t}}$.*

Proof. Given integers k, m and r, let $n_{\mathbb{F}}(k, m, r)$ be the minimal number n such that the factorization statement in Theorem 1.7 holds for the parameters k, m and r, and now let $n_0 := n_{\mathbb{F}}(k, m, r^{\mathrm{GL}(\mathbb{F}^k)})$, and let $n := n_{\mathbb{F}}(k, n_0, r^{M^k_{n_0,k}(\mathbb{F})})$. We claim that n works. Fix any r-coloring $f : M^k_{n,n}(\mathbb{F}) \to r$. Let $P \in \mathcal{E}_{n,n_0}(\mathbb{F})$ be arbitrary. We define the coloring $c : M^k_{n,k}(\mathbb{F}) \to r^{M^k_{n_0,k}(\mathbb{F})}$ by

$$c(A) := (f(A\cdot B^{\mathrm{t}}\cdot P^{\mathrm{t}}))_{B\in M^k_{n_0,k}(\mathbb{F})}.$$

The coloring c is well defined because $A\cdot B^{\mathrm{t}}\cdot P^{\mathrm{t}}$ is an $n\times n$-matrix of rank k. Let $R \in \mathcal{E}_{n,n_0}$ and $c_0 : \mathrm{GL}(\mathbb{F}^k) \to r^{M^k_{n_0,k}(\mathbb{F})}$ be such that $c(R\cdot A) = c_0(\tau(A))$ for every $A \in M^k_{n_0,k}(\mathbb{F})$. Define now the "adjoint" coloring $d : M^k_{n_0,k}(\mathbb{F}) \to r^{\mathrm{GL}(\mathbb{F}^k)}$ by

$$d(B) := (c_0(\Gamma)(B))_{\Gamma\in\mathrm{GL}(\mathbb{F}^k)}.$$

Let $S \in \mathcal{E}_{n_0,m}(\mathbb{F})$ and $d_0 : \mathrm{GL}(\mathbb{F}^k) \to r^{\mathrm{GL}(\mathbb{F}^k)}$ be such that $d(S \cdot B) = d_0(\tau(B))$ for every $B \in M^k_{m,k}(\mathbb{F})$. We claim that $R_0 = R \cdot Q$ and $R_1 := P \cdot S$ work, where $Q \in \mathcal{E}_{n_0,m}(\mathbb{F})$ is arbitrary. Finally, let $g : \mathrm{GL}(\mathbb{F}^k) \to r$ be defined by $g(\Gamma) = d_0(\Gamma_0)(\Gamma_1)$, where $\Gamma = \Gamma_1 \cdot \Gamma_0^{\mathrm{t}}$ is an arbitrary decomposition with $\Gamma_0, \Gamma_1 \in \mathrm{GL}(\mathbb{F}^k)$. Notice that

$$\begin{aligned} d_0(\Gamma_0)(\Gamma_1) =& d(S \cdot P_0 \cdot \Gamma_0)(\Gamma_1) = c_0(\Gamma_1)(S \cdot P_0 \cdot \Gamma_0) = c(R \cdot P_1 \cdot \Gamma_1)(S \cdot P_0 \cdot \Gamma_0) = \\ =& f(R \cdot P_1 \cdot \Gamma_1 \cdot \Gamma_0^{\mathrm{t}} \cdot P_0^{\mathrm{t}} \cdot R_1^{\mathrm{t}}) = f(R \cdot P_1 \cdot \Gamma \cdot P_0^{\mathrm{t}} \cdot R_1^{\mathrm{t}}) \end{aligned}$$

where $P_0 \in \mathcal{E}_{m,k}(\mathbb{F})$ and $P_1 \in \mathcal{E}_{n_0,k}(\mathbb{F})$ are arbitrary. So, g does not depend on the decomposition $\Gamma = \Gamma_1 \cdot \Gamma_0^{\mathrm{t}}$. Similarly one proves that $g(\tau^{(2)}(A)) = f(R_0 \cdot A \cdot R_1^{\mathrm{t}})$ for all $A \in M^k_{m,m}(\mathbb{F})$. □

1.2.2. *Uniqueness.* We see that in a natural way the factors we presented are unique. We introduce the abstract notion of Ramsey factor in this context.

Definition 1.13. Given $\mu : M^k_{\infty,k}(\mathbb{F}) \to X$, X finite, and $\mathcal{A} \subseteq \bigcup_{n,m} M_{n,m}(\mathbb{F})$, we say that the couple $(\mu, \mathcal{A})$ is a *k-Ramsey factor* when

i) $\mu(M^k_{\infty,k}(\mathbb{F})) = X$.
ii) $\mu(R \cdot A) = \mu(A)$ for every $A \in M^k_{m,k}(\mathbb{F})$ and every $R \in \mathcal{A} \cap M^m_{n,m}(\mathbb{F})$.
iii) For every $m, r \in \mathbb{N}$ there is some $n \in \mathbb{N}$ such that for every r-coloring c of $M^k_{n,k}(\mathbb{F})$ there is $R \in \mathcal{A} \cap M^m_{n,m}(\mathbb{F})$ such that μ is a factor of c in $R \cdot M^k_{m,k}(\mathbb{F})$.

We call X the *set of colors* of μ, denoted by X_μ.

It follows that $(\tau, \mathcal{E})$ is a k-Ramsey factor, and it is the minimal one in the following precise sense.

Proposition 1.14. *Suppose that $(\mu, \mathcal{A})$, $(\nu, \mathcal{B})$ are k-Ramsey factors.*
a) $|X_\mu| \geqslant |\mathrm{GL}(\mathbb{F}^k)| = \prod_{j=0}^{k-1}(|\mathbb{F}|^k - |\mathbb{F}|^j)$.
b) If $\mathcal{A} \subseteq \mathcal{B}$, then there is a surjection $\theta : X_\mu \to X_\nu$ such that $\mu \circ \theta = \nu$.
c) If $\mathcal{A} = \mathcal{B}$, then there is a bijection $\theta : X_\mu \to X_\nu$ such that $\mu \circ \theta = \nu$.

Proof. a): In fact, we prove that if $(\mu, \mathcal{A})$ satisfies iii) of Definition 1.13, then $|X_\mu| \geqslant |\mathrm{GL}(\mathbb{F}^k)| = \prod_{j=0}^{k-1}(|\mathbb{F}|^k - |\mathbb{F}|^j)$. Find the corresponding n in iii) for $m = k$ and $r = |\mathrm{GL}(\mathbb{F}^k)|$. Fix an enumeration $\mathrm{GL}(\mathbb{F}^k) = \{\Delta_j\}_{j<r}$, let $c : M^k_{n,k}(\mathbb{F}) \to r$ be the coloring $c(A) := j$ if $\tau(A) = \Delta_j$. By iii), there are $R \in \mathcal{A} \cap M^k_{n,k}(\mathbb{F})$ and $\theta : X_\mu \to \mathrm{GL}(\mathbb{F}^k)$ such that $\tau(R \cdot A) = \theta(\mu(R \cdot A))$ for every $A \in M^k_{k,k}(\mathbb{F})$. It is easy to see that $\tau : R \cdot M^k_{k,k}(\mathbb{F}) \to \mathrm{GL}(\mathbb{F}^k)$ is surjective, hence θ is also surjective. *b)* is proved similarly. *c)*: From *b)* we have that $|X_\mu| = |X_\nu|$, and θ in *b)* must be a bijection. □

2. Matrices and Grassmannians over $\mathbb{R}, \mathbb{C}$

We present factorization results of compact colorings of matrices and Grassmannians over the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$. There are several such results, depending on the chosen metric on the objects we color. These factorizations are approximate, because, as we deal with infinite fields, it is easily seen that the exact ones are not true; on the other hand, they apply to arbitrary colorings given by Lipschitz mappings with values in a compact metric space. Given $\alpha, \beta \in \mathbb{N} \cup \{\infty\}$, the collection of matrices $M_{\alpha,\beta}(\mathbb{F})$ can be naturally turned into a metric space by fixing two norms $\mathtt{m}$ and $\mathtt{n}$ on $\mathbb{F}^\alpha$ and $\mathbb{F}^\beta$, respectively, and identifying a matrix $A \in M_{\alpha,\beta}$ with the linear operator $T_A : \mathbb{F}^\beta \to \mathbb{F}^\alpha$, $T_A(x) := A \cdot x$, x as column vector (i.e., a $\beta \times 1$-matrix). This allows to define the norm $\|A\|_{\mathtt{m},\mathtt{n}} := \|T_A\|_{(\mathbb{F}^\beta,\mathtt{m}),(\mathbb{F}^\alpha,\mathtt{n})}$, and the corresponding distance $d_{\mathtt{m},\mathtt{n}}(A, B) := \|A - B\|_{\mathtt{m},\mathtt{n}} = \|T_A - T_B\|_{(\mathbb{F}^\beta,\mathtt{m}),(\mathbb{F}^\alpha,\mathtt{n})}$. Also, in this way each full rank $\alpha \times k$-matrix A defines a norm $\nu(A)$ on $\mathbb{F}^k$, $\nu(A)(x) := \mathtt{n}(A \cdot x)$. When $\mathtt{m}$ is a norm on $\mathbb{F}^\infty$, by identifying each $\mathbb{F}^k$ with $\langle u_j \rangle_{j<k}$, let $\mathtt{m}_k$

be the norm on $\mathbb{F}^k$, $\mathtt{m}_k((a_j)_{j<k}) := \mathtt{m}(\sum_{j<k} a_j u_j)$. When there is no possible misunderstanding, we will write $d_{\mathtt{m}}$ to denote $d_{\mathtt{m}_\beta,\mathtt{m}_\alpha}$.

Recall that in general, given two normed spaces $X = (V, \mathtt{m})$ and $Y = (W, \mathtt{n})$, $\mathcal{L}(X, Y)$ denotes the space of continuous (equivalently bounded) linear operators from X to Y, that is again a normed space by considering the norm $\|T\| := \sup_{x\in \mathrm{Ball}(X)} \mathtt{n}(T(x))$, where $\mathrm{Ball}(X) = \{x \in X : \mathtt{m}(x) \leqslant 1\}$ denotes the unit ball of X. Let $\mathcal{L}^k(X, Y)$ is the set of those operators of rank k. Since when V is finite dimensional every linear mapping from V to W is automatically continuous, in this case, we will use also $\mathcal{L}(V, W)$ and $\mathcal{L}^k(V, W)$, to denote the collection of linear mappings from V to W, and those of rank k, respectively. By an *isometric embedding* we mean a linear mapping $T : X \to Y$ such that $\mathtt{n}(T(x)) = \mathtt{m}(x)$ for every $x \in X$. The space of these operators is denoted by $\mathrm{Emb}(X, Y)$.

Of particular importance will be the p-norms. Recall that for every $1 \leqslant p \leqslant \infty$, ℓ_p^n is the normed space $(\mathbb{F}^n, \|\cdot\|_p)$, where $\|(a_j)_{j<n}\|_p := (\sum_{j<n} |a_j|^p)^{1/p}$ for $p < \infty$ and $\|(a_j)_{j<n}\|_\infty := \max_{j<n} |a_j|$. Similarly one defines the p-norms on $\mathbb{F}^\infty$, that we denote as $\ell_p^\infty := (\mathbb{F}^\infty, \|\cdot\|_p)$, and their completions are usually denoted by ℓ_p, for $p < \infty$ and by c_0, when $p = \infty$.

Roughly speaking, our factorization theorem for full rank matrices (Theorem 2.8) states that every coloring of such matrices, endowed with the p-metrics for $p \in [1, +\infty]\backslash 2(\mathbb{N} + 2)$ is "approximately determined" by the corresponding ν described above.

Similarly, once a norm $\mathtt{m}$ is fixed in $\mathbb{F}^\alpha$, $\mathrm{Gr}(k, \mathbb{F}^\alpha)$ turns into a metric space by considering a corresponding Hausdorff metric (see (2) below), and each k-dimensional subspace V of $\mathbb{F}^\alpha$ determines a member of the *Banach-Mazur* compactum $\mathcal{B}_k$, that is, the isometry class $\tau_{\mathtt{m}}(V)$ of all k-dimensional normed spaces isometric to $(V, \mathtt{m})$. We prove that when choosing p-norms on each $\mathbb{F}^n$ for n large enough, any coloring of the k-Grassmannians of $\mathbb{F}^n$ is approximately determined by $\tau_{\mathtt{m}}$ on some $\mathrm{Gr}(V, k)$. We introduce a more appropriate terminology, in particular we extend the type of colorings to work with. A *metric coloring* of a pseudo-metric space M is a 1-Lipschitz map c from M to a metric space (K, d_K). We will say that c is a K-coloring. A *compact coloring* is a metric coloring whose target space is a compact metric space. For a subset X of a metric space (K, d_K) and $\varepsilon > 0$, the ε-fattening $X_\varepsilon = \{p \in K :$ there is some $q \in X$ with $d(p, q) \leqslant \varepsilon\}$.

The *oscillation* $\mathrm{osc}(c \upharpoonright F)$ of a compact coloring $c : M \to (K, d_K)$ on a subset F of M is the supremum of $d_K(c(y), c(y'))$ where y, y' range within F. When $\mathrm{osc}(c \upharpoonright F) \leqslant \varepsilon$ we also say that c *ε-stabilizes* on F, or that F is *ε-monochromatic* for c. A *finite* (or *discrete*) coloring of M is a function c from M to a finite set X; in the particular case when the target space is a natural number r (identified with the set $\{0, 1, \ldots, r-1\}$ of its predecessors), we will say that c is an r-coloring. Given a finite coloring $c : M \to X$ and $\varepsilon \geqslant 0$, we say that a subset F of M is ε-monochromatic for c, or that c ε-stabilizes on F, if there exists some $x \in X$ such that F is included in the ε-neighborhood $(c^{-1}(x))_\varepsilon$ of $c^{-1}(x)$. Notice that when X is a finite metric space and c is assumed to be 1-Lipschitz, we have two notions of ε-monochromatic sets: one by considering c as a finite coloring, and another by considering c as a 1-Lipschitz coloring. Because of this, we will always emphasize which kind of coloring we mean each time. In general, every ε-monochromatic set of c considered as finite coloring is 2ε-monochromatic for c as a compact coloring, but the converse implication is not always true. However, when (M, d) is a bounded pseudo-metric space, given finite coloring $c : M \to X$ it is possible to define a compact coloring $\widetilde{c}$ on X such that ε-monochromatic sets for $\widetilde{c}$ are 2ε-monochromatic for c; to see this, let λ be the diameter of M, and fix a finite coloring $c : M \to X$. Let $X' := c(M)$ and let $K := [-\lambda, \lambda]^{X'}$ be endowed with the sup-distance $d_K((\alpha_x)_{x\in X'}, (\beta_x)_{x\in X'}) := \max_{x\in X'} |\alpha_x - \beta_x|$. Let $\widetilde{c} : M \to K$, $\widetilde{c}(p) := (d(p, c^{-1}(x)))_{x\in X'}$. This is a 1-Lipschitz mapping that has the property we want. In this way, it is proved in [6, Proposition 5.9] and [4, Proposition 2.13] that several approximate

Ramsey properties associated to those colorings turn to be equivalent. To simplify the notation, when $\varepsilon = 0$ we will omit the use of the prefix "0-".

Definition 2.1 (Approximate factors)**.** Let (M, d_M), (N, d_N) and (P, d_P) be metric spaces, $\varepsilon > 0$, and $c : (M, d_M) \to (N, d_N)$ and $\pi : (M, d_M) \to (P, d_P)$ be metric colorings, i.e., 1-Lipschitz maps. We say that π is an *ε-approximate factor* (or simply *ε-factor*) of c if there is some metric coloring $\widetilde{c} : (P, d_P) \to (N, d_N)$ such that

$$\sup_{x \in M} d_N(c(x), \widetilde{c}(\pi(x))) \leqslant \varepsilon.$$

That is, "up to ε" $c = \widetilde{c} \circ \pi$. Given $M_0 \subseteq M$ we say that π is an *ε-factor of c in M_0* if $\pi \restriction_{M_0} : M_0 \to P$ is an ε-factor of $c \restriction M_0$, i.e., there is some metric coloring $\widetilde{c} : P \to N$ such that $\sup_{x \in M_0} d_N(c(x), \widetilde{c}(\pi(x))) \leqslant \varepsilon$.

2.1. **The statements. Ramsey factors.** As discussed above, given norms $\mathtt{m}, \mathtt{n}$ on $\mathbb{F}^m$ and $\mathbb{F}^n$ respectively, we regard $M_{n,m}$ as a metric space by considering a $n \times m$-matrix A as the particular representation of a linear operator T_A in the unit bases of suitable normed spaces $(\mathbb{F}^m, \mathtt{m})$ and $(\mathbb{F}^n, \mathtt{n})$, and then by considering the corresponding operator norm.

2.1.1. *Full rank matrices.* Given a vector space V, let $\mathcal{N}_V$ be the set of all norms on V, endowed with the topology of pointwise convergence.

Proposition 2.2. *When* $\dim V < \infty$*, a compatible metric on* $\mathcal{N}_V$ *is*

$$\omega(\mathtt{m}, \mathtt{n}) = \omega_V(\mathtt{m}, \mathtt{n}) := \log \max\{\|\mathrm{Id}\|_{(V,\mathtt{m}),(V,\mathtt{n})}, \|\mathrm{Id}\|_{(V,\mathtt{n}),(V,\mathtt{m})}\},$$

that will be called the intrinsic metric *on* $\mathcal{N}_V$*. Moreover,* $(\mathcal{N}_V, \omega)$ *has the Heine-Borel property, that is every* ω*-bounded and closed subset of* $\mathcal{N}_V$ *is compact.*

Proof. Fix a linear basis $(v_k)_{k<d}$ of V, and define the norm $\mathtt{n}_1(\sum_{k<d} a_k v_k) := \sum_{k<d} |a_k|$. Let us see first that ω is a metric on $\mathcal{N}_V$ such that $(\mathcal{N}_V, \omega)$ has the Heine-Borel property. It is a well-know and fundamental fact that the norms on a finite dimensional space V are all equivalent, that is, given $\mathtt{m}, \mathtt{n} \in \mathcal{N}_V$ there is $C \geqslant 1$ such that $C^{-1}\mathtt{n}(v) \leqslant \mathtt{m}(v) \leqslant C\mathtt{n}(v)$ for every $v \in V$: it follows from the Heine-Borel Theorem (after identifying V with $\mathbb{F}^d$ via the basis $(v_k)_{k<d}$) that the unit sphere $\mathrm{Sph}(V, \mathtt{n}_1) := \{v \in V : \mathtt{n}_1(v) = 1\}$ is compact. Let $\mathtt{m} \in \mathcal{N}_V$. Since $\mathtt{m}(v) \leqslant \sum_{k<d} |a_k| \mathtt{m}(v_k) \leqslant (\max_{k<d} \mathtt{m}(v_k))\mathtt{n}_1(v)$, it follows that $\mathtt{m} : (V, \mathtt{n}_1) \to [0, \infty[$ is continuous, so $\min_{\mathtt{n}_1(v)=1} \mathtt{m}(v) = K > 0$ exists. This means that given a nonzero $v \in V$ we have that $\mathtt{m}(v/\mathtt{n}_1(v)) \geqslant K$, i.e. $\mathtt{m}(v) \geqslant K\mathtt{n}_1(v)$. Hence $\mathtt{m}$ and $\mathtt{n}_1$ are equivalent, and consequently any two norms on V are equivalent.

Let $\omega_0(\mathtt{m}, \mathtt{n}) := \max\{\|\mathrm{Id}\|_{(V,\mathtt{m}),(V,\mathtt{n})}, \|\mathrm{Id}\|_{(V,\mathtt{n}),(V,\mathtt{m})}\}$ be the multiplicative version of ω. It follows from the composition rule $\|T \circ U\| \leqslant \|T\| \cdot \|U\|$ that $\omega_0(\mathtt{m}, \mathtt{p}) \leqslant \omega_0(\mathtt{m}, \mathtt{n}) \cdot \omega_0(\mathtt{n}, \mathtt{p})$, hence ω satisfies the triangle inequality, and that ω is positive because $1 = \|\mathrm{Id}\|_{(V,\mathtt{m}),(V,\mathtt{m})} \leqslant \|\mathrm{Id}\|_{(V,\mathtt{m}),(V,\mathtt{n})} \cdot \|\mathrm{Id}\|_{(V,\mathtt{n}),(V,\mathtt{m})}$, and consequently $\omega_0(\mathtt{m}, \mathtt{n}) = \max\{\|\mathrm{Id}\|_{(V,\mathtt{m}),(V,\mathtt{n})}, \|\mathrm{Id}\|_{(V,\mathtt{n}),(V,\mathtt{m})}\} \geqslant 1$. Also, if we have $\omega(\mathtt{m}, \mathtt{n}) = 0$, then $\omega_0(\mathtt{m}, \mathtt{n}) = 1$, and consequently, $\mathtt{m} = \mathtt{n}$. Suppose that $\mathcal{C} \subseteq \mathcal{N}_V$ is closed and bounded, and let $C > 0$ be such that $\mathcal{C}$ is included in the ω-ball of center $\mathtt{n}_1$ and radius C. Let $\widehat{\mathcal{C}} := \{\mathtt{m} \restriction \mathrm{Ball}(V, \mathtt{n}_1) : \mathtt{m} \in \mathcal{C}\}$. Note that $\mathtt{m}(v) \leqslant \exp(C)\mathtt{n}_1(v)$ for every $v \in V$ and $\mathtt{m} \in \mathcal{C}$, so it follows that $\widehat{\mathcal{C}}$ is a set of real-valued $\exp(C)$-Lipschitz functions defined on the compact metric space $(\mathrm{Ball}(V, \mathtt{n}_1), d)$, where $d(v, w) := \mathtt{n}_1(v - w)$. It follows that $\widehat{\mathcal{C}}$ is equicontinuous and pointwise bounded, so it follows from the Arzelà-Ascoli Theorem that $\widehat{\mathcal{C}}$ is compact. Let $(\mathtt{m}_n)_n$ be a sequence in $\mathcal{C}$, set $f_n := \mathtt{m}_n \restriction \mathrm{Ball}(V, \mathtt{n}_1) \in \widehat{\mathcal{C}}$ for each $n \in \mathbb{N}$, and let $(f_{n_l})_{l \in \mathbb{N}}$ be a converging subsequence with limit $f : \mathrm{Ball}(V, \mathtt{n}_1) \to [0, \infty[$. By pointwise convergence, it follows that $f(v + w) \leqslant f(v) + f(w)$ if $v, w, v + w \in \mathrm{Ball}(V, \mathtt{n}_1)$ and that $f(tv) = |t| f(v)$ if $v, tv \in \mathrm{Ball}(V, \mathtt{n}_1)$. Define now $\mathtt{m}(v) := \mathtt{n}_1(v) \cdot f(v/\mathtt{n}_1(v))$ for every nonzero $v \in V$, and $\mathtt{m}(0) := 0$. We check the subadditivity of $\mathtt{m}$. Let $v, w \in V$ be such that

$v \neq -w$ (otherwise, $\mathtt{m}(v+w) = 0 \leqslant \mathtt{m}(v) + \mathtt{m}(w)$ trivially), and let $\lambda \geqslant 1$ be such that $\mathtt{n}_1(v), \mathtt{n}_1(w) \leqslant \lambda \mathtt{n}_1(v+w)$. We have that $f((v+w)/(\lambda \mathtt{n}_1(v+w))) \leqslant f(v/(\lambda \mathtt{n}_1(v+w))) + f(w/(\lambda \mathtt{n}_1(v+w))) = (\mathtt{n}_1(v)/(\lambda \mathtt{n}_1(v+w)))f(v/\mathtt{n}_1(v)) + (\mathtt{n}_1(w)/(\lambda \mathtt{n}_1(v+w)))f(w/\mathtt{n}_1(w))$, or in other words $\lambda \mathtt{n}_1(v+w) f((v+w)/(\lambda \mathtt{n}_1(v+w))) \leqslant \mathtt{m}(v) + \mathtt{m}(w)$. But then, $\mathtt{m}(v+w) = \mathtt{n}_1(v+w) f((v+w)/\mathtt{n}_1(v+w)) = \lambda \mathtt{n}_1(v+w) f((v+w)/(\lambda \mathtt{n}_1(v+w))) \leqslant \mathtt{m}(v) + (w)$. Similarly one shows that $\mathtt{m}(tv) = |t|\mathtt{m}(v)$ for every $v \in V$ and $t \in \mathbb{F}$. In particular we have that $\mathtt{m}(v) = f(v)$ if $\mathtt{n}_1(v) \leqslant 1$. Also, given a nonzero $v \in V$, $\mathtt{m}_n(v/\mathtt{n}_1(v)) \geqslant \exp(-C)$ for every n, and hence, $\mathtt{m}(v) \geqslant \exp(-C)\mathtt{n}_1(v) > 0$. This means that $\mathtt{m} \in \mathcal{N}_V$. Let us see that $\omega_0(\mathtt{m}_{n_l}, \mathtt{m}) \to_{l\to\infty} 1$: Let $K > 0$ be such that $\mathtt{n}_1(v) \leqslant K\mathtt{m}(v)$ for every $v \in V$. Given $\varepsilon > 0$, let l_0 be such that for every $v \in V$ and every $l \geqslant l_0$ we have that $|\mathtt{m}_{n_l}(v) - \mathtt{m}(v)| \leqslant (\varepsilon/K)\mathtt{n}_1(v)$, and consequently, $|\mathtt{m}_{n_l}(v) - \mathtt{m}(v)| \leqslant \varepsilon \mathtt{m}(v)$. From this it follows that $(1-\varepsilon)\mathtt{m}(v) \leqslant \mathtt{m}_{n_l}(v) \leqslant (1+\varepsilon)\mathtt{m}(v)$ for every $v \in V$ and $l \geqslant l_0$, and consequently $\omega_0(\mathtt{m}_{n_l}, \mathtt{m}) \leqslant \max\{1+\varepsilon, 1/(1-\varepsilon)\} = 1/(1-\varepsilon)$ for every $l \geqslant l_0$. Since $\varepsilon > 0$ was arbitrary, we obtain that $\omega_0(\mathtt{m}_{n_l}, \mathtt{m}) \to_{l\to\infty} 1$. Finally, since $\mathcal{C}$ is closed for the ω-topology, $\mathtt{m} \in \mathcal{C}$.

Let us see that ω_0, hence ω, defines the topology of pointwise convergence. Suppose that $(\mathtt{m}_n)_n$ and $\mathtt{m}$ are norms in V. It is easy to see from the definition of ω_0 that if $\omega_0(\mathtt{m}_n, \mathtt{m}) \to_{n\to\infty} 1$, then $\mathtt{m}_n \to_{n\to\infty} \mathtt{m}$ pointwise. Suppose now that $\mathtt{m}_n \to_{n\to\infty} \mathtt{m}$ pointwise, and let us see first that $\{\mathtt{m}_n\}_n$ is ω-bounded: Let n_0 be such that $|\mathtt{m}_n(v_k) - \mathtt{m}(v_k)| \leqslant 1$ for every $n \geqslant n_0$ and $k < d$. It follows that for such $n \geqslant n_0$ and every $v = \sum_{k<d} a_k v_k$ we have that $\mathtt{m}_n(v) \leqslant \sum_{k<d} |a_k|\mathtt{m}_n(v_k) \leqslant \sum_{k<d} |a_k||\mathtt{m}_n(v_k) - \mathtt{m}(v_k)| + \sum_{k<d} |a_k|\mathtt{m}(v_k) \leqslant (1 + \max_{k<d} \mathtt{m}(v_k))\mathtt{n}_1(v)$. Therefore, there is some $K > 0$ such that $\mathtt{m}_n(v) \leqslant K\mathtt{n}_1(v)$ for all $v \in V$. On the other hand, working towards a contradiction, suppose that there is no $L > 0$ such that $\mathtt{n}_1(v) \leqslant L\mathtt{m}_n(v)$ for all $v \in V$ and $n \in \mathbb{N}$. This means that for each $l \in \mathbb{N}$ there exist $n_l \in \mathbb{N}$ and v_l such that $\mathtt{n}_1(v_l) = 1$ and $\mathtt{m}_{n_l}(v_l) < 1/l$. Since $\mathrm{Ball}(V, \mathtt{n}_1)$ is compact, without loss of generality, we may assume that $(v_l)_l$ converges to some v with $\mathtt{n}_1(v) = 1$, and that $(n_l)_l$ is strictly increasing or constant with value n. None of the cases can happen: if $(n_l)_l$ is strictly increasing, then $\mathtt{m}(v) = \lim_{l\to\infty} \mathtt{m}_{n_l}(v) \leqslant \lim_{l\to\infty} \mathtt{m}_{n_l}(v_l) + \lim_{l\to\infty} \mathtt{m}_{n_l}(v - v_l) \leqslant K \lim_{l\to\infty} \mathtt{n}_1(v - v_l) = 0$, and this is impossible since $v \neq 0$. If $n_l = n$ for all l, then it follows that $\mathtt{m}_n(v) = \lim_{l\to\infty} \mathtt{m}_n(v_l) \leqslant \lim_{l\to\infty} 1/l = 0$, again impossible.

By the Heine-Borel property of $(\mathcal{N}_V, \omega)$, it follows that every subsequence of $(\mathtt{m}_n)_n$ has a further ω-convergent subsequence; all these limits must be $\mathtt{m}$ because by hypothesis $(\mathtt{m}_n)_n$ converges to $\mathtt{m}$. Hence $(\mathtt{m}_n)_n$ ω-converges to $\mathtt{m}$. □

In particular, each closed ω-ball is compact. Given a normed space $E = (W, \|\cdot\|)$, let $\mathcal{N}_V(E)$ be the collection of norms $\mathtt{m}$ on V such that there exists a linear isometry $T : (V, \mathtt{m}) \to E$. In general, $\mathcal{N}_V(E)$ is not closed in $\mathcal{N}_V$, although in some natural cases is. We will write $\mathcal{N}_\alpha$ to denote $\mathcal{N}_{\mathbb{F}^\alpha}$

Definition 2.3. Suppose that V is finite dimensional, $E = (W, \|\cdot\|)$ a normed space. Let

$$\nu_{V,E} : \mathcal{L}^{\dim V}(V, W) \to \mathcal{N}_V(E)$$

be the mapping that assigns to a 1-1 linear mapping $T : V \to W$ the norm $\nu_{V,E}(T)$ on V, defined by $(\nu_{V,E}(T))(x) := \|T(x)\|$, that is, the norm on V that makes T an isometric embedding. With a slight abuse of notation, we also write $\nu_{k,(\mathbb{F}^\alpha,\|\cdot\|)}$ to denote the mapping $A \in M^k_{\alpha,k} \mapsto \nu_{\mathbb{F}^k,(\mathbb{F}^\alpha,\|\cdot\|)}(T_A)$ that assigns to a such matrix A the norm defined for each $x \in \mathbb{F}^k$ by $(\nu_{k,E}(A))(x) := \|A \cdot x\|$.

Given a finite dimensional normed space $X = (X, \|\cdot\|_X)$ and a normed space $E = (V, \|\cdot\|_E)$, we define on $\mathcal{N}_X(E) \times \mathcal{N}_X(E)$ the *E-extrinsic function*

$$\partial_{X,E}(\mathtt{m}, \mathtt{n}) := \inf\{\|T - U\|_{X,E} : T \in \mathrm{Emb}((X, \mathtt{m}), E),\, U \in \mathrm{Emb}((X, \mathtt{n}), E)\}.$$

So, $\partial_{X,E}(\mathtt{m},\mathtt{n})$ computes the minimal distance $d_{X,E}(T,U)$ between possible representations of $\mathtt{m}$ and $\mathtt{n}$, $\nu_{X,E}(T)=\mathtt{m}$, $\nu_{X,E}(U)=\mathtt{n}$. In general $\partial_{X,E}$ is not a compatible metric. Note that $\partial_{X,E}(\mathtt{m},\mathtt{n})>0$ for every $\mathtt{m}\neq\mathtt{n}$. To see this, given $\varepsilon>0$ and $x\neq 0$, choose $T\in\mathrm{Emb}((X,\mathtt{m}),E)$ and $U\in\mathrm{Emb}((X,\mathtt{n}),E)$ such that $\|T-U\|_{X,E}\leqslant\varepsilon/\|x\|_X$. It follows that $|\mathtt{m}(x)-\mathtt{n}(x)|=|\|T(x)\|-\|U(x)\||\leqslant\|(T-U)(x)\|\leqslant\varepsilon$, and since $\varepsilon>0$ is arbitrary, we obtain that $\mathtt{m}(x)=\mathtt{n}(x)$. This means that $\partial_{X,E}$ is a metric exactly when $\partial_{X,E}$ satisfies the triangle inequality. The following is easy to prove.

Proposition 2.4. *If $\partial_{X,E}$ is a compatible metric on $\mathcal{N}_X(E)$, then $\nu_{X,E}:(\mathcal{L}^{\dim X}(X,E),d_{X,E})\to(\mathcal{N}_X(E),\partial_{X,E})$ is 1-Lipschitz.* □

Recall that given a linear operator $T:X\to Y$ between normed spaces X and Y,

$$\|T\|=\min\{\lambda\geqslant 0\ :\ T(\mathrm{Ball}(X))\subseteq\lambda\cdot\mathrm{Ball}(Y)\},$$

and when X is finite dimensional, let

$$\mathtt{r}_{-1}(T)=\min\{\lambda\geqslant 0\ :\ \mathrm{Ball}(T(X))\subseteq\lambda\cdot T(\mathrm{Ball}(X))\},$$

that is well-defined: consider the quotient mapping $\widetilde{T}:X/\ker T\to TX$. Since TX is finite dimensional, it is a Banach space, hence by the open mapping Theorem, it follows that $\widetilde{T}$ is an isomorphism, and it is not difficult to see that $\mathtt{r}_{-1}(T)=\|U\|$, where U is the inverse of $\widetilde{T}$. By definition, $\|T\|\cdot\mathtt{r}_{-1}(T)\geqslant 1$, and when T is 1-1, $\mathtt{r}_{-1}(T)=\|U\|$, where $U:TX\to X$ is the inverse operator of T, hence if $T:X\to Y$ is an isomorphism, $\mathtt{r}_{-1}(T)=\|T^{-1}\|$. Given $\alpha,\beta\in\mathbb{N}\cup\{\infty\}$, and a norm $\mathtt{m}\in\mathcal{N}_\infty$, let $M_{\alpha,\beta}^k(\mathtt{m};\lambda)$ be the collection of matrices in $M_{\alpha,\beta}^k$ such that the corresponding linear operator $T_A:(\mathbb{F}^\beta,\mathtt{m})\to(\mathbb{F}^\alpha,\mathtt{m})$ satisfies that $\|T_A\|,\mathtt{r}_{-1}(T_A)\leqslant\lambda$.

$$\frac{1}{\lambda}\mathrm{Ball}(\mathrm{Im}\,T_A)\subseteq T_A(\mathrm{Ball}(\mathbb{F}^\beta,\mathtt{m}))\subseteq\lambda\mathrm{Ball}(\mathbb{F}^\alpha,\mathtt{m}).$$

Let also $M_{\alpha,k}^k(\mathtt{m};{<}\lambda)=\bigcup_{1\leqslant\mu<\lambda}M_{\alpha,k}^k(\mathtt{m},\mu)$, that is, the matrices $A\in M_{\alpha,k}^k$ such $\|T_A\|,\mathtt{r}_{-1}(T_A)<\lambda$. Observe that $M_{\alpha,\beta}^k(\mathtt{m};{<}1)=\varnothing$, and that the boundary $M_{\alpha,k}^k(\mathtt{m};1)$ is the collection of matrices A defining isometric embeddings $T_A:(\mathbb{F}^k,\mathtt{m})\to(\mathbb{F}^\infty,\mathtt{m})$, that will be denoted by $\mathcal{E}_{\alpha,k}(\mathtt{m})$, and $\mathcal{E}(\mathtt{m}):=\bigcup_{n\geqslant m}\mathcal{E}_{n,m}(\mathtt{m})$. The following is easy to prove, and highlights the interest of this collection.

Proposition 2.5. *Let $R\in M_{\alpha,m}^m$.*

a) The multiplication by R operator $\mu_R:(M_{m,k}^k,d_{\mathtt{m}})\to(M_{\alpha,k}^k,d_{\mathtt{m}})$, $A\mapsto\mu_R(A):=R\cdot A$ defines an isometry if and only if $R\in\mathcal{E}_{\alpha,m}(\mathtt{m})$.

b) If $R\in\mathcal{E}_{\alpha,m}(\mathtt{m})$, then $\nu_{k,(\mathbb{F}^\alpha,\mathtt{m})}\circ\mu_R=\nu_{k,(\mathbb{F}^m,\mathtt{m})}$. □

Proof. a): Suppose that X is a normed space of finite dimension k, Y,Z normed spaces with $\dim Y\geqslant k$ and suppose that $U\in\mathcal{L}(Y,Z)$ is such that the composition operator $T\in\mathcal{L}^k(X,Y)\mapsto U\circ T\in\mathcal{L}^k(X,Z)$ is an isometry with respect to the norm metrics. Let us prove that U must be an isometry. Fix a non-zero vector $y\in Y$. Let $(x_j)_{j<k}$ be a normalized basis of X, and let $(x_j^*)_{j<k}$ be its biorthogonal sequence. Let also $(y_j)_{j<k}$ be a linearly independent sequence in Y with $y_0=y$. For each $n\geqslant 1$, let $T_n(x)=f_0(x)y+(1/n)\sum_{j=1}^{k-1}x_j^*(x)y_j$, where $f_0:=x_0^*/\|x_0^*\|$. It is easy to see that T_n, $(1/n)T_n$ are $1-1$. Then, $\|U\circ T_n-U\circ(1/n)\cdot T_n\|-\|U\circ T_n\|\to_n 0$, and $\|U\circ T_n-U\circ(1/n)\cdot T_n\|-\|T_n\|=\|T_n-(1/n)\cdot T_n\|-\|T_n\|\to_n 0$, hence $\|U\circ T_n\|-\|T_n\|\to_n 0$. Since we have that $\|T_n\|\to_n\|f_0\|^*\|y\|$, $\|U\circ T_n\|\to_n\|f_0\|^*\|U(y)\|$ and $\|f_0\|^*=1$, we obtain that $\|U(y)\|=\|y\|$. *b)* is trivial. □

Given a normed space $E=(\mathbb{F}^\infty,\|\cdot\|_E)$, let $\mathcal{N}_k(E;\lambda)$ ($\mathcal{N}_k(E;{<}\lambda)$) be the closed (resp. open) ball of $\mathcal{N}_k(E)$ with respect to the multiplicative metric ω_0 centered on the norm $\|\cdot\|_E$ in $\mathbb{F}^k$ and with radius λ, i.e., $\mathcal{N}_k(E;\lambda)=\{\mathtt{n}\in\mathcal{N}_k(E)\ :\ \omega(\mathtt{n},\|\cdot\|_E\restriction\langle u_j\rangle_{j<k})\leqslant\log\lambda\}$, similarly for

$\mathcal{N}_k(E;<\lambda)$. In the next, recall that two metrics d_1 and d_2 on a set X are *uniformly equivalent* when the identity $\mathrm{Id}:(X,d_1)\to(X,d_2)$ is an uniform homeomorphism.

Proposition 2.6. *a)* $\mathcal{N}_k(E;\lambda)=\nu_{k,E}(M^k_{\alpha,k}(\|\cdot\|_E;\lambda))$ *and* $\mathcal{N}_k(E;<\lambda)=\nu_{k,E}(M^k_{\alpha,k}(\|\cdot\|_E;<\lambda))$.
b) If $\partial_{X,E}$ *and* ω *are uniformly equivalent on* ω*-bounded subsets of* $\mathcal{N}_X(E)$*, then every* ω*-bounded set is* $\partial_{X,E}$*-totally bounded, thus, the* $\partial_{(\mathbb{F}^k,\mathtt{m}),E}$*-completion of* $\mathcal{N}_k(E;\lambda)$ *and* $\mathcal{N}_k(E;<\lambda)$ *are compact.*

Proof. b): Suppose that $A\subseteq\mathcal{N}_X(E)$ is ω-bounded. Since $(\mathcal{N}_X,\omega)$ has the Heine-Borel property, A is ω-totally bounded. $\partial_{X,E}$ is uniformly equivalent to ω on A, so A is $\partial_{X,E}$-totally bounded. □

Definition 2.7 (Ramsey factors for full-rank matrices)**.** Let $\mathtt{m}$ be a norm on $\mathbb{F}^\infty$, set $E_\alpha:=(\mathbb{F}^\alpha,\mathtt{m})$ for every $\alpha\leqslant\infty$. We say that $\mathtt{m}$ *produces Ramsey factors* for compact colorings of full rank matrices, when

i) ∂_{E_k,E_∞} is a compatible metric on $\mathcal{N}_k(E_\infty)$ uniformly equivalent to ω on ω-bounded sets.
ii) Given $k,m\in\mathbb{N}$, $\varepsilon>0$, $\lambda>1$ and a compact metric space (K,d_K) there is $n\in\mathbb{N}$ such that for every 1-Lipschitz coloring $c:(M^k_{n,k}(\mathtt{m};\lambda),d_{\mathtt{m}})\to(K,d_K)$ there is $R\in\mathcal{E}_{n,m}(\mathtt{m})$ such that the restriction $\nu_{\mathbb{F}^k,E_\infty}:M^k_{\infty,k}(\mathtt{m};<\lambda)\to\mathcal{N}_k(E_\infty;<\lambda)$ is an ε-factor of c in $R\cdot M^k_{m,k}(\mathtt{m};<\lambda)$; that is, there is a 1-Lipschitz coloring $\widetilde{c}:(\mathcal{N}_k(E_\infty;<\lambda),\partial_{E_k,E_\infty})\to(K,d_K)$ such that $d_K(c(R\cdot A),\widetilde{c}(\nu_p(A)))\leqslant\varepsilon$ for every $A\in M^k_{m,k}(\mathtt{m};<\lambda)$.

Theorem 2.8 (p-Factorization of colorings of full rank matrices over $\mathbb{R}$, $\mathbb{C}$)**.** *For* $1\leqslant p\leqslant\infty$*,* $p\notin 2\mathbb{N}+4$*, the* p*-norm* $\|\cdot\|_p\in\mathcal{N}_\infty$ produces Ramsey factors *for compact colorings of full rank matrices.*

This result is a consequence of a Ramsey-like property of the class of finite dimensional subspaces of the spaces ℓ_p for those values of p. This property is called the steady approximate Ramsey property (SARP$^+$) (see Definition 3.1), and, in fact, it characterizes the norms on $\mathbb{F}^\infty$ that produce Ramsey factors for compact colorings of full rank matrices. We explain this in Theorem 3.3.

2.1.2. *Grassmannians.* Given a normed space $E=(V,\|\cdot\|_E)$ the k-Grassmannian $\mathrm{Gr}(k,V)$ of V is naturally a topological space, as it can be identified with the subspace of the equivalence classes of linearly independent sequences $(x_j)_{j<k}$ in the topological quotient of E^k by the relation $(x_j)_{j<k}\sim(y_j)_{j<k}$ iff $\langle x_j\rangle_{j<k}=\langle y_j\rangle_{j<k}$. If in addition E is separable, this turns $\mathrm{Gr}(k,E):=\mathrm{Gr}(k,V)$ into a Polish space. A natural compatible metric is the *gap (or opening) metric* (see [13]), $\Lambda_{k,E}(U,W)$ defined as the Hausdorff distance, with respect to the norm metric in E, between the unit balls $\mathrm{Ball}(U,\|\cdot\|_E)$ and $\mathrm{Ball}(W,\|\cdot\|_E)$, that is,

$$\begin{aligned}\Lambda_{k,E}(U,W)&:=\max\{\max_{u\in\mathrm{Ball}(U,\|\cdot\|_E)}d_{\|\cdot\|_E}(u,\mathrm{Ball}(W,\|\cdot\|_E)),\max_{w\in\mathrm{Ball}(W,\|\cdot\|_E)}d_{\|\cdot\|_E}(w,\mathrm{Ball}(U,\|\cdot\|_E))\}=\\&=\max\{\max_{u\in\mathrm{Ball}(U,\|\cdot\|_E)}\min_{w\in\mathrm{Ball}(W,\|\cdot\|_E)}\|u-w\|_E,\max_{w\in\mathrm{Ball}(W,\|\cdot\|_E)}\min_{u\in\mathrm{Ball}(U,\|\cdot\|_E)}\|w-u\|_E\}.\end{aligned}\tag{2}$$

Since Minkowski's Theorem states that every compact and convex subset K of a finite dimensional vector space is the convex hull of its extreme points $\mathcal{E}(K)$, we can rewrite the opening metric as

$$\begin{aligned}\Lambda_{k,E}(U,W)&=\max\{\max_{u\in\mathrm{Sph}(U,\|\cdot\|_E)}d_{\|\cdot\|_E}(u,\mathrm{Ball}(W,\|\cdot\|_E)),\max_{w\in\mathrm{Sph}(W,\|\cdot\|_E)}d_{\|\cdot\|_E}(w,\mathrm{Ball}(U,\|\cdot\|_E)\}=\\&=\max\{\max_{u\in\mathcal{E}(\mathrm{Ball}(U,\|\cdot\|_E))}d_{\|\cdot\|_E}(u,\mathrm{Ball}(W,\|\cdot\|_E)),\max_{w\in\mathcal{E}(\mathrm{Ball}(W,\|\cdot\|_E))}d_{\|\cdot\|_E}(w,\mathrm{Ball}(U,\|\cdot\|_E)\}\end{aligned}\tag{3}$$

where $\mathrm{Sph}(X) := \{x \in X : \|x\|_X = 1\}$ is the unit sphere of a normed space X. Let $\mathrm{GL}(V) \curvearrowright \mathcal{N}_V$ be the canonical action

$$(\Delta \cdot \mathtt{m})(v) := \mathtt{m}(\Delta^{-1}(v))$$

for every $v \in V$. Notice that the intrinsic metric ω is invariant under this action. Let $\mathcal{B}_V := \mathcal{N}_V/\mathrm{GL}(V)$ be the quotient space. Since ω is invariant under the action, it follows that the quotient of ω,

$$\widetilde{\omega}([\mathtt{m}],[\mathtt{n}]) := \inf_{\Delta,\Gamma\in\mathrm{GL}(V)} \omega(\Delta\cdot\mathtt{m},\Gamma\cdot\mathtt{n}) = \inf_{\Delta\in\mathrm{GL}(V)} \omega(\Delta\cdot\mathtt{m},\mathtt{n}) \tag{4}$$

defines a compatible metric on the quotient $\mathcal{B}_V$, called the *quotient metric*: First of all, let us see that the classes $[\mathtt{m}]$ are closed: For suppose that $\mathtt{n}$ is in the closure of $[\mathtt{m}]$, and choose $(\Delta_j)_{j\in\mathbb{N}}$ in $\mathrm{GL}(V)$ such that $\omega(\Delta_j\cdot\mathtt{m},\mathtt{n}) \downarrow_j 0$. Observe that $\|\Delta\|_{(V,\mathtt{m}),(V,\mathtt{n})} = \|\mathrm{Id}\|_{(V,\mathtt{m}),(V,\Delta^{-1}\cdot\mathtt{n})} = \|\mathrm{Id}\|_{(V,\Delta\cdot\mathtt{m}),(V,\mathtt{n})}$, so $\omega(\Delta_j\cdot\mathtt{m},\mathtt{n}) = \log\max\{\|\Delta_j\|_{(V,\mathtt{m}),(V,\mathtt{n})}, \|\Delta_j^{-1}\|_{(V,\mathtt{n}),(V,\mathtt{m})}\}$. Fix a basis $(v_j)_{j<k}$ of V. Let $K > 0$ be such that for large enough j we have that $\{\Delta_j(v_l)\}_{l<k} \subseteq K\cdot\mathrm{Ball}(V,\mathtt{n})$, so we can find $L = \{\xi_j\}_j \subseteq \mathbb{N}$ infinite such that $(\Delta_{\xi_j}(v_l))_{j\in\mathbb{N}}$ converges to w_l for every $l < k$. Define $\Delta(\sum_{j<k} a_j v_j) := \sum_{j<k} a_j w_j$. Then $\Delta_j \to_{j\in L} \Delta$ pointwise, and for $v \in V$, $\mathtt{n}(\Delta(v)) = \lim_{j\to\infty}\mathtt{n}(\Delta_{\xi_j}(v)) \geqslant \lim_{j\to\infty}(\|\Delta_{\xi_j}^{-1}\|_{(V,\mathtt{n}),(V,\mathtt{m})})^{-1}\mathtt{m}(v) \geqslant \mathtt{m}(v)$. This means that $\Delta \in \mathrm{GL}(\mathbb{F}^k)$. Similarly one shows that $\mathtt{n}(\Delta(v)) \leqslant \mathtt{m}(v)$ for every v. Hence, $\mathtt{n} = \Delta\cdot\mathtt{m}$.

In a similar way one shows that the infimum in (4) is a minimum, so if $\widetilde{\omega}([m],[n]) = 0$, then there is $\Delta \in \mathrm{GL}(V)$ such that $\omega(\Delta\cdot\mathtt{m},\mathtt{n}) = 0$, i.e. $\mathtt{n} = \Delta\cdot\mathtt{m}$.

In addition, $\mathcal{B}_V$ has the Heine-Borel property, because $(\mathcal{N}_V,\omega)$ has this property. It is a fundamental fact, called as Auerbach Lemma (see [1, Problem 12.1]) that given a norm $\mathtt{m}$ on V of dimension k there is a linear transformation Δ such that $(\Delta(u_j))_{j<k}$ is an Auerbach basis of $(V,\mathtt{m})$, i.e., a normalized sequence such that $\mathtt{m}(\sum_{j<k} a_j\Delta(u_j)) \geqslant \max_{j<k}|a_j|$ for every sequence of scalars $(a_j)_{j<k}$. Given two norms $\mathtt{m},\mathtt{n} \in \mathcal{N}_V$, let $\Delta,\Gamma \in \mathrm{GL}(V)$ be such that $(\Delta(u_j))_{j<k}$ and $(\Gamma(u_j))_{j<k}$ are Auerbach bases of $(V,\mathtt{m})$ and of $(V,\mathtt{n})$, respectively. It follows that given $v = \sum_{j<k} a_j u_j$, $(\Delta^{-1}\cdot\mathtt{m})(\sum_{j<k} a_j u_j) = \mathtt{m}(\sum_{j<k} a_j\Delta(u_j)) \leqslant (\sum_{j<k}\mathtt{m}(\Delta(u_j)))\max_{j<k}|a_j| \leqslant k\mathtt{n}(\sum_{j<k} a_j\Gamma(u_j)) = k(\Gamma^{-1}\cdot\mathtt{n}(v))$ and similarly, $(\Gamma^{-1}\cdot\mathtt{m})(v) \leqslant k(\Delta^{-1}\cdot\mathtt{n})(v)$. In other words, $\omega(\Delta^{-1}\cdot\mathtt{m},\Gamma^{-1}\cdot\mathtt{n}) \leqslant \log k$, consequently the diameter of $\mathcal{B}_V$ is at most $\log k$. Since $(\mathcal{B}_V,\widetilde{\omega})$ has the Heine-Borel property, it is compact, called the *Banach-Mazur compactum.* The quotient metric $\widetilde{\omega}$ is 2-Lipschitz equivalent to the well-known *Banach-Mazur metric*

$$d_{\mathrm{BM}}([\mathtt{m}],[\mathtt{n}]) := \log \inf_{\Delta\in\mathrm{GL}(V)} \|\Delta\|_{(V,\mathtt{m}),(V,\mathtt{n})}\cdot\|\Delta^{-1}\|_{(V,\mathtt{n}),(V,\mathtt{m})}. \tag{5}$$

To see this, we rewrite $d_{\mathrm{BM}}([\mathtt{m}],[\mathtt{n}]) = \log\inf_{\Delta\in\mathrm{GL}(V)}\|\mathrm{Id}\|_{(V,\Delta\cdot\mathtt{m}),(V,\mathtt{n})}\cdot\|\mathrm{Id}\|_{(V,\mathtt{n}),(V,\Delta\cdot\mathtt{m})}$. Since $\omega_0(\Delta\cdot\mathtt{m},\mathtt{n}) \geqslant 1$ (because $\omega = \log\omega_0$ is a metric), it follows that $\|\mathrm{Id}\|_{(V,\Delta\cdot\mathtt{m}),(V,\mathtt{n})}\cdot\|\mathrm{Id}\|_{(V,\mathtt{n}),(V,\Delta\cdot\mathtt{m})} \leqslant \omega_0(\Delta\cdot\mathtt{m},\mathtt{n})^2$, and consequently, $d_{\mathrm{BM}}([\mathtt{m}],[\mathtt{n}]) \leqslant \log\inf_{\Delta\in\mathrm{GL}(V)}(\omega_0(\Delta\cdot\mathtt{m},\mathtt{n})^2) = 2\widetilde{\omega}([\mathtt{m}],[\mathtt{n}])$. Now given $\Delta \in \mathrm{GL}(V)$, we define $\Gamma := \Delta/\|\Delta\|_{(V,\mathtt{m}),(V,\mathtt{n})}$ and we have that $\max\{\|\Gamma\|_{(V,\mathtt{m}),(V,\mathtt{n})},\|\Gamma^{-1}\|_{(V,\mathtt{n}),(V,\mathtt{m})}\} = \max\{1,\|\Delta\|_{(V,\mathtt{m}),(V,\mathtt{n})}\cdot\|\Delta^{-1}\|_{(V,\mathtt{n}),(V,\mathtt{m})}\} = \|\Delta\|_{(V,\mathtt{m}),(V,\mathtt{n})}\cdot\|\Delta^{-1}\|_{(V,\mathtt{n}),(V,\mathtt{m})}$, hence, taking the corresponding infimums we obtain that $\widetilde{\omega}([\mathtt{m}],[\mathtt{n}]) \leqslant d_{\mathrm{BM}}([\mathtt{m}],[\mathtt{n}])$.

Notice that the infimum in (5) is also a minimum, as for $\widetilde{\omega}$. Let $\mathcal{B}_V(E)$ denote the Banach-Mazur classes corresponding to norms in $\mathcal{N}_V(E)$, or, in other words, the isometric types of finite dimensional subspaces of E of the same dimension as V. In general, $\mathcal{B}_V(E)$ is not compact, although obviously $\overline{\mathcal{B}_V(E)}$ is compact, since $\mathcal{B}_V$ is compact. We write $\mathcal{B}_k$, $\mathcal{B}_k(E)$ and $\mathrm{Gr}(k,E)$ to denote $\mathcal{B}_{\mathbb{F}^k}$, $\mathcal{B}_{\mathbb{F}^k}(E)$ and $\mathrm{Gr}(k,V)$, respectively.

Definition 2.9. Let $\tau_{k,E} : \mathrm{Gr}(k,E) \to \mathcal{B}_k(E)$ be the mapping that assigns to each k-dimensional normed subspace W of E the isometric type of $(W,\|\cdot\|_E)$.

In other words, for $W \in \mathrm{Gr}(k, E)$, $\tau_{k,E}(W) = [\nu_{k,E}(T)]_{\mathrm{BM}}$ for some 1-1 linear function $T : \mathbb{F}^k \to W$ such that $\mathrm{Im}\, T = W$. We define the *E-Kadets* mapping $\gamma_{k,E}$ on $\mathcal{B}_k(E) \times \mathcal{B}_k(E)$ by

$$\gamma_{k,E}([\mathtt{m}], [\mathtt{n}]) := \inf\{\Lambda_{k,E}(U, W) \, : \, U, W \in \mathrm{Gr}(k, E),\ \tau_{k,E}(U) = [\mathtt{m}],\ \tau_{k,E}(W) = [\mathtt{n}]\}.$$

Definition 2.10. $\gamma_{k,E}$ is the *E-Kadets metric* when it is uniformly equivalent to $d_{\mathrm{B}M}$ in $\mathcal{B}_k(E)$.

In the literature the Kadets metric γ corresponds to the metric γ_E for Grassmannians of any universal space E for separable Banach spaces, for example for the space of continuous functions on the unit interval $C[0, 1]$, or for the Gurarij space $\mathbb{G}$ (see Appendix A). For more information on the Kadets metric we refer the reader to [13]. In general, there is always the following relation between $\gamma_{k,E}$ and d_{BM}.

Proposition 2.11. *For every normed space E, every $k \geqslant 2$ and every $V, W \in \mathrm{Gr}(k, E)$,*

$$d_{\mathrm{BM}}(\tau_{k,E}(V), \tau_{k,E}(W)) \leqslant 3k \log k \Lambda_{k,E}(V, W).$$

Proof. Suppose first that $\Lambda_{k,E}(V, W) \geqslant (3k)^{-1}$. Since the diameter of $\mathcal{B}_k$ is at most $\log(k)$, we obtain that $d_{\mathrm{BM}}(\tau_{k,E}(V), \tau_{k,E}(W)) \leqslant \log(k) \leqslant 3k \log(k)\Lambda_{k,E}(V, W)$. Suppose now that $\Lambda_{k,E}(V, W) < 1/(3k)$. Let $(x_j)_{j<k}$ be an Auerbach basis of $(V, \| \cdot \|_E)$. For each $j < k$, let $y_j \in \mathrm{Ball}((W, \| \cdot \|_E))$ be such that $\|x_j - y_j\|_E \leqslant \Lambda_{k,E}(V, W)$. Since

$$\begin{aligned}\| \sum_{j<k} \lambda_j y_j\|_E \geqslant& \| \sum_{j<k} \lambda_j x_j\|_E - \| \sum_{j<k} \lambda_j (x_j - y_j)\|_E \geqslant \| \sum_{j<k} \lambda_j x_j\|_E - k\Lambda_{k,E}(V, W) \max_{j<k} |\lambda_j| \geqslant \\ \geqslant& (1 - k\Lambda_{k,E}(V, W))\| \sum_{j<k} \lambda_j x_j\|_E > 0. \end{aligned} \tag{6}$$

we obtain that $(y_j)_{j<k}$ is a basis of W and $T : V \to W$, $T(x_j) := y_j$, $j < k$ is invertible. In addition, from (6) we have that

$$\|T\|_{(V,\|\cdot\|_E),(W,\|\cdot\|_E)} \leqslant 1 + k\Lambda_{k,E}(V, W) \text{ and } \|T^{-1}\|_{(W,\|\cdot\|_E),(V,\|\cdot\|_E)} \leqslant \frac{1}{1 - k\Lambda_{k,E}(V, W)}$$

We use that $(1 + x)/(1 - x) \leqslant \exp(9x/4)$ if, in particular, $0 \leqslant x \leqslant 1/3$, and that $\log k \geqslant 3/4$ for $k \geqslant 2$ to conclude that

$$\begin{aligned} d_{\mathrm{BM}}(\tau_{k,E}(V), \tau_{k,E}(W)) \leqslant& \log \left(\|T\|_{(V,\|\cdot\|_E),(W,\|\cdot\|_E)} \cdot \|T^{-1}\|_{(W,\|\cdot\|_E),(V,\|\cdot\|_E)}\right) \leqslant \\ \leqslant& \frac{9}{4} k\Lambda_E(V, W) \leqslant 3k \log k\Lambda_{k,E}(V, W). \end{aligned}$$

□

Proposition 2.12. *If $\gamma_{k,E}$ is Kadets, $\tau_{k,E} : (\mathrm{Gr}(k, E), \Lambda_E) \to (\mathcal{B}_k(E), \gamma_{k,E})$ is 1-Lipschitz.* □

Given $E = (\mathbb{F}^\alpha, \mathtt{m})$, we write $\mathrm{Gr}_{\mathtt{m}}(k, \mathbb{F}^\alpha)$ to denote the set of k-dimensional subspaces W of $\mathbb{F}^\alpha$ so that $(W, \mathtt{m})$ is isometric to $(\mathbb{F}^k, \mathtt{m})$, i.e., $\tau_{k,E}(W) = [\mathtt{m} \upharpoonright \langle u_j \rangle_{j<k}]$. The next explains the interest of $\mathrm{Gr}_{\mathtt{m}}(k, \mathbb{F}^\alpha)$ and it is proved similarly to Proposition 2.5.

Proposition 2.13. *Fix $k \leqslant m$ and $W \in \mathrm{Gr}(m, E)$.*

a) An invertible linear operator $\theta : W \to \mathbb{F}^m$ defines an isometry $\Theta : (\mathrm{Gr}(k, W), \Lambda_{k,E}) \to (\mathrm{Gr}(k, \mathbb{F}^m), \Lambda_{k,E})$, $V \mapsto \theta(V)$, if and only if $\theta : (W, \mathtt{m}) \to (\mathbb{F}^m, \mathtt{m})$ is an isometry.

b) $\tau_{k,(W,\mathtt{m})} = \tau_{k,E} \upharpoonright \mathrm{Gr}(k, W)$.

Definition 2.14 (Ramsey factors for Grassmannians)**.** A norm $\mathtt{m}$ on $\mathbb{F}^\infty$, $E = (\mathbb{F}^\infty, \mathtt{m})$, *produces Ramsey factors* for compact colorings of Grassmannians when

i) $\gamma_{k,E}$ is uniformly equivalent to d_{BM} on $\mathcal{B}_k(E)$.
ii) For every $k \in \mathbb{N}$, $\varepsilon > 0$, and every compact metric (K, d_K) there is n such that for every 1-Lipschitz $c : (\mathrm{Gr}(k, \mathbb{F}^n), \Lambda_{k,E}) \to (K, d_K)$ there is $V \in \mathrm{Gr}_{\mathtt{m}}(m, \mathbb{F}^n)$ such that $\tau_{k,E}$ is an ε-factor of c in $\mathrm{Gr}(k, V)$.

Remark 2.15. The definition of Kadets demands that $\gamma_{k,E}$ is uniform equivalent with d_{BM} on $\mathcal{B}_k(E)$, not only that they are topologically equivalent. In this way, the metric space $(\mathcal{B}_k(E), \gamma_{k,E})$ is pre-compact, that is its metric completion is compact, as it is the topological closure of $\mathcal{B}_k(E)$ in $\mathcal{B}_k$. This implies that $\tau_{k,E} : (\mathrm{Gr}(k, E), \Lambda_E) \to (\widehat{\mathcal{B}_k(E), \gamma_{k,E}})$ is a compact coloring and ii) in Definition 2.14 can be restated as:

ii') For every $k, m \in \mathbb{N}$, $\varepsilon > 0$, and every compact metric (K, d_K) there is n such that for every 1-Lipschitz $c : (\mathrm{Gr}(k, \mathbb{F}^n), \Lambda_{k,E}) \to (K, d_K)$ there is $V \in \mathrm{Gr}_{\mathtt{m}}(m, \mathbb{F}^n)$ and a 1-Lipschitz $\widetilde{c} : (\widehat{\mathcal{B}_k(E), \gamma_{k,E}}) \to (K, d_K)$ such that $\sup_{W \in \mathrm{Gr}(k,V)} d_K(\widetilde{c}(\tau_{k,E}(W)), c(W)) \leqslant \varepsilon$.

Theorem 2.16 (Factorization of Grassmannians over $\mathbb{R}$, $\mathbb{C}$)**.** *For $1 \leqslant p \leqslant \infty$, $p \notin 2\mathbb{N} + 4$, the p-norm $\|\cdot\|_p \in \mathcal{N}_\infty$* produces Ramsey factors *for compact colorings of Grassmannians.*

Geometrically, the previous result states that restrictions of compact colorings of Grassmannians depend on shapes (or more precisely on the equivalence classes of shapes) of their unit balls. This result, like the corresponding one for full rank matrices in Theorem 2.8, is a consequence of the steady approximate Ramsey property of the family of finite dimensional subspaces of the those ℓ_p-spaces, done in Theorem 3.3 *a)*.

The following statement can be considered as a version of the Graham-Leeb-Rothschild Theorem for the fields $\mathbb{R}, \mathbb{C}$. Recall that ℓ_p^∞ is the vector space $\mathbb{F}^\infty$ of eventually zero sequences $(a_j)_{j \in \mathbb{N}}$ in $\mathbb{F}$ endowed with the p-norm.

Corollary 2.17 (Graham-Leeb-Rothschild for $\mathbb{R}$, $\mathbb{C}$)**.** *For every $1 \leqslant p \leqslant \infty$, $p \notin 2\mathbb{N} + 4$, every $k, m \in \mathbb{N}$, every $\varepsilon > 0$ and every compact metric space (K, d_K) there is n such that every compact coloring $c : (\mathrm{Gr}(k, \mathbb{F}^n), \Lambda_{k,\ell_p^n}) \to (K, d_K)$ ε-stabilizes on $\mathrm{Gr}(k, W)$ for some $W \in \mathrm{Gr}(m, \mathbb{F}^n)$.*

Proof. It is a well-known fact that for every $\varepsilon > 0$ and $m \in \mathbb{N}$ there is some r_0 such that for every $r \geqslant r_0$ the space ℓ_p^r has a subspace X of dimension m that is $1 + \varepsilon$-isomorphic to ℓ_2^m, that is, there is an isomorphism $T : X \to \ell_2^m$ such that $\|T\| \cdot \|T^{-1}\| \leqslant 1 + \varepsilon$. (see for example, [16, Section 5]). Fix now the parameters p, k, m, $\varepsilon > 0$ and (K, d_K). Let $\delta > 0$ be such that if $[\mathtt{m}], [\mathtt{n}] \in \mathcal{B}_k(\ell_p^\infty)$ are such that $d_{\mathrm{BM}}([\mathtt{m}], [\mathtt{n}]) < \delta$ then $\gamma_{k,\ell_p^\infty}([\mathtt{m}], [\mathtt{n}]) \leqslant \varepsilon/3$. Let r be such that ℓ_p^r contains a subspace of dimension m that is $(1 + \delta/2)$-isomorphic to ℓ_2^m. Let n be the "Ramsey number" corresponding to the parameters k, r, $\varepsilon/3$ and K. We claim that n works: for suppose that $c : (\mathrm{Gr}(k, \mathbb{F}^n), \Lambda_{k,E}) \to (K, d_K)$ is 1-Lipschitz. Let $V \in \mathrm{Gr}(r, \mathbb{F}^n)$ be isometric to ℓ_p^r and a 1-Lipschitz mapping $\widetilde{c} : (\mathcal{B}_k(\ell_p^\infty), \gamma_{k,\ell_p^\infty}) \to (K, d_K)$ such that $\sup_{Z \in \mathrm{Gr}(k,V)} d_K(\widetilde{c}(\tau_{k,\ell_p^\infty}(Z)), c(Z)) \leqslant \varepsilon/3$. Let $W \subseteq V$ be a m-dimensional subspace $(1+\delta/2)$-isomorphic to ℓ_2^m. Fix $Y, Z \in \mathrm{Gr}(k, W)$. Both spaces are obviously $(1+\delta/2)$-isomorphic to ℓ_2^k, so it follows that

$$d_{\mathrm{BM}}(\tau_{k,\ell_p^\infty}(Y), \tau_{k,\ell_p^\infty}(Z)) \leqslant d_{\mathrm{BM}}(\tau_{k,\ell_p^\infty}(Y), [\|\cdot\|_2]) + d_{\mathrm{BM}}(\tau_{k,\ell_p^\infty}(Z), [\|\cdot\|_2]) \leqslant \delta.$$

By the choice of δ, we obtain that $\gamma_{k,\ell_p^\infty}(\tau_{k,\ell_p^\infty}(Y), \tau_{k,\ell_p^\infty}(Z)) \leqslant \varepsilon/3$. Hence

$$\begin{aligned} d_K(c(Y), c(Z)) \leqslant & d_K(\widetilde{c}(\tau_{k,\ell_p^\infty}(Y)), c(Y)) + d_K(\widetilde{c}(\tau_{k,\ell_p^\infty}(Z)), c(Z)) + d_K(\widetilde{c}(\tau_{k,\ell_p^\infty}(Y)), \widetilde{c}(\tau_{k,\ell_p^\infty}(Z))) \\ \leqslant & \frac{2\varepsilon}{3} + \gamma_{k,\ell_p^\infty}(\tau_{k,\ell_p^\infty}(Y), \tau_{k,\ell_p^\infty}(Z)) \leqslant \varepsilon. \end{aligned}$$

□

Remark 2.18. Recall that Dvoretzky's Theorem asserts that any finite-dimensional normed space X of dimension r contains $1 + \varepsilon$-isomorphic copies of ℓ_2^m with m proportional to $\log(r)$ and to a fixed function of ε (see [1, Theorem 12.3.6]). This implies that the following version of Corollary 2.17 remains true for every norm on $\mathbb{F}^\infty$: For every norm $\mathtt{m}$ on $\mathbb{F}^\infty$, every $k, m \in \mathbb{N}$, every $\varepsilon > 0$ and every compact metric space (K, d_K) there is n such that every compact coloring $c : (\mathrm{Gr}(k, \mathbb{F}^n), d'_{\mathrm{BM}}) \to (K, d_K)$ ε-stabilizes on $\mathrm{Gr}(k, W)$ for some

$W \in \mathrm{Gr}(m, \mathbb{F}^n)$. Here we are considering the pseudometric d'_{BM} on $\mathrm{Gr}(k, \mathbb{F}^\infty)$ defined by $d'_{\mathrm{BM}}(V, W) := d_{\mathrm{BM}}(\tau_{k,(\mathbb{F}^\infty,\mathtt{m})}(V), \tau_{k,(\mathbb{F}^\infty,\mathtt{m})}(W))$. Its proof is similar to that of the previous corollary, skipping the part relating the Banach-Mazur metric and the corresponding Kadets metric.

2.1.3. *Square matrices.* Given a vector space V, let V^* be its (algebraic) dual, the vector space of linear functions $f : V \to \mathbb{F}$; given $\Delta \in \mathrm{GL}(V)$, we denote by $\Delta^* \in \mathrm{GL}(V^*)$ the *adjoint operator* of Δ, defined by the rule $(\Delta^*(f))(v) := f(\Delta(v))$ for every $f \in V^*$ and every $v \in V$. In the particular case of $V = \mathbb{F}^n$, we have that the biorthogonal sequence $(u_j^*)_{j<n}$ of the unit basis $(u_j)_{j<n}$ defined by $u_j^*((a_k)_{k<n}) := a_j$ is a basis of $(\mathbb{F}^n)^*$ and $\sum_{j<n} a_j u_j \mapsto \sum_{j<n} a_j u_j^*$ is the canonical identification of $(\mathbb{F}^n)^*$ with $\mathbb{F}^n$. The sequence $(u_j^*)_{j<n}$ will be called the unit basis of $(\mathbb{F}^n)^*$. It follows easily that if $A = (a_{ij})_{i,j<n}$ is the matrix representing $\Delta \in \mathrm{GL}(\mathbb{F}^n)$ in the unit basis of $\mathbb{F}^n$, then Δ^* is represented in the corresponding unit basis of $(\mathbb{F}^n)^*$ by the transpose $A^* = (a_{ji})_{i,j<n}$ of A. A similar result is true in any finite dimensional space by replacing the unit basis by a given basis and its biorthogonal.

If in addition $X = (V, \mathtt{m})$ is a normed space, X^* will denote the (normed) dual space $\mathcal{L}(X, (\mathbb{F}, |\cdot|))$, that is, the vector space of continuous linear functionals $f : V \to \mathbb{F}$ endowed with the *dual norm* $\mathtt{m}^*(f) := \sup_{\mathtt{m}(x)\leqslant 1} |f(x)|$.

Let $\mathrm{GL}(V) \curvearrowright \mathcal{N}_{V^*}$ be the canonical action

$$(\Delta \cdot \mathtt{n})(f) := \mathtt{n}(\Delta^*(f))$$

for $f \in V^*$. Observe that the dual mapping $\cdot^* : (\mathcal{N}_V, \omega) \to (\mathcal{N}_{V^*}, \omega)$ is a $\mathrm{GL}(V)$-equivariant isometry, that is, given $\mathtt{m} \in \mathcal{N}_V$, $(\Delta \cdot \mathtt{m})^* = \Delta \cdot \mathtt{m}^*$. Let $\mathrm{GL}(V) \curvearrowright \mathcal{N}_{V^*} \times \mathcal{N}_V$ be the action

$$\Delta \cdot (\mathtt{m}, \mathtt{n}) := (\Delta \cdot \mathtt{m}, \Delta \cdot \mathtt{n})$$

and let $\mathcal{D}_V$ be the quotient space $(\mathcal{N}_{V^*} \times \mathcal{N}_V) /\!\!/ \mathrm{GL}(V)$. With the compatible metric

$$\omega_2((\mathtt{m}, \mathtt{n}), (\mathtt{p}, \mathtt{q})) := \omega(\mathtt{m}, \mathtt{p}) + \omega(\mathtt{n}, \mathtt{q})$$

the product $\mathcal{N}_{V^*} \times \mathcal{N}_V$ has the Heine-Borel property, so $\mathcal{D}_V$ with the corresponding quotient metric $\widetilde{\omega}_2$,

$$\widetilde{\omega}_2([(\mathtt{m}, \mathtt{n})], [(\mathtt{p}, \mathtt{q})]) = \inf_{\Delta \in \mathrm{GL}(V)} \omega_2(\Delta \cdot (\mathtt{m}, \mathtt{n}), (\mathtt{p}, \mathtt{q}))$$

also has this property. Similarly to the case of d_{BM} and of $\widetilde{\omega}$ the infimum defining $\widetilde{\omega}_2$ is in fact a minimum. Given a normed space E, let $\mathcal{D}_V(E) := (\mathcal{N}_{V^*}(E) \times \mathcal{N}_V(E)) /\!\!/ \mathrm{GL}(V)$. Its orbits will be denoted by $[\mathbf{m}] = [(\mathtt{m}_0, \mathtt{m}_1)]$. In the next $E := (\mathbb{F}^\alpha, \|\cdot\|_E)$ and $\mathcal{D}_k$, $\mathcal{D}_k(E)$ denote $\mathcal{D}_{\mathbb{F}^k}$ and $\mathcal{D}_{\mathbb{F}^k}(E)$, respectively.

Definition 2.19. Let $\nu^2_{k,E} : M^k_\alpha \to \mathcal{D}_k(E)$ be the function that assigns to an α-squared matrix A of rank k, the class $\mathrm{GL}(\mathbb{F}^k)$-orbit of the pair $(\nu_{k,E}(B), \nu_{k,E}(C))$ for $B, C \in M^k_{\alpha,k}$ with $A = B \cdot C^*$.

The fact that $\nu^2_{k,E}$ is well defined follows from the full-rank factorization of matrices.

Proposition 2.20. $[(\nu_{(\mathbb{F}^k)^*,E}(T_0), \nu_{\mathbb{F}^k,E}(T_1))] = \{(\nu_{(\mathbb{F}^k)^*,E}(U_0), \nu_{\mathbb{F}^k,E}(U_1)) : U_0 \circ U_1^* = T_0 \circ T_1^*\}$.

Proof. If $U_0, U_1 : \mathbb{F}^k \to E$ are linear operators of rank k then $T_0 \circ T_1^* = U_0 \circ U_1^*$ if and only if there is some $\Delta \in \mathrm{GL}(\mathbb{F}^k)$ such that $U_0 = T_0 \circ \Delta^{-1}$ and $U_1 = T_1 \circ \Delta^*$. □

We define on $\mathcal{D}_k(E) \times \mathcal{D}_k(E)$ the function $\mathfrak{d}_{k,E}$

$$\mathfrak{d}_{k,E}([\mathbf{m}], [\mathbf{n}])) := \inf_{T,U} \|T - U\|_{E^*,E}$$

where the infimum is over bounded linear mappings $T, U : E^* \to E$ of rank k admitting decompositions $T = T_0 \circ T_1^*$ and $U = U_0 \circ U_1^*$ with $T_0, U_0 : (\mathbb{F}^k)^* \to E$ and $T_1, U_1 : \mathbb{F}^k \to E$ of rank k for $j = 0, 1$ and such that $(\nu_{(\mathbb{F}^k)^*,E}(T_0), \nu_{\mathbb{F}^k,E}(T_1)) \in [\mathbf{m}]$ and $(\nu_{(\mathbb{F}^k)^*,E}(U_0), \nu_{\mathbb{F}^k,E}(U_1)) \in [\mathbf{n}]$. The following is easy to prove.

Proposition 2.21. *If $\mathfrak{d}_{k,E}$ is a metric on $\mathcal{D}_k(E)$, then $\nu^2_{k,E} : (M^k_\alpha, d_{E^*,E}) \to (\mathcal{D}_k(E), \mathfrak{d}_{k,E})$ is 1-Lipschitz.* □

Given a norm $\mathtt{m} \in \mathcal{N}_{\mathbb{F}^\alpha}$, let $M^k_\alpha(\mathtt{m}; \lambda)$ be the collection of $A \in M^k_\alpha$ such that $\|T_A\|, \mathtt{r}_{-1}(T_A) \leqslant \lambda$, where $T_A : ((\mathbb{F}^\alpha)^*, \mathtt{m}^*) \to (\mathbb{F}^\alpha, \mathtt{m})$ is the linear operator $T_A(\sum_j a_j u^*_j) = A \cdot (a_j)_j$. The next is the version of Proposition 2.5 for square matrices.

Proposition 2.22. *Let $L \in M^m_{\alpha,m}$ and $R \in M^m_{m,\alpha}$.*

a) The multiplication by L and R function $\mu_{L,R} : (M^k_m, d_{E^*,E}) \to (M^k_\alpha, d_{E^*,E})$, $A \in M^k_m \mapsto \mu_{L,R}(A) := L \cdot A \cdot R \in M^k_\alpha$ is an isometry if and only if there is some $\alpha \neq 0$ such that $\alpha L, (1/\alpha)R^* \in \mathcal{E}_{\alpha,m}(\|\cdot\|_E)$.

b) If $L, R^* \in \mathcal{E}_{\alpha,m}(\|\cdot\|_E)$, then $\nu^2_{k,(\mathbb{F}^\alpha,\|\cdot\|_E)} \circ \mu_{L,R} = \nu^2_{k,(\mathbb{F}^m,\|\cdot\|_E)}$. □

Proof. Suppose that X, Y, Z are finite dimensional spaces with $k \leqslant \dim X \leqslant \dim Y, \dim Z$. Let $L \in \mathcal{L}(X, Z)$ and $R \in \mathcal{L}(Y, X^*)$ be such that $T \in \mathcal{L}^k(X^*, X) \mapsto L \circ T \circ R \in \mathcal{L}(Y, Z)$ is an isometry. Proceeding as in the proof of Proposition 2.5, we fix a normalized basis $(x_j)_{j<l}$ of X such that $R^*(x_0) \neq 0$, and let $(x^*_j)_{j<l}$ be its biorthogonal sequence in X^*. For a given $p \in X$, $p \neq 0$, let $(p_j)_{j<k}$ be a linearly independent sequence in X with $p_0 = p$. For each $n \geqslant 1$, let $T_n, T : X^* \to X$ be defined for $x^* \in X^*$ by $T_n(x^*) = x^*(x_0)p + (1/n)\sum_{j=1}^{k-1} x^*(x_j)p_j$. Then $T_n, (1/n)T_n \in \mathcal{L}^k(X^*, X)$, because $\langle T_n(x^*_j)\rangle_{j<k} = \langle p_j\rangle_{j<k}$, and $\|L \circ T_n \circ R - L \circ (1/n) \cdot T_n \circ R\| - \|L \circ T_n \circ R\| \to_n 0$, and $\|L \circ T_n \circ R - L \circ (1/n) \cdot T_n \circ R\| - \|T_n\| = \|T_n - (1/n) \cdot T_n\| - \|T_n\| \to_n 0$, hence $\|L \circ T_n \circ R\| - \|T_n\| \to_n 0$. We have that $\|T_n\| \to_n \|x_0\|\|p\|$ and $\|L \circ T_n \circ R\| \to_n \|R^*(x_0)\|^*\|L(p)\|$. This shows that $\|L(p)\| = (1/\|R^*(x_0)\|^*)\|p\|$, and since $p \neq 0$ was arbitrary, we obtain that αL is an isometric embedding for $\alpha = \|R^*(x_0)\|^*$. Observe that $T \in \mathcal{L}^k(X^*, X) \mapsto (L \circ T \circ R)^* = R^* \circ T^* \circ L^*$ and $T \in \mathcal{L}^k(X^*, X) \mapsto T^* \in \mathcal{L}^k(X^*, X)$ are isometries, so $T \mapsto R^* \circ T \circ L^*$ is an isometry, and we have just proved that there is $\beta \neq 0$ such that βR^* is an isometric embedding. Finally choosing $T \neq U \in \mathcal{L}^k(X^*, X)$, we have that $\|T - U\| = \|L \circ T \circ R - L \circ U \circ R\| = \alpha\beta\|T - U\|$ hence $\alpha\beta = 1$.*b)* is trivial. □

Given $\lambda \geqslant 1$, let

$$\mathcal{D}_k(\lambda) := \{[(\mathtt{m}, \mathtt{n})] \in \mathcal{D}_k \ : \ \omega_{(\mathbb{F}^k)^*}(\mathtt{m}, \mathtt{n}^*) \leqslant \log \lambda\},$$
$$\mathcal{D}_k(<\lambda) := \{[(\mathtt{m}, \mathtt{n})] \in \mathcal{D}_k \ : \ \omega_{(\mathbb{F}^k)^*}(\mathtt{m}, \mathtt{n}^*) < \log \lambda\}$$

and let $\mathcal{D}_k(E; \lambda) := \mathcal{D}_k(\lambda) \cap \mathcal{D}_k(E)$, and $\mathcal{D}_k(E; <\lambda) := \mathcal{D}_k(<\lambda) \cap \mathcal{D}_k(E)$.

Proposition 2.23. *a) $\mathcal{D}_k(\lambda)$ and $\mathcal{D}_k(<\lambda)$ are well defined.*
b) $\mathcal{D}_k(\lambda)$ is compact.
c) $\mathcal{D}_k(E; \lambda) = \nu^2_{k,E}(M^k_\alpha(\|\cdot\|_E; \lambda))$ and $\mathcal{D}_k(E; <\lambda) = \nu^2_{k,E}(M^k_\alpha(\|\cdot\|_E; <\lambda))$.

Proof. a) follows from the fact that for $\Delta \in \mathrm{GL}(\mathbb{F}^k)$ we have that $\omega_{(\mathbb{F}^k)^*}(\Delta \cdot \mathtt{m}, (\Delta \cdot \mathtt{n})^*) = \omega_{(\mathbb{F}^k)^*}(\Delta \cdot \mathtt{m}, \Delta \cdot \mathtt{n}^*) = \omega_{(\mathbb{F}^k)^*}(\mathtt{m}, \mathtt{n}^*)$. *b)*: It is clear that $\mathcal{D}_k(\lambda)$ is closed, so by the Heine-Borel property of $(\mathcal{D}_k, \widetilde{\omega}_2)$, we just have to prove that $\mathcal{D}_k(\lambda)$ is $\widetilde{\omega}_2$-bounded: Fix $[(\mathtt{m}, \mathtt{n})], [(\mathtt{p}, \mathtt{q})] \in \mathcal{D}_k(\lambda)$, let $\Delta \in \mathrm{GL}(\mathbb{F}^k)$ be such that $\omega_{\mathbb{F}^k}(\Delta \cdot \mathtt{n}, \mathtt{q}) \leqslant \log(k)$. Then it follows that $\omega_{(\mathbb{F}^k)^*}(\Delta \cdot \mathtt{m}, \mathtt{p}) \leqslant \omega_{(\mathbb{F}^k)^*}(\Delta \cdot \mathtt{m}, \Delta \cdot \mathtt{n}^*) + \omega_{(\mathbb{F}^k)^*}(\Delta \cdot \mathtt{n}^*, \mathtt{q}^*) + \omega_{(\mathbb{F}^k)^*}(\mathtt{p}, \mathtt{q}^*) \leqslant 2\log\lambda + \omega_{\mathbb{F}^k}(\Delta \cdot \mathtt{n}, \mathtt{q}) \leqslant \log(k\lambda^2)$. *c)* will be proved in Lemma 3.13. □

Definition 2.24 (Ramsey factors for square matrices)**.** A norm $\mathtt{m}$ on $\mathbb{F}^\infty$ *produces Ramsey factors* for compact colorings of square matrices if

i) $\mathfrak{d}_{k,E_\infty}$ is a compatible metric on $\mathcal{D}_k(E)$ uniformly equivalent to ω_2 on ω_2-bounded sets.
ii) Given $k, m \in \mathbb{N}$, real numbers $\varepsilon > 0$, $\lambda \geqslant 1$, and a compact metric space (K, d_K), there is $n \in \mathbb{N}$ such that for every 1-Lipschitz coloring $c : (M^k_n(\mathtt{m}; \lambda), d_{E^*_\infty,E_\infty}) \to (K, d_K)$ there are $R_0, R_1 \in \mathcal{E}_{n,m}(\mathtt{m})$ such that the restriction $\nu^2_{k,E_\infty} : M^k_\infty(\mathtt{m}; <\lambda) \to \mathcal{D}_k(E_\infty; <\lambda)$ is an ε-factor of c in $R_0 \cdot M^k_m(\mathtt{m}, <\lambda) \cdot R^*_1$.

Theorem 2.25 (Factorization of colorings of square matrices over $\mathbb{R}$, $\mathbb{C}$)**.** *For* $1 \leqslant p \leqslant \infty$, $p \notin 2\mathbb{N}+4$, *the* p*-norm* $\|\cdot\|_p \in \mathcal{N}_\infty$ produces Ramsey factors *for compact colorings of square matrices.*

This result is again a consequence of the steady approximate Ramsey property of the family of finite dimensional subspaces of the ℓ_p-spaces for $p \notin 2\mathbb{N}+4$, done in Theorem 3.3 *a)*.

2.1.4. *Uniqueness.* We see now how when the metric on matrices/Grassmannians is fixed there are not so many options of being a Ramsey factor. Suppose that $\mathtt{m} \in \mathcal{N}_\infty$, $k \in \mathbb{N}$ and $\lambda > 1$. A $(k, \mathtt{m}, \lambda)$-Ramsey factor is a pair $(\mu, \mathcal{A})$ where $\mu : (M_{\infty,k}^k(\mathtt{m}; {<}\lambda), d_{\mathtt{m}}) \to (K_\mu, d_\mu)$ is a 1-Lipschitz coloring to a compact metric space (K_μ, d_μ), $\mathcal{A} \subseteq \mathcal{E}(\mathtt{m})$, and

i) The image of μ is dense in K_μ.
ii) $\mu(RA) = \mu(A)$ for every $R \in \mathcal{A} \cap M_{n,m}^m$ and $A \in M_{m,k}^k(\mathtt{m}; \lambda)$.
iii) For every m, $\varepsilon > 0$ and every compact metric (L, d_L) there is $n \in \mathbb{N}$ such that if $c : (M_{n,k}^k(\mathtt{m}; \lambda), d_{\mathtt{m}}) \to (L, d_L)$ is a 1-Lipschitz coloring then there is some $R \in \mathcal{A}$ such that the restriction $\mu : R \cdot M_{m,k}^k(\mathtt{m}; {<}\lambda) \to (K_\mu, d_\mu)$ is an ε-factor of c in $R \cdot M_{m,k}^k(\mathtt{m}; {<}\lambda)$.

Suppose that $\mathtt{m}$ produces Ramsey factors for full rank matrices and set $E_\alpha := (\mathbb{F}^\alpha, \mathtt{m})$ for $\alpha \leqslant \infty$. We have that $\nu_{\mathbb{F}^k, E_\infty}(M_{\infty,k}^k(\mathtt{m}; {<}\lambda)) = \mathcal{N}_k(E; {<}\lambda)$ is ∂_{E_k, E_∞}-totally bounded: This is because ∂_{E_k,E_∞} and ω are, by hypothesis, uniformly equivalent to ω on $\mathcal{N}_k(E_\infty; {<}\lambda)$, and this set is ω-totally bounded because is an ω-bounded set of $\mathcal{N}_k$. This implies that the completion $\widehat{\mathcal{N}_k(E_\infty; {<}\lambda)}$ is a compact space. Then it is obvious from the definition of producing Ramsey factors that $\nu_{\mathbb{F}^k,E_\infty} : M_{\infty,k}^k(\mathtt{m}; {<}\lambda) \to \widehat{\mathcal{N}_k(E_\infty; {<}\lambda)}$ is a $(k, \mathtt{m}, \lambda)$-Ramsey factor, and in fact is the minimal one.

Proposition 2.26. *Suppose that* $\mathtt{m}$ *produces Ramsey factors for full rank matrices, and suppose that* $(\mu, \mathcal{A})$ *is a* $(k, \mathtt{m}, \lambda)$*-Ramsey factor.*

a) There is some surjective 1-Lipschitz mapping $\theta : K_\mu \to \widehat{\mathcal{N}_k(E_\infty; {<}\lambda)}$ *such that* $\nu_{\mathbb{F}^k,E_\infty} = \theta \circ \mu$.

b) If $\mathcal{A} = \mathcal{E}(\mathtt{m})$*, then there is a surjective isometry* $\theta : K_\mu \to \widehat{\mathcal{N}_k(E_\infty; {<}\lambda)}$ *such that* $\nu_{\mathbb{F}^k,E_\infty} = \theta \circ \mu$.

Proof. a): For each integer m we can find 1-Lipschitz mappings $\theta_m : (K_\mu, d_\mu) \to \widehat{\mathcal{N}_k(E_\infty; {<}\lambda)}$ and $R_m \in \mathcal{E}_{n_m,m}(\mathtt{m})$ such that $\partial_{\mathbb{F}^k,E_\infty}(\theta_m(\mu(R_m \cdot A)), \nu_{\mathbb{F}^k,E_\infty}(R_m \cdot A)) \leqslant 1/2^m$ for every $A \in M_{m,k}^k(\mathtt{m}; {<}\lambda)$. Since $\mathcal{A} \subseteq \mathcal{E}(\mathtt{m})$, by the coherence properties of μ and $\nu_{\mathbb{F}^k,E_\infty}$, we obtain that

$$\partial_{\mathbb{F}^k,E_\infty}(\theta_m(\mu(A)), \nu_{k,E_\infty}(A)) \leqslant 1/2^m \text{ for every } A \in M_{m,k}^k(\mathtt{m}; {<}\lambda). \tag{7}$$

Given $A \in M_{\infty,k}^k(\mathtt{m}; {<}\lambda)$, set $x := \mu(A)$, and let n be such that $A \in M_{n,k}$. Then we know from (7) that $(\theta_m(\mu(A)))_{m \geqslant n}$ is a Cauchy sequence, and let $\theta(x) \in \widehat{\mathcal{N}_k(E_\infty; {<}\lambda)}$ be its limit. Notice that $\theta(\mu(A)) = \nu_{\mathbb{F}^k,E_\infty}(A)$, so, since $\nu_{\mathbb{F}^k,E_\infty}(M_{\infty,k}^k(\mathtt{m}; {<}\lambda))$ is dense in $\widehat{\mathcal{N}_k(E_\infty; {<}\lambda)}$, we can conclude that θ is onto. *b):* proceeding similarly as in *a)*, now using the hypothesis $\mathcal{A} = \mathcal{E}(\mathtt{m})$, one can produce a surjective 1-Lipschitz mapping $\varrho : \widehat{\mathcal{N}_k(E_\infty; {<}\lambda)} \to K_\mu$ such that $\mu = \varrho \circ \nu_{\mathbb{F}^k,E_\infty}$. From *a)* we can find a 1-Lipschitz mapping such that $\theta : K_\mu \to \widehat{\mathcal{N}_k(E_\infty; {<}\lambda)}$ such that $\nu_{\mathbb{F}^k,E_\infty} = \theta \circ \mu$. Hence, $\mu = \varrho \circ \theta \circ \mu$, and since μ has dense range, it follows that $p = \varrho \circ \theta(p)$ for every $p \in K_\mu$, and consequently θ is an isometry. □

With the obvious definitions of Ramsey factors for Grassmannians and for square matrices, the corresponding statements on τ_{k,E_∞} and ν_{k,E_∞}^2 are also true.

Remark 2.27. For some norms $\mathtt{m}$, for example the p-norms, the completion $\widehat{\mathcal{N}_k(E; {<}\lambda)}$ is exactly $\mathcal{N}_k(E; \lambda)$. A sufficient condition is that for every finite dimensional subspaces X and Y of E_∞ there is a finite dimensional subspace Z of E_∞ that has isometric copies X_0 and Y_0 of X and Y, respectively, such that $X_0 \cap Y_0 = \{0\}$.

3. The proofs: Approximate Ramsey properties and extreme amenability

In Ramsey theory, the usual strategy to prove that a list of colorings is the canonical one is, given a coloring of a class of embeddings, use the Ramsey property for an appropriate good class of embeddings and an enlarged number of colors that take now into account the transformation necessary to make an arbitrary embedding a good one. This is exactly what we did for full rank matrices over a finite field. On the approximate case, one may follow the same direct approach and obtain similar results to the ones we presented, but now obliged to deal with several approximation arguments that make the proofs somehow unnecessarily complicated. Instead, our approach is to apply a topological principe that is equivalent to a strong version of an *approximate Ramsey property*, and that makes the computations much more clear. This is the *extreme amenability* of the group of linear isometries of appropriate Banach spaces that locally are like ℓ_p^∞, for $p \notin 2\mathbb{N}+4$. We introduce some relevant terminology and concepts. Recall that a Banach space is a complete normed space. Given Banach spaces $X = (X, \|\cdot\|_X)$ and $Y = (Y, \|\cdot\|_Y)$, and given $\delta \geqslant 0$, let $\mathrm{Emb}_\delta(X,Y)$ be the collection of all linear functions $T : X \to Y$ such that $(1+\delta)^{-1}\|x\|_X \leqslant \|T(x)\|_Y \leqslant (1+\delta)\|x\|_X$. Notice that when $\dim X = k < \infty$ this definition corresponds to $\mathcal{L}^k_{1+\delta}(X,Y)$ presented before. We endow $\mathrm{Emb}_\delta(X,Y)$ with the norm metric $\|T-U\|$. The following concept was introduced in [6, Definition 5.1] (see also [3] ,[4]).

Definition 3.1. A family $\mathcal{G}$ of finite dimensional normed spaces has the *Steady Approximate Ramsey Property*$^+$ (SARP$^+$) when for every $k \in \mathbb{N}$ and every $\varepsilon > 0$ there is $\delta := \delta(k,\varepsilon) > 0$ such that if $X, Y \in \mathcal{G}$ and $\dim X = k$, then there exists $Z \in \mathcal{G}$ such that every 1-Lipschitz coloring $c : \mathrm{Emb}_\delta(X,Z) \to [0,1]$ ε-stabilizes on $\gamma \circ \mathrm{Emb}_\delta(X,Y)$ for some $\gamma \in \mathrm{Emb}(Y,Z)$.

The (SARP$^+$) of the classes $\{\ell_p^n\}_n$ can be seen as a multidimensional Borsuk-Ulam principle (see [6, §5.1.1] for more information). Also, the (SARP$^+$) of a family $\mathcal{F}$ is a strong form of amalgamation for $\mathcal{F}$: recall that $\mathcal{F}$ is an *amalgamation class* when $\{0\} \in \mathcal{F}$ and for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there is $\delta > 0$ such that if $X \in \mathcal{F}$ has dimension k, $Y, Z \in \mathcal{F}$, and $\gamma \in \mathrm{Emb}_\delta(X,Y)$, $\eta \in \mathrm{Emb}_\delta(X,Z)$, then there are $V \in \mathcal{F}$, $i \in \mathrm{Emb}(Y,V)$ and $j \in \mathrm{Emb}(Z,V)$ such that $\|i\circ\gamma - j\circ\eta\| \leqslant \varepsilon$. It is not difficult to see that if $\mathcal{F}$ has the (SARP$^+$) then it is an amalgamation class (see [6, Proposition 5.7] and [4, Claim 2.13.1] for a stable version of it).

To an amalgamation class $\mathcal{G}$ it corresponds a unique separable "generic" Banach space E whose family of finite dimensional substructures, denoted by $\mathrm{Age}(E)$, is minimal containing $\mathcal{G}$. This is the content of the Fraïssé correspondence on the category of Banach spaces. We write $\mathcal{G}_E$ to denote the class of subspaces of E that are isometric to some element of $\mathcal{G}$, and $\overline{\mathcal{G}}^{\subseteq}$ to denote the class of subspaces of elements of $\mathcal{G}$; let $\mathcal{G}_\equiv$ be the class of those spaces X being isometric to some element of $\mathcal{G}$, and we say that $\mathcal{G}$ is hereditary if $Y \in \mathcal{G}$, and $\mathrm{Emb}(X,Y) \neq \varnothing$, then $X \in \mathcal{G}_\equiv$. Finally, $\mathcal{G} \preceq \mathcal{H}$ means that every space in $\mathcal{G}$ is isometric to some element of $\mathcal{H}$, and $\mathcal{G} \equiv \mathcal{H}$ to denote that $\mathcal{G} \preceq \mathcal{H} \preceq \mathcal{G}$. Note that if $\mathcal{G} \equiv \mathcal{H}$, then $\mathcal{G}$ has the (SARP$^+$) (is an amalgamation class) if and only if $\mathcal{H}$ has the (SARP$^+$) (resp. is an amalgamation class).

Theorem 3.2 (Fraïssé correspondence; [4], [6])**.** *Let $\mathcal{G}$ be a class of finite dimensional normed spaces.*

a) If $\mathcal{G}$ is an amalgamation class, then there is a unique separable Banach space E, called the $\mathcal{G}$-Fraïssé limit*, and denoted by* $\mathrm{FLim}\,\mathcal{G}$*, such that $\overline{\mathcal{G}_E}^{\subseteq}$ is Λ_E-dense in* $\mathrm{Age}(E)$ *and E is* Fraïssé*, that is for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there is $\delta > 0$ such that the natural action* $\mathrm{Iso}(E) \curvearrowright \mathrm{Emb}_\delta(X,E)$ *is ε-transitive for every $X \in \mathrm{Age}(E)$ of dimension k (that is, given $\gamma, \eta \in \mathrm{Emb}_\delta(X,E)$ there is $g \in \mathrm{Iso}(E)$ such that $\|g \circ \eta - \gamma\| \leqslant \varepsilon$).*

b) The following are equivalent:

i) *$\mathcal{G}$ is hereditary amalgamation class that is d_{BM}-compact, that is, for every k, the collection of classes $[\mathtt{m}]$ of norms $\mathtt{m} \in \mathcal{N}_k$ such that $(\mathbb{F}^k, \mathtt{m}) \in \mathcal{G}_{\equiv}$ is a closed subset of $\mathcal{B}_k$.*
ii) *There is a unique separable Fraïssé Banach space E such that $\mathrm{Age}(E) \equiv \mathcal{G}$.*

This can be considered as the Banach space version of the Fraïssé correspondence of first order structures, that, for example, interprets several Random graphs (Rado, Henson graphs), Boolean algebras (the countable atomless one), or metric spaces (the rational Urysohn space) as Fraïssé limits. The known examples of families having the (SARP$^+$) are related to the p-norms:

- $\{\ell_p^n\}_n$ for all $1 \leqslant p \leqslant \infty$: For $p = 2$, this is a consequence of the fact, via the Kechris-Pestov-Todorcevic (KPT) correspondence (see [4, Theorem 2.12], [6, Theorem 5.10]), that the unitary group $\mathrm{Iso}(\ell_2)$ is extremely amenable, proved by M. Gromov and V. Milman [11], and the fact that $\{\ell_2^n\}_n$ is an amalgamation class (see for example [6, Example 2.4.]). The case $1 \leqslant p \neq 2 < \infty$ follows from the approximate Ramsey property of this class, proved in [6] and the result of G. Schechtman in [21] stating that $\{\ell_p^n\}_n$ are amalgamation classes. The case $p = \infty$ is proved in [4] (see also [2]) using the dual Ramsey Theorem.
- $\mathrm{Age}(L_p[0,1])$ for $p \notin 2\mathbb{N} + 4$: This is a byproduct of the extreme amenability of $\mathrm{Iso}(L_p[0,1])$, proved by T. Giordano and V. Pestov in [7], the (KPT) correspondence, and the fact that $\mathrm{Age}(L_p[0,1])$ is an amalgamation class, proved in [6]. On the other direction, when $p \in 2\mathbb{N}+4$, it is shown in [6, Proposition 2.10.] that $\mathrm{Age}(L_p[0,1])$ does not have the (SARP$^+$) because in these spaces there are finite dimensional isometric subspaces, one well complemented and the other badly complemented.
- $\mathcal{F} = \mathrm{Age}(C[0,1])$: This is proved in [4] (see also [2]), directly using injective envelopes and some approximations, or as a byproduct of the (SARP$^+$) of the family $\{\ell_\infty^n\}_n$ and the Kechris-Pestov-Todorcevic correspondence for Banach spaces.

The (SARP$^+$) characterizes norms on $\mathbb{F}^\infty$ that produce Ramsey factors.

Theorem 3.3. *Let $\mathtt{m}$ be a norm on $\mathbb{F}^\infty$, $E := (\mathbb{F}^\infty, \mathtt{m})$.*

a) *If $\mathrm{Age}(E)$ has the (SARP$^+$), then $\mathtt{m}$ produces Ramsey factors for compact colorings of full-rank matrices, Grassmannians and square matrices.*
b) *If $\mathtt{m}$ produces Ramsey factors for compact colorings of full rank matrices, $\mathrm{Age}(E)$ has the (SARP$^+$).*

To prove *b)* we will use the following.

Lemma 3.4. *Let $\mathtt{m}$ be a norm on $\mathbb{F}^\infty$ that produces Ramsey factors for compact colorings of full rank matrices, set $E := (\mathbb{F}^\infty, \mathtt{m})$. For every $k, m, r \in \mathbb{N}$, $\varepsilon > 0$, and $\lambda \in]1, \infty[$ there is some $n \in \mathbb{N}$ such that for every discrete coloring $c : M_{n,k}^k(\mathtt{m}; \lambda) \to r$ there is some $R \in \mathcal{E}_{n,m}(\mathtt{m})$ such that*

$$R \cdot B \in (c^{-1}(c(R \cdot A)))_{\partial_{(\mathbb{F}^k, \mathtt{m}), E}(\nu_{\mathbb{F}^k, E}(A), \nu_{\mathbb{F}^k, E}(B)) + \varepsilon} \text{ for every } A, B \in M_{m,k}^k(\mathtt{m}; {<}\lambda) \tag{8}$$

Proof. Fix the parameters $k, m, r \in \mathbb{N}$, ε, and λ. Let $n \in \mathbb{N}$ be the outcome of property *ii)* in Definition 2.7 when applied to k, m, $\varepsilon/2$, λ and the compact metric space $K := 2\lambda\,\mathrm{Ball}(\ell_\infty^r)$. We claim that n works. Fix $c : M_{n,k}^k(\mathtt{m}; \lambda) \to r$ and let $f : M_{n,k}^k(\mathtt{m}; \lambda) \to K$, $f(A) := (d(A, c^{-1}(j)))_{j<r}$, where for coherence we declare that $d(A, \varnothing) := 2\lambda$. It is clear that f is 1-Lipschitz, so there is some $R \in \mathcal{E}_{n,m}(\mathtt{m})$ and a 1-Lipschitz $\widetilde{f} : \mathcal{N}_k(E; {<}\lambda) \to K$ such that $d_K(\widetilde{f}(\nu_{\mathbb{F}^k,E}(A)), f(R \cdot A)) \leqslant \varepsilon/2$ for every $A \in M_{m,k}^k(\mathtt{m}; {<}\lambda)$. Fix $A, B \in M_{m,k}^k(\mathtt{m}; {<}\lambda)$. Then, $d_K(f(R \cdot A), f(R \cdot B)) \leqslant \partial_{(\mathbb{F}^k, \mathtt{m}), E}(\nu_{\mathbb{F}^k,E}(A), \nu_{\mathbb{F}^k,E}(B)) + \varepsilon$. Thus, if $j := c(R \cdot A)$, then the j^{th}-coordinate of $f(R \cdot A)$ is zero, hence, the j^{th}-coordinate of $c(R \cdot B)$ must satisfy that $d(R \cdot B, c^{-1}(j)) \leqslant \partial_{(\mathbb{F}^k, \mathtt{m}), E}(\nu_{\mathbb{F}^k,E}(A), \nu_{\mathbb{F}^k,E}(B)) + \varepsilon$, as desired. □

Proof of b) of Theorem 3.3. The proof of *a)* is more involved, and it will be done in several steps later. Let $\mathcal{F}$ be the collection of all normed spaces of the form $(\mathbb{F}^k, \mathtt{n})$ where $\mathtt{n} \in \mathcal{N}_k(E)$ is such that $\omega(\mathtt{n}, \|\cdot\|_1) \leqslant \log k$. Since the diameter of the Banach-Mazur compactum $\mathcal{B}_k$ is at most $\log k$, it follows that $\mathcal{F} \equiv \mathrm{Age}(E)$, that is, every finite dimensional $X \subseteq E$ has an isometric copy in $\mathcal{F}$. This means that the (SARP$^+$) of $\mathcal{F}$ and of $\mathrm{Age}(E)$ are equivalent. Moreover, we prove the following equivalent discrete version of the (SARP$^+$) (see [6, Proposition 5.9] and [4, Proposition 2.13] for a stable version of it):

For every k and $\varepsilon > 0$ there is a $\delta > 0$ such that for every $r \in \mathbb{N}$ and $X, Y \in \mathcal{F}$ with $\dim X = k$ there is $Z \in \mathcal{F}$ such that every discrete coloring $c : \mathrm{Emb}_\delta(X, Z) \to r$ has an ε-monochromatic set of the form $R \circ \mathrm{Emb}_\delta(X, Y)$ for some $R \in \mathrm{Emb}(Y, Z)$.

Fix a dimension k and $\varepsilon > 0$. Notice that the collection of spaces in $\mathcal{F}$ of dimension k is a ω-bounded set, so, by hypothesis, the metrics $\partial_{(\mathbb{F}^k, \mathtt{m}), E}$ and ω are uniformly equivalent on $\mathcal{M} := \{\mathtt{n} \in \mathcal{N}_k(E) \,:\, (\mathbb{F}^k, \mathtt{n}) \in \mathcal{F}\}$. Let $\delta > 0$ be such that if $\mathtt{n}, \mathtt{p} \in \mathcal{M}$ are such that $\omega(\mathtt{n}, \mathtt{p}) \leqslant \delta$, then $\partial_{(\mathbb{F}^k, \mathtt{m}), E}(\mathtt{n}, \mathtt{p}) < \varepsilon/2$. We claim that δ works. For suppose that $X = (\mathbb{F}^k, \mathtt{n}), Y = (\mathbb{F}^m, \mathtt{p}) \in \mathcal{F}$ are such that $\mathrm{Emb}_\delta(X, Y) \neq \varnothing$, and $r \in \mathbb{N}$. Let $m_0 \geqslant m$ and $C \in M_{m_0, m}^m$ be such that $\mathtt{p} = \nu_{\mathbb{F}^m, (\mathbb{F}^{m_0}, \mathtt{m})}(C)$, and let $\lambda > 1$ be such that

$$T_C \circ \mathrm{Emb}_\delta(X, Y) \subseteq \{T_B \,:\, B \in M_{m_0, k}^k(\mathtt{m}; {<}\lambda)\}.$$

We use Lemma 3.4 for the parameters $k, m_0, r + 1$, $\varepsilon/2$ and λ to find a corresponding n; set $Z := (\mathbb{F}^n, \mathtt{m})$. Now suppose that $c : \mathrm{Emb}_\delta(X, Z) \to r$. We define $\widehat{c} : M_{n,k}^k(\mathtt{m}; \lambda) \to r + 1$ by $\widehat{c}(A) = c(T_A)$ if $T_A \in \mathrm{Emb}_\delta(X, Z)$ and by $\widehat{c}(A) = r$ otherwise. Let $R \in \mathcal{E}_{n, m_0}(\mathtt{m})$ be such that (8) holds. Let $\gamma := T_R \circ T_C \in \mathrm{Emb}(Y, Z)$. We see that $\gamma \circ \mathrm{Emb}_\delta(X, Y)$ is ε-monochromatic for c:

Fix some auxiliary $\eta \in \mathrm{Emb}_\delta(X, Y)$, let $A \in M_{m_0, k}^k(\mathtt{m}; {<}\lambda)$ be such that $T_A = T_C \circ \eta$, and let $j := c(T_R \circ T_C \circ \eta) = \widehat{c}(R \cdot A)$. Now given $\xi \in \mathrm{Emb}_\delta(X, Y)$, let us see that $\gamma \circ \xi \in (c^{-1}(j))_\varepsilon$: let $B \in M_{m,k}^k(\mathtt{m}; {<}\lambda)$ be such that $T_B = T_C \circ \xi$, and set $\mathtt{q} := \nu_{\mathbb{F}^k, E}(B)$. Since $\eta \in \mathrm{Emb}_\delta(X, Y)$, it follows that $(1 + \delta)^{-1}\mathtt{n}(x) \leqslant \mathtt{m}(T_C \circ \eta(x)) = \mathtt{q}(x) \leqslant (1 + \delta)\mathtt{n}(x)$ for every $x \in \mathbb{F}^k$, and consequently, $\omega(\mathtt{n}, \mathtt{q}) \leqslant \delta$. Since $\mathtt{n} := \nu_{\mathbb{F}^k, E}(A)$, we obtain by the choice of $\delta > 0$ that $\partial_{(\mathbb{F}^k, \mathtt{m}), E}(\nu_{\mathbb{F}^k, E}(A), \nu_{\mathbb{F}^k, E}(B)) \leqslant \varepsilon/2$. This together with (8) gives that $R \cdot B \in (\widehat{c}^{-1}(j))_\varepsilon$, so there must be $D \in M_{n,k}^k(\mathtt{m}; \lambda)$ such that $\widehat{c}(D) = j$ and such that $\|T_D - \gamma \circ \xi\| = \|T_D - T_R \circ T_B\| = d_{\mathtt{m}}(D, R \cdot B) \leqslant \varepsilon$; since $j < r$, $T_D \in \mathrm{Emb}_\delta(X, Z)$, so $c(T_D) = \widehat{c}(D) = j$ and we are done. □

The proof of *a)* of Theorem 3.3 has two main parts. The first one (Theorem 3.8) uses the fact that if $\mathcal{F}$ has the (SARP$^+$) and it is hereditary, then the isometry group G of the Fraïssé limit $\mathrm{FLim}\,\mathcal{F}$ is extremely amenable with its topology of pointwise convergence. This property will be used as infinitary principles can be used to conclude in Corollary 3.11 and Theorem 3.12, via compactness arguments, the finitary ones (e.g. infinite vs finite Ramsey, Hindman vs Folkman theorem, etc.). The fixed point property of G will naturally provide abstract Ramsey factors that are G-quotients. The second part of the argument is to see that these G-quotients are in fact the desired Ramsey factors.

Recall that a topological group G is called *extremely amenable* when every continuous action of G on a compact Hausdorff space has a fixed point. There is a useful characterization of extreme amenability in terms of factors through quotients that we pass to explain.

Let (M, d) be a metric space, and let $G \curvearrowright M$ be a continuous action by isometries. We write $[p]_G$ to denote the closure of the G-orbit of $p \in M$, and $M /\!\!/ G$ to denote the space of closures of G-orbits of M. Since G acts by isometries the formula

$$\widetilde{d}^G([p], [q]) := \inf\{d_M(p_0, q_0) \,:\, p_0 \in [p],\ q_0 \in [q]\}$$

defines the quotient pseudometric induced by the quotient map $\pi : M \mapsto M /\!\!/ G$, and as we consider *closures* of orbits, $\widetilde{d}^G$ is a metric. It is easily seen that $\widetilde{d}^G$ is complete if d is complete.

Given a compact metric space (K, d_K), let $\mathrm{Lip}((M, d_M), (K, d_K))$ be the collection of all 1-Lipschitz colorings from M to K. With the topology of pointwise convergence the collection $\mathrm{Lip}((M, d_M), (K, d_K))$ is a compact space, which is metrizable when (M, d_M) is separable. The continuous action $G \curvearrowright (M, d_M)$ induces a natural continuous action $G \curvearrowright \mathrm{Lip}((M, d_M), (K, d_K))$, defined by setting $(g \cdot c)(p) := c(g^{-1} \cdot p)$ for every $c \in \mathrm{Lip}((M, d_M), (K, d_K))$ and $p \in M$. This is the aforementioned characterization (see [4]).

Proposition 3.5. *Suppose that G is a Polish group, and d_G is a left-invariant compatible metric on G. The following assertions are equivalent.*

i) G is extremely amenable.

ii) The left translation of G on (G, d_G) is finitely oscillation stable [19, Definition 1.1.11], that is, for every 1-Lipschitz coloring $c : G \to [0, 1]$ and every $F \subseteq G$ finite and $\varepsilon > 0$ there is some $g \in G$ such that $\mathrm{Osc}(c \upharpoonright g \cdot F) \leqslant \varepsilon$.

iii) For every action by isometries $G \curvearrowright M$ of G on a metric space (M, d_M), and for every 1-Lipschitz coloring $c : (M, d_M) \to (K, d_K)$ of (M, d_M), there exists a 1-Lipschitz coloring $\widehat{c} : M /\!\!/ G \to K$ such that for every finite $F \subseteq M$ and $\varepsilon > 0$ there is some $g \in G$ such that $d_K(c(p), \widehat{c}([p]_G)) < \varepsilon$ for every $p \in g \cdot F$.

iv) The same as iii) where F is compact.

Proof. The equivalence of *i)* and *ii)* can be found in [19, Theorem 2.1.11]. The implication *iii)⇒ii)* is immediate, since orbit space $G /\!\!/ G$ is one point. We now establish the implication *i)⇒iv)*: Fix a 1-Lipschitz $c : (M, d_M) \to (K, d_K)$. Let L be the closure of the G-orbit of c in $\mathrm{Lip}((M, d_M), (K, d_K))$. By the extreme amenability of G, there is some $c_\infty \in L$ such that $G \cdot c_\infty = \{c_\infty\}$, so we can define the quotient K-coloring $\widehat{c}([p]_G) := c_\infty(p)$. Let $F \subseteq M$ be compact, and let $D \subseteq F$ be a finite $\varepsilon/3$-dense subset of F. Since c_∞ is in the closure of the G-orbit of c, we can find $g \in G$ such that $d_K(c_\infty(p), (g^{-1} \cdot c)(p)) < \varepsilon/3$ for every $p \in D$. Fix $p \in F$ and let $q \in D$ be such that $d_M(p, q) < \varepsilon/3$. Both c_∞ and c are 1-Lipschitz, so it follows that

$$\begin{aligned} d_K(c_\infty(p), (g^{-1} \cdot c)(p)) \leqslant & d_K(c_\infty(p), c_\infty(q)) + d_K(c_\infty(q), (g^{-1} \cdot c)(q)) + \\ & + d_K((g^{-1} \cdot c)(p), (g^{-1} \cdot c)(q)) < 2 d_M(p, q) + \frac{\varepsilon}{3} < \varepsilon. \end{aligned}$$

Consequently, given $p \in F$, then $d_K(\widehat{c}([g \cdot p]_G), c(g \cdot p)) = d_K(c_\infty(p), c(g \cdot p)) < \varepsilon$. □

We apply this characterization to groups of linear isometries of a Banach space. Given two Banach spaces X and Y, recall that $\mathcal{L}(X, Y)$ is the Banach space of all bounded linear operators $T : X \to Y$, endowed with the operator norm $\|T\| := \sup_{\|x\|_X = 1} \|Tx\|_Y$. The *adjoint* $T^* : Y^* \to X^*$ of T, defined for $f \in Y^*$ as the function $T^*(f) : X \to \mathbb{F}$, $(T^*(f))(x) := f(T(x))$ for every $x \in X$, satisfies, as consequence of the Hahn-Banach Theorem, that $\|T^*\| = \|T\|$. Recall that when T is a finite rank operator, we have defined $\mathtt{r}_{-1}(T) := \min\{a \geqslant 0 : \mathrm{Ball}(TX) \subseteq aT(\mathrm{Ball}(X))\}$. Notice that $T^* : Y^* \to X^*$ is a finite-rank operator when T is finite-rank, and it holds that $\mathtt{r}_{-1}(T^*) = \mathtt{r}_{-1}(T)$. To see this, choose a basis $(y_j)_{j<k}$ of TX, and for each $j < k$, define $f_j : X \to TX$ for x as the j^{th}-coordinate of $T(x)$ in the basis $(y_l)_{l<k}$, i.e. by $T(x) = \sum_{j<k} f_j(x) y_j$. This decomposition of T is written as $T = \sum_{j<k} f_j \otimes y_j$, hence it follows that $T^* = \sum_{j<k} \delta_{y_j} \otimes f_j$, where $\delta_{y_j} : Y^* \to \mathbb{F}$ is the evaluation functional $f \mapsto f(y_j)$. Let us check the equality of corresponding $\mathtt{r}_{-1}$'s. Recall that the annihilator $Z^\perp$ of a closed subspace Z of X is the closed subspace of X^* consisting of the functionals of X that vanish on Z. Then it is a well-known fact that the canonical mappings $\theta_0 : Z^\perp \to (X/Z)^*$, $(\theta_0(f))([x]) := f(x)$ for $[x] \in X/Z$ and $f \in Z^\perp$, and $\theta_1 : Z^* \to X^*/Z^\perp$, $\theta_1(g) := [h]$, where $h \in X^*$ extends $g \in Z^*$ (by Hahn-Banach Theorem), are both surjective linear isometries (see for example [5, Proposition 2.7]). It is not difficult to see that $\widetilde{T^*} \circ \theta_1 = \theta_0^{-1} \circ (\widetilde{T})^*$, where $\widetilde{T} : X / \ker T \to TX$ and $\widetilde{T^*} : Y^* / \ker T^* \to T^*(Y^*)$ are

the corresponding quotient mappings of T and T^* (see the paragraph after Proposition 2.4). Consequently $\mathtt{r}_{-1}(T) = \|\widetilde{T}^{-1}\| = \|(\widetilde{T}^{-1})^*\| = \|((\widetilde{T})^*)^{-1}\| = \|((\widetilde{T})^*)^{-1} \circ \theta_0\| = \|\theta_1^{-1} \circ (\widetilde{T^*})^{-1}\| = \|(\widetilde{T^*})^{-1}\| = \mathtt{r}_{-1}(T^*)$.

Notice also that when X is finite dimensional, $T : X \to Y$ is 1-1 and $\|T\| = \mathtt{r}_{-1}(T) = 1$ corresponds to T being an isometric embedding. The collection of such maps is denoted by $\mathrm{Emb}(X, Y)$.

In general, given normed spaces F and G, let $\mathcal{L}_\lambda(F, G)$, $\mathcal{L}_{<\lambda}(F, G)$, be the set of all $T \in \mathcal{L}(F, G)$ with finite dimensional image such that $\|T\|, \mathtt{r}_{-1}(T) \leqslant \lambda$, resp. $< \lambda$. Let $\mathcal{L}^k(F, G)$ be the set of all $T \in \mathcal{L}(F, G)$ whose image is k-dimensional, and let $\mathcal{L}^k_\lambda(F, G) = \mathcal{L}_\lambda(F, G) \cap \mathcal{L}^k(F, G)$, $\mathcal{L}^k_{<\lambda}(F, G) = \mathcal{L}_{<\lambda}(F, G) \cap \mathcal{L}^k(F, G)$. Let $\mathcal{L}^{k,\mathrm{w}^*}(F^*, F)$ be the metric space of operators $T \in \mathcal{L}^k(F^*, F)$ such that T admits a full rank decomposition, i.e., when $T = T_0 \circ T_1^*$ for some $T_0 \in \mathcal{L}^k((\mathbb{F}^k)^*, F)$ and $T_1 \in \mathcal{L}^k(\mathbb{F}^k, F)$. Let $\mathcal{L}^{k,\mathrm{w}^*}_\lambda(F^*, F) := \mathcal{L}^{k,\mathrm{w}^*}(F^*, F) \cap \mathcal{L}_\lambda(F^*, F)$.

Notice that when X is finite dimensional every linear operator $T : X^* \to X$ of rank k has a such factorization $T = T_0 \circ T_1^*$, as we pointed out before for the full rank decomposition of square matrices of rank k. For a Banach space X, a finite-rank operator T has this factorization exactly when T is a w*-to-norm continuous linear operators from X^* to X. Recall that the w^*-topology on X^* is the topology induced by the inclusion $X^* \subseteq C_p(X, \mathbb{F})$, the space of continuous functions from X to $\mathbb{F}$ with the topology of pointwise convergence. Suppose that T is w*-to-norm continuous. Since the dual unit ball $\mathrm{Ball}(X^*)$ is w^*-compact by the Banach-Alaoglu Theorem, it follows that $T(\mathrm{Ball}(X^*))$ is norm-compact, and this happens only if $T(X^*)$ is finite dimensional. As we have seen before, there is a linearly independent sequence $(x_j)_{j<k}$ in X and functionals $\{\varphi_j\}_{j<k} \subseteq (X^*)^*$ such that $T = \sum_{j<k} \varphi_j \otimes x_j$. Since we are assuming that T is w^*-to-norm continuous, it follows that each $\varphi_j = \delta_{y_j}$ for some $y_j \in X$ (see for example [5, Theorem 3.16]). It is easy to see that $T = T_0 \circ T_1^*$, where $T_0(u_j^*) := x_j$ and $T_1(u_j) := y_j$ for every $j < k$ and $(u_j)_{j<k}$ and $(u_j^*)_{j<k}$ are the unit bases of $\mathbb{F}^k$ and $(\mathbb{F}^k)^*$, respectively. Conversely, if $T_0(u_j) = x_j$, and $T_1(u_j) = y_j$ for every $j < k$ then $T = T_0 \circ T_1^* = \sum_{j<k} \delta_{y_j} \otimes x_j$ is w^*-to-norm continuous. From this characterization is easy to see that the dual operator of a w*-to-norm continuous finite-rank operator $T : X^* \to X$ is again a w*-to-norm continuous finite-rank operator, since if $T = \sum_{j<k} \delta_{y_j} \otimes x_j$, then $T^* = \sum_{j<k} \delta_{x_j} \otimes y_j$.

With a slight abuse of notation, let $\nu^2_{k,X} : \mathcal{L}^{k,\mathrm{w}^*}(X^*, X) \to \mathcal{D}_k(X)$ be defined by $\nu^2_{k,X}(T) := [\nu_{(\mathbb{F}^k)^*,X}(T_0), \nu_{\mathbb{F}^k,X}(T_1)]$, where $T = T_0 \circ T_1^*$ is an arbitrary decomposition with $T_0 \in \mathcal{L}^k((\mathbb{F}^k)^*, X)$ and $T_1 \in \mathcal{L}^k(\mathbb{F}^k, X)$.

Definition 3.6. Let $\mathrm{Iso}(E) \curvearrowright \mathcal{L}(X, E)$ be the canonical action by isometries $g \cdot T := g \circ T$, $\mathrm{Iso}(E)^2 \curvearrowright \mathcal{L}^{k,\mathrm{w}^*}(E^*, E)$ be the canonical action by isometries $(g, h) \cdot T := g \circ T \circ h^*$ for $(g, h) \in \mathrm{Iso}(E)^2$ and $T \in \mathcal{L}^{k,\mathrm{w}^*}(E^*, E)$, and let $\mathrm{Iso}(E) \curvearrowright \mathrm{Gr}(k, E)$ be the canonical action by isometries $g \cdot V := g(V)$.

Note that $\mathcal{L}^k_\lambda(X, E)$, $\mathcal{L}^k_{<\lambda}(X, E)$, and $\mathcal{L}^{k,\mathrm{w}^*}_\lambda(X^*, X)$, $\mathcal{L}^{k,\mathrm{w}^*}_{<\lambda}(X^*, X)$ are $\mathrm{Iso}(E)$-closed and $\mathrm{Iso}(E)^2$-closed, respectively. The next readily follows from Proposition 3.5, using the fact that if G is extremely amenable, then G^2 is also extremely amenable (see [19, Corollary 6.2.10.]). Its asymptotic version in Corollary 3.11 together with Theorem 3.12 will prove *a)* of Theorem 3.3.

Lemma 3.7. *Suppose that X, E are Banach spaces, X is finite-dimensional, and* $\mathrm{Iso}(E)$ *is extremely amenable. Let $k \in \mathbb{N}$, $\varepsilon > 0$, $1 \leqslant \lambda$, $Y \in \mathrm{Age}(E)_\equiv$, and let (K, d_K) be a compact metric space.*

a) For every 1-Lipschitz coloring $c : (\mathcal{L}^k_{<\lambda}(X, E), d_{X,E}) \to (K, d_K)$ there is $R \in \mathrm{Emb}(Y, E)$ such that the quotient map $\pi : \mathcal{L}^k_{<\lambda}(X, E) \to \mathcal{L}^k_{<\lambda}(X, E) /\!\!/ \mathrm{Iso}(E)$ is an ε-factor of c in $R \circ \mathcal{L}^k_{<\lambda}(X, Y)$; that is, there is some 1-Lipschitz coloring $\widetilde{c} : \mathcal{L}^k_{<\lambda}(X, E) /\!\!/ \mathrm{Iso}(E) \to K$ such that $d_K(\widetilde{c}(\pi(T)), c(R \circ T)) \leqslant \varepsilon$ for every $T \in \mathcal{L}^k_{<\lambda}(X, Y)$

b) For every 1-Lipschitz coloring $c : (\mathrm{Gr}(k, E), \Lambda_E) \to (K, d_K)$ *there exists* $V \in \mathrm{Gr}(\dim Y, E)$ *with* $(V, \|\cdot\|_E)$ *is isometric to* Y *such that the quotient map* $\pi : \mathrm{Gr}(k, E) \to \mathrm{Gr}(k, E) /\!\!/ \mathrm{Iso}(E)$ *is an* ε*-factor of* c *in* $\mathrm{Gr}(k, V)$*; that is, there is some 1-Lipschitz coloring* $\widetilde{c} : \mathrm{Gr}(k, E) /\!\!/ \mathrm{Iso}(E) \to K$ *such that* $d_K(\widetilde{c}(\pi(W)), c(W)) \leqslant \varepsilon$ *for every* $W \in \mathrm{Gr}(k, V)$.

c) For every 1-Lipschitz coloring $c : (\mathcal{L}^{k,\mathrm{w}^*}_{<\lambda}(E^*, E), d_{E^*,E}) \to (K, d_K)$ *there are* $R_0, R_1 \in \mathrm{Emb}(Y, E)$ *such that the quotient map* $\pi : \mathcal{L}^{k,\mathrm{w}^*}_{<\lambda}(E^*, E) \to \mathcal{L}^{k,\mathrm{w}^*}_{<\lambda}(E^*, E) /\!\!/ \mathrm{Iso}(E)^2$ *is an* ε*-factor of* c *in* $R_0 \circ \mathcal{L}^{k,\mathrm{w}^*}_{<\lambda}(Y^*, Y) \circ R_1^*$*; that is, there is some 1-Lipschitz coloring* $\widetilde{c} : \mathcal{L}^{k,\mathrm{w}^*}_{<\lambda}(E^*, E) /\!\!/ \mathrm{Iso}(E)^2 \to K$ *such that* $d_K(\widetilde{c}(\pi(T)), c(R_0 \circ T \circ R_1^*)) \leqslant \varepsilon$ *for every* $T \in \mathcal{L}^{k,\mathrm{w}^*}_{<\lambda}(Y^*, Y)$. □

The relationship between the (SARP$^+$) of a class of finite dimensional normed spaces and the extreme amenability of the isometry group of its Fraïssé limit is the next mix of the Fraïssé and the Kechris-Pestov-Todorcevic correspondences, that we took from [6, Corollary 5.11].

Theorem 3.8. *If* $\mathcal{G}$ *is an hereditary family with the* (SARP$^+$)*, then the Banach-Mazur closure of* $\mathcal{G}$ *also has the* (SARP$^+$) *and the Fraïssé limit* $\mathrm{FLim}\,\mathcal{G}$ *is a Fraïssé Banach space whose isometry group is extremely amenable with its strong operator topology.* □

Definition 3.9. Given a normed space $E := (\mathbb{F}^\infty, \mathtt{m})$ such that $\mathrm{Age}(E)$ is an amalgamation class, we write $\widehat{E}$ to denote, the Fraïssé limit $\mathrm{FLim}\,\mathrm{Age}(E)$.

The following is an important properties of Fraïssé limits (see [6, Proposition 2.13] for a proof).

Proposition 3.10. *Suppose that* $E = (\mathbb{F}^\infty, \mathtt{m})$ *is such that* $\mathrm{Age}(E)$ *is an amalgamation class and* F *is a separable Banach space. The following are equivalent:*

a) $\mathrm{Age}(F) \preceq \mathrm{Age}(E)$

b) F *can be isometrically embedded into* E.

In particular, E *can be isometrically embedded.*

We will use the following notation. Given a normed space $E = (\mathbb{F}^\infty, \mathtt{m})$ and $n \in \mathbb{N}$, we set $E_n := (\langle u_j \rangle_{j<n}, \mathtt{m})$, and given a normed space X we write $\mathrm{Age}(X)_\mathtt{m}$ to denote the collection of subspaces of X isometric to some E_n. The following is the asymptotic version of Lemma 3.7.

Corollary 3.11. *Suppose that* $E = (\mathbb{F}^\infty, \mathtt{m})$ *is such that* $\mathrm{Age}(E)$ *has the* (SARP$^+$)*, and* $\mathrm{Age}(\widehat{E})_\mathtt{m}$ *is an amalgamation class. Let* $k, m \in \mathbb{N}$, $\varepsilon > 0$ *and* $\lambda \geqslant 1$*. Given a compact metric space* (K, d_K) *there is* $X \in \mathrm{Age}(\widehat{E})_\mathtt{m}$ *such that*

1) for every 1-Lipschitz coloring $c : \mathcal{L}^k_\lambda(E_k, X) \to K$ *there is* $R \in \mathrm{Emb}(E_m, X)$ *such that the quotient map* $\pi : \mathcal{L}^k_{<\lambda}(E_k, \widehat{E}) \to \mathcal{L}^k_{<\lambda}(E_k, \widehat{E}) /\!\!/ \mathrm{Iso}(\widehat{E})$ *is an* ε*-factor of* c *in* $R \circ \mathcal{L}^k_{<\lambda}(E_k, E_m)$*;*

2) for every 1-Lipschitz coloring $c : (\mathrm{Gr}(k, X), \Lambda_X) \to K$ *there is* $V \in \mathrm{Gr}(m, X) \cap \mathrm{Age}(\widehat{E})_\mathtt{m}$ *such that the quotient map* $\pi : \mathrm{Gr}(k, \widehat{E}) \to \mathrm{Gr}(k, \widehat{E}) /\!\!/ \mathrm{Iso}(\widehat{E})$ *is an* ε*-factor of* c *in* $\mathrm{Gr}(k, V)$*;*

3) for every 1-Lipschitz coloring $c : \mathcal{L}^{k,\mathrm{w}^*}_\lambda(X^*, X) \to K$ *there are* $R_0, R_1 \in \mathrm{Emb}(E_m, X)$ *such that the quotient map* $\pi : \mathcal{L}^{k,\mathrm{w}^*}_{<\lambda}(\widehat{E}^*, \widehat{E}) \to \mathcal{L}^{k,\mathrm{w}^*}_{<\lambda}(\widehat{E}^*, \widehat{E}) /\!\!/ \mathrm{Iso}(\widehat{E})^2$ *is an* ε*-factor of* c *in* $R_0 \circ \mathcal{L}^{k,\mathrm{w}^*}_{<\lambda}(E_m^*, E_m) \circ R_1^*$.

Proof. Suppose for the sake of contradiction, that, for some parameters $k, m \in \mathbb{N}$, $\varepsilon > 0$ and $\lambda \geqslant 1$, and some compact space (K, d_K), there is no such $X \in \mathrm{Age}(\widehat{E})_\mathtt{m}$ satisfying *1), 2)* or *3)*. For each $X \in \mathrm{Age}(\widehat{E})$, and each $\gamma > 0$, let $A_{X,\gamma}$ be the collection of all $Y \in \mathrm{Age}(\widehat{E})_\mathtt{m}$ such that there is some $\widetilde{X} \subseteq Y$ with $\dim \widetilde{X} = \dim X$ and such that $\Lambda_{\widehat{E}}(X, \widetilde{X}) \leqslant \gamma$. In other words, $A_{X,\gamma}$ consists of all finite dimensional $Y \subseteq \widehat{E}$ isometric to some E_n that "ε-almost" contain X. It is easy to see that $\{A_{X,\gamma}\}_{X,\gamma}$ is a family of subsets of $\mathrm{Age}(\widehat{E})_\mathtt{m}$ with the finite intersection property, so let $\mathcal{U}$ be a non principal ultrafilter on $\mathrm{Age}(\widehat{E})_\mathtt{m}$ containing all $A_{X,\gamma}$. Since $\mathcal{U}$ is an ultrafilter there is $j = 1, 2, 3$ such that the set $B_j := \{X \in \mathrm{Age}(\widehat{E})_{\mathtt{m}} : X$ does not satisfy $j)\}$ belongs to $\mathcal{U}$. Suppose that $j = 1$. For each $X \in B_1$ there exists a 1-Lipschitz coloring $c_X : \mathcal{L}^k_\lambda(E_k, X) \to K$ providing a counterexample. For each $T \in \mathcal{L}^k_{<\lambda}(E_k, \widehat{E})$, let $c(T) \in K$ be defined as follows. We say that $c(T) = p \in K$ if and only if for every $\gamma > 0$ one has that

$$C_\gamma(T) := \{X \in B_1 : T \in (\mathcal{L}^k_\lambda(E_k, X))_\gamma \text{ and } d_K(p, c_X\left((T)_\gamma \cap \mathcal{L}^k_\lambda(E_k, X)\right)) \leqslant \gamma\} \in \mathcal{U},$$

where $(T)_\gamma$ is the γ-fattening of T with respect to the norm-metric. This is well defined: fix $\gamma > 0$, and let $K = \bigcup_{i\in I} V_i^{(\gamma)}$ be a finite open covering by d_K-balls of radius γ. Let $0 < \delta = \delta_\gamma \leqslant \varepsilon$ be the Lebesgue number associated to this covering, that is, δ satisfies that every $C \subseteq K$ of diameter at most δ is included in one of the balls of the covering. Notice that for small enough δ', if $X \in A_{T(E_k),\delta'}$, then $T \in (\mathcal{L}^k_\lambda(E_k, X))_{\delta/2}$: for suppose that $X \in A_{T(E_k),\delta'}$, and let $Z \subseteq X$ be such that $\Lambda_{\widehat{E}}(T(E_k), Z) \leqslant \delta'$. For each $j < k$, choose $x_j \in \mathrm{Ball}(Z) \subseteq \mathrm{Ball}(X)$ such that $\|T(u_j)/\|T(u_j)\| - x_j\|_{\widehat{E}} \leqslant \gamma$, and define $S : E_k \to X$ by $S(u_j) := \|T(u_j)\| \cdot x_j$, $j < k$. Then, given $x := \sum_{j<k} a_j u_j \in E_k$, $\|S(x) - T(x)\| \leqslant \sum_{j<k} |a_j|\|T(u_j) - x_j\| \leqslant \Gamma \cdot \mathtt{m}(x)\delta'\|T\|$, where $\Gamma := \|\mathrm{Id}\|_{(E_k,\mathtt{m}),(E_k,\|\cdot\|_1)}$. Hence, $\|S - T\| \leqslant C\delta'\|T\|$, and $\|S\| \leqslant (1 + \Gamma \cdot \delta')\|T\|$, while $\|S(x)\| \geqslant \|T(x)\| - \|T(x) - S(x)\| \geqslant ((1/\mathtt{r}_{-1}(T)) - \Gamma \cdot \|T\| \cdot \delta')\mathtt{m}(x)$ for every $x \in E_k$. Since we are assuming that $\|T\|, \mathtt{r}_{-1}(T) < \lambda$, for small enough δ' we can have $\|S - T\| \leqslant \delta/2$, $\|S\| \leqslant \lambda$ and $(1/\mathtt{r}_{-1}(T)) - \Gamma \cdot \|T\| \cdot \delta' \geqslant 1/\lambda$, i.e. $\mathtt{r}_{-1}(S) \leqslant \lambda$. Since c_X is 1-Lipschitz, the d_K-diameter of $c_X\left((T)_\varepsilon \cap \mathcal{L}^k_\lambda(E_k, X)\right)$ is at most 2ε for every $\varepsilon > 0$. This means that for $\delta' < \delta/2$,

$$A_{T(E_k),\delta'} \cap B_1 \subseteq \bigcup_{i\in I} \{X \in B_1 : \text{ s. t. } T \in (\mathcal{L}^k_\lambda(E_k, X))_{\frac{\delta}{2}} \text{ and } c_X\left((T)_{\frac{\delta}{2}} \cap \mathcal{L}^k_\lambda(E_k, X)\right) \subseteq V_i^{(\gamma)}\}.$$

Since $A_{T(E_k),\delta'} \cap B_1 \in \mathcal{U}$ and I is finite, there is some $i_\gamma \in I$ such that

$$\{X \in B_1 : \text{such that } T \in (\mathcal{L}^k_\lambda(E_k, X))_{\frac{\delta}{2}} \text{ and } c_X\left((T)_{\frac{\delta}{2}} \cap \mathcal{L}^k_\lambda(E_k, X)\right) \subseteq V_{i_\gamma}^{(\gamma)}\} \in \mathcal{U}.$$

It follows that the family of closed balls $\{\overline{V_{i_\gamma}^{(\gamma)}}\}_{\gamma>0}$ has the finite intersection property, so since K is compact, $\bigcap_{\gamma>0} \overline{V_{i_\gamma}^{(\gamma)}} \neq \emptyset$, and this intersection must be a single point p because each closed ball $\overline{V_{i_\gamma}^{(\gamma)}}$ has radius γ. Such p is such that $c(T) = p$ by definition.

It is easy to see that c is a 1-Lipschitz coloring. Let $\pi : \mathcal{L}^k_{<\lambda}(E_k, \widehat{E}) \to \mathcal{L}^k_{<\lambda}(E_k, \widehat{E}) /\!\!/ \mathrm{Iso}(\widehat{E})$ be the quotient mapping. By Lemma 3.7 there exist $S \in \mathrm{Emb}(E_m, \widehat{E})$ and a 1-Lipschitz coloring $\widehat{c} : (\mathcal{L}^k_{<\lambda}(E_k, \widehat{E}) /\!\!/ \mathrm{Iso}(\widehat{E}), \widehat{d}) \to (K, d_K)$ such that $d_K(c(S \circ T), \widehat{c}(\pi(T)) \leqslant \varepsilon/2$ for every $T \in \mathcal{L}^k_{<\lambda}(E_k, E_m)$. Since $\mathrm{Age}(\widehat{E})_{\mathtt{m}}$ is an amalgamation class and since $\{A_{X,\gamma}\}_{X,\gamma} \subseteq \mathcal{U}$, we obtain that for $\gamma > 0$, the set

$$D_\gamma := \{Y \in \mathrm{Age}(\widehat{E})_{\mathtt{m}} : \text{ there is } S_Y \in \mathrm{Emb}(E_m, Y) \text{ such that } \|S_Y - S\| \leqslant \gamma\} \in \mathcal{U}$$

For a fixed $\gamma > 0$, let $F \subseteq \mathcal{L}^k_{<\lambda}(E_k, E_m)$ be a finite γ-dense subset of it. Observe that if $Y \in D_\gamma \cap \bigcap_{T\in F} C_\gamma(S\circ T)$, then for each $T \in F$ there is $\gamma_T \in \mathcal{L}^k_\lambda(E_k, Y)$ such that $\|\gamma_T - S\circ T\| \leqslant \gamma$, $d_K(c(S\circ T), c_Y(\gamma_T)) \leqslant \gamma$, and there is some $S_Y \in \mathrm{Emb}(E_m, Y)$ such that $\|S\circ T - S_Y \circ T\| \leqslant \gamma\|T\|$ for every $T \in F$; consequently, $d_K(c_Y(S_Y\circ T), c(S\circ T)) \leqslant \gamma(2 + \|T\|) \leqslant \gamma(2 + \lambda)$ for every $T \in F$. Since F is γ-dense, given $T \in \mathcal{L}^k_{<\lambda}(E_k, E_m)$, choose $T' \in F$ such that $\|T - T'\| \leqslant \gamma$, and using that both c and c_Y are 1-Lipschitz, and that S and S_Y are isometric embeddings, we get that $d_K(c_Y(S_Y \circ T), c(S \circ T)) \leqslant \gamma(4 + \lambda)$. γ is arbitrary, so we obtain that

$$\{Y \in \mathrm{Age}(\widehat{E})_{\mathtt{m}} : \text{there is } S_Y \in \mathrm{Emb}(E_m, Y) \text{ with } \max_{T\in\mathcal{L}^k_{<\lambda}(E_k,E_m)} d_K(c_Y(S_Y \circ T), c(S \circ T)) \leqslant \frac{\varepsilon}{2}\}$$

belongs to $\mathcal{U}$. Y belonging to the previous set, S_Y and $\widehat{c}$ contradicts the assumption that c_Y is a counterexample, because given $T \in \mathcal{L}^k_{<\lambda}(E_k, E_m)$ we have that $\pi(T) = \pi(S_Y \circ T)$, hence it follows that

$$d_K(\widehat{c}(\pi(T)), c_Y(S_Y \circ T)) = d_K(\widehat{c}(\pi(S_Y \circ T)), c_Y(S_Y \circ T)) \leqslant d_K(\widehat{c}(\pi(S \circ T)), c(S \circ T)) + \\ + d_K(c_Y(S_Y \circ T), c(S \circ T)) \leqslant \varepsilon$$

The cases $j = 2, 3$ are proved similarly, so we leave the details to the reader. □

3.1. **Orbit spaces for Fraïssé Banach spaces.** We see that the orbit spaces considered in Corollary 3.11 *1*, *2*, and *3*, are homeomorphic to $\mathcal{N}_k(\widehat{E})$, $\mathcal{B}_k(\widehat{E})$ and $\mathcal{D}_k(\widehat{E})$, respectively. We also show that the $\widehat{E}$-extrinsic metrics extend the corresponding E-extrinsic ones, finishing the proof of Theorem 3.3 *a)*.

Theorem 3.12. *Suppose $E = (\mathbb{F}^\infty, \|\cdot\|_E)$ is such that* $\mathrm{Age}(E)$ *is an amalgamation class, X is a finite dimensional normed space and $k \in \mathbb{N}$. Then,*

a) The quotient mapping $\widehat{\nu}_{X,E} : \mathcal{L}^k(X, \widehat{E}) /\!\!/ \mathrm{Iso}(\widehat{E}) \to \mathcal{N}_X(\widehat{E})$ is an homeomorphism that is a uniform homeomorphism when restricted to $A \subseteq \mathcal{L}^k(X, \widehat{E}) /\!\!/ \mathrm{Iso}(\widehat{E})$ such that $\widehat{\nu}_{X,E}(A) \subseteq \mathcal{N}_X(\widehat{E})$ is ω-bounded. Consequently, $\partial_{X,\widehat{E}}$ is a compatible metric on $\mathcal{N}_X(\widehat{E})$ that is uniformly equivalent to ω on ω-bounded sets. Moreover, $\partial_{X,E} = \partial_{X,\widehat{E}}$ on $\mathcal{N}_X(E)$, and $\mathcal{N}_X(E)$ is dense in $\mathcal{N}_X(\widehat{E})$ and

b) The quotient mapping $\widetilde{\tau}_{k,\widehat{E}} : \mathrm{Gr}(k, \widehat{E}) /\!\!/ \mathrm{Iso}(\widehat{E}) \to \mathcal{B}_k(\widehat{E})$ is a uniform homeomorphism, and consequently, $\gamma_{k,\widehat{E}}$ and d_{BM} are uniformly equivalent on $\mathcal{B}_k(\widehat{E})$. Moreover $\gamma_{k,\widehat{E}} = \gamma_{k,E}$ on $\mathcal{B}_k(E)$, and $\mathcal{B}_k(E)$ is dense in $\mathcal{B}_k(\widehat{E})$.

c) The quotient mapping $\widetilde{\nu^2}_{k,\widehat{E}} : \mathcal{L}^{k,\mathrm{w}^*}((\widehat{E})^*, \widehat{E}) /\!\!/ \mathrm{Iso}(\widehat{E})^2 \to \mathcal{D}_k(\widehat{E})$ is an homeomorphism then is a uniform homeomorphism when restricted to $A \subseteq \mathcal{L}^{k,\mathrm{w}^*}((\widehat{E})^*, \widehat{E}) /\!\!/ \mathrm{Iso}(\widehat{E})^2$ such that $\widehat{\nu}^2_{k,E}(A) \subseteq \mathcal{D}_k(\widehat{E})$ is $\widetilde{\omega}_2$-bounded. Consequently, $\mathfrak{d}_{k,\widehat{E}}$ is a compatible metric on $\mathcal{D}_k(\widehat{E})$ that is uniformly equivalent to $\widetilde{\omega}_2$ on $\widetilde{\omega}_2$-bounded sets. Moreover, $\mathfrak{d}_{k,E} = \mathfrak{d}_{k,\widehat{E}}$ on $\mathcal{D}_k(E)$, and $\mathcal{D}_k(E)$ is dense in $\mathcal{D}_k(\widehat{E})$.

Proof. a): Suppose that $\dim X = k$. Recall that we consider $\mathcal{N}_X$ with its natural topology of pointwise convergence. The mapping $\nu_{X,\widehat{E}} : \mathcal{L}^k(X, \widehat{E}) \to \mathcal{N}_X(\widehat{E})$ is continuous, because the convergence in norm implies pointwise convergence. We see that $\nu_{X,\widehat{E}}(T) = \nu_{X,\widehat{E}}(U)$ if and only if $[T] = [U]$, that is, when the closed orbits of T and U are equal. The reverse implication is clear; now suppose that $\nu_{X,\widehat{E}}(T) = \nu_{X,\widehat{E}}(U)$. Let $Y := T(X)$ be endowed with the $\widehat{E}$-norm, and let $\theta : Y \to X$ be the inverse of $T : X \to Y$. Then $U \circ \theta \in \mathrm{Emb}(Y, \widehat{E})$; so, given $\varepsilon > 0$, there is a global isometry α of $\widehat{E}$ such that $\|U \circ \theta - \alpha \upharpoonright Y\| \leqslant \varepsilon$, or equivalently, $\|U - \alpha \circ T\| \leqslant \varepsilon$. Since ε is arbitrary, we obtain that $U \in [T]$. We show that the quotient mapping $\widetilde{\nu}_{X,\widehat{E}} : \mathcal{L}^k(X, \widehat{E}) /\!\!/ \mathrm{Iso}(\widehat{E}) \to \mathcal{N}_X(\widehat{E})$ is a homeomorphism. Suppose that $(\mathtt{m}_j)_j$ is a converging sequence in $\mathcal{N}_X(\widehat{E})$ with limit $\mathtt{m} \in \mathcal{N}_X(\widehat{E})$. For each j, let $T_j \in \mathcal{L}^k(X, \widehat{E})$ be such that $\nu_{X,\widehat{E}}(T_j) = \mathtt{m}_j$, and let $T \in \mathcal{L}^k(X, \widehat{E})$ be such that $\nu_{X,\widehat{E}}(T) = \mathtt{m}$.

Claim 3.12.1. $([T_j])_j$ is a Cauchy sequence.

Notice that it follows from this, and the fact that the quotient metric $\widetilde{d}_{X,\widehat{E}}$ is complete (here we use that X and $\widehat{E}$ are Banach spaces), that $([T_j])_j$ converges to some $[U]$; by the continuity of $\nu_{X,\widehat{E}}$ we have that $\nu_{X,\widehat{E}}(U) = \mathtt{m} = \nu_{X,\widehat{E}}(T)$, so $([T_j])_j$ converges to $[U] = [T]$. Also, suppose that A is a bounded subset of $\mathcal{N}_X(\widehat{E})$. The space $\widehat{E}$ is Fraïssé, so its age $\mathrm{Age}(\widehat{E})$ is ω-closed in $\mathcal{N}_X$ (because $\mathcal{B}_X(\widehat{E})$ is closed in $\mathcal{B}_X$; see Theorem 3.2 *b)*). Hence the closure $\overline{A}$ is compact and it is included in $\mathcal{N}_X(\widehat{E})$. Hence, the restriction $\widetilde{\nu}_{X,\widehat{E}} : \widetilde{\nu}^{-1}_{X,\widehat{E}}(\overline{A}) \to \overline{A}$ is a uniform homeomorphism.

Let us prove the previous claim.

Proof of Claim: Set $Y := T(X)$, normed as subspace of $\widehat{E}$, and let $\theta : Y \to X$ be the inverse isometry of $T : X \to Y$, and fix $\varepsilon > 0$; since $\widehat{E}$ is Fraïssé, there is some $\delta > 0$ such that the canonical action $\mathrm{Iso}(\widehat{E}) \curvearrowright \mathrm{Emb}_\delta(Y, \widehat{E})$ is ε-transitive; let j_0 be such that $T_j \circ \theta \in \mathrm{Emb}_\delta(Y, \widehat{E})$ for every $j \geqslant j_0$; this means that if $j_1, j_2 \geqslant j_0$, then there is $\alpha \in \mathrm{Iso}(\widehat{E})$ such that $\|T_{j_1} - \alpha \circ T_{j_2}\| = \|T_{j_1} \circ \theta - \alpha \circ T_{j_2} \circ \theta\| \leqslant \varepsilon$, hence $\widetilde{d}_{X,\widehat{E}}([T_{j_1}], [T_{j_2}]) \leqslant \varepsilon$. □

Since $\partial_{X,\widehat{E}}(\widetilde{\nu}_{X,\widehat{E}}(T), \widetilde{\nu}_{X,\widehat{E}}(U)) = \widetilde{d}_{X,\widehat{E}}([T], [U])$, we obtain that $\partial_{X,\widehat{E}}$ is compatible metric on $\mathcal{N}_X(\widehat{E})$ this is uniformly equivalent to ω on ω-bounded sets.

We see now that $\partial_{X,E} = \partial_{X,\widehat{E}}$ on $\mathcal{N}_X(E)^2$. Since E is isometrically embedded into $\widehat{E}$ (see Proposition 3.10), we have that $\partial_{X,\widehat{E}}(\cdot,\cdot) \leqslant \partial_{X,E}(\cdot,\cdot)$ on $\mathcal{N}_X(E)^2$.

Claim 3.12.2. Suppose that for each $j < n$ X_j is a finite dimensional normed space and $T_j \in \mathcal{L}^{\dim X}(X_j, \widehat{E})$ is such that $\nu_{X_j,\widehat{E}}(T_j) \in \mathcal{N}_X(E)$. For every $\varepsilon > 0$ there is some $V \in \mathrm{Age}(E)_{\widehat{E}}$, and $T_j' \in \mathcal{L}^{\dim X_j}(X_j, V)$ for $j < n$, and $g \in \mathrm{Iso}(\widehat{E})$ such that $\nu_{X_j,E}(T_j') = \nu_{X_j,\widehat{E}}(T_j)$ and $\|g \circ T_j - T_j'\|_{X_j,\widehat{E}} \leqslant \varepsilon$ for every $j < n$.

From the claim, given $T \in \mathrm{Emb}((X, \mathtt{m}), \widehat{E})$, $U \in \mathrm{Emb}((X, \mathtt{n}), \widehat{E})$, and $\varepsilon > 0$, choose $V \in \mathrm{Age}(E)_{\widehat{E}}$, $T' \in \mathrm{Emb}((X, \mathtt{m}), V)$, $U' \in \mathrm{Emb}((X, \mathtt{m}), V)$ and $g \in \mathrm{Iso}(\widehat{E})$ as in the claim. Then,

$$\|T' - U'\|_{X,V} = \|T' - U'\|_{X,\widehat{E}} \leqslant \|T' - g\circ T\|_{X,\widehat{E}} + \|g\circ T - g\circ U\|_{X,\widehat{E}} + \|U' - g\circ U\|_{X,\widehat{E}} \leqslant 2\varepsilon + \|T - U\|_{X,\widehat{E}}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain that $\partial_{X,E}(\mathtt{m}, \mathtt{n}) \leqslant \|T - U\|_{X,\widehat{E}}$, and since T, U are arbitrary with $\nu_{X,\widehat{E}}(T) = \mathtt{m}$ and $\nu_{X,\widehat{E}}(U) = \mathtt{n}$, we obtain that $\partial_{X,E}(\mathtt{m}, \mathtt{n}) \leqslant \partial_{X,\widehat{E}}(\mathtt{m}, \mathtt{n})$.

Proof of Claim: To simplify the notation, for each $j < n$ we set $\mathtt{m}_j := \nu_{X_j,\widehat{E}}(T_j)$, $H_j := (X_j, \mathtt{m}_j)$, and $K_j := \|\mathrm{Id}_{X_j}\|_{X_j,H_j}$. We use the fact that $\mathrm{Age}(E)$ is an amalgamation class to find $\delta > 0$ such that for every Y, Z, V that can be isometrically embedded into E, with $\dim Y = k$, and every $\gamma \in \mathrm{Emb}_\delta(Y, Z)$, $\eta \in \mathrm{Emb}_\delta(Y, V)$ there is W that can be isometrically embedded into E and $i \in \mathrm{Emb}(Z, W)$, $j \in \mathrm{Emb}(V, W)$ such that $\|i \circ \gamma - j \circ \eta\| \leqslant \varepsilon$. Since $\mathrm{Age}(E)_{\widehat{E}}$ is $\Lambda_{\widehat{E}}$-dense in $\mathrm{Age}(\widehat{E})$, we can find $Z \in \mathrm{Age}(E)_{\widehat{E}}$ such that there is $\theta \in \mathrm{Emb}_\delta(Y, Z)$ such that $\|\theta - i_{Y,\widehat{E}}\| \leqslant \varepsilon$, where $Y := \sum_{j<n} \mathrm{Im}\, T_j$. By definition, each $\mathtt{m}_j$ are the norms on X_j that make $T_j : H_j \to \widehat{E}$ isometric embeddings, i.e., $T_j \in \mathrm{Emb}(H_j, \widehat{E})$. Consequently, $U_j := \theta \circ T_j \in \mathrm{Emb}_\delta(H_j, Z)$. Then,

$$\|U_j - T_j\|_{X_j,\widehat{E}} \leqslant \|\theta - i_{Y,\widehat{E}}\|_{Y,\widehat{E}} \|T_j\| \leqslant \varepsilon \|T_j\|.$$

We use that $\mathrm{Age}(E)$ has the amalgamation property to find $V \in \mathrm{Age}(E)_{\widehat{E}}$, $I \in \mathrm{Emb}(Z, V)$, $T_j' \in \mathrm{Emb}(H_j, V)$ for $j < n$ such that $\|T_j' - I \circ U_j\|_{H_j,V} \leqslant \varepsilon$. Thus,

$$\|T_j' - I \circ U_j\|_{X_j,\widehat{E}} \leqslant \|T_j' - I \circ U_j\|_{H_j,\widehat{E}} K_j \leqslant \varepsilon K_j.$$

Since $\widehat{E}$ is Fraïssé, there is some $g \in \mathrm{Iso}(\widehat{E})$ such that $\|g \upharpoonright Z - I\| \leqslant \varepsilon$. Then, for every $j < n$,

$$\begin{aligned}\|g \circ T_j - T_j'\|_{X_j,\widehat{E}} \leqslant & \|g \circ T_j - g \circ U_j\|_{X_j,\widehat{E}} + \|g \circ U_j - I \circ U_j\|_{X_j,\widehat{E}} + \|I \circ U_j - T_j'\|_{X_j,\widehat{E}} \leqslant \\ \leqslant & \varepsilon K_j + \varepsilon \|U_j\| + \varepsilon \leqslant \varepsilon K_j + \varepsilon \|T_j\| + \varepsilon^2 K_j + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the claim is proved. □

Finally, because $\mathrm{Age}(E)_{\widehat{E}}$ is $\Lambda_{\widehat{E}}$-dense in $\mathrm{Age}(\widehat{E})$, it follows that $\bigcup_{Y \in \mathrm{Age}(E)_{\widehat{E}}} \mathcal{L}^k(X, Y)$ is dense in $\mathcal{L}^k(X, Y)$, so, $\mathcal{N}_X(E)$ is dense in $\mathcal{N}_X(\widehat{E})$.

b): We have seen in Proposition 2.11 that $d_{\mathrm{BM}}(\tau_{k,X}(V), \tau_{k,X}(W)) \leqslant 3k \log k \Lambda_{k,X}(V, W)$ for every normed space X and $V, W \in \mathrm{Gr}(k, X)$, and this implies that $\tau_{k,\widehat{E}}$ is uniformly continuous. Let us see now that $\widetilde{\tau}_{k,\widehat{E}}$ is 1-1: Suppose that $\tau_{k,\widehat{E}}(V) = \tau_{k,\widehat{E}}(W)$. Since $\widehat{E}$ is a Fraïssé Banach space, for a given $\varepsilon > 0$ we can find an isometry $g \in \mathrm{Aut}(\widehat{E})$ such that $\Lambda_{\widehat{E}}(V, g\cdot W) < \varepsilon$, and hence $V \in [W]$.

Now, given $\varepsilon > 0$, we work to find $\delta > 0$ such that if $V, W \in \mathrm{Gr}(k, \widehat{E})$ are such that $d_{\mathrm{BM}}(\widetilde{\tau}_{k,\widehat{E}}(V), \widetilde{\tau}_{k,\widehat{E}}(W)) \leqslant \delta$, then $\widetilde{\Lambda}_{k,\widehat{E}}([V],[W]) = \gamma_{k,E}(\tau_{k,\widehat{E}}(V), \tau_{k,\widehat{E}}(W)) \leqslant \varepsilon$.

Claim 3.12.3. Suppose that X, Y are normed spaces and $\gamma, \eta \in \mathrm{Emb}_\delta(X, Y)$ are such that $\|\gamma - \eta\| \leqslant \varepsilon$. Then, $\Lambda_Y(\operatorname{Im}\gamma, \operatorname{Im}\eta) \leqslant (1+\delta)(\varepsilon+\delta)+\delta$.

Proof of Claim: Fix $v \in \mathrm{Ball}(\operatorname{Im}\gamma)$. Let $x \in X$ be such that $\gamma(x) = v$. Then $\|x\| \leqslant 1+\delta$. Since $\|\eta(x)\| \leqslant \|\eta\|\|x\| \leqslant (1+\delta)^2$, $w := \eta(x)/(1+\delta)^2 \in \mathrm{Ball}(\operatorname{Im}\eta)$ and

$$\|w - v\| \leqslant \|\eta(x) - \gamma(x)\| + \|\eta(x) - \frac{\eta(x)}{(1+\delta)^2}\| \leqslant \varepsilon(1+\delta) + \|\eta(x)\|\frac{\delta^2+2\delta}{(1+\delta)^2} = (1+\delta)(\varepsilon+\delta)+\delta.$$

□

Since $\widehat{E}$ is Fraïssé, there is some $\delta > 0$ such that if $\gamma \in \mathrm{Emb}_\delta(V, E)$ then there is some $g \in \mathrm{Iso}(\widehat{E})$ such that $\|g \restriction V - \gamma\| \leqslant \varepsilon/2$. Finally, let $0 < \delta_0 \leqslant \delta$ be such that $(1+\delta_0)(\varepsilon/2 + \delta_0) + \delta_0 \leqslant \varepsilon$. We claim that δ_0 works. For suppose that $V, W \in \mathrm{Gr}(k, \widehat{E})$ are such that $d_{\mathrm{BM}}(\widetilde{\tau}_{k,\widehat{E}}(V), \widetilde{\tau}_{k,\widehat{E}}(W)) \leqslant \delta_0$. Choose $\gamma \in \mathrm{Emb}_{\delta_0}(V, W)$, and let $g \in \mathrm{Iso}(\widehat{E})$ be such that $\|g \restriction V - \gamma\| \leqslant \varepsilon/2$. It follows from the Claim 3.12.3 that

$$\widetilde{\Lambda}_{k,\widehat{E}}([V],[W]) \leqslant \Lambda_{\widehat{E}}(W, g(V)) = \Lambda_{\widehat{E}}(\operatorname{Im}\gamma, g(V)) \leqslant (1+\delta_0)(\frac{\varepsilon}{2}+\delta_0)+\delta_0 \leqslant \varepsilon.$$

Since $\widetilde{\tau}_{k,\widehat{E}}$ is a uniform homeomorphism, the next claim gives that $\gamma_{k,\widehat{E}}$ and d_{BM} are uniformly equivalent on $\mathcal{B}_k(\widehat{E})$.

Claim 3.12.4. $\gamma_{k,\widehat{E}}(\tau_{k,\widehat{E}}(V), \tau_{k,\widehat{E}}(W)) = \widetilde{\Lambda}_{k,\widehat{E}}([V],[W])$ for every $V, W \in \mathrm{Gr}(k, \widehat{E})$, where $\widetilde{\Lambda}_{k,\widehat{E}}$ is the quotient metric on $\mathrm{Gr}(k,\widehat{E}) /\!\!/ \mathrm{Iso}(\widehat{E})^2$.

Proof of Claim: Given $V, W \in \mathrm{Gr}(k, \widehat{E})$, we have that $\gamma_{k,E}$ is the infimum of $\Lambda_{k,\widehat{E}}(V', W')$ where $\tau_{k,\widehat{E}}(V') = \tau_{k,\widehat{E}}(V)$ and $\tau_{k,\widehat{E}}(W') = \tau_{k,\widehat{E}}(W)$, so it is the infimum of $\Lambda_{k,\widehat{E}}(V', W')$ where $V' \in [V]$ and $W' \in W$, and this is equal to $\widetilde{\Lambda}_{k,\widehat{E}}([V],[W])$. □

The fact that $\gamma_{k,\widehat{E}}$ coincides with $\gamma_{k,E}$ on $\mathcal{B}_k(E)$ is proved similarly to the corresponding fact in *a)*. We leave the details to the reader. Finally, we see that $\mathcal{B}_k(E)$ is dense in $\mathcal{B}_k(\widehat{E})$. Fix $[\mathtt{m}] \in \mathcal{B}_k(\widehat{E})$, and $\varepsilon > 0$. Choose a k-dimensional subspace X of $\widehat{E}$ such that X is isometric to $(\mathbb{F}^k, \mathtt{m})$. Since $\mathrm{Age}(E)_{\widehat{E}}$ is $\Lambda_{\widehat{E}}$-dense in $\mathrm{Age}(\widehat{E})$ we can find $Y \in \mathrm{Age}(E)_{\widehat{E}}$ such that $\Lambda_{\widehat{E}}(Y, X) \leqslant \varepsilon$. Let $\mathtt{n} \in \mathcal{N}_k$ be such that $(\mathbb{F}^k, \mathtt{n})$ is isometric to Y. Since $Y \in \mathrm{Age}(E)_{\widehat{E}}$, $\mathtt{n} \in \mathcal{N}_k(E)$, and $\gamma_{k,\widehat{E}}([\mathtt{m}],[\mathtt{n}]) \leqslant \Lambda_{\widehat{E}}(Y, X) \leqslant \varepsilon$. This shows that $\mathcal{B}_k(E)$ is dense in $\mathcal{B}_k(\widehat{E})$ because $\gamma_{k,\widehat{E}}$ is a compatible metric in $\mathcal{B}_k(\widehat{E})$.

c): We start with the continuity of $\nu^2_{k,\widehat{E}}$. Suppose that $T_n \to_n T$ in norm. Then, $\mathrm{Im}(T_n) \to \mathrm{Im}(T)$ in the opening distance $\Lambda_{\widehat{E}}$. Now fix a basis $(x_j)_{j<k}$ of $\operatorname{Im} T$, and let $(y_j)_{j<k}$ be a linearly independent sequence in $\widehat{E}$ such that $T = T_0 \circ T_1^*$, where $T_0 : (\mathbb{F}^k)^* \to \widehat{E}$ is defined by $T_0(u_j^*) = x_j$ and $T_1 : \mathbb{F}^k \to \widehat{E}$ by $T_1(u_j) = y_j$. Since we know that $\mathrm{Im}(T_n) \to \mathrm{Im}(T)$ in the opening distance $\Lambda_{\widehat{E}}$, and since a finite sequence sufficiently close to a linearly independent sequence is also linearly independent, for large enough n we can choose a basis $\{x_j^n\}_{j<k}$ of $\operatorname{Im} T_n$ such that $x_j^n \to_n x_j$ for every $j < k$. Similarly, we define $T_0^n : (\mathbb{F}^k)^* \to \widehat{E}$, $T_0^n(u_j^*) := x_j^n$, and let $T_1^n : \mathbb{F}^k \to \widehat{E}$ be such that $T_n = T_0^n \circ (T_1^n)^*$. Let $X := ((\mathbb{F}^k)^*, \mathtt{m}_0)$, where $\mathtt{m}_0 := \nu_{(\mathbb{F}^k)^*,\widehat{E}}(T_0)$. Since $T_0^n \to_n T_0$, by continuity of $\nu_{X,\widehat{E}}$, it follows that $\nu_{X,\widehat{E}}(T_0^n) \to_n \nu_{X,\widehat{E}}(T_0)$. On the other hand, T_0, is 1-1, so

$$\begin{aligned}\|(T_1^n)^* - (T_1)^*\|_{(\widehat{E})^*,X} =&\|T_0 \circ (T_1^n)^* - T_0 \circ T_1^*\|_{(\widehat{E})^*,\widehat{E}} \leqslant \\ \leqslant&\|T_n - T\|_{(\widehat{E})^*,\widehat{E}} + \|T_0 \circ (T_1^n)^* - T_0^n \circ (T_1^n)^*\|_{(\widehat{E})^*,\widehat{E}} \leqslant \\ \leqslant&\|T_n - T\|_{(\widehat{E})^*,\widehat{E}} + \|T_0^n - T_0\|_{X,\widehat{E}} \cdot \|(T_1^n)^*\|_{(\widehat{E})^*,X}. \end{aligned} \tag{9}$$

Since $T_n \to_n T$ and $T_0^n \to T_0$ both in norm, from (9) we will obtain that $T_1^n \to_n T_1$ in norm once we show that the numerical sequence $(\|(T_1^n)^*\|_{(\widehat{E})^*,X})_n$ is bounded. To see this, for each n let $\gamma_n > 0$ be the maximal γ such that $\|T_0^n(x)\|_{\widehat{E}} \geqslant \gamma\|x\|_X$ for every $x \in (\mathbb{F}^k)^*$; that is, $1/\gamma_n$ is the value $\mathtt{r}_{-1}$ of the bounded operator T_0^n when considered from X to $\widehat{E}$. Since $T_0^n \to T_0$ in norm and since $\mathtt{r}_{-1}(T_0) = 1$, it follows that $\gamma_n \to_n 1$. Then,

$$\|(T_1^n)^*\|_{(\widehat{E})^*,X} \leqslant (1/\gamma_n)\|T_0^n \circ (T_1^n)^*\|_{(\widehat{E})^*,\widehat{E}} = (1/\gamma_n)\|T_n\|_{(\widehat{E})^*,\widehat{E}},$$

and using that $\gamma_n \to_n 1$ and $\|T_n\| \to \|T\|$, we obtain that $(\|(T_1^n)^*\|_{(\widehat{E})^*,X})_n$ is bounded. As we said above, we have that hat $T_1^n \to_n T_1$ in norm, and consequently, $\nu_{X,\widehat{E}}(T_1^n) \to_n \nu_{X,\widehat{E}}(T_1)$.

Suppose now that $\nu^2_{k,\widehat{E}}(T) = \nu^2_{k,\widehat{E}}(U)$. Decompose $T = T_0 \circ T_1^*$ and $U = U_0 \circ U_1^*$ in a way that $\nu_{(\mathbb{F}^k)^*,\widehat{E}}(T_0) = \nu_{(\mathbb{F}^k)^*,\widehat{E}}(U_0)$ and $\nu_{\mathbb{F}^k,\widehat{E}}(T_1) = \nu_{\mathbb{F}^k,\widehat{E}}(U_1)$. As in the proof of *a)*, we can find $g, h \in \mathrm{Iso}(\widehat{E})$ such that $\|g \circ T_0 - U_0\| \leqslant \varepsilon/(2\|U_1\|)$ and $\|h \circ T_1 - U_1\| \leqslant \varepsilon/(2\|T_0\|)$. Hence,

$$\begin{aligned}\|g \circ T \circ h^* - U\| \leqslant&\|g \circ T_0 \circ T_1^* \circ h^* - g \circ T_0 \circ U_1^*\| + \|g \circ T_0 \circ U_1^* - U_0 \circ U_1^*\| \leqslant \\ \leqslant&\|g\| \cdot \|T_0\| \cdot \|h \circ T_1 - U_1\| + \|g \circ T_0 - U_0\| \cdot \|U_1\| \leqslant \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $[U] = [T]$. We see now that $\widetilde{\nu}^2_{k,\widehat{E}}$ is a homeomorphism. Suppose that $\widetilde{\nu}^2_{k,\widehat{E}}([T_n]) \to_n \widetilde{\nu}^2_{k,\widehat{E}}([T]) \in \mathcal{D}_k(\widehat{E})$. Our goal is to find a subsequence of $([T_n])_n$ that converges to $[T]$: Fix $X := (\mathbb{F}^k, \|\cdot\|_X)$. We first decompose $T = T_0 \circ T_1^*$, with $T_0 \in \mathcal{L}^k(X^*, \widehat{E})$ and $T_1 \in \mathcal{L}^k(X, \widehat{E})$, a subsequence $(T_{n_m})_m$ and decompositions $T_{n_m} = T_0^m \circ (T_1^m)^*$ in a way that both $\omega(\nu_{X^*,\widehat{E}}(T_0^m), \nu_{X^*,\widehat{E}}(T_0)) < m^{-1}$ and $\omega(\nu_{X,\widehat{E}}(T_1^m), \nu_{X,\widehat{E}}(T_1)) < m^{-1}$ for every $m \in \mathbb{N}$. It follows from *a)* that $[T_0^m] \to [T_0]$ and $[T_1^m] \to_m [T_1]$. This easily implies that $[T_{n_m}] \to_m [T]$.

The fact that the restrictions $\widetilde{\nu}^2_{k,\widehat{E}} : (\widetilde{\nu}^2_{k,\widehat{E}})^{-1}(A) \to A$ are uniform homeomorphisms when A is $\widetilde{\omega}_2$-bounded follows from the Heine-Borel property of $(\mathcal{D}_k, \widetilde{\omega}_2)$. Let us check now that $\mathfrak{d}_{k,\widehat{E}}$ coincides with $\mathfrak{d}_{k,E}$ on $\mathcal{D}_k(E)$. Since $\mathcal{L}^{k,\mathrm{w}^*}(E^*, E) \subseteq \mathcal{L}^{k,\mathrm{w}^*}((\widehat{E})^*, \widehat{E})$ we obtain that $\mathfrak{d}_{k,\widehat{E}}(\cdot,\cdot) \leqslant \mathfrak{d}_{k,E}(\cdot,\cdot)$. For the other inequality we use the next.

Claim 3.12.5. Suppose that $T, U \in \mathcal{L}^{k,\mathrm{w}^*}((\widehat{E})^*, \widehat{E})$ are such that $\nu^2_{k,\widehat{E}}(T) = [\mathbf{m}]$ and $\nu^2_{k,\widehat{E}}(U) = [\mathbf{n}]$ both belong to $\mathcal{D}_k(E)$. For every $\varepsilon > 0$ there are $V \in \mathrm{Age}(E)_{\widehat{E}}$, $g \in \mathrm{Iso}(\widehat{E})$ and $T', U' \in \mathcal{L}^{k,\mathrm{w}^*}(V^*, V)$ such that $\nu^2_{k,V}(T') = [\mathbf{m}]$, $\nu^2_{k,V}(U') = [\mathbf{n}]$ and

$$\|g \circ T \circ h^* - i \circ T' \circ r\|_{(\widehat{E})^*,\widehat{E}}, \|g \circ U \circ h - i \circ U' \circ r\|]_{(\widehat{E})^*,\widehat{E}} \leqslant \varepsilon,$$

where $i : V \to \widehat{E}$ is the canonical inclusion, and $r : E^* \to V^*$ is the canonical restriction.

Proof of Claim: Decompose $T = T_0 \circ T_1^*$ and $U = U_0 \circ U_1^*$ with $T_0, U_0 \in \mathcal{L}^k((\mathbb{F}^k)^*, \widehat{E})$, $T_1, U_1 \in \mathcal{L}^k(\mathbb{F}^k, \widehat{E})$, and set $\mathtt{m}_0 := \nu_{(\mathbb{F}^k)^*,\widehat{E}}(T_0)$, $\mathtt{m}_1 := \nu_{\mathbb{F}^k,\widehat{E}}(T_1)$, $\mathtt{n}_0 := \nu_{(\mathbb{F}^k)^*,\widehat{E}}(U_0)$, $\mathtt{n}_1 := \nu_{\mathbb{F}^k,\widehat{E}}(U_1)$. Fix a norm on $\mathbb{F}^k$, $X = (\mathbb{F}^k, \|\cdot\|_X)$ and $\delta > 0$. Since $T_0 \in \mathrm{Emb}(((\mathbb{F}^k)^*, \mathtt{m}_0), \widehat{E})$, $T_1 \in \mathrm{Emb}((\mathbb{F}^k, \mathtt{m}_1), \widehat{E})$, $U_0 \in \mathrm{Emb}(((\mathbb{F}^k)^*, \mathtt{n}_0), \widehat{E})$, $U_1 \in \mathrm{Emb}((\mathbb{F}^k, \mathtt{n}_1), \widehat{E})$ and by hypothesis $\mathtt{m}_0, \mathtt{n}_0 \in \mathcal{N}_{(\mathbb{F}^k)^*}(E)$ and $\mathtt{m}_1, \mathtt{n}_1 \in \mathcal{N}_{\mathbb{F}^k}(E)$, we can use Claim 3.12.2 to find $V \in \mathrm{Age}(E)_{\widehat{E}}$, $g \in \mathrm{Iso}(\widehat{E})$ and operators $T_0' \in \mathrm{Emb}(((\mathbb{F}^k)^*, \mathtt{m}_0), V)$, $T_1' \in \mathrm{Emb}((\mathbb{F}^k, \mathtt{m}_1), V)$, $U_0' \in \mathrm{Emb}(((\mathbb{F}^k)^*, \mathtt{n}_0), V)$, $U_1' \in \mathrm{Emb}((\mathbb{F}^k, \mathtt{n}_1), V)$ such that $\|g \circ T_0 - T_0'\|_{X^*,\widehat{E}}$, $\|g \circ U_0 - U_0'\|_{X^*,\widehat{E}}$, $\|g \circ T_1 - T_1'\|_{X,\widehat{E}}$ and $\|g \circ U_1 - U_1'\|_{X,\widehat{E}}$ are all at most ε. Let $T' := T_0' \circ (T_1')^*$ and $U' := U_0' \circ (U_1')^*$. Then,

$$\begin{aligned}\|g\circ T\circ g^* - i\circ T'\circ r\|_{(\widehat{E})^*,\widehat{E}} \leqslant &\|g\circ T_0\circ T_1^*\circ g^* - T_0'\circ T_1^*\circ g^*\|_{(\widehat{E})^*,\widehat{E}}+\\ &+\|T_0'\circ T_1^*\circ g^* - T_0'\circ (T_1')^*\circ r\|_{(\widehat{E})^*,\widehat{E}}\leqslant\\ \leqslant&\|g\circ T_0 - T_0'\|_{X^*,\widehat{E}}\cdot\|T_1\|_{X,\widehat{E}}+\\ &+\|T_0'\|_{X^*,\widehat{E}}\cdot\|T_1^*\circ g^* - (T_1')^*\circ r\|_{(\widehat{E})^*,X^*}\leqslant\\ \leqslant&\varepsilon\|T_1\|_{X,\widehat{E}}+\varepsilon(\|T_0\|_{X^*,\widehat{E}}+\varepsilon),\end{aligned}$$

and similarly one shows that $\|g\circ U\circ g^* - i\circ U'\circ r\|_{(\widehat{E})^*,\widehat{E}} \leqslant \varepsilon\|U_1\|_{X,\widehat{E}}+\varepsilon(\|U_0\|_{X^*,\widehat{E}}+\varepsilon)$. Since $\varepsilon>0$ is arbitrary, the claim is proved. □

So, given $T, U \in \mathcal{L}^{k,\mathrm{w}^*}((\widehat{E})^*,\widehat{E})$ such that $\nu^2_{k,\widehat{E}}(T) = [\mathbf{m}], \nu^2_{k,\widehat{E}}(U) = [\mathbf{n}] \in \mathcal{D}_k(E)$, and $\varepsilon>0$, we use the previous claim to find the corresponding $V \in \mathrm{Age}(E)_{\widehat{E}}$, $g\in \mathrm{Iso}(\widehat{E})$ and $T', U' \in \mathcal{L}^{k,\mathrm{w}^*}(V^*,V)$. Let $\gamma\in\mathrm{Emb}(V,E)$, and set $T_0 := \gamma\circ T'\circ\gamma^*$ and $U_0 := \gamma\circ U'\circ\gamma^*$. Then,

$$\|T_0-U_0\|_{E^*,E} = \|T'-U'\|_{V^*,V} = \|i\circ T'\circ r - i\circ U'\circ r\|_{E^*,E}\leqslant \|T-U\|_{(\widehat{E})^*,\widehat{E}}+2\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, $\mathfrak{d}_{k,E}([\mathbf{m}],[\mathbf{n}])\leqslant\|T-U\|_{(\widehat{E})^*,\widehat{E}}$, and since T,U are arbitrary such that $\nu^2_{k,\widehat{E}}(T) = [\mathbf{m}]$ and $\nu^2_{k,\widehat{E}}(U) = [\mathbf{n}]$, we obtain that $\mathfrak{d}_{k,E}([\mathbf{m}],[\mathbf{n}])\leqslant\mathfrak{d}_{k,\widehat{E}}([\mathbf{m}],[\mathbf{n}])$.

Finally, $\mathcal{D}_k(E)$ is dense in $\mathcal{D}_k(\widehat{E})$ because, by *a)*, $\mathcal{N}_X(E)$ is dense in $\mathcal{N}_X(\widehat{E})$ for every X. □

We finish with the following fact on bounded sets considered before.

Lemma 3.13. *Suppose that E is a normed space, $k\in\mathbb{N}$ and $\lambda\geqslant 1$. Then $\nu^2_{k,E}(\mathcal{L}^{k,\mathrm{w}^*}_\lambda(E^*,E)) = \mathcal{D}_k(E;\lambda)$ and $\nu^2_{k,E}(\mathcal{L}^{k,\mathrm{w}^*}_{<\lambda}(E^*,E)) = \mathcal{D}_k(E;{<}\lambda)$.*

Proof. We will use the following simple fact.

Claim 3.13.1. If $Y = (V,\|\cdot\|_Y)$ is a normed space of dimension k, then $(\nu_{Y,E}(T))^*(f) = \min\{\|g\|_{E^*} : T^*(g) = f\}$ for every $T\in\mathcal{L}^k(Y,E)$ and $f\in Y^*$.

Proof of Claim: Fix $T\in\mathcal{L}^k(Y,E)$ and $f\in Y^*$. We know that $T: Z\to E$ is an isometry where $Z := (V,\mathtt{n})$ with $\mathtt{n} := \nu_{V,E}(T)$. Set $H := (T(Y),\|\cdot\|_E)$, and $U: H\to Z$ be the inverse of T. Let $g_0\in H^*$ be such that $U^*(g_0) = f$, and use the Hahn-Banach Theorem to extend g_0 to $g\in E^*$ in a way that $\|g\| = \|g_0\|$. It is easily seen that $T^*(g) = f$, and since $\|T^*\|_{E^*,Z^*} = \|T\|_{Z,E} = 1$, we obtain the desired equality. □

Fix $[(\mathtt{m}_0,\mathtt{m}_1)]\in\mathcal{D}_k(E;\lambda)$, that is, $\|\mathrm{Id}\|_{((\mathbb{F}^k)^*,\mathtt{m}_0),((\mathbb{F}^k)^*,\mathtt{m}_1^*)}, \|\mathrm{Id}\|_{((\mathbb{F}^k)^*,\mathtt{m}_1^*),((\mathbb{F}^k)^*,\mathtt{m}_0)},\leqslant\lambda$. Choose $T_0\in\mathcal{L}^k((\mathbb{F}^k)^*,E)$ and $T_1\in\mathcal{L}^k(\mathbb{F}^k,E)$ such that $\nu_{(\mathbb{F}^k)^*,E}(T_0) = \mathtt{m}_0$ and $\nu_{\mathbb{F}^k,E}(T_1) = \mathtt{m}_1$. We claim that $T := T_0\circ T_1^*\in\mathcal{L}^{k,\mathrm{w}^*}_\lambda(E^*,E)$. Given $g\in E^*$, since $T_0: ((\mathbb{F}^k)^*,\mathtt{m}_0)\to E$ and $T_1: (\mathbb{F}^k,\mathtt{m}_1)\to E$ are isometric embeddings,

$$\|T_0(T_1^*(g))\|_E = \mathtt{m}_0(T_1^*(g))\leqslant\lambda\mathtt{m}_1^*(T_1^*(g))\leqslant\lambda\|g\|_{E^*},$$

hence, $\|T\|\leqslant\lambda$. Let us see now that $\mathtt{r}_{-1}(T)\leqslant\lambda$, i.e. $(1/\lambda)\cdot\mathrm{Ball}(\mathrm{Im}(T))\subseteq T(\mathrm{Ball}(E^*))$. So, suppose that $\|T(g)\|_E\leqslant\lambda^{-1}$. It follows that $\mathtt{m}_0(T_1^*(g)) = \|T_0(T_1^*(g))\|_E\leqslant\lambda^{-1}$, and hence $\mathtt{m}_1^*(T_1^*(g))\leqslant 1$. By Claim 3.13.1, there is $h\in E^*$ such that $T_1^*(h) = T_1^*(g)$ and $\|h\|_{E^*}\leqslant 1$. Similarly one shows that $\nu^2_{k,E}(\mathcal{L}^{k,\mathrm{w}^*}_{<\lambda}(E^*,E)) = \mathcal{D}_k(E;{<}\lambda)$. □

Appendix A. Extrinsic metrics for $p = \infty$

The case $p = \infty$ is special because the Fraïssé limit that corresponds to ℓ_∞^∞ is a universal space, the Gurarij space $\mathbb{G}$ [12]. We are going to see that the $\mathbb{G}$-extrinsic metrics are Lipschitz equivalent to the intrinsic ones on bounded sets. We start by analyzing $\partial_{X,\mathbb{G}}$. Given a finite dimensional normed space $X = (X, \|\cdot\|_X)$, another compatible, more geometrical, metric on $\mathcal{N}_X$ is the next. Having in mind that a norm is completely determined by its dual unit ball, given $\mathtt{m}, \mathtt{n} \in \mathcal{N}_X$, let

$$\alpha_X(\mathtt{m}, \mathtt{n}) := d_{\mathcal{H},\|\cdot\|_{X^*}}(\mathrm{Ball}((X, \mathtt{m})^*), \mathrm{Ball}((X, \mathtt{n})^*)),$$

where $d_{\mathcal{H},\|\cdot\|_{X^*}}(\cdot,\cdot)$ is the Hausdorff distance with respect to the norm distance induced by $\|\cdot\|_{X^*}$. In other words, $\alpha_X(\mathtt{m}, \mathtt{n})$ measures the $d_{\|\cdot\|_X^*}$-distance between the unit balls of $(X^*, \mathtt{m}^*)$ and of $(X^*, \mathtt{n}^*)$. Notice that since the unit balls of finite dimensional normed spaces are compact and convex, it follows from the Minkowski Theorem that

$$\alpha_X(\mathtt{m}, \mathtt{n}) = \max \left\{ \sup_{f \in \mathrm{Ext}(\mathrm{Ball}((X,\mathtt{m})^*))} d_{\|\cdot\|_{X^*}}(f, \mathrm{Ball}((X, \mathtt{m})^*)), \right.$$
$$\left. \sup_{g \in \mathrm{Ext}(\mathrm{Ball}((X,\mathtt{n})^*))} d_{\|\cdot\|_{X^*}}(g, \mathrm{Ball}((X, \mathtt{m})^*)) \right\}$$

where $\mathrm{Ext}(K)$ is the collection of extreme points of a given compact and convex subset $K \subseteq X$. In the next we write $\mathrm{Sph}(X) = \{x \in X : \|x\|_X = 1\}$ to denote the unit sphere of X.

Proposition A.1. *Let $X = (X, \|\cdot\|_X)$ be a finite dimensional normed space and let $\mathtt{m}, \mathtt{n} \in \mathcal{N}_X$.*

a) If $\mathtt{m}, \mathtt{n} \in \mathcal{N}_X(\ell_\infty^\infty)$, then $\partial_{X,\ell_\infty^\infty}(\mathtt{m}, \mathtt{n}) = \alpha_X(\mathtt{m}, \mathtt{n})$. Consequently, in general, $\partial_{X,\mathbb{G}}(\mathtt{m}, \mathtt{n}) = \alpha_X(\mathtt{m}, \mathtt{n})$.

b) If $\mathtt{m}, \mathtt{n} \in \mathcal{N}_X$ satisfy $\omega(\mathtt{m}, \|\cdot\|_X), \omega(\mathtt{n}, \|\cdot\|_X) \leqslant \log \lambda$, then $\lambda^{-1} \cdot \omega(\mathtt{m}, \mathtt{n}) \leqslant \alpha_X(\mathtt{m}, \mathtt{n}) \leqslant \lambda \cdot \omega(\mathtt{m}, \mathtt{n})$.

Proof. a): Fix $\varepsilon > 0$, and let $T, U \in \mathcal{L}(X, \ell_\infty^\infty)$ be such that $\nu_{X,\ell_\infty^\infty}(T) = \mathtt{m}$, $\nu_{X,\ell_\infty^\infty}(U) = \mathtt{n}$, and $\|T - U\|_{X,\ell_\infty^\infty} \leqslant \partial_{X,\ell_\infty^\infty}(\mathtt{m}, \mathtt{n}) + \varepsilon$. Given $f \in \mathrm{Ball}((X, \mathtt{m})^*)$, if $g \in \mathrm{Ball}((\ell_\infty^\infty)^*)$ is such that $T^*(g) = f$, then $d_{X^*}(f, \mathrm{Ball}((X, \mathtt{n})^*)) \leqslant \|f - U^*(g)\|_{X^*} \leqslant \|T^* - U^*\| \leqslant \partial_{X,\ell_\infty^\infty}(\mathtt{m}, \mathtt{n}) + \varepsilon$. Similarly one shows that $d_{X^*}(g, \mathrm{Ball}((X, \mathtt{m})^*)) \leqslant \partial_{X,\ell_\infty^\infty}(\mathtt{m}, \mathtt{n}) + \varepsilon$ for every $g \in \mathrm{Ball}((X, \mathtt{n})^*)$. Since $\varepsilon > 0$ is arbitrary, it follows that $\alpha_X(\mathtt{m}, \mathtt{n}) \leqslant \partial_{X,\ell_\infty^\infty}(\mathtt{m}, \mathtt{n})$.

Let us show now that $\partial_{X,\ell_\infty^\infty}(\mathtt{m}, \mathtt{n}) \leqslant \alpha_X(\mathtt{m}, \mathtt{n})$: Fix $T, U \in \mathcal{L}(X, \ell_\infty^\infty)$ such that $\nu_{X,\ell_\infty^\infty}(T) = \mathtt{m}$, $\nu_{X,\ell_\infty^\infty}(U) = \mathtt{n}$. Choose $n \in \mathbb{N}$ such that $\mathrm{Im}\, T, \mathrm{Im}\, U \subseteq \langle u_j \rangle_{j<n}$. Recall that $\|\sum_{j<n} a_j u_j\|_\infty = \max_{j<n} |a_j|$ so $Y := (\langle u_j \rangle_{j<n}, \|\cdot\|_\infty)$ is isometric to ℓ_∞^n, hence $Y^* = \langle u_j^* \rangle_{j<n}$ is isometric to $(\ell_\infty^n)^* = \ell_1^n$ and the extreme points of $\mathrm{Ball}(Y^*)$ are $\{\pm u_j^*\}_{j<n}$. Since $T : (X, \mathtt{m}) \to Y$ and $U : (X, \mathtt{n}) \to Y$ are isometric embeddings, i.e. $\|T\| = \mathtt{r}_{-1}(T) = \|U\| = \mathtt{r}_{-1}(U)$, it follows that $\|T^*\| = \mathtt{r}_{-1}(T^*) = \|U^*\| = \mathtt{r}_{-1}(U^*)$, and this exactly means that the restrictions $T^* : \mathrm{Ball}(Y^*) \to \mathrm{Ball}((X^*, \mathtt{m}^*))$ and $U^* : \mathrm{Ball}(Y^*) \to \mathrm{Ball}((X^*, \mathtt{n}^*))$ are continuous affine surjections. This implies that $\mathrm{Ext}(\mathrm{Ball}((X^*, \mathtt{m}^*))) \subseteq T^*(\mathrm{Ext}(\mathrm{Ball}(Y^*))) = \{\pm T^*(u_j^*)\}_{j<n}$ and $\mathrm{Ext}(\mathrm{Ball}((X^*, \mathtt{m}^*))) \subseteq \{\pm U^*(u_j^*)\}_{j<n}$. Resuming, we have just seen that in this case

$$\alpha_X(\mathtt{m}, \mathtt{n}) = \max_{j<n} \max\{d_{X^*}(T^*(u_j^*), U^*(\mathrm{Ball}(Y^*))), d_{X^*}(U^*(u_j^*), T^*(\mathrm{Ball}(Y^*))).\}$$

Here we have used that $d_{X^*}(v, K) = d_{X^*}(-v, K)$ if $K \subseteq X^*$ is symmetric, that is, if satisfies that $K = -K$. For every $0 \leqslant j < n$ choose $f_j, g_j \in \mathrm{Ball}(Y^*)$ such that $\|T^*(u_j^*) - U^*(g_j)\|_{X^*} = d_{X^*}(T^*(u_j^*), U^*(\mathrm{Ball}(Y^*)))$ and $\|U^*(u_j^*) - T^*(f_j)\|_{X^*} = d_{X^*}(U^*(u_j^*), T^*(\mathrm{Ball}(Y^*)))$.

Set $Z := (\langle u_j \rangle_{j<2n}, \|\cdot\|_\infty)$, that is isometric to ℓ_∞^{2n}. Let $\xi, \eta \in \mathrm{Emb}(Y, Z)$ be defined dually by $\xi^*(u_j^*) := u_j^*$, $\xi^*(u_{n+j}^*) := f_j^*$, $\eta^*(u_j^*) := g_j$ and $\eta^*(u_{n+j}^*) = u_j^*$ for $0 \leqslant j < n$. Then,

$$\begin{aligned}\partial_{X,\ell_\infty^\infty}(\mathtt{m},\mathtt{n}) \leqslant & \|\xi \circ T - \eta \circ U\|_{X,Z} = \|T^* \circ \xi^* - U^* \circ \eta^*\|_{Z^*,X^*} = \\ & = \max_{j<2n} \|T^*(\xi^*(u_j^*)) - U^*(\eta^*(u_j^*))\| = \alpha_X(\mathtt{m},\mathtt{n}).\end{aligned}$$

Here we have used that the norm of a bounded operator $\gamma : \ell_1^k \to E$ is $\|\gamma\| = \max_{j<k} \|T(u_j)\|$ where $(u_j)_{j<k}$ is the unit basis of ℓ_1^k.

b): Notice that $\omega(\mathtt{m},\mathtt{n}) = \omega(\mathtt{m}^*,\mathtt{n}^*)$, because $\|\mathrm{Id}\|_{(X,\mathtt{m}),(X,\mathtt{n})} = \|\mathrm{Id}\|_{(X^*,\mathtt{n}^*),(X^*,\mathtt{m})}$. So, it suffices to prove that if $\omega(\|\cdot\|_X, \mathtt{m}), \omega(\|\cdot\|_X, \mathtt{n}) \leqslant \log\lambda$, then

$$\frac{1}{\lambda} \cdot \omega(\mathtt{m},\mathtt{n}) \leqslant d_{\mathcal{H},\|\cdot\|_X}(\mathrm{Ball}(X,\mathtt{m}), \mathrm{Ball}(X,\mathtt{n})) \leqslant \lambda \cdot \omega(\mathtt{m},\mathtt{n}) \tag{10}$$

To simplify the notation we set $d(\mathtt{m},\mathtt{n}) := d_{\mathcal{H},\|\cdot\|_X}(\mathrm{Ball}(X,\mathtt{m}), \mathrm{Ball}(X,\mathtt{n}))$. We assume that $d(\mathtt{m},\mathtt{n}) > 0$ since otherwise $\mathtt{m} = \mathtt{n}$ and the inequalities to check are trivially true. Let us show the first inequality in (10). Without of generality we assume that there is $x \in \mathrm{Sph}(X,\mathtt{m})$ such that $0 < d(\mathtt{m},\mathtt{n}) = d_X(x, \mathrm{Ball}(X,\mathtt{n}))$. Then $\mathtt{n}(x) > 1$ and

$$\begin{aligned} d(\mathtt{m},\mathtt{n}) = & d_X(x, \mathrm{Ball}(X,\mathtt{n})) \leqslant \|x - \frac{x}{\mathtt{n}(x)}\|_X = \|x\|_X \left|1 - \frac{1}{\mathtt{n}(x)}\right| \leqslant \lambda\left(1 - \frac{1}{\mathtt{n}(x)}\right) \leqslant \\ \leqslant & \lambda\left(1 - \frac{1}{\exp(\omega(\mathtt{m},\mathtt{n}))}\right) \leqslant \lambda\omega(\mathtt{m},\mathtt{n}).\end{aligned}$$

Let us prove the second inequality in (10). Fix $x \in X$ such that $\mathtt{m}(x) = 1$, and let $y \in X$ be such that $\mathtt{n}(y) \leqslant 1$ and $\|x - y\|_X \leqslant d(\mathtt{m},\mathtt{n})$. It follows that

$$\mathtt{n}(x) \leqslant \mathtt{n}(y) + \mathtt{n}(x-y) \leqslant 1 + \lambda\|x-y\|_X \leqslant 1 + \lambda d(\mathtt{m},\mathtt{n}) \leqslant \exp(\lambda \cdot d(\mathtt{m},\mathtt{n})).$$

Since x with $\mathtt{m}(x) = 1$ was arbitrary, it follows that $\|\mathrm{Id}\|_{(X,\mathtt{m}),(X,\mathtt{n})} \leqslant \exp(\lambda \cdot d(\mathtt{m},\mathtt{n}))$, and similarly one shows that $\|\mathrm{Id}\|_{(X,\mathtt{n}),(X,\mathtt{m})} \leqslant \exp(\lambda \cdot d(\mathtt{m},\mathtt{n}))$. Consequently, $\omega(\mathtt{m},\mathtt{n}) \leqslant \lambda d(\mathtt{m},\mathtt{n})$. □

We see now that the $\mathbb{G}$-Kadets mapping $\gamma_k = \gamma_{k,\mathbb{G}}$ is Lipschitz equivalent to the Banach-Mazur metric on $\mathcal{B}_k$.

Corollary A.2. *d_{BM} and $\gamma_{k,\mathbb{G}}$ are Lipschitz equivalent on $\mathcal{B}_k$. In fact, for $k \geqslant 2$ and $\mathtt{m},\mathtt{n} \in \mathcal{N}_k$,*

$$\frac{1}{3k\log k} d_{\mathrm{BM}}([\mathtt{m}],[\mathtt{n}]) \leqslant \gamma([\mathtt{m}],[\mathtt{n}]) \leqslant (\log k) d_{\mathrm{BM}}([\mathtt{m}],[\mathtt{n}]).$$

Proof. We have seen in Proposition 2.11 that $d_{\mathrm{BM}}([\mathtt{m}],[\mathtt{n}]) \leqslant 3k\log k\gamma_{k,E}([\mathtt{m}],[\mathtt{n}])$ for every normed space E and $\mathtt{m},\mathtt{n} \in \mathcal{N}_k(E)$, so we only have to prove the first inequality above. Fix two norms $\mathtt{m},\mathtt{n} \in \mathcal{N}_k$, set $X := (\mathbb{F}^k, \mathtt{m})$ and $Y := (\mathbb{F}^k, \mathtt{n})$. The following result is a slight modification of [18, Proposition 6.2].

Claim A.2.1. Suppose that F and G are two finite-dimensional normed spaces, and $T : F \to G$ is a 1-1 linear operator such that $\|T\| \geqslant 1$. There is a normed space H, $I \in \mathrm{Emb}(F,H)$ and $J \in \mathrm{Emb}(G,H)$ such that:

i) If $\|T^{-1}\| \geqslant 1$, then $\|I - J \circ T\| \leqslant \|T\| \cdot \|T^{-1}\| - 1$.
ii) If $\dim F = \dim G$ and $\|T\| = 1$, then $\Lambda_H(\mathrm{Im}\, I, \mathrm{Im}\, J) \leqslant \|T^{-1}\| - 1$.

Proof of Claim: Fix a 1-1 linear operator $T : F \to G$. On the cartesian product $F \times G$ we define the seminorm

$$\mathtt{m}(x,y) := \max\left\{\left\|\frac{Tx}{\|T\|} + y\right\|_G, \max_{g\in D}\left|\frac{g}{\|T\|}(y) + \frac{(T^*g)(x)}{\|T^*g\|_{F^*}}\right|\right\},$$

where D is chosen so that $(T^*)^{-1}(\|T^{-1}\|^{-1}\cdot \mathrm{Ext}(\mathrm{Ball}(F^*)))\subseteq D\subseteq \mathrm{Ball}(G^*)$, and where for a compact convex set K, $\mathrm{Ext}(K)$ is the set of extreme points of K. Let H by the quotient of $F\times G$ by the kernel of $\mathtt{m}$ with its quotient norm

$$[\mathtt{m}]([(x,y)]) := \inf_{\mathtt{m}(x_0,y_0)=0} \mathtt{m}(x_0+x,y_0+y),$$

and let $I:F\to H$, $J:G\to H$ be the two canonical injections $I(x):=[(x,0)]$, $J(y):=[(0,y)]$. Let us see first that I and J are isometric embeddings. Fix $x\in F$ and $y\in G$, and let us check that $[\mathtt{m}]([(x,0)])=\|x\|_F$ and $[\mathtt{m}]([(0,y)])=\|y\|_G$: suppose that $\mathtt{m}(x_0,y_0)=0$. Then $\|T(x+x_0)/\|T\|+y_0\|_G=\|T(x)/\|T\|\|_G\leqslant\|x\|_F$, because $T(x_0)/\|T\|+y_0=0$, and similarly given $g\in D$, $|g(y_0)/\|T\|+((T^*g)(x)+(T^*g)(x_0))/\|T^*g\|_{F^*}|=|(T^*g)(x)/\|T^*g\|_{F^*}|\leqslant\|x\|_F$. This proves that $\inf_{\mathtt{m}(x_0,y_0)=0}\mathtt{m}(x_0+x,y_0)\leqslant\|x\|_F$. On the other hand, it is a well-known that the extreme points of the dual unit ball norm the space (see for example [5, Fact 3.45]), so we can choose $h\in\mathrm{Ext}(\mathrm{Ball}(F^*))$ such that $h(x)=\|x\|_F$, and let $g:=(T^*)^{-1}(h)/\|T^{-1}\|$. It follows that $T^*g(x)/\|T^*g\|=(h(x)/\|T^{-1}\|)/(1/\|T^{-1}\|)=\|x\|_F$, hence $\mathtt{m}(x,0)\geqslant\|x\|_F$, and consequently $\inf_{\mathtt{m}(x_0,y_0)=0}\mathtt{m}(x_0+x,y_0)=\|x\|_F$. In a similar way, for (x_0,y_0) with $\mathtt{m}(x_0,y_0)=0$, $\|Tx_0/\|T\|+y_0+y\|_G=\|y\|_G$ and $|(g(y_0+y)/\|T\|+(T^*g)(x_0)/\|T^*(g)\||=|g(y)/\|T\||\leqslant\|y\|_G/\|T\|\leqslant\|y\|_G$, so $\mathtt{m}(x_0,y_0+y)=\|y\|_G$. Suppose that $\|T^{-1}\|\geqslant 1$. Fix $x\in\mathrm{Ball}(F)$, and let us see that $[\mathtt{m}]([(x,-Tx)])\leqslant\mathtt{m}(x,-Tx)\leqslant\|T\|\|T^{-1}\|-1$: We have that $\|Tx/\|T\|-Tx\|_G\leqslant\|x\|_F(\|T\|-1)\leqslant\|T\|-1\leqslant\|T\|\|T^{-1}\|-1$, while given $g\in D$ we have that $|g(Tx)/\|T\|-T^*g(x)/\|T^*g\||=|g(Tx)||1/\|T\|-1/\|T^*g\||\leqslant\|T\||1/\|T\|-1/\|T^*g\||=\|T\|/\|T^*g\|-1\leqslant\|T\|\|T^{-1}\|-1$, where we have used that $1/\|T^{-1}\|\leqslant\|T^*g\|\leqslant\|T\|$.

ii): Suppose that in addition $\dim F=\dim G$ and that $\|T\|=1$. We have seen in (3) that

$$\Lambda_H(\mathrm{Im}\, I,\mathrm{Im}\, J)=\max\{\max_{v\in\mathrm{Sph}(\mathrm{Im}\, I)} d_H(v,\mathrm{Ball}(\mathrm{Im}\, J)),\max_{v\in\mathrm{Sph}(\mathrm{Im}\, J)} d_H(v,\mathrm{Ball}(\mathrm{Im}\, I)),$$

so we fix first $z\in\mathrm{Sph}(I(F))$, that is $z=I(x)$ for some $x\in\mathrm{Sph}(F)$. It follows from i) that $\|I(x)-J(T(x))\|\leqslant\|T^{-1}\|-1$, and since $\|J(T(x))\|\leqslant\|x\|\leqslant 1$, it follows that $d_H(I(x),\mathrm{Ball}(J(T(F))))\leqslant\|T^{-1}\|-1$. Now suppose $z\in\mathrm{Sph}(JG)$, and let $y\in\mathrm{Sph}(G)$ be such that $z=Jy$. Since $T:F\to G$ is 1-1 and $\dim F=\dim G$ it follows that T is a bijection, so let $x\in F$ be such that $Tx=y$. Note that $1=\|Tx\|_G\leqslant\|x\|_F\leqslant\|T^{-1}\|$, and let us see that $\mathtt{m}((x/\|T^{-1}\|),-y)\leqslant\|T^{-1}\|-1$: on one side $\|Tx/\|T^{-1}\|-y\|=\|Tx\||1/\|T^{-1}\|-1|\leqslant\|T^{-1}\|-1$, and on the other, given $g\in D$ we have that $|g(y)-g(Tx)/(\|T^{-1}\|\|T^*g\|)|=\|T^*g(x)\||1-1/(\|T^{-1}\|\|T^*g\|)|\leqslant\|T^*g\|\|T^{-1}\|-1\leqslant\|T^{-1}\|-1$. $\square$

Let $T:X\to Y$ be such that $\|T\|\cdot\|T^{-1}\|=\exp(d_{\mathrm{BM}}([\mathtt{m}],[\mathtt{n}]))$, and without loss of generality, we assume that $\|T\|=1$. We apply Claim A.2.1 to T, and we obtain a normed space Z and isometric embeddings $I:X\to Z$ and $J:Y\to Z$ such that (b) holds, that is, $\Lambda_Z(\mathrm{Im}\, I,\mathrm{Im}\, J)\leqslant\exp(d_{\mathrm{BM}}([\mathtt{m}],[\mathtt{n}]))-1$. Since $d_{BM}([\mathtt{m}],[\mathtt{n}])\leqslant\log k$, it follows that $\exp(d_{\mathrm{BM}}([\mathtt{m}],[\mathtt{n}]))-1\leqslant\log k\cdot d_{\mathrm{BM}}([\mathtt{m}],[\mathtt{n}])$. Thus $\gamma([\mathtt{m}],[\mathtt{n}])\leqslant\Lambda_Z(X,Y)\leqslant\log k\cdot d_{\mathrm{BM}}([\mathtt{m}],[\mathtt{n}])$. $\square$

We conclude by proving that $\mathfrak{d}_{k,\mathbb{G}}$ is Lipschitz equivalent to the following intrinsically defined metric on $\mathcal{D}_k(\lambda)$. Recall that ω_2 is the compatible metric $\omega_2((\mathtt{m}_0,\mathtt{m}_1),(\mathtt{n}_0,\mathtt{n}_1)):=\omega(\mathtt{m}_0,\mathtt{n}_0)+\omega(\mathtt{m}_1,\mathtt{n}_1)$, and that $\widetilde{\omega}_2$ is the corresponding quotient metric on $\mathcal{D}_k$.

We start with the following quantitative versions of the continuity of $\nu_{X,Y}$ and of $\nu^2_{k,Y}$. In order to simplify the notation, given a linear operator $T:X\to Y$ on $X=(V,\mathtt{m})$ and $Y=(W,\mathtt{m})$, we will write sometimes $\|T\|_{\mathtt{m},Y}$, $\|T\|_{\mathtt{m},\mathtt{n}}$, $\|T\|_{X,\mathtt{n}}$ to denote the norm $\|T\|_{X,Y}$.

Lemma A.3. *Let Y be a normed space.*

a) Suppose that X is a finite dimensional normed space, and $T,U\in\mathcal{L}^{\dim X}(X,Y)$, are such that

$$\|T-U\|\leqslant 1/\|\mathrm{Id}\|_{\nu_{X,Y}(T),\|\cdot\|_X}.$$

Then,

$$\omega(\nu_{X,Y}(T),\nu_{X,Y}(U))\leqslant\|\mathrm{Id}\|_{\nu_{X,Y}(T),\|\cdot\|_X}\cdot\|T-U\|.$$

b) Suppose that $T, U \in \mathcal{L}_{\lambda}^{k,\mathrm{w}^*}(X^*, X)$ are such that

$$2(e^{\omega(\mathtt{m}_0,\mathtt{m}_1^*)} + \lambda k e^{2\omega(\mathtt{m}_0,\mathtt{m}_1^*)})\|T-U\| \leqslant 1 \tag{11}$$

for some (every) $(\mathtt{m}_0,\mathtt{m}_1) \in \nu_{k,Y}^2(T)$. Then,

$$\widetilde{\omega}_2(\nu_{k,Y}^2(T), \nu_{k,Y}^2(U)) \leqslant 2(2e^{\omega(\mathtt{m}_0,\mathtt{m}_1^*)} + 3\lambda k e^{2\omega(\mathtt{m}_0,\mathtt{m}_1^*)})\|T-U\|. \tag{12}$$

Proof. a): Fix $x \in X$. Then,

$$\begin{aligned}\mathtt{n}(x) = \|U(x)\|_Y \leqslant \|T-U\|_{X,Y}\cdot\|x\|_X + \|T(x)\|_Y \leqslant \|T-U\|_{X,Y}\cdot\|\mathrm{Id}\|_{\mathtt{m},\|\cdot\|_X}\mathtt{m}(x) + \mathtt{m}(x) = \\ =(1+\|T-U\|_{X,Y}\|\mathrm{Id}\|_{\mathtt{m},\|\cdot\|_X})\mathtt{m}(x),\end{aligned}$$

and similarly one shows that $\mathtt{m}(x) \leqslant (1/(1-\|T-U\|_{X,Y}\|\mathrm{Id}\|_{\mathtt{m},\|\cdot\|_X}))\mathtt{n}(x)$. This implies that $\exp(\omega(\mathtt{m},\mathtt{n})) \leqslant (1/(1-\|T-U\|_{X,Y}\|\mathrm{Id}\|_{\mathtt{m},\|\cdot\|_X}))$, and since we are assuming that $\|T-U\| \leqslant 1/\|\mathrm{Id}\|_{\mathtt{m},\|\cdot\|_X}$, we obtain the desired inequality $\omega(\mathtt{m},\mathtt{n}) \leqslant \|\mathrm{Id}\|_{\mathtt{m},\|\cdot\|_X}\cdot\|T-U\|$.

b): Let $(x_j)_{j<k}$ be an Auerbach basis of $\mathrm{Im}\, T$. For each $j<k$ choose $f_j \in X^*$ such that $T(f_j) = x_j$ and $\|f_j\|_{X^*} \leqslant \lambda$. This is possible since we are assuming that $\mathtt{r}_{-1}(T) \leqslant \lambda$.

Let $T_0, U_0 : (\mathbb{F}^k)^* \to X$ be linearly defined by $T_0(u_j^*) := T(f_j) = x_j$ and $U_0(u_j^*) := U(f_j) =: y_j$, and let $\mathtt{m}_0 := \nu_{(\mathbb{F}^k)^*,X}(T_0)$.

Claim A.3.1. $\|T_0 - U_0\|_{\mathtt{m}_0,X} \leqslant \lambda k\|T-U\|_{X^*,X}$. Consequently, U_0 is 1-1.

Proof of Claim: Fix $v := \sum_{j<k} a_j u_j^* \in (\mathbb{F}^k)^*$, and set $f := \sum_{j<k} a_j f_j$. Then, using that $(x_j)_{j<n}$ is an Auerbach basis,

$$\begin{aligned}\|T_0(v) - U_0(v)\|_X =&\|T(f) - U(f)\|_X \leqslant \|T-U\|\|f\| \leqslant \|T-U\|\sum_{j<k}|a_j|\cdot\|f_j\|_{X^*} \leqslant \\ \leqslant&\|T-U\|k\lambda\max_{j<k}|a_j| \leqslant \|T-U\|k\lambda\|\sum_{j<k}a_jx_j\|_X = k\lambda\|T-U\|\mathtt{m}_0(v).\end{aligned}$$

Let us see now that U_0 is 1-1: Suppose that $v \in (\mathbb{F}^k)^*$ is non-zero. Then

$$\begin{aligned}\|U_0(v)\|_X \geqslant&\|T_0(v)\|_X - \|T_0-U_0\|_{\mathtt{m}_0,X}\mathtt{m}_0(v) = (1-\|T_0-U_0\|_{\mathtt{m}_0,X})\mathtt{m}_0(v) \geqslant \\ \geqslant&(1-k\lambda\|T-U\|)\mathtt{m}_0(v) > 0,\end{aligned}$$

where the last inequality follows from the hypothesis in (11). □

Let $\mathtt{n}_0 := \nu_{(\mathbb{F}^k)^*,X}(U_0)$.

Claim A.3.2. $\omega(\mathtt{m}_0,\mathtt{n}_0) \leqslant 2\lambda k\|T-U\|$.

Proof of Claim: Fix $v \in (\mathbb{F}^k)^*$. Then, $\mathtt{n}_0(v) = \|U_0(v)\|_X \leqslant \|T_0(v)\|_X + \|T_0 - U_0\|_{\mathtt{m}_0,X}\mathtt{m}_0(v) \leqslant (1+k\lambda\|T-U\|)\mathtt{m}_0(v)$, and this shows that $\|\mathrm{Id}\|_{\mathtt{m}_0,\mathtt{n}_0} \leqslant 1+k\lambda\|T-U\|$. Similarly one shows that

$$\|\mathrm{Id}\|_{\mathtt{n}_0,\mathtt{m}_0} \leqslant \frac{1}{1-k\lambda\|T-U\|}, \tag{13}$$

and by (11) we have that $\|\mathrm{Id}\|_{\mathtt{n}_0,\mathtt{m}_0} \leqslant 1/(1-k\lambda\|T-U\|) \leqslant 1+2k\lambda\|T-U\|$. This means that $\omega(\mathtt{m}_0,\mathtt{n}_0) \leqslant \log(1+2k\lambda\|T-U\|) \leqslant 2k\lambda\|T-U\|$. □

Let now $T_1, U_1 \in \mathcal{L}^k(\mathbb{F}^k, X)$ be such that $T = T_0 \circ T_1^*$ and $U = U_0 \circ U_1^*$. Set $\mathtt{m}_1 := \nu_{\mathbb{F}^k,X}(T_1)$ and $\mathtt{n}_1 := \nu_{\mathbb{F}^k,X}(U_1)$.

Claim A.3.3. $\omega(\mathtt{m}_1,\mathtt{n}_1) \leqslant 4\|\mathrm{Id}\|_{\mathtt{m}_0,\mathtt{m}_1^*}((1+k\lambda\|\mathrm{Id}\|_{\mathtt{m}_1^*,\mathtt{m}_0})\|T-U\|_{X^*,X})$

Proof of Claim: First, using that $\|T_1^*\|_{X^*,\mathtt{m}_1^*} = \|T_1\|_{\mathtt{m}_1,X} = 1$,

$$\begin{aligned}\|T_1^* - U_1^*\|_{X^*,\mathtt{n}_0} =&\|U_0 \circ T_1^* - U_0 \circ U_1^*\|_{X^*,X} \leqslant \|T - U\|_{X^*,X} + \|U_0 \circ T_1^* - T_0 \circ T_1^*\|_{X^*,X} \leqslant \\ \leqslant&\|T - U\|_{X^*,X} + \|U_0 - T_0\|_{\mathtt{m}_0,X} \cdot \|T_1^*\|_{X^*,\mathtt{m}_0} = \\ \leqslant&\|T - U\|_{X^*,X} + \|U_0 - T_0\|_{\mathtt{m}_0,X} \cdot \|T_1^*\|_{X^*,\mathtt{m}_1^*}\|\mathrm{Id}\|_{\mathtt{m}_1^*,\mathtt{m}_0} = \\ =&\|T - U\|_{X^*,X} + \|U_0 - T_0\|_{\mathtt{m}_0,X}\|\mathrm{Id}\|_{\mathtt{m}_1^*,\mathtt{m}_0}.\end{aligned}$$

Now using the Claim A.3.1, $\|T_1^* - U_1^*\|_{X^*,\mathtt{m}_0} \leqslant (1 + k\lambda\|\mathrm{Id}\|_{\mathtt{m}_1^*,\mathtt{m}_0})\|T - U\|_{X^*,X}$. By this, (13), and the hypothesis (11),

$$\begin{aligned}\|T_1 - U_1\|_{\mathtt{m}_1,X} =&\|T_1^* - U_1^*\|_{X^*,\mathtt{m}_1^*} \leqslant \|T_1^* - U_1^*\|_{X^*,\mathtt{n}_0}\|\mathrm{Id}\|_{\mathtt{n}_0,\mathtt{m}_0}\|\mathrm{Id}\|_{\mathtt{m}_0,\mathtt{m}_1^*} \leqslant \\ \leqslant&\frac{(1 + k\lambda\|\mathrm{Id}\|_{\mathtt{m}_1^*,\mathtt{m}_0})\|T - U\|_{X^*,X}}{1 - k\lambda\|T - U\|_{X^*,X}} \cdot \|\mathrm{Id}\|_{\mathtt{m}_0,\mathtt{m}_1^*} \leqslant \\ \leqslant&2((1 + k\lambda\|\mathrm{Id}\|_{\mathtt{m}_1^*,\mathtt{m}_0})\|T - U\|_{X^*,X}) \cdot \|\mathrm{Id}\|_{\mathtt{m}_0,\mathtt{m}_1^*}.\end{aligned}$$

From this we proceed as in the proof of the Claim A.3.2 to obtain that

$$\omega(\mathtt{m}_1, \mathtt{n}_1) \leqslant 4\|\mathrm{Id}\|_{\mathtt{m}_0,\mathtt{m}_1^*}((1 + k\lambda\|\mathrm{Id}\|_{\mathtt{m}_1^*,\mathtt{m}_0})\|T - U\|_{X^*,X}).$$

From Claim A.3.2, Claim A.3.3 and the fact that $\|\mathrm{Id}\|_{\mathtt{m}_0,\mathtt{m}_1^*} \cdot \|\mathrm{Id}\|_{\mathtt{m}_1^*,\mathtt{m}_0} \geqslant 1$, we obtain the desired estimation in (12). □

□

We have the following (not sharp) inequalities.

Proposition A.4. *For $k \geqslant 2$ the metrics $\mathfrak{d}_{k,\ell_\infty^\infty}$ and $\widetilde{w}_2$ are Lipschitz equivalent on $\mathcal{D}_k(\ell_\infty^\infty; \lambda)$. In fact, for every $[\mathbf{m}], [\mathbf{n}] \in \mathcal{D}_k(\ell_\infty^\infty; \lambda)$ we have that*

$$\frac{1}{16\log(\lambda k)k\lambda^3}\widetilde{\omega}_2([\mathbf{m}], [\mathbf{n}]) \leqslant \mathfrak{d}_{k,\ell_\infty^\infty}([\mathbf{m}], [\mathbf{n}]) \leqslant k^2\lambda^3\widetilde{\omega}_2([\mathbf{m}], [\mathbf{n}]). \tag{14}$$

Proof. We first estimate the $\widetilde{\omega}_2$-diameter of $\mathcal{D}_k(\ell_\infty^\infty, \lambda)$.

Claim A.4.1. For every $[(\mathtt{m}_0, \mathtt{m}_1)], [(\mathtt{n}_0, \mathtt{n}_1)] \in \mathcal{D}_k(\ell_\infty^\infty; \lambda)$ one has that

$$\widetilde{\omega}_2([(\mathtt{m}_0, \mathtt{m}_1)], [(\mathtt{n}_0, \mathtt{n}_1)]) \leqslant 2(\log\lambda + \max\{d_{\mathrm{BM}}([\mathtt{m}_0], [\mathtt{n}_0]), d_{\mathrm{BM}}([\mathtt{m}_1], [\mathtt{n}_1])\}). \tag{15}$$

Consequently, $\mathrm{diam}(\mathcal{D}_k(\ell_\infty^\infty; \lambda)) \leqslant 2\log(\lambda k)$.

Proof of Claim: Given $[(\mathtt{m}_0, \mathtt{m}_1)], [(\mathtt{n}_0, \mathtt{n}_1)] \in \mathcal{D}_k(\ell_\infty^\infty; \lambda)$, we have that $\omega(\mathtt{m}_0, \mathtt{m}_1^*), \omega(\mathtt{n}_0, \mathtt{n}_1^*) \leqslant \log(\lambda)$. Choose $\Delta \in \mathrm{GL}(\mathbb{F}^k)$ with $\widetilde{\omega}([\mathtt{m}_0], [\mathtt{n}_0]) = \omega(\mathtt{m}_0, \Delta \cdot \mathtt{n}_0)$. Then,

$$\begin{aligned}\omega(\mathtt{m}_1, \Delta \cdot \mathtt{n}_1) =&\omega(\mathtt{m}_1^*, \Delta \cdot \mathtt{n}_1^*) \leqslant \omega(\mathtt{m}_1^*, \mathtt{m}_0) + \omega(\mathtt{m}_0, \Delta \cdot \mathtt{n}_0) + \omega(\Delta \cdot \mathtt{n}_0, \Delta \cdot \mathtt{n}_1^*) \leqslant \\ \leqslant&2\log\lambda + \widetilde{\omega}([\mathtt{m}_0], [\mathtt{n}_0]) \leqslant 2\log\lambda + d_{\mathrm{BM}}([\mathtt{m}_0], [\mathtt{n}_0]).\end{aligned}$$

Similarly one shows that $\omega(\mathtt{m}_0, \Delta \cdot \mathtt{n}_0) \leqslant 2\log\lambda + d_{\mathrm{BM}}([\mathtt{m}_1], [\mathtt{n}_1])$ if $\Delta \in \mathrm{GL}(\mathbb{F}^k)$ is such that $\widetilde{\omega}([\mathtt{m}_1], [\mathtt{n}_1]) = \omega(\mathtt{m}_1, \Delta \cdot \mathtt{n}_1)$. From here we get easily the inequality in (15). □

Let us see now that $\widetilde{\omega}_2([\mathbf{m}], [\mathbf{n}]) \leqslant 16\log(\lambda k)k\lambda^3\mathfrak{d}_{k,\ell_\infty^\infty}([\mathbf{m}], [\mathbf{n}])$ for $[\mathbf{m}], [\mathbf{n}] \in \mathcal{D}_k(\ell_\infty^\infty; \lambda)$: Suppose first that $\mathfrak{d}_{k,\ell_\infty^\infty}([\mathbf{m}], [\mathbf{n}]) < 1/(8k\lambda^3)$. For a fixed $\varepsilon > 0$, choose $T, U \in \mathcal{L}_\lambda^{k,\mathrm{w}^*}(\ell_1^\infty, \ell_\infty^\infty)$ such that $\nu_{k,\ell_\infty^\infty}^2(T) = [\mathbf{m}] = [(\mathtt{m}_0, \mathtt{m}_1)]$, $\nu_{k,\ell_\infty^\infty}^2(U) = [\mathbf{n}]$ and

$$\|T - U\| \leqslant \min\{\mathfrak{d}_{k,\ell_\infty^\infty}([\mathbf{m}], [\mathbf{n}]) + \varepsilon, \frac{1}{8k\lambda^3}\}.$$

We have that $[\mathbf{m}] \in \mathcal{D}_k(\lambda)$, that is, $\omega(\mathtt{m}_0, \mathtt{m}_1^*) \leqslant \log\lambda$. It follows that $2(\exp(\omega(\mathtt{m}_0, \mathtt{m}_1^*)) + \lambda k \exp(2\omega(\mathtt{m}_0, \mathtt{m}_1^*))) \leqslant 2(\lambda + k\lambda^3) \leqslant 8k\lambda^3$ and consequently T, U, $\mathtt{m}_0$ and $\mathtt{m}_1$ satisfy (11). This implies by *b)* in Proposition A.3 that

$$\widetilde{\omega}_2([\mathbf{m}], [\mathbf{n}]) \leqslant 2(2e^{\omega(\mathtt{m}_0, \mathtt{m}_1^*)} + 3\lambda k e^{2\omega(\mathtt{m}_0, \mathtt{m}_1^*)})\|T - U\| \leqslant 2(2\lambda + 3k\lambda^3)\|T - U\| \leqslant 8k\lambda^3\|T - U\|.$$

Since $\varepsilon > 0$ is arbitrary, we get that $\widetilde{\omega}_2([\mathbf{m}], [\mathbf{n}]) \leqslant 8k\lambda^3 \mathfrak{d}_{k,\ell_\infty^\infty}([\mathbf{m}], [\mathbf{n}])$. Now suppose that $\mathfrak{d}_{k,E}([\mathbf{m}], [\mathbf{n}]) \geqslant 1/(8k\lambda^3)$. By the Claim A.4.1, we have that $\operatorname{diam}(\mathcal{D}_k(\ell_\infty^\infty; \lambda)) \leqslant 2\log(\lambda k)$. Hence, $\widetilde{\omega}_2([\mathbf{m}], [\mathbf{n}]) \leqslant 2\log(\lambda k) \cdot (8k\lambda^3)\mathfrak{d}_{k,\ell_\infty^\infty}([\mathbf{m}], [\mathbf{n}]) = 16\log(\lambda k) k\lambda^3 \mathfrak{d}_{k,\ell_\infty^\infty}([\mathbf{m}], [\mathbf{n}])$.

It rests to show the second inequality in (14).

Claim A.4.2. For every $[\mathbf{m}], [\mathbf{n}] \in \mathcal{D}_k(\ell_\infty^\infty; \lambda)$ one has that $\mathfrak{d}_{k,\ell_\infty^\infty}([\mathbf{m}], [\mathbf{n}]) \leqslant k^2\lambda^3 \widetilde{\omega}_2([\mathbf{m}], [\mathbf{n}])$.

Proof of Claim: As we have pointed out before, the infimum defining the metric $\widetilde{\omega}_2$ is a minimum, so let $\Delta \in \mathrm{GL}(\mathbb{F}^k)$ be such that $\widetilde{\omega}_2([(\mathtt{m}_0, \mathtt{m}_1)], [(\mathtt{n}_0, \mathtt{n}_1)]) = \omega(\mathtt{m}_0, \Delta \cdot \mathtt{n}_0) + \omega(\mathtt{m}_1, \Delta \cdot \mathtt{n}_1)$. We use Proposition A.1 *a)* to find $T_0, U_0 \in \mathcal{L}((\mathbb{F}^k)^*, \ell_\infty^\infty)$, $T_1, U_1 \in \mathcal{L}(\mathbb{F}^k, \ell_\infty^\infty)$ such that:

v) $\mathtt{m}_0 = \nu_{(\mathbb{F}^k)^*, \ell_\infty^\infty}(T_0)$, $\Delta \cdot \mathtt{n}_0 = \nu_{(\mathbb{F}^k)^*, \ell_\infty^\infty}(U_0)$, $\mathtt{m}_1 = \nu_{\mathbb{F}^k, \ell_\infty^\infty}(T_1)$ and $\Delta \cdot \mathtt{m}_1 = \nu_{\mathbb{F}^k, \ell_\infty^\infty}(U_1)$.

vi) $\alpha_{((\mathbb{F}^k)^*, \mathtt{m}_1^*)}(\mathtt{m}_0, \Delta \cdot \mathtt{n}_0) = \|T_0 - U_0\|_{((\mathbb{F}^k)^*, \mathtt{m}_1^*), \ell_\infty^\infty}$ and $\alpha_{((\mathbb{F}^k)^*, \mathtt{m}_0)}(\mathtt{m}_1^*, \Delta \cdot \mathtt{n}_1^*) = \|T_1 - U_1\|_{((\mathbb{F}^k)^*, \mathtt{m}_0), \ell_\infty^\infty}$.

Let $T := T_0 \circ T_1^*$, $U := U_0 \circ U_1^*$. It follows that $\nu^2([T]) = [\mathbf{m}]$, $\nu^2([U]) = [\mathbf{n}]$, and $\mathfrak{d}_{k,\ell_\infty^\infty}([\mathbf{m}], [\mathbf{n}]) \leqslant \|T - U\|_{\ell_1^\infty, \ell_\infty^\infty}$. Now let $g \in \mathrm{Sph}(\ell_1^\infty)$. Then,

$$\begin{aligned}
\|(T - U)(g)\|_{\ell_\infty^\infty} \leqslant & \|T_0(T_1^* - U_1^*)(g)\|_{\ell_\infty^\infty} + \|(T_0 - U_0)(U_1^*(g))\|_{\ell_\infty^\infty} \leqslant \\
\leqslant & \|T_0\|_{((\mathbb{F}^k)^*, \mathtt{m}_0), \ell_\infty^\infty} \|T_1^* - U_1^*\|_{\ell_1^\infty, ((\mathbb{F}^k)^*, \mathtt{m}_0)} + \|T_0 - U_0\|_{(\mathbb{F}^k, \mathtt{m}_1)^*, \ell_\infty^\infty} \|U_1^*\|_{\ell_1^\infty, (\mathbb{F}^k, \mathtt{m}_1)^*} = \\
= & \alpha_{((\mathbb{F}^k)^*, \mathtt{m}_0)}(\mathtt{m}_1^*, \Delta \cdot \mathtt{n}_1^*) + \alpha_{(\mathbb{F}^k, \mathtt{m}_1)^*}(\mathtt{m}_0, \Delta \cdot \mathtt{n}_0) \qquad (16)
\end{aligned}$$

We want to estimate the α quantities above by the corresponding ω ones by using Proposition A.1 *b)*. It follows by the estimation on the diameter $\operatorname{diam}(\mathcal{D}_k(\ell_\infty^\infty; \lambda)) \leqslant 2\log(\lambda k)$ on Claim A.4.1, and the fact that $\omega(\mathtt{m}_0, \mathtt{m}_1^*) \leqslant \log(\lambda)$,

$$\begin{aligned}
\max\{\|\mathrm{Id}\|_{\Delta \cdot \mathtt{n}_1^*, \mathtt{m}_0}, \|\mathrm{Id}\|_{\Delta \cdot \mathtt{n}_0, \mathtt{m}_1^*}\} \leqslant & \max\{\|\mathrm{Id}\|_{\mathtt{m}_1^*, \mathtt{m}_0} \cdot \|\mathrm{Id}\|_{\Delta \cdot \mathtt{n}_1^*, \mathtt{m}_1^*}, \|\mathrm{Id}\|_{\mathtt{m}_0, \mathtt{m}_1^*} \cdot \|\mathrm{Id}\|_{\Delta \cdot \mathtt{n}_0, \mathtt{m}_0}\} \leqslant \\
\leqslant & \lambda \max\{\|\mathrm{Id}\|_{\Delta \cdot \mathtt{n}_1^*, \mathtt{m}_1^*}, \|\mathrm{Id}\|_{\Delta \cdot \mathtt{n}_0, \mathtt{m}_0}\} = \lambda \max\{\|\mathrm{Id}\|_{\Delta \cdot \mathtt{n}_1, \mathtt{m}_1}, \|\mathrm{Id}\|_{\Delta \cdot \mathtt{n}_0, \mathtt{m}_0}\} \leqslant \\
\leqslant & \lambda e^{\widetilde{\omega}_2([\mathbf{m}], [\mathbf{n}])} \leqslant k^2\lambda^3,
\end{aligned}$$

and similarly one shows that $\max\{\|\mathrm{Id}\|_{\mathtt{m}_0, \Delta \cdot \mathtt{n}_1^*}, \|\mathrm{Id}\|_{\mathtt{m}_1^*, \Delta \cdot \mathtt{n}_0}\} \leqslant k^2\lambda^3$. This means that

$$\omega(\mathtt{m}_0, \Delta \cdot \mathtt{m}_1^*), \omega(\mathtt{m}_1^*, \Delta \cdot \mathtt{n}_0) \leqslant k^2\lambda^3.$$

It follows from this, the inequality in (16) and Proposition A.1 *b)* that

$$\mathfrak{d}_{k,\ell_\infty^\infty}([\mathbf{m}], [\mathbf{n}]) \leqslant \|T - U\|_{\ell_1^\infty, \ell_\infty^\infty} \leqslant k^2\lambda^3(\omega(\mathtt{m}_1, \Delta \cdot \mathtt{n}_1) + \omega(\mathtt{m}_0, \Delta \cdot \mathtt{n}_0)) \leqslant k^2\lambda^3 \widetilde{\omega}_2([\mathbf{m}], [\mathbf{n}]). \quad \square \quad \square$$

We do not know a similar explicit description of the extrinsic metrics for p's other than ∞.

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