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ON A SOBOLEV CRITICAL PROBLEM FOR THE SUPERPOSITION OF A LOCAL AND NONLOCAL OPERATOR WITH THE "WRONG SIGN"

STEFANO BIAGI, SERENA DIPIERRO, ENRICO VALDINOCI, AND EUGENIO VECCHI

Abstract. We study a critical problem for an operator of mixed order obtained by the superposition of a Laplacian with a fractional Laplacian.

The main novelty is that we consider a mixed operator of the form $-\Delta - \gamma(-\Delta)^s$, namely we suppose that the fractional Laplacian has the "wrong sign" and can be seen as a nonlocal perturbation of the purely local case, which is needed to produce a nontrivial solution of the critical problem.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ (with $n \geq 3$) be an open and bounded set with smooth enough (say Lipschitz) boundary $\partial\Omega$. We consider the following problem

$$\begin{cases} -\Delta u = \gamma\,(-\Delta)^s u + |u|^{2^*-2}u & \text{in } \Omega, \\ u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.1}$$

The operator $(-\Delta)^s$ is the fractional Laplacian, defined, for $s \in (0,1)$, as

$$(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy.$$

The parameter γ is assumed to be positive and suitably small (namely, $\gamma \in (0, C_{emb})$, with a notation that will be clarified here below). As usual, $2^* := \frac{2n}{n-2}$ is the Sobolev critical exponent. Our aim is to prove that under the previous assumption on γ, problem (1.1) admits a nontrivial weak solution in the appropriate variational space. The cases $n = 1$ and $n = 2$ are usually considered separately. Indeed, the case $n = 2$ leads to critical nonlinearities of exponential type closely related to the Moser-Trudinger inequality.

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- *Assumptions and motivations.* It is known, see e.g. [16], that the Sobolev space $H^1(\mathbb{R}^n)$ continuously embeds into the fractional Sobolev space $H^s(\mathbb{R}^n)$ for $s \in (0,1)$, i.e. there exists a positive constant $C > 0$ (depending on s and n) such that

$$C\,[u]_s^2 \le \|u\|^2_{H^1(\mathbb{R}^n)}, \tag{1.2}$$

for every $u \in H^1(\mathbb{R}^n)$. Here and in what follows, we use the notation

$$[u]_s := \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \right)^{1/2}$$

for the Gagliardo seminorm of u.

Together with the classical Poincaré inequality, formula (1.2) shows that there exists $C > 0$ (depending on Ω as well) such that

$$C[u]_s^2 \le \int_\Omega |\nabla u|^2\,dx = \|\nabla u\|^2_{L^2(\Omega)}, \tag{1.3}$$

for every $u \in H^1_0(\Omega)$. Hence, we can define

$$C_{emb} := \inf\left\{ \|\nabla u\|^2_{L^2(\Omega)} : u \in H^1_0(\Omega),\ [u]_s^2 = 1 \right\}.$$

The assumption $\gamma \in (0, C_{emb})$ can now be explained: we are interested in dealing with a local and nonlocal operator *with the wrong sign*, namely

$$-\Delta - \gamma\,(-\Delta)^s, \tag{1.4}$$

but at the same time we need some restrictions on the coefficient γ so that the operator is positive definite. As a "historical" comment, we are unsure who first introduced the terminology "wrong sign" to denote superpositions of operators with coefficients of different signs; however, this jargon already appears in [6]. The jargon "mixed local and nonlocal operator" was also formerly used in [7, 24]. Mixed operators with the "wrong sign" were also studied specifically in [30] and referred to as "local and nonlocal operators" (however, [30] cannot include the specific case of a Laplacian minus the fractional Laplacian, see the assumptions after (1.1) there). We also mention [29, 27, 13] where the authors refer to similar operators as *indefinite* or *nonpositive* operators.

An extensive study of Lévy measures associated with integro-differential operators was presented in [26]. In this spirit, one can also consider superposition operators averaged over a measure. The case of a signed measure can be seen as a generalization to a continuum (or to an infinite series) of the sum of two operators in which the lower-order one possesses the wrong sign: for this setting, see [17, 18, 19, 20, 21, 23, 30, 31, 3, 4, 5].

In any case, the study of operators of mixed integer and fractional order is classical (see e.g. [14] in relation with the maximum principle), but it has been recently experiencing an intense revival (see e.g., among the many, [28, equation (1.2)], [33, Section 3.2], [25, Remark 5.6], and [1, equation (2)]).

The main motivation behind the study of problem (1.1) comes from the recent works [8, 2]. The starting point is the following simple observation: let $\Omega \subseteq \mathbb{R}^n$ be an open set and $s \in (0,1)$, then there exists a constant $\mathcal{S}_{n,s}(\Omega) > 0$ such that

$$\mathcal{S}_{n,s}(\Omega)\|u\|^2_{L^{2^*}(\mathbb{R}^n)} \le \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + [u]_s^2, \tag{1.5}$$

for every $u \in C^\infty_0(\mathbb{R}^n)$. This is clearly a consequence of the classical Sobolev inequality.

In principle, the constant $\mathcal{S}_{n,s}(\Omega)$ could depend on n, s and Ω, but this is actually not the case (see [8, Theorem 1.1]), indeed

$$\mathcal{S}_{n,s}(\Omega) = S_n$$

where S_n is the best constant for the classical Sobolev inequality, achieved by the Aubin-Talenti functions.

Moreover, the energy functional

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}[u]_s^2 \tag{1.6}$$

exhibits a lack of scaling invariance, namely

$$E(u_t) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{t^{2s-2}}{2}[u]_s^2, \tag{1.7}$$

where $u_t(x) := t^{(n-2)/2}u(tx)$ is the rescaled function which preserves the H_0^1-norm of u. This second simple remark allows to show that the best constant in the local-nonlocal Sobolev inequality (1.5) is never achieved, see [8, Theorem 1.2].

Since the first variation of the energy functional E in (1.6) is given by the local-nonlocal operator

$$-\Delta + (-\Delta)^s,$$

it becomes natural to attack the study of Sobolev critical problems in bounded domains, with leading operator $-\Delta + (-\Delta)^s$. The first results in this direction are contained in [8, 9, 10] and are all inspired by the seminal paper [12], where the authors showed the importance of adding some kind of perturbation (linear, superlinear, sublinear) when looking for positive weak solutions. In all the mentioned papers [8, 9, 10] the main difficulties (and differences from the purely local case) come once again from the lack of scaling invariance in (1.7), especially when one has to compute the fractional seminorm of a suitably localized version of the Aubin-Talenti functions, see e.g. (iv) in Lemma 3.2 below.

A natural question (raised to us by S. Terracini) is what happens when one considers a Sobolev critical problem with a leading operator of the form (1.4). This could also be interpreted in the spirit of [12] as a nonlocal perturbation of the purely local problem. Our main result reads as follows:

Theorem 1.1. *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary. Then, the following assertions hold:*

1. (***High-dimensional case***) *If $n \geq 5$, for every $\gamma \in (0, C_{emb})$ there exists a nontrivial weak solution $u \in \mathcal{X}^{1,2}(\Omega)$ to problem (1.1).*
2. (***Low-dimensional case***) *If $n = 3, 4$, there exists $\gamma^* \in [0, C_{emb})$ such that problem (1.1) has a nontrivial weak solution $u \in \mathcal{X}^{1,2}(\Omega)$ for every $\gamma \in (\gamma^*, C_{emb})$.*

We mention [17, Corollary 5.4] for a result close in spirit to Theorem 1.1 here. The setting in [17] is however structurally different from the one presented here and relies on the analysis of the Dancer-Fučík spectrum. Also, the coefficients a and b in [17] were considered as strictly positive, while the setting here would correspond to $a = b = 0$, which was not explicitly addressed in [17].

We postpone to Section 2, and in particular to (2.5), the exhaustive discussion on the function space $\mathcal{X}^{1,2}(\Omega)$ and several equivalent definitions. For the moment, we just point out that one of the equivalent definitions reads

$$\mathcal{X}^{1,2}(\Omega) = \big\{u \in H^1(\mathbb{R}^n) : u|_\Omega \in H^1_0(\Omega) \text{ and } u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\big\}.$$

The proof of Theorem 1.1 is contained in Section 3.

2. Preliminary results

Notations. Throughout the paper, we tacitly exploit all the notation listed below; we thus refer the Reader to this list for any non-standard notation encountered.

- $\Omega \subset \mathbb{R}^n$ is an open and bounded set with Lipschitz boundary $\partial\Omega$.
- Given any set E, $|E|$ denotes its n-dimensional Lebesgue measure.
- Given any set E, E^c denotes its complement (in $\mathbb{R}^n$), namely $E^c := \mathbb{R}^n \setminus E$.
- Given $x_0 \in \mathbb{R}^n$ and $r > 0$, we denote by $B_r(x_0)$ the Euclidean open ball of center x_0 and radius r. When $x_0 = 0$, we will omit it and simply write B_r.
- Given any two vectors $v, w \in \mathbb{R}^n$, we denote by $v \cdot w$ the usual scalar product.
- Given any $p \in [1, n)$ we denote by p^* the associated Sobolev exponent (with respect to Euclidean space $\mathbb{R}^n$), that is
$$p^* := \frac{np}{n-p}.$$
- Given any $r \in (1, +\infty)$, we denote by r' the conjugate exponent of r in the usual Hölder inequality, that is,
$$r' := \frac{r}{r-1}.$$
- Given a reflexive Banach space V with dual space V', we denote by $\langle \cdot, \cdot \rangle$ the duality product.
- Given a measurable set E, we denote by χ_E its characteristic function.
- Given any function v, we denote the positive and negative part of v by
$$v^+ := \max\{v, 0\} \quad \text{and} \quad v^- := -\min\{v, 0\}. \tag{2.1}$$
- S_n will denote the best constant in the classical Sobolev inequality which is achieved by the family of Aubin-Talenti functions defined, for every $y \in \mathbb{R}^n$ and $\varepsilon > 0$, as
$$V_{\varepsilon,y}(x) := \frac{\varepsilon^{(n-2)/2}}{(\varepsilon^2 + |x-y|^2)^{(n-2)/2}}. \tag{2.2}$$
We will avoid writing the dependence on y when $y = 0$.

We now set the adequate functional setting to study problem (1.1). We refer to e.g. [8, Section 2] for more details.

Let $s \in (0, 1)$ and let $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We set

$$[u]_s := \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{1/2}, \tag{2.3}$$

and we refer to $[u]_s$ as the *Gagliardo seminorm* of u (of order s).

Let $\varnothing \neq \Omega \subseteq \mathbb{R}^n$ (here $n \geq 3$ would suffice) be an open set, not necessarily bounded. We define the function space $\mathcal{X}^{1,2}(\Omega)$ as

$$\mathcal{X}^{1,2}(\Omega) := \overline{C^\infty_0(\Omega)}^{\,\rho(\cdot)}$$

i.e. as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\rho(u) := \Big(\|\nabla u\|^2_{L^2(\mathbb{R}^n)} + [u]_s^2\Big)^{1/2}. \tag{2.4}$$

We stress that the norm $\rho(\cdot)$ is induced by the scalar product

$$\langle u, v\rangle_\rho := \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx + \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dx\, dy,$$

and therefore $\mathcal{X}^{1,2}(\Omega)$ is a Hilbert space.

For our purposes, we are interested in the case Ω bounded. Recalling the continuous embedding given by (1.2), and combining it with the classical Poincaré inequality, we find that $\rho(\cdot)$ and the full H^1-norm in $\mathbb{R}^n$ are actually equivalent on the space $C_0^\infty(\Omega)$, and hence

$$\begin{aligned}\mathcal{X}^{1,2}(\Omega) &= \overline{C_0^\infty(\Omega)}^{\,\|\cdot\|_{H^1(\mathbb{R}^n)}} \\ &= \big\{u \in H^1(\mathbb{R}^n) :\, u|_\Omega \in H_0^1(\Omega) \text{ and } u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\big\}.\end{aligned} \tag{2.5}$$

We stress that in [21, equation (2.10)], the authors chose a different norm modeled on the energy associated to the local-nonlocal operator. In our notation their choice reads as

$$\Big(\|\nabla u\|^2_{L^2(\mathbb{R}^n)} - \gamma[u]_s^2\Big)^{1/2}.$$

It turns out that their choice is actually equivalent to the one taken in the present paper, thanks to [17, Lemma 2.1].

Remark 2.1. On account of (2.5), we derive the following useful facts.

i) $\mathcal{X}^{1,2}(\Omega)$ is *continuously embedded into* $L^p(\Omega)$ for every $1 \le p \le 2^*$; moreover, if $1 \le p < 2^*$, this embedding is *compact*.

ii) Taking into account the continuous embedding in (1.2), and since this embedding *is actually compact* (see, e.g., [22, Theorem 1.3], applied here with $p := \widetilde{p} := 2$, $s := 1$ and $\widetilde{s} := s$), we get that, for all $s \in (0,1)$,

$$\mathcal{X}^{1,2}(\Omega) \text{ is compactly embedded into } H^s(\mathbb{R}^n).$$

iii) Moreover, it is easily seen that

$$C_{emb} = \inf\big\{\|\nabla u\|^2_{L^2(\Omega)} :\, u \in \mathcal{X}^{1,2}(\Omega),\, [u]_s^2 = 1\big\}, \tag{2.6}$$

and this infimum is achieved, that is, there exists a function $\Phi_0 \in \mathcal{X}^{1,2}(\Omega)$ such that

$$[\Phi_0]_s^2 = 1 \quad \text{and} \quad \|\nabla \Phi_0\|^2_{L^2(\Omega)} = C_{emb}.$$

We end this section by giving the precise definition of *weak solution* to (1.1).

Definition 2.2. We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is a *weak solution* to problem (1.1) if it satisfies the following properties:

1) $u \in \mathcal{X}^{1,2}(\Omega)$;
2) for every test function $v \in \mathcal{X}^{1,2}(\Omega)$, we have

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx - \gamma \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dx\, dy = \int_\Omega |u|^{2^*-2} uv \, dx.$$

3. Proof of Theorem 1.1

We follow the ideas presented in [12] combined with the crucial expansion provided by Proposition 3.4. To this aim, let us first define the functional

$$Q_\gamma(u) := \int_\Omega |\nabla u|^2\,dx - \gamma\,[u]_s^2, \quad u \in \mathcal{X}^{1,2}(\Omega), \tag{3.1}$$

and let us consider the following constrained minimization problem

$$S(\gamma) := \inf\left\{Q_\gamma(u) : u \in \mathcal{X}^{1,2}(\Omega), \|u\|_{L^{2^*}(\Omega)} = 1\right\}. \tag{3.2}$$

Obviously, by formally choosing $\gamma = 0$, the above minimization problem reduces to finding the best Sobolev constant S_n in Ω, which is known to be never achieved; as a consequence, we have the following upper bound for $S(\gamma)$:

$$S(\gamma) \le S_n \quad \text{for every } 0 < \gamma < C_{emb}.$$

We also notice that, since we are assuming $0 < \gamma < C_{emb}$ (recall that C_{emb} is the constant defined in (2.6)), for every $u \in \mathcal{X}^{1,2}(\Omega)$ we have

$$\begin{aligned} Q_\gamma(u) &= \int_\Omega |\nabla u|^2\,dx - \gamma\,[u]_s^2 \ge \left(1 - \frac{\gamma}{C_{emb}}\right)\int_\Omega |\nabla u|^2\,dx \\ &\ge \left(1 - \frac{\gamma}{C_{emb}}\right) S_n > 0. \end{aligned}$$

Gathering the last two formulas in display, we conclude that

$$0 < \left(1 - \frac{\gamma}{C_{emb}}\right) S_n \le S(\gamma) \le S_n \quad \text{for every } 0 < \gamma < C_{emb}. \tag{3.3}$$

Now, the key tool for the proof of Theorem 1.1 is the following.

Proposition 3.1. *Let $n \ge 3$. If $S(\gamma) < S_n$, the infimum in (3.2) is achieved. In this case, if $u_0 \in \mathcal{X}^{1,2}(\Omega)$ achieves this infimum, then*

$$\widetilde{u} = S(\gamma)^{1/(2^*-2)} u_0 \tag{3.4}$$

is a nontrivial weak solution to problem (1.1).

Proof. The argument and the notation that we will adopt is heavily influenced by the original one used by Brezis and Nirenberg [12]. In particular, since we will consider a minimizing sequence denoted by u_j, we also denote by $o(1)$ quantities that tend to zero as $j \to +\infty$. To ease the readability, we split the proof into two steps.

Step I): We first prove that, if $S(\gamma) < S_n$, then the infimum in (3.2) is achieved. For this, let u_j be a minimizing sequence of functions in $\mathcal{X}^{1,2}(\Omega)$. In particular, it holds that $\|u_j\|_{L^{2^*}(\Omega)} = 1$ for every $j \in \mathbb{N}$. In this way, we have that

$$\lim_{j\to+\infty} Q_\gamma(u_j) = S(\gamma). \tag{3.5}$$

Moreover, since $\gamma \in (0, C_{emb})$, it follows that

$$\left(1 - \frac{\gamma}{C_{emb}}\right)\int_\Omega |\nabla u_j|^2\,dx \le Q_\gamma(u_j). \tag{3.6}$$

From this and (3.5), we deduce that u_j is a bounded sequence in the Hilbert space $\mathcal{X}^{1,2}(\Omega)$. Therefore, up to subsequences and using Remark 2.1-ii), we infer the existence of $u \in \mathcal{X}^{1,2}(\Omega)$ such that

- $u_j \to u$ weakly in $\mathcal{X}^{1,2}(\Omega)$;
- $[u_j - u]_s \to 0$;
- $u_j \to u$ a.e. in Ω,

as $j \to +\infty$. We stress that the latter implies that $\|u\|_{L^{2^*}(\Omega)} \le 1$.

Now, since $\|u_j\|_{L^{2^*}(\Omega)} = 1$ for every $j \in \mathbb{N}$, it follows that

$$\int_\Omega |\nabla u_j|^2\,dx \ge S_n, \qquad \text{for every } j \in \mathbb{N},$$

and hence, recalling (3.5), we get

$$\begin{aligned}\gamma\,[u]_s^2 &= \gamma \lim_{j\to+\infty} [u_j]_s^2 = \limsup_{j\to+\infty} \left(\int_\Omega |\nabla u_j|^2\,dx - Q_\gamma(u_j)\right)\\ &\ge \limsup_{j\to+\infty} \left(S_n - Q_\gamma(u_j)\right) = S_n - S(\gamma) > 0.\end{aligned}$$

This shows that $u \not\equiv 0$.

Set now $v_j := u_j - u$ and notice that, as $j \to +\infty$,

$$\begin{aligned}&\|u\|^2_{H^1_0(\Omega)} + \|v_j\|^2_{H^1_0(\Omega)} - \gamma[u]_s^2\\ &\quad= \|u\|^2_{H^1_0(\Omega)} + \|u_j\|^2_{H^1_0(\Omega)} + \|u\|^2_{H^1_0(\Omega)} - 2\int_\Omega \nabla u\cdot\nabla u_j\,dx - \gamma[u]_s^2\\ &\quad= 2\|u\|^2_{H^1_0(\Omega)} + Q_\gamma(u_j) + \gamma\big([u_j]_s^2 - [u]_s^2\big) - 2\left(\|u\|^2_{H^1_0(\Omega)} + o(1)\right)\\ &\quad= S(\gamma) + o(1).\end{aligned} \tag{3.7}$$

Moreover, since $v_j \to 0$ weakly in $\mathcal{X}^{1,2}(\Omega)$ (hence, v_j is bounded in $\mathcal{X}^{1,2}(\Omega)$), it follows that v_j is bounded in $L^{2^*}(\Omega)$ as well. This allows to use the Brezis-Lieb Lemma in [11] to get that, as $j \to +\infty$,

$$1 = \|u_j\|^{2^*}_{L^{2^*}(\Omega)} = \|u + v_j\|^{2^*}_{L^{2^*}(\Omega)} = \|u\|^{2^*}_{L^{2^*}(\Omega)} + \|v_j\|^{2^*}_{L^{2^*}(\Omega)} + o(1).$$

This yields, as $j \to +\infty$,

$$\begin{aligned}1 &= \left(\|u\|^{2^*}_{L^{2^*}(\Omega)} + \|v_j\|^{2^*}_{L^{2^*}(\Omega)} + o(1)\right)^{2/2^*}\\ &\le \|u\|^2_{L^{2^*}(\Omega)} + \|v_j\|^2_{L^{2^*}(\Omega)} + o(1)\\ &\le \|u\|^2_{L^{2^*}(\Omega)} + \frac{1}{S_n}\int_\Omega |\nabla v_j|^2\,dx + o(1).\end{aligned}$$

Recalling that $S(\gamma) > 0$, we find that,

$$S(\gamma) \le S(\gamma)\,\|u\|^2_{L^{2^*}(\Omega)} + \frac{S(\gamma)}{S_n}\int_\Omega |\nabla v_j|^2\,dx + o(1) \quad \text{as } j \to +\infty,$$

and thus, by using (3.7),

$$\|u\|^2_{H^1_0(\Omega)} + \|v_j\|^2_{H^1_0(\Omega)} - \gamma[u]_s^2 \le S(\gamma)\,\|u\|^2_{L^{2^*}(\Omega)} + \frac{S(\gamma)}{S_n}\int_\Omega |\nabla v_j|^2\,dx + o(1).$$

From this, and exploiting that $S(\gamma) < S_n$, we get that, as $j \to +\infty$,

$$\|u\|^2_{H^1_0(\Omega)} - \gamma[u]_s^2 \le S(\gamma)\|u\|^2_{L^{2^*}(\Omega)} + o(1). \tag{3.8}$$

Now, since $u \not\equiv 0$, we can define

$$u_0 := \frac{u}{\|u\|_{L^{2^*}(\Omega)}}$$

and, plugging it into (3.8) and taking the limit as $j \to +\infty$, we obtain

$$S(\gamma) \le Q_\gamma(u_0) \le S(\gamma).$$

This says, that u_0 achieves the infimum in (3.2).

Step II): We now turn to show that, if u_0 achieves the infimum in (3.2), then there exists a nontrivial solution of problem (1.1), given by (3.4).

To this end, we first observe that, if u_0 achieves the infimum in (3.2), by applying the Lagrange Multiplier Theorem we can infer the existence of $\mu \in \mathbb{R}$ such that, for every $\psi \in \mathcal{X}^{1,2}(\Omega)$,

$$\begin{aligned} &\int_\Omega \nabla u_0 \cdot \nabla \psi \, dx - \gamma \iint_{\mathbb{R}^{2n}} \frac{(u_0(x)-u_0(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}} \, dx\, dy \\ &\qquad = \mu \int_\Omega |u_0|^{2^*-2} u_0 \psi \, dx. \end{aligned} \tag{3.9}$$

In particular, we can choose $\psi := u_0$ in (3.9) and this yields

$$Q_\gamma(u_0) = \mu \int_\Omega |u_0|^{2^*} \, dx = \mu,$$

which, recalling that $Q_\gamma(u_0) = S(\gamma)$, shows that

$$\mu = S(\gamma).$$

Setting now $\widetilde{u} := k u_0$, with $k := S(\gamma)^{1/(2^*-2)}$, we get

$$\begin{aligned} &\int_\Omega \nabla \widetilde{u} \cdot \nabla \psi \, dx - \gamma \iint_{\mathbb{R}^{2n}} \frac{(\widetilde{u}(x)-\widetilde{u}(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}} \, dx\, dy \\ &\qquad = S(\gamma)^{1/(2^*-2)+1} \int_\Omega |u_0|^{2^*-2} u_0 \psi \, dx \\ &\qquad = \int_\Omega |\widetilde{u}|^{2^*-2} \widetilde{u} \psi \, dx, \end{aligned}$$

for every $\psi \in \mathcal{X}^{1,2}(\Omega)$, and this shows that $\widetilde{u}$ solves (1.1). □

In view of Proposition 3.1, the path towards establishing the existence of a weak solution to problem (1.1) is laid out: indeed, it remains to show that

$$S(\gamma) < S_n. \tag{3.10}$$

To prove this, we need to distinguish between two cases, the high-dimensional case $n \ge 5$ and the low-dimensional case $n = 3, 4$, which we now deal with separately.

1) The high-dimensional case $n \ge 5$. In this first case, we are able to prove that (3.10) actually holds *for every* $\gamma \in (0, C_{emb})$ by using an *ad-hoc* competitor function in (3.2). To begin with, we have the following technical results.

Lemma 3.2. *Assume that $0 \in \Omega$. Let $\delta > 0$ be such that $B_{4\delta} \Subset \Omega$. Let $\varphi \in C_0^\infty(\mathbb{R}^n, [0,1])$ be such that $\varphi \equiv 1$ in B_δ and $\varphi \equiv 0$ in $B_{2\delta}^c$.*

For every $\varepsilon > 0$, let

$$U_\varepsilon(x) := \varphi(x)\, V_\varepsilon(x), \quad \text{for all } x \in \mathbb{R}^n, \tag{3.11}$$

where V_ε is as in (2.2) with $y = 0$.

Then, the following holds:

(i) $\dfrac{\|\nabla V_\varepsilon\|^2_{L^2(\mathbb{R}^n)}}{\|V_\varepsilon\|^2_{L^{2^*}(\mathbb{R}^n)}} = S_n$;

(ii) $\|\nabla U_\varepsilon\|^2_{L^2(\Omega)} = K_1 + O(\varepsilon^{n-2})$, *as* $\varepsilon \to 0^+$;

(iii) $\|U_\varepsilon\|^{2^*}_{L^{2^*}(\Omega)} = K_2 + O(\varepsilon^n)$, *as* $\varepsilon \to 0^+$, *where* $\dfrac{K_1}{K_2^{1-2/n}} = S_n$;

(iv) $[U_\varepsilon]^2_s = O(\varepsilon^{2-2s})$, *as* $\varepsilon \to 0^+$.

Proof. The validity of (i) follows from the classical result by Talenti [34]. For (ii) and (iii) we refer to [12, Lemma 1.1], while (iv) is contained in the proof of [8, Lemma 4.10]. □

We stress that the assumption $n \geq 5$ is actually not needed in the previous Lemma 3.2: indeed, the typical term which exhibits different behaviours according to the dimension n is $\|U_\varepsilon\|_{L^2(\mathbb{R}^n)}$. The restriction on n becomes important in the upcoming estimate.

Lemma 3.3. *Let $n \geq 5$, $C_0 > 0$ and $v \in C^1(\mathbb{R}^n)$ be such that*

$$|v(x)| + |x||\nabla v(x)| \leq C_0 \min\left\{1, \frac{1}{|x|^{n-2}}\right\}, \qquad \text{for every } x \in \mathbb{R}^n \setminus \{0\}.$$

Then, there exists $C = C(n, C_0, s) > 0$ such that

$$\iint_{\mathbb{R}^n \times B_R^c} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \leq \frac{C}{R^{2s}} \qquad \text{for all } R \geq 1. \tag{3.12}$$

Proof. Let $\rho \in \left(0, \frac{R}{2}\right]$. We remark that, if $y \in B_R^c$ and $z \in B_\rho(y)$,

$$|z| \geq |y| - |z-y| > |y| - \rho \geq |y| - \frac{R}{2} > \frac{|y|}{2},$$

and therefore, if $x \in B_\rho(y)$, we have that

$$|v(x) - v(y)| \leq \sup_{z \in B_\rho(y)} |\nabla v(z)|\, |x-y| \leq \sup_{z \in B_\rho(y)} \frac{C_0\, |x-y|}{|z|^{n-1}} \leq \frac{C\, |x-y|}{|y|^{n-1}},$$

where we freely rename $C > 0$ from line to line.

Consequently,

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times B_R^c} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad\le \iint_{\substack{\mathbb{R}^n\times B_R^c\\ \{|x-y|\le\rho\}}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + 2\iint_{\substack{\mathbb{R}^n\times B_R^c\\ \{|x-y|>\rho\}}} \frac{|v(x)|^2+|v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad\le C\iint_{\substack{\mathbb{R}^n\times B_R^c\\ \{|x-y|\le\rho\}}} \frac{|x-y|^{2-n-2s}}{|y|^{2n-2}}\,dx\,dy\\
&\qquad + C\iint_{\substack{\mathbb{R}^n\times B_R^c\\ \{|x-y|>\rho\}}} \min\left\{1,\frac{1}{|x|^{2n-4}}\right\}\frac{dx\,dy}{|x-y|^{n+2s}}\\
&\qquad + C\iint_{\substack{\mathbb{R}^n\times B_R^c\\ \{|x-y|>\rho\}}} \frac{dx\,dy}{|y|^{2n-4}|x-y|^{n+2s}}\\
&\quad\le \frac{C\rho^{2-2s}}{R^{n-2}} + \frac{C}{\rho^{2s}}\int_{\mathbb{R}^n}\min\left\{1,\frac{1}{|x|^{2n-4}}\right\}dx + \frac{C}{R^{n-4}\rho^{2s}}\\
&\quad\le \frac{C}{\rho^{2s}}\left(\frac{\rho^2}{R^{n-2}}+1+\frac{1}{R^{n-4}}\right).
\end{aligned}$$

In particular, choosing $\rho := \frac{R}{2}$, the left-hand side of (3.12) is controlled by

$$\frac{C}{R^{2s}}\left(\frac{1}{R^{n-4}}+1\right)$$

and the desired result follows. □

Proposition 3.4. *Let $n\ge 5$ and let U_ε be as in (3.11). Then,*

$$[U_\varepsilon]_s^2 = \varepsilon^{2-2s}\,[V_1]_s^2 + O(\varepsilon^2), \qquad \text{as } \varepsilon\to 0^+.$$

Proof. The proof follows the lines of that of [8, Lemma 4.10]. Recall that $\delta>0$ is such that $B_{4\delta}\subset\Omega$ and define the sets

$$\begin{aligned}
\mathbb{D} &:= \left\{(x,y)\in\mathbb{R}^{2n} : x\in B_\delta, y\in B_\delta^c \text{ and } |x-y|>\delta/2\right\},\\
\mathbb{E} &:= \left\{(x,y)\in\mathbb{R}^{2n} : x\in B_\delta, y\in B_\delta^c \text{ and } |x-y|\le\delta/2\right\}.
\end{aligned}$$

With this notation, we get that

$$\begin{aligned}
\iint_{\mathbb{R}^{2n}} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &= \iint_{B_\delta\times B_\delta} \frac{|V_\varepsilon(x)-V_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad + 2\iint_{\mathbb{D}} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad + 2\iint_{\mathbb{E}} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&\quad + \iint_{B_\delta^c\times B_\delta^c} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.
\end{aligned}$$

Following the computations of [32, Proposition 21], one can infer that the last three integrals of the above identity behave like $O(\varepsilon^{n-2})$ as $\varepsilon\to 0^+$. See also [15] for similar computations with $p\neq 2$.

Let us now improve the estimate performed in [8] for the first integral (recall Lemma 3.2-(iv)). We perform the change of variables $x := \varepsilon\xi$ and $y := \varepsilon\eta$ to see that

$$\iint_{B_\delta\times B_\delta} \frac{|V_\varepsilon(x)-V_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \iint_{B_{\delta/\varepsilon}\times B_{\delta/\varepsilon}} \frac{|V_\varepsilon(\varepsilon\xi)-V_\varepsilon(\varepsilon\eta)|^2}{|\varepsilon\xi-\varepsilon\eta|^{n+2s}}\,\varepsilon^{2n}\,d\xi\,d\eta. \tag{3.13}$$

We also observe that, for all $z\in\mathbb{R}^n$,

$$V_\varepsilon(\varepsilon z) = \frac{\varepsilon^{(n-2)/2}}{(\varepsilon^2+|\varepsilon z|^2)^{(n-2)/2}} = \frac{\varepsilon^{-(n-2)/2}}{(1+|z|^2)^{(n-2)/2}} = \varepsilon^{-(n-2)/2}V_1(z).$$

Plugging this information into (3.13), we infer that

$$\iint_{B_\delta\times B_\delta} \frac{|V_\varepsilon(x)-V_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \varepsilon^{2-2s}\iint_{B_{\delta/\varepsilon}\times B_{\delta/\varepsilon}} \frac{|V_1(\xi)-V_1(\eta)|^2}{|\xi-\eta|^{n+2s}}\,d\xi\,d\eta. \tag{3.14}$$

Moreover, we notice that

$$\iint_{\mathbb{R}^{2n}\setminus\left(B_{\delta/\varepsilon}\times B_{\delta/\varepsilon}\right)} \frac{|V_1(\xi)-V_1(\eta)|^2}{|\xi-\eta|^{n+2s}}\,d\xi\,d\eta \le 2\iint_{\mathbb{R}^n\times B^c_{\delta/\varepsilon}} \frac{|V_1(\xi)-V_1(\eta)|^2}{|\xi-\eta|^{n+2s}}\,d\xi\,d\eta.$$

Now, since V_1 satisfies the assumptions of Lemma 3.3, choosing $R := \delta/\varepsilon$, it follows that there exists $\widetilde{C}>0$ (depending on δ but independent of ε) such that

$$\iint_{\mathbb{R}^{2n}\setminus\left(B_{\delta/\varepsilon}\times B_{\delta/\varepsilon}\right)} \frac{|V_1(\xi)-V_1(\eta)|^2}{|\xi-\eta|^{n+2s}}\,d\xi\,d\eta \le \widetilde{C}\varepsilon^{2s}.$$

From this and (3.14), we conclude that

$$\begin{aligned}
&\iint_{B_\delta\times B_\delta} \frac{|V_\varepsilon(x)-V_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&= \varepsilon^{2-2s}\left(\iint_{\mathbb{R}^{2n}} \frac{|V_1(\xi)-V_1(\eta)|^2}{|\xi-\eta|^{n+2s}}\,d\xi\,d\eta - \iint_{\mathbb{R}^{2n}\setminus\left(B_{\delta/\varepsilon}\times B_{\delta/\varepsilon}\right)} \frac{|V_1(\xi)-V_1(\eta)|^2}{|\xi-\eta|^{n+2s}}\,d\xi\,d\eta\right)\\
&= \varepsilon^{2-2s}[V_1]_s^2 + O(\varepsilon^2).
\end{aligned}$$

Combining all the estimates, we get the desired result. □

Thanks to the previous results, we can now prove that the family U_ε decreases the energy of $Q_\gamma(\cdot)$ below the threshold S_n, that is, (3.10) holds.

Proposition 3.5. *Let $n\ge 5$, and let $\gamma\in(0,C_{emb})$. Then, there exists a function $v\in\mathcal{X}^{1,2}(\Omega)$ such that $Q_\gamma(v)<S_n$.*

Proof. Let U_ε be as in (3.11). By Lemma 3.2 and Proposition 3.4, it follows that, as $\varepsilon \to 0^+$,

$$\begin{aligned} Q_\gamma\left(\frac{U_\varepsilon}{\|U_\varepsilon\|_{L^{2^*}(\Omega)}}\right) &= \frac{1}{\|U_\varepsilon\|^2_{L^{2^*}(\Omega)}}\left(\int_\Omega |\nabla U_\varepsilon|^2\,dx - \gamma\,[U_\varepsilon]_s^2\right)\\ &= \frac{1}{K_2^{2/2^*}+O(\varepsilon^n)}\left(K_1+O(\varepsilon^{n-2})-\gamma[V_1]_s^2\varepsilon^{2-2s}+O(\varepsilon^2)\right)\\ &= \left(\frac{1}{K_2^{2/2^*}}+O(\varepsilon^n)\right)\left(K_1+O(\varepsilon^{n-2})-\gamma[V_1]_s^2\varepsilon^{2-2s}+O(\varepsilon^2)\right)\\ &= S_n - \frac{\gamma[V_1]_s^2}{K_2^{2/2^*}}\,\varepsilon^{2-2s}+o(\varepsilon^{2-2s}). \end{aligned}$$

This entails the desired result. □

With Proposition 3.5 at hand, we can immediately prove the first assertion in our main Theorem 1.1.

Proof (of Theorem 1.1 - (1)). Assume that $n \geq 5$ and let $\gamma \in (0, C_{emb})$. Owing to Proposition 3.5, we know that $S(\gamma) < S_n$; hence, by Proposition 3.1 we conclude that $S(\gamma)$ is achieved, and there exists a weak solution of problem (1.1). □

2) The low-dimensional case $n = 3, 4$**.** In this second case, even if we can still rely on Proposition 3.1 (which holds for every $n \geq 3$), we *cannot use* the estimate in Proposition 3.4; rather, we make use of a continuity argument to show that

$$S(\gamma) < S_n \quad \text{for every } \gamma^* < \gamma < C_{emb},$$

for some (not explicit) $\gamma^* \in [0, C_{emb})$.

To this end, we consider $S(\cdot)$ as a *function of* γ, defined in the interval $\mathcal{J} := (0, C_{emb})$. Owing to the definition of $S(\gamma)$ in (3.2), we see that

$$S(\lambda) \leq S(\mu) \quad \text{for every } 0 < \mu \leq \lambda < C_{emb}, \tag{3.15}$$

that is, $S(\cdot)$ is non-increasing on $\mathcal{J}$.

Moreover, we have the following:

Lemma 3.6. *The function $S(\cdot)$ is continuous on $\mathcal{J}$. Furthermore,*

$$\lim_{\gamma\to 0^+} S(\gamma) = S_n \qquad \text{and} \qquad \lim_{\gamma\to C_{emb}} S(\gamma) = 0. \tag{3.16}$$

Proof. To ease the readability, we split the proof into two steps.

Step I): We begin by proving the continuity of $S(\cdot)$ in $\mathcal{J}$. To this end we first observe that, since $S(\cdot)$ is non-increasing, for every $\gamma \in \mathcal{J}$ we have

$$\exists\ S(\gamma-) := \lim_{\mu\to\gamma^-} S(\mu) \in \mathbb{R} \quad \text{and} \quad \exists\ S(\gamma+) := \lim_{\mu\to\gamma^+} S(\mu) \in \mathbb{R}.$$

We now fix $\gamma_0 \in \mathcal{J}$ and $r > 0$ such that $[\gamma_0 - r, \gamma_0 + r] \subseteq \mathcal{J}$ and we show that $S(\cdot)$ is continuous from both the left and the right at γ_0, that is,

$$S(\gamma_0-) = S(\gamma_0+) = S(\gamma_0).$$

To prove the left-continuity, we proceed as follows. First of all, given any $\varepsilon \in (0,1)$, by the definition of $S(\gamma_0)$ we infer that there exists $u = u_{\varepsilon,\gamma_0} \in \mathcal{X}^{1,2}(\Omega)$ such that

$$\|u\|_{L^{2^*}(\Omega)} = 1 \qquad \text{and} \qquad S(\gamma_0) \le Q_{\gamma_0}(u) < S(\gamma_0) + \frac{\varepsilon}{2}. \tag{3.17}$$

From this, using the monotonicity of $S(\cdot)$, for every $\gamma \le \gamma_0$ we obtain

$$\begin{aligned} 0 \le S(\gamma) - S(\gamma_0) &\le Q_\gamma(u) - S(\gamma_0) \\ &= \big(Q_{\gamma_0}(u) - S(\gamma_0)\big) + (\gamma_0 - \gamma)[u]_s^2 < \frac{\varepsilon}{2} + (\gamma_0 - \gamma)[u]_s^2. \end{aligned} \tag{3.18}$$

Moreover, exploiting (2.6),

$$Q_{\gamma_0}(u) = \|\nabla u\|^2_{L^2(\Omega)} - \gamma_0 [u]_s^2 \ge \left(1 - \frac{\gamma_0}{C_{emb}}\right) \|\nabla u\|^2_{L^2(\Omega)}.$$

Therefore, using again (2.6) and recalling (3.17),

$$\begin{aligned} [u]_s^2 &\le \frac{1}{C_{emb}} \|\nabla u\|^2_{L^2(\Omega)} \le \frac{1}{C_{emb}} \left(1 - \frac{\gamma_0}{C_{emb}}\right)^{-1} Q_{\gamma_0}(u) \\ &< \frac{1}{C_{emb} - \gamma_0} \left(S(\gamma_0) + \frac{\varepsilon}{2}\right) \le \frac{S(\gamma_0) + 1}{C_{emb} - \gamma_0}. \end{aligned}$$

From this and (3.3) we thus obtain that

$$[u]_s^2 < \frac{S_n + 1}{C_{emb} - \gamma_0} \le \frac{S_n + 1}{C_{emb} - \gamma_0 - r} =: \Theta(\gamma_0). \tag{3.19}$$

Using this estimate together with (3.18), we find that

$$0 \le S(\gamma) - S(\gamma_0) < \frac{\varepsilon}{2} + (\gamma_0 - \gamma)\Theta(\gamma_0).$$

As a consequence, setting

$$\delta = \delta_{\varepsilon,\gamma_0} := \min\left\{ r, \frac{\varepsilon}{2\Theta(\gamma_0)} \right\} > 0, \tag{3.20}$$

we conclude that, for every $\gamma \in (\gamma_0 - \delta, \gamma_0]$,

$$0 \le S(\gamma) - S(\gamma_0) < \varepsilon$$

and this proves that $S(\cdot)$ is continuous from the left at γ_0.

As regards the continuity from the right, we proceed essentially as above, but we exploit in a more crucial way the uniform estimate (3.19). First of all, given $\varepsilon > 0$ and $\gamma \in [\gamma_0, \gamma_0 + r]$, we let $u = u_{\varepsilon,\gamma} \in \mathcal{X}^{1,2}(\Omega)$ be such that

$$\|u\|_{L^{2^*}(\Omega)} = 1 \qquad \text{and} \qquad S(\gamma) \le Q_\gamma(u) < S(\gamma) + \frac{\varepsilon}{2}.$$

From this, by the monotonicity of $S(\cdot)$, we obtain

$$\begin{aligned} 0 \le S(\gamma_0) - S(\gamma) &\le Q_{\gamma_0}(u) - S(\gamma) \\ &= \big(Q_\gamma(u) - S(\gamma)\big) + (\gamma - \gamma_0)[u]_s^2 \quad < \frac{\varepsilon}{2} + (\gamma - \gamma_0)[u]_s^2. \end{aligned}$$

Furthermore, by arguing exactly as above to obtain the estimate in (3.19) (with γ_0 here replaced by γ), we obtain that

$$[u]_s^2 < \frac{1}{C_{emb} - \gamma} \left(S(\gamma) + \frac{\varepsilon}{2}\right) \le \frac{S_n + 1}{C_{emb} - \gamma} < \frac{S_n + 1}{C_{emb} - \gamma_0 - r} = \Theta(\gamma_0).$$

As a consequence, taking δ as in (3.20), we conclude that, for every $\gamma \in [\gamma_0, \gamma_0+\delta_\varepsilon)$,

$$0 \le S(\gamma_0) - S(\gamma) < \varepsilon.$$

We then conclude that $S(\cdot)$ is also continuous from the right at γ_0, and thus, by the arbitrariness of γ_0, $S(\cdot) \in C(\mathcal{J})$.

Step II): We now prove the limits in (3.16). The limit of $S(\cdot)$ as $\gamma \to 0^+$ is a direct consequence of (3.3), hence we now focus on the limit as $\gamma \to C_{emb}$.

We recall Remark 2.1-iii) and we take $\Phi_0 \in \mathcal{X}^{1,2}(\Omega)$ such that

$$[\Phi_0]_s^2 = 1 \qquad \text{and} \qquad \|\nabla \Phi_0\|^2_{L^2(\Omega)} = C_{emb}.$$

Then, we have that, for every $\gamma \in \mathcal{J}$,

$$0 \le S(\gamma) \le Q_\gamma(\Phi_0) = (C_{emb} - \gamma)[\Phi_0]_s^2 = C_{emb} - \gamma.$$

Hence, by letting $\gamma \to C_{emb}$ in the above estimate, we obtain the desired limit. This ends the proof. □

Thanks to Lemma 3.6, we can prove the second assertion in Theorem 1.1.

Proof (of Theorem 1.1 - (2)). Assume that $n = 3, 4$ and recall the notation $\mathcal{J} = (0, C_{emb})$.

By (3.3), we know that $S(\gamma) > 0$ for all $\gamma \in \mathcal{J}$. Thus, owing to (3.15) and Lemma 3.6, we have that

$$\text{either } S(\mathcal{J}) = (0, S_n) \text{ or } S(\mathcal{J}) = (0, S_n]. \tag{3.21}$$

Concerning this statement, we point that, in our setting, we are not able to exclude that $S_n \in S(\mathcal{J})$, as we cannot guarantee that $S(\gamma) < S_n$ on $\mathcal{J}$, and this is the reason for which the second option in (3.21) cannot be excluded.

In any case, it follows from (3.21) that

$$\{\gamma \in \mathcal{J} : S(\gamma) < S_n\} \neq \varnothing.$$

As a consequence, we can define

$$\gamma^* := \inf\{\gamma \in \mathcal{J} : S(\gamma) < S_n\}.$$

We also deduce from (3.21) that $\gamma^* \in [0, C_{emb})$.

Moreover,

i) $S(\gamma) = S_n$ for all $\gamma \in (0, \gamma^*)$;
ii) $0 < S(\gamma) < S_n$ for all $\gamma \in (\gamma^*, C_{emb})$.

Hence, for every fixed $\gamma \in (\gamma^*, C_{emb})$, we can apply Proposition 3.1, ensuring that $S(\gamma)$ is achieved and that there exists a weak solution of problem (1.1). □

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