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Interfaces with other disciplines

A new bivariate approach for modeling the interaction between stock volatility and interest rate: An application to S&P500 returns and options

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ABSTRACT

The GARCH models developed so far do not take into account the interaction between the volatility of asset returns and the dynamics of the interest rate. In this paper, we propose a bivariate GARCH model in which interest rate movements and asset price volatility are fully coupled. This approach yields explicit and simple to implement recursion formulas for the moment generating function, which can be exploited to compute option prices by applying the fast Fourier transform or other convolution techniques. We perform a thorough and comprehensive empirical analysis based on real S&P500 return and option data showing the usefulness and robustness of the suggested methodology. Both in-sample and out-of-sample results reveal the superiority of our approach over the GARCH model with constant interest rates.

1. Introduction

Since the seminal papers of Engle (1982) and Bollerslev (1986), ARCH and GARCH models, as well as their nonlinear specifications, see Nelson (1991) and Glosten et al. (1993), have become standard statistical tools for capturing the well-known serial correlation of the conditional variance of asset returns and modeling volatility clustering. A remarkable advantage of GARCH-type models is that they can be easily estimated and filtered leveraging past return observations. In fact, unlike stochastic volatility models, which, due to the unobservable variance, require complex simulation-based techniques (such as Markov chain Monte Carlo methods, see Shephard (2005) for a comprehensive review), GARCH models can be readily estimated by maximum likelihood (ML), since the conditional variance is a measurable function of past observations.

Motivated by the advantages of GARCH models, Duan (1995) was the first to provide an option pricing framework combining the theory developed by Black and Scholes (1973) and Merton (1973) and GARCH models with normal innovations. However, as argued by Duan (1995), since the multi-period distribution of the traditional GARCH process is not known, the calculation of GARCH option prices requires a Monte Carlo approach, see for instance Menn and Rachev (2005) and Badescu et al. (2015). Nevertheless, standard Monte Carlo simulation procedures are very time-consuming, especially if a high degree of accuracy is desired, and, as Duan and Simonato (1998) pointed out, they could also lead to simulated samples that violate the martingale property of the discounted asset prices under the risk-neutral measure. A general overview of the pricing performances of various nonlinear GARCH specifications can be found in Christoffersen and Jacobs (2004).

To avoid the use of the aforementioned simulation-based procedures, Heston and Nandi (2000) introduce an alternative nonlinear GARCH model (hereafter *GARCH-HN*) with normal innovations that yields closed-form recursions for pricing options. This approach has been extended in several directions, see for instance Christoffersen et al. (2006), Christoffersen et al. (2008) and Escobar-Anel et al. (2021).

A major drawback of all these works is that the interest rate is assumed to be constant or predetermined, even if several empirical studies document that the stochastic dynamics of stock returns and of the interest rate are closely intertwined. For example, Fama (1981) argues that returns on the stock market are negatively correlated with the expected inflation, which, in turn, is strongly interconnected with the interest rate. Campbell (1987) provides evidence that the interest rate term structure plays a crucial role in predicting excess returns on the US stock market. Furthermore, Breen et al. (1989) find a statistically significant and negative relationship between the monthly Treasury-Bill return and a value weighted index of stocks of the NYSE. Very recently, using panel Johansen cointegration analysis, Eldomiaty et al. (2020) show that interest rates and stock prices are positively correlated, and that changes in the former Granger cause significant variations in the latter.

To analyze the dynamic properties of the short-term interest rate and its volatility, a variety of models have been proposed in both the discrete and the continuous time setting. Chan et al. (1992) presents a

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model in continuous time that encompasses the popular Vasicek (1977) and Cox et al. (1985) square-root models as particular cases.

Continuous time models with stochastic interest rates are also very popular for option pricing. Amin and Jarrow (1991) propose a four-factor model to price foreign currency options under stochastic interest rates. Choi and Marcozzi (2003) develop a finite element method for foreign currency options when interest rates are stochastic. Trolle and Schwartz (2008) price interest rates derivatives using a general stochastic volatility model. Grzelak and Oosterlee (2011) and Recchioni and Sun (2016) extend the popular Heston stochastic volatility model to allow for stochastic interest rates. Moreover, Recchioni and Tedeschi (2017) introduce an interest rate model with stochastic volatility to describe government bonds. Grzelak et al. (2012) present an extension of stochastic volatility models based on a stochastic Hull–White interest rate component. For a review of continuous time models for stochastic interest rates the reader is referred to Brigo and Mercurio (2007) and Schmidt (2011).

Amin and Bodurtha (1995) combine a discrete version of the Heath et al. (1992) term structure model with the binomial model. Ho et al. (1997) use multivariate binomial trees and extrapolation techniques to value American-option in a stochastic interest rate environment. Brenner et al. (1996), considering an Euler–Maruyama discrete time approximation of the Chan et al. (1992) model, assume that the volatility of the interest rate follows a GARCH-type specification. Furthermore, Cvsa and Ritchken (2001) provide an analytic solution for pricing bonds under a general GARCH-level dependent interest rate model. More recently, to reduce possible model misspecifications, Hou and Suardi (2011) have proposed a semiparametric GARCH alternative approach for the interest rate volatility. In sum, the clear advantage of GARCH specifications is that they can be easily filtered and estimated, while still allowing for complex dynamic features such as time-varying volatility and asymmetric effects. Yet, GARCH models that exploit the close relationship between the volatility of stock returns and the dynamics of the interest rate are still lacking.

To fill this gap, in this paper we propose a bivariate GARCH framework to simultaneously model the coupled movements of asset returns and of the short-term interest rate. In particular, returns are modeled by a GARCH process in which the drift component that depends on the interest rate is assumed to follow a separate GARCH specification. Such an approach allows us to incorporate the correlation between the volatility of the returns and the volatility of the interest rate while still providing closed-form expressions for option valuation. To the best of our knowledge, no other studies consider a GARCH model in which the drift of the stock price is described by a GARCH specification itself. It is also worth noting that our methodology encompasses the *GARCH-HN* model as a special case.

We test the proposed model on both the S&P500 time series and a rich sample of Call and Put S&P500 option values, and we show that it fits realized prices much better than the more traditional *GARCH-HN* specification and the component GARCH of Christoffersen et al. (2008) (hereafter *GARCH-C*), which incorporates both short- and long-term volatilities. Then, the model developed in this paper turns out to be an efficient, tractable and easy to implement approach for managing asset price risk and valuing options.

The remainder of the paper is organized as follows: In Section 2, we provide some empirical insights about the dynamics of the S&P500 Index and the 3-month Treasury Bill interest rate, which further motivate the proposed model. In Section 3, we present our bivariate GARCH approach with dynamic interest rates. In Section 4, we perform an empirical analysis in which we estimate the model and compare it with the *GARCH-HN*. In Section 5, we derive analytical recursion formulas for the moment generating function and use them to fit and forecast prices of Call and Put S&P500 options. Finally, Section 6 concludes. All the mathematical proofs are reported in Appendix.

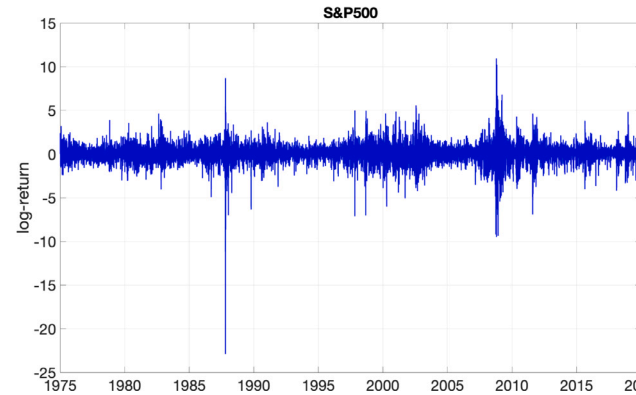

Fig. 1. S&P500 log-return time series from January 01, 1975 to December 31, 2022 (12522 daily observations).

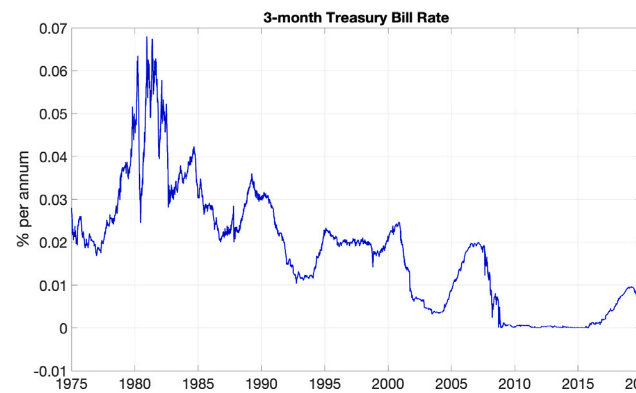

Fig. 2. 3-month Treasury Bill interest rate time series from January 01, 1975 to December 31, 2022 (12522 daily observations).

2. S&P500 return and short-term interest rate empirics

In this Section we start by reporting some descriptive statistics of the S&P500 daily log-returns (in percentage) and the 3-month Treasury Bill interest rate time series. We collect data from January 01, 1975 to December 31, 2022 (12522 daily observations), which are displayed in Figs. 1 and 2.

Fig. 1 clearly shows the time-varying feature of the volatility of the S&P500 index. In particular, as the time series is very long, we note several volatility clusters, which tend to start with the upward/downward variations of the interest rate (see Fig. 2). For example, the clusters beginning in the years 1983, 1987, 1998, 2008, 2016 and 2019 follow either an upward or a downward movement of the interest rate. Such a stylized fact is also shown by Fig. 3, where we plot the first-differenced series of the 3-month Treasury Bill interest rate. As we can easily see, the interest rate exhibits conditional heteroskedasticity. In what follows, we will show that all of these empirical findings are well-captured by the model introduced in Section 3.

Table 1 reports descriptive statistics for the three time series. In particular, the S&P500 return is significantly and negatively skewed, whereas the 3-month Treasury Bill interest rate is significantly and positively skewed. Its first-differenced series, however, does not have a significant skewness coefficient. All the time series have a kurtosis coefficient that is significantly higher than that of the normal distribution.

To further motivate the model proposed in this paper from an empirical standpoint, we have computed the empirical volatility of both the S&P500 returns and the first-differenced 3-month Treasury Bill interest rate, along with the their correlation, using a rolling window

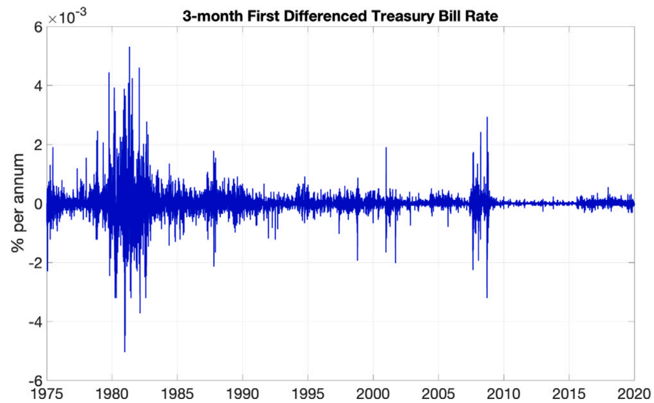

Fig. 3. First-differenced 3-month Treasury Bill interest rate time series from January 01, 1975 to December 31, 2022 (12522 daily observations).

Table 1
Descriptive statistics of the S&P500 daily log-return (y_t), 3-month Treasury Bill interest rate (r_t) and first-differenced interest rate (Δr_t).

Variable	*Mean*	*Std deviation*	*Skewness*	*Kurtosis*
y_t	0.032	1.089	−1.083	28.686
r_t	4.271	3.527	0.775	3.433
Δr_t	−0.002	0.089	0.401	40.044

of 50 days. The results are displayed in Fig. 4. We observe that the two volatility time series show some commonalities. For instance, common peaks appear around the years 1988, 2003 and 2008, and overall there is a significant amount of correlation between the volatilities, ranging from −0.63 to 0.63.

3. The model

In this Section, we introduce a novel bivariate GARCH model to capture the simultaneous time-variations of the volatility of log-returns and the dynamics of the volatility of the short-term interest rate. This model will be labeled *Bivariate GARCH with Dynamic Interest Rate (B-GARCH-DIR)*.

Let S_t denotes the price of an asset at time t and let us define $x_t = \ln(S_t)$ and $y_t = \ln(S_t) - \ln(S_{t-1})$. According to Heston and Nandi (2000), we specify the daily log-price process $\{x_t\}_{t\in\mathbb{Z}}$ as follows:

$$x_{t+1} = x_t + r_{t+1} + \lambda_h h_{t+1} + \sqrt{h_{t+1}} Z_{t+1}, \qquad Z_{t+1} \sim \mathcal{N}(0,1), \tag{1}$$

where r_{t+1} and h_{t+1} are the time-varying short-term interest rate and the conditional variance, respectively, λ_h is the premium parameter for the volatility risk and $\mathcal{N}(0,1)$ denotes the standard normal distribution.

To model the dynamic behavior of the short-term interest rate, we follow Chan et al. (1992) and Brenner et al. (1996) and use the following discrete time process

$$\Delta r_{t+1} = r_{t+1} - r_t = \bar{r} + \beta_r r_t + \lambda_v v_t + \sqrt{v_t}\epsilon_t, \qquad \epsilon_t \sim \mathcal{N}(0,1). \tag{2}$$

The drift of the process $\{r_{t+1}\}_{t\in\mathbb{Z}}$ is captured by $\bar{r} + \beta_r r_t$, while v_t represents the conditional variance of r_t, and the innovations ϵ_t and Z_t are independent. For the conditional variance process v_t we adopt a nonlinear GARCH dynamics as in Heston and Nandi (2000), so that

$$v_{t+1} = \omega_v + \beta_v v_t + \alpha_v \left(\epsilon_t - \psi_v \sqrt{v_t}\right)^2. \tag{3}$$

It is well-known that the parameter ψ_v captures the leverage effect, i.e., the asymmetric response of the conditional variance.

Finally, we model the conditional variance of the return process by an augmented version of the previous nonlinear GARCH, namely

$$h_{t+1} = \omega_h + \beta_h h_t + \alpha_h \left(Z_t - \gamma_h \sqrt{h_t}\right)^2 + \kappa_h \left(\epsilon_t - \psi_v \sqrt{v_t}\right)^2. \tag{4}$$

Table 2
Maximum likelihood estimation results. The estimation period spans from January 01, 1975 to December 31, 2022 (12 522 daily observations). The standard errors reported in parenthesis are computed by inverting the negative Hessian matrix evaluated at the optimum parameter values.

	B-GARCH-DIR	*GARCH-HN*	*GARCH-C*
Interest rate			
$\bar{r}$	−0.000 (0.004)	(−)	(−)
β_r	0.000 (0.001)	(−)	(−)
λ_v	0.000 (0.028)	(−)	(−)
ω_v	0.001** (0.000)	(−)	(−)
α_v	0.001** (0.000)	(−)	(−)
β_v	0.909** (0.335)	(−)	(−)
ψ_v	-0.234** (0.019)	(−)	(−)
S&P500 log-return			
λ_h	0.015* (0.009)	−0.018 (0.037)	−0.016 (0.036)
ω_h	-0.002** (0.001)	-0.013** (0.000)	(−)
α_h	0.044** (0.001)	0.044** (0.000)	0.045** (0.000)
β_h	0.895** (0.015)	0.903** (0.006)	0.661** (0.004)
γ_h	1.257** (0.171)	1.215** (0.141)	2.045** (0.246)
κ_h	0.004** (0.001)	(−)	(−)
ω_q	(−)	(−)	0.102** (0.001)
α_q	(−)	(−)	0.019** (0.001)
β_q	(−)	(−)	0.989** (0.013)
γ_q	(−)	(−)	0.653** (0.198)
Volatility persistence			
Interest rate	0.909	(−)	(−)
S&P500 log-return	0.965	0.968	0.989
In-Sample log-likelihood			
Interest rate	89 728.48	(−)	(−)
S&P500 log-return	−16388.52	−16508.97	−16427.12
Out-of-sample log-likelihood			
S&P500 log-return	−772.57	−782.12	−776.89

** Denote statistical significance at the 0.01 levels, respectively.
* Denote statistical significance at the 0.05 levels, respectively.

Note that, in the dynamics of h_{t+1}, we have explicitly included the leverage term of v_t through the additional parameter κ_h. By substituting (4) in (1), and by the orthogonality of $\{Z_t\}_{t\in\mathbb{Z}}$ and $\{\epsilon_t\}_{t\in\mathbb{Z}}$, we can easily compute

$$\mathbb{C}\mathrm{ov}_t[y_{t+1}, r_{t+1}] = v_t(1 - 2\lambda_h \kappa_h \psi_v),$$

where $\mathbb{C}\mathrm{ov}_t[\cdot,\cdot]$ denotes the conditional covariance given the information available at time t.

4. Maximum likelihood (ML) estimation

The model described in Section 3 can be estimated by standard ML, see Bollerslev and Wooldridge (1992). In fact, upon Eq. (2), the log-likelihood function of the interest rate at time $t+1$ conditional on the information at time t is given by

$$\ell_{r,t+1}(\theta_r) = -\frac{1}{2}\ln(2\pi v_t) - \frac{\left(\Delta r_{t+1} - \bar{r} - \beta_r r_t - \lambda_v v_t\right)^2}{2 v_t},$$

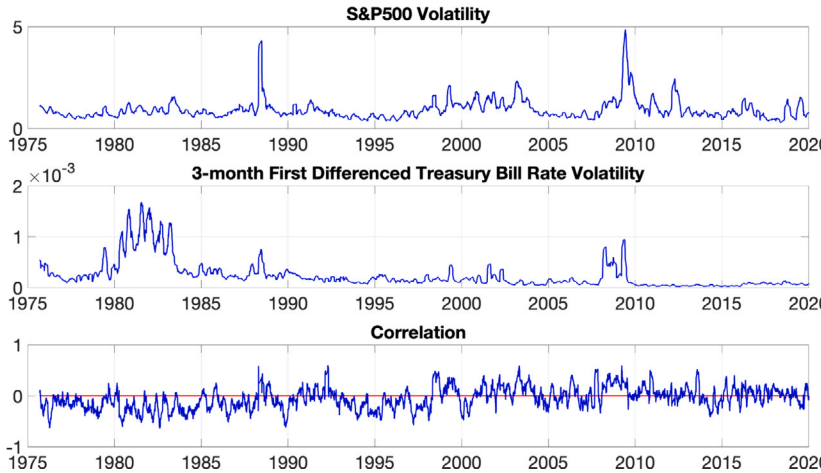

Fig. 4. Empirical volatilities and correlation computed using a rolling window of 50 days. Empirical volatility of the S&P500 log-return (top panel) and the first-differenced 3-month Treasury Bill (middle panel) time series, and their correlation. The time series span from January 01, 1975 to December 31, 2022 (12522 daily observations).

where $\theta_r = (\bar{r}, \beta_r, \lambda_v, \omega_v, \alpha_v, \beta_v, \psi_v)^\top$. Similarly, Eq. (1) yields the log-likelihood function of the return process at time t conditional on the information at time $t-1$

$$\ell_{y,t}(\theta_x) = -\frac{1}{2}\ln(2\pi h_t) - \frac{\left(y_t - r_t - \lambda_h h_t\right)^2}{2h_t},$$

where $\theta_x = (\lambda_h, \omega_h, \alpha_h, \beta_h, \kappa_h)^\top$. Then, the log-likelihood function of the bivariate model reads

$$\ell_{r,y}(\theta) = \sum_{t=1}^{T}(\ell_{r,t}(\theta_r) + \ell_{y,t}(\theta_x)), \tag{5}$$

where $\theta = (\theta_r^\top, \theta_y^\top)^\top$. As the noise processes $\{Z_t\}_{t\in\mathbb{Z}}$ and $\{\epsilon_t\}_{t\in\mathbb{Z}}$ are uncorrelated, we can maximize (5) in two steps, that is, first we maximize the sum of $\ell_{r,t}(\theta_r)$ with respect to θ_r and then the sum $\ell_{y,t}(\theta_x)$ with respect to θ_y.

4.1. Estimation results

We have applied the model developed in Section 3 to the S&P500 daily time series from January 01, 1975 to December 31, 2022. The short-term interest rate is proxied with the 3-month Treasury Bill rate. The data for the S&P500 are retrieved from Refinitiv Datastream, whereas the data for the 3-month Treasury Bill rate are gathered from the Federal Funds Effective Rate (FRED) dataset. The ML estimation results are presented in Table 2, where we show both the in-sample and the out-of-sample likelihoods. As mentioned earlier, to compute the out-of-sample likelihoods, we re-estimate all the GARCH models using the time series data in the period from January 01, 1975 to December 31, 2020, and compute the out-of-sample log-likelihood on the time period from January 01, 2021 to December 31, 2022.

We start by considering the coefficients of the *B-GARCH-DIR* for the interest rate. The parameters $\bar{r}$ and β_r are very close to zero and are not statistically significant, implying that the degree of mean reversion is very weak. In particular, the fact that β_r is not statistically significant would suggest that the considered interest rate time series has a unit root. On the other hand, in the literature there is no clear and definitive consensus on whether interest rate time series are stationary (see, e.g., Engle and Granger (1987), Rose (1988), Perron (1989), Rapach and Weber (2004)). Therefore, to shed further light on this, we perform some unit root tests. In particular, the augmented Dickey–Fuller test provides a statistic equal to -1.907, failing to reject the unit root hypothesis. However, the plot in Fig. 3, the parameters ω_v, α_v, β_v and ψ_v, and the interest rate volatility persistence in Table 2 show a strong evidence of conditional heteroscedasticity. This is confirmed by the Engle's test, which we performed up to the order ten and gave a p-value $< 1.0e^{-10}$ (see Engle (1982)). Therefore, we use a more appropriate unit root test, namely, the one presented by Seo (1999), who has shown that, if a time series is conditional heteroscedastic, the asymptotic distribution of the t-statistic for the autoregressive unit root is a mixture of the Dickey–Fuller t-distribution and the standard normal. Then, the test proposed by Seo (1999) is more robust then the Dickey–Fuller test for establishing if the considered interest rate time series is stationary. The resulting statistic is -2.535, which, according to the critical values provided by Seo (1999), allows us to reject the unit root hypothesis.

Next, we focus on the estimated coefficients of the *B-GARCH-DIR* process for the return, described by Eqs. (1) and (4). All the coefficients are statistically significant. The volatility persistence is very high, showing a value of 0.966, which is a typical feature of the S&P500 log-returns. Finally, it is also worth noting that the parameter κ_h, which is not present in standard GARCH models, is statistically significant, which highlights that the leverage term of v_t (the conditional variance of the interest rate error term) is an important predictor of the log-return variance.

As benchmarks, we have also estimated the *GARCH-HN* and the *GARCH-C* models. Each of these approaches performs worse, both in terms of in-sample and out-of-sample log-likelihood, confirming again that for the S&P500 Index there is an advantage to using a model that also incorporates information about the 3-month Treasury Bill interest rate time series.

Figs. 5 and 6 display the filtered one-step ahead conditional volatilities of the S&P500 log-returns and the 3-month Treasury Bill interest rate, respectively. The volatility paths look rather different between the three models. In fact, we observe that both the *GARCH-HN* and the *GARCH-C* models do not provide a good reproduction for most of the volatility dynamics in the S&P500 log-returns time series, especially in the period from the early 1970's to the 1990's, as they fail to capture several high-volatility regimes. Finally, Fig. 6 shows the significant activity of the time-varying volatility of the 3-month Treasury Bill interest rate, which is captured by our *B-GARCH-DIR*.

4.1.1. Other non-linear GARCH models

As suggested by an anonymous Referee, we have compared the in-sample performances of our model with other popular non-linear GARCH models, reporting the log-likelihood in Table 3. Specifically we consider the *GJR-GARCH* of Glosten et al. (1993), the *AGARCH* originally considered by Engle (1990) and the *APARCH* as proposed by Ding et al. (1993). Furthermore, we also report the out-of-sample log-likelihood computed using a procedure analogous to that detailed in Section 4.1. Again, our *B-GARCH-DIR* still provides slightly higher likelihood values, for both in-sample and out-of-sample data.

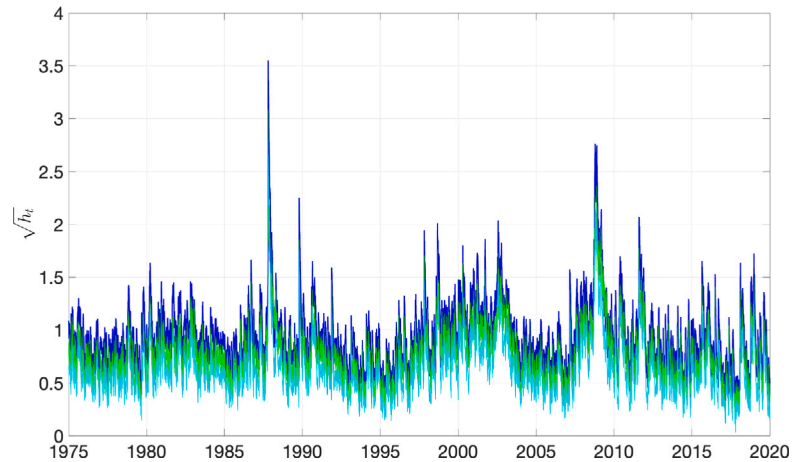

Fig. 5. Filtered conditional volatility of the S&P500 log-return from January 01, 1975 to December 31, 2022 (12522 daily observations), *B-GARCH-DIR* (blue line), *GARCH-C* (green line) and *GARCH-HN* (cyan line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

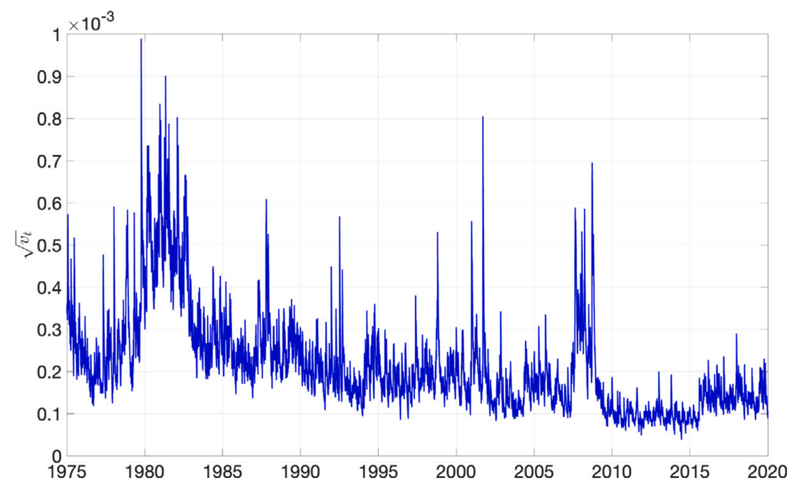

Fig. 6. Filtered conditional volatility of the 3-month Treasury Bill interest rate from January 01, 1975 to December 31, 2022 (12522 daily observations).

Table 3
Maximum likelihood estimation results. The estimation period spans from January 01, 1975 to December 31, 2022 (12 522 daily observations).

	GJR-GARCH	*AGARCH*	*APARCH*
In-sample log-likelihood			
S&P500 log-return	−16398.72	−16400.91	−16439.37
Out-of-sample log-likelihood			
S&P500 log-return	−785.38	−795.14	−789.95

5. Option valuation

In this Section, we apply the model presented in Section 3 to option pricing. To this aim, we first derive the moment generating function of the bivariate process $\{x_t, r_t\}$, and then we use it to develop suitable pricing formulas. In particular, a closed-form solution for European Call and Put options is obtained based on the inversion theorem of Gil-Pelaez (1951). Moreover, we show that the problem of pricing options under the bivariate model (1)–(4) can be reduced to computing a univariate expectation, which can be easily done by applying fast convolution techniques (see, e.g., Fang and Oosterlee (2009) and Jackson et al. (2007)).

We conduct both an in-sample and an out-of-sample exercise in which we consider Call and Put options on the S&P500 Index. Again, both the *GARCH-HN* and *GARCH-C* models are used as benchmarks.

5.1. The moment generating function

From the analytical standpoint, a great advantage of our model is that the moment generating function has an exponentially-affine structure which is available in closed-form and can be computed through a set of recursive coefficients, as stated in the next proposition.

Proposition 1. *Let us define*

$$Y_{t,T} = \begin{cases} r_{t+1} + \cdots + r_T & t < T, \\ 0 & t = T. \end{cases} \tag{6}$$

Moreover, let us consider the moment generating function of the process $\{x_t, Y_{t,T}\}$

$$f(t,T,\phi,\xi) = \mathbb{E}_t[\exp\{\phi x_T + \xi Y_{t,T}\}].$$

Then, we have

$$f(t,T,\phi,\xi) = \exp\left\{\phi x_t + A_t + B_t h_{t+1} + C_t r_{t+1} + D_t v_{t+1}\right\}, \tag{7}$$

where the coefficients A_t*,* B_t*,* C_t *and* D_t *satisfy the following backward recursions:*

$$\begin{aligned} A_t =& A_{t+1} + B_{t+1}\omega_h + C_{t+1}\bar{r} + D_{t+1}\omega_v - \frac{1}{2}\ln(1 - 2B_{t+1}\alpha_h) \\ &-\frac{1}{2}\ln[1 - 2(B_{t+1}\kappa_h + D_{t+1}\alpha_v)], \end{aligned} \tag{8}$$

$$B_t = B_{t+1}\beta_h + \phi(\lambda_h + \gamma_h) + \frac{\phi(\phi - 2\gamma_h)/2 + B_{t+1}\alpha_h\gamma_h^2}{1 - 2B_{t+1}\alpha_h}, \tag{9}$$

$$C_t = C_{t+1}(\beta_r + 1) + \phi + \xi, \tag{10}$$

$$D_t = C_{t+1}\lambda_v + D_{t+1}\beta_v + C_{t+1}\psi_v + \frac{C_{t+1}(C_{t+1} - 2\psi_v)/2 + (B_{t+1}\kappa_h + D_{t+1}\alpha_v)\psi_v^2}{1 - 2(B_{t+1}\kappa_h + D_{t+1}\alpha_v)}, \tag{11}$$

with final conditions $A_T = 0$, $B_T = 0$, $C_T = 0$ and $D_T = 0$.

Proof. See Appendix. □

5.2. Option pricing

To compute option prices we need to risk-neutralize the model (1)–(4). To risk-neutralize the equations in (1) and (4), we follow the same approach as in Heston and Nandi (2000), according to which the risk premium λ_h and the γ_h parameter are simply replaced by $\lambda_h^* = -1/2$ and $\gamma_h^* = \gamma_h + \lambda_h + 1/2$, respectively. Among the many probability measures that can describe the risk-neutral dynamics of the interest rate, for the sake of simplicity, we choose the same under which we specified the model, that is, we assume that processes (2) and (3) are already risk-neutralized.

To this aim, we assume that the market premium for the interest rate is zero, i.e., that Eqs. (2) is already written under the risk-neutral measure. Note that the GARCH process (2)–(3) is calibrated based on the time series of the 3-Month Treasury Bills, which can reasonably be considered risk-free since they are a US government security with a very short maturity.

From now on, $f(t, T, \phi, \xi)$ will denote the moment generating function of risk-neutralized process $\{x_t, r_t\}$.

Proposition 2. *The price at time t of a European Call option with maturity T and strike price K is given by*

$$C(S_t, t) = f(t, T, 0, -1) \int_{-\infty}^{+\infty} \max(e^{x_T} - K, 0) p_t^{\star\star}(x_T) \mathrm{d}x_T, \tag{12}$$

where

$$p_t^{\star\star}(x_T) = \int_{-\infty}^{+\infty} \frac{e^{-Y_{t,T}} p(x_T, Y_{t,T} | x_t, r_t)}{f(t, T, 0, -1)} \mathrm{d}Y_{t,T}, \tag{13}$$

and $p(x_T, Y_{t,T} | x_t, r_t)$ is the probability density function of the risk-neutralized bivariate process $\{x_T, Y_{t,T}\}$ given x_t and r_t. Similarly, for a Put option we have

$$P(S_t, t) = f(t, T, 0, -1) \int_{-\infty}^{+\infty} \max(K - e^{x_T}, 0) p_t^{\star\star}(x_T) \mathrm{d}x_T. \tag{14}$$

Proof. See Appendix. □

Formulas (12) and (14) require knowledge of the univariate probability density function $p_t^{\star\star}(x_T)$, which, unfortunately, is not available in closed-form. Nevertheless, we can easily obtain the moment generating function associated to it, as shown in the following proposition.

Proposition 3. *The moment generating function associated to the probability density function in (13) is as follows:*

$$f^{\star\star}(t, T, \phi) = \int_{-\infty}^{+\infty} e^{\phi x_T} p_t^{\star\star}(x_T) \mathrm{d}x_T = \frac{f(t, T, \phi, -1)}{f(t, T, 0, -1)}. \tag{15}$$

Proof. See Appendix. □

Based on relations (12), (14) and (15), we can efficiently compute the price of Call and Put options by using past convolution techniques. Moreover, it is also possible to obtain an integral representation of Call and Put option values by applying the inversion theorem of Gil-Pelaez (1951), as in the following proposition.

Proposition 4. *The prices of European Call and Put options are given by*

$$\begin{aligned} C(S_t, t) = {} & f(t, T, 1, -1) \left[\frac{1}{2} - \frac{1}{2\pi} \int_0^{+\infty} \frac{K^{i\phi} f(t, T, -i\phi + 1, -1) - K^{-i\phi} f(t, T, i\phi + 1, -1)}{i\phi f(t, T, 1, -1)} \mathrm{d}\phi \right] \\ & - K f(t, T, 0, -1) \left[\frac{1}{2} - \frac{1}{2\pi} \int_0^{+\infty} \frac{K^{i\phi} f(t, T, -i\phi, -1) - K^{-i\phi} f(t, T, i\phi, -1)}{i\phi f(t, T, 0, -1)} \mathrm{d}\phi \right], \end{aligned} \tag{16}$$

$$\begin{aligned} P(S_t, t) = {} & K f(t, T, 0, -1) \left[\frac{1}{2} + \frac{1}{2\pi} \int_0^{+\infty} \frac{K^{i\phi} f(t, T, -i\phi, -1) - K^{-i\phi} f(t, T, i\phi, -1)}{i\phi f(t, T, 0, -1)} \mathrm{d}\phi \right] \\ & - f(t, T, 1, -1) \left[\frac{1}{2} + \frac{1}{2\pi} \int_0^{+\infty} \frac{K^{i\phi} f(t, T, -i\phi + 1, -1) - K^{-i\phi} f(t, T, i\phi + 1, -1)}{i\phi f(t, T, 1, -1)} \mathrm{d}\phi \right], \end{aligned} \tag{17}$$

where i is the imaginary unit.

Proof. See Appendix. □

5.2.1. Price sensitivities

Based on Eqs. (16) and (17) in Proposition 4 we can easily calculate the first- and second-order derivatives of the option price with respect to the underlying asset price. According to (7) we have

$$\frac{\partial f(t, T, \phi, \xi)}{\partial S_t} = \frac{\phi}{S_t} f(t, T, \phi, \xi).$$

Thus, by straightforward differentiation in (16), we obtain

$$\begin{aligned} \Delta(S_t, t) := {} & \frac{\partial C(S_t, t)}{\partial S_t} \\ = {} & \frac{1}{S_t} \left[\frac{1}{2} f(t, T, 1, -1) - \frac{1}{2\pi} \int_0^{+\infty} \frac{(-i\phi + 1) K^{i\phi} f(t, T, -i\phi + 1, -1) - (i\phi + 1) K^{-i\phi} f(t, T, i\phi + 1, -1)}{i\phi} \mathrm{d}\phi \right. \\ & \left. + K \frac{1}{2\pi} \int_0^{+\infty} \frac{-i\phi K^{i\phi} f(t, T, -i\phi, -1) - i\phi K^{-i\phi} f(t, T, i\phi, -1)}{i\phi} \mathrm{d}\phi \right], \end{aligned} \tag{18}$$

and

$$\begin{aligned} \Gamma(S_t, t) := {} & \frac{\partial^2 C(S_t, t)}{\partial S_t^2} \\ = {} & \frac{1}{S_t^2} \left[\frac{1}{2\pi} \int_0^{+\infty} \frac{(-i\phi + 1) K^{i\phi} f(t, T, -i\phi + 1, -1) - (i\phi + 1) K^{-i\phi} f(t, T, i\phi + 1, -1)}{i\phi} \mathrm{d}\phi \right. \\ & - K \frac{1}{2\pi} \int_0^{+\infty} \frac{-i\phi K^{i\phi} f(t, T, -i\phi, -1) - i\phi K^{-i\phi} f(t, T, i\phi, -1)}{i\phi} \mathrm{d}\phi \\ & - \frac{1}{2\pi} \int_0^{+\infty} \frac{(-i\phi + 1)^2 K^{i\phi} f(t, T, -i\phi + 1, -1) - (i\phi + 1)^2 K^{-i\phi} f(t, T, i\phi + 1, -1)}{i\phi} \mathrm{d}\phi \\ & \left. + K \frac{1}{2\pi} \int_0^{+\infty} \frac{(-i\phi)^2 K^{i\phi} f(t, T, -i\phi, -1) - (i\phi)^2 K^{-i\phi} f(t, T, i\phi, -1)}{i\phi} \mathrm{d}\phi \right]. \end{aligned} \tag{19}$$

Analogous relations can be retrieved for the sensitivities of the put option price by differentiating (17).

In Figs. 7 and 8 we plot $\Delta(S_t, t)$ and $\Gamma(S_t, t)$ computed using Eqs. (18) and (19), respectively, as a function of the strike price K and the

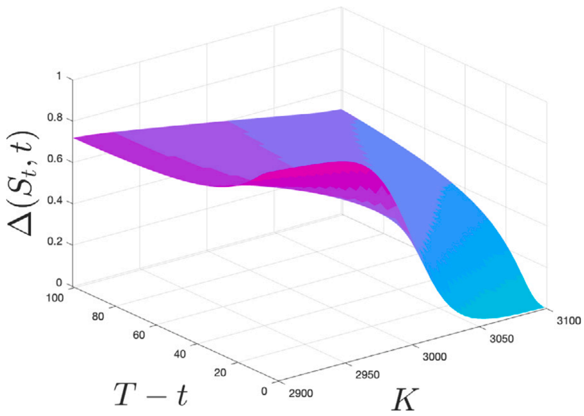

Fig. 7. Delta surface.

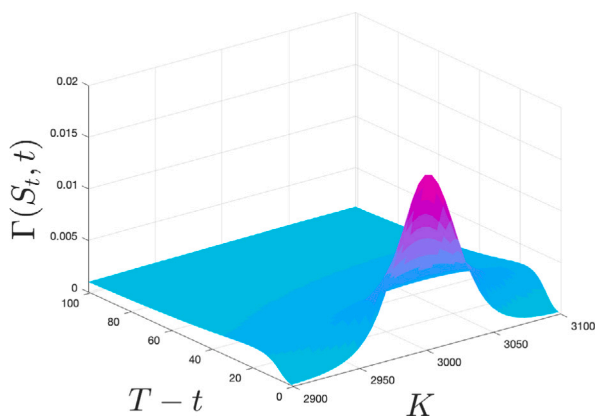

Fig. 8. Gamma surface.

option maturity T. The model parameters are those reported in Table 2. Moreover, we set $S_t = 3000$.

5.3. Joint model estimation

Several studies document the importance of calibrating asset pricing models based on both stock and option values, see Chernov and Ghysels (2000), Pan (2002), Santa-Clara and Yan (2010). As argued by Ornthanalai (2014), fitting and evaluating a dynamic model with both of the two sources of uncertainty can be challenging. However, we note that the joint estimation method proposed by Ornthanalai (2014) can be easily adapted for our bivariate process, and provides a feasible and efficient procedure for gaining more insights about option pricing performances.

To apply this method, we need to define a suitably weighted log-likelihood function which takes into account both the daily time series of log-returns and the panel of option contracts. While for the daily returns the log-likelihood $\ell_{y,r}(\theta)$ is already available in (5), further assumptions are required for the option pricing error structure. Formally, if we have N option prices, we define the error on the ith option price, u_i, $i = 1, \dots, N$, as follows:

$$u_i = \frac{IV_i^{MKT} - IV_i^{MOD}}{IV_i^{MKT}}, \tag{20}$$

where IV_i^{MKT} and IV_i^{MOD} denote the market and the model implied volatilities of the ith option, respectively, computed according to the popular Black–Scholes model, see Black and Scholes (1973). Then, following Christoffersen et al. (2012), Ornthanalai (2014) and Ballestra et al. (2023), the likelihood associated to the error on the implied volatility is computed by using a Gaussian specification:

$$\ell_u(\theta) = -\frac{N}{2}(2\pi\sigma_u^2) - \frac{1}{2}\sum_{i=1}^{N}\frac{u_i^2}{\sigma_u^2}.$$

It is worth noting that the number N of data points available in the option panel could be different from the number T of daily returns. Therefore, according to Christoffersen et al. (2012), Ornthanalai (2014) and Ballestra et al. (2023), to assign an equal weight to returns and option prices, we consider the following weighted joint (total) log-likelihood:

$$\ell_{joint}(\theta) = \frac{T+N}{2}\frac{\ell_{r,y}(\theta)}{T} + \frac{T+N}{2}\frac{\ell_u(\theta)}{N}. \tag{21}$$

5.4. Option valuation results

We consider European Call and Put options on the S&P500 index, with data retrieved from Thomson Reuters Eikon Datastream. We select the S&P500 options expiring from 2016 to 2022, and we take the time series of their daily prices from January 14, 2016 to January 14, 2023. We conduct both an in-sample analysis, based on the options expiring from 2016 2020, and an out-of-sample exercise, where we use the options with maturity in years 2021 and 2022. The joint estimation results are reported in Table 4.

As a common approach, we apply several filters to obtain the final panel of Call and Put option contracts. Specifically, following Bormetti et al. (2020) and Ballestra et al. (2023), we keep only the options with time-to-maturity between 14 and 365 days. Moreover, as in Christoffersen et al. (2012), we only consider out-of-the-money Call and Put options (we define the moneyness as S_t/K), and, following Bakshi et al. (1997), we remove price quotes lower than 3/8$. Finally, as in Christoffersen et al. (2012), for the options expiring in year 2020 we exclude illiquid quotes by selecting only the six most liquid strikes at each maturity (we do not apply this last filter for the options expiring in year 2021, in order to have a significant amount of option prices for the out-of-sample analysis).

To asses the performances of the four competing models, we follow Majewski et al. (2015) and Alitab et al. (2019), and we employ the (percentage) implied volatility root mean square error:

$$IVRMSE(\%) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(IV_i^{MKT} - IV_i^{MOD})^2} \times 100. \tag{22}$$

Finally, for the *B-GARCH-DIR* and both the *GARCH-HN* and *GARCH-C*, we evaluate the model implied option prices based on formulas (12) and (14), and by numerical integration.

We first discuss the in-sample results, which are reported in Table 5. The overall pricing performance (Panel A) indicates that our *B-GARCH-DIR* provides a much better fit than the *GARCH-HN* and better than the *GARCH-C*. In fact, for the *B-GARCH-DIR* the in-sample *IVRMSE* is equal to 11.35%, whereas for the *GARCH-HN* and the *GARCH-C* it is equal to 16.99% and to 12.52, respectively. Moreover, the *B-GARCH-DIR* emerges as superior to both the *GARCH-HN* and the *GARCH-C* even if we split the options by moneyness (Panel B) or maturity (Panel C). In particular, for options with large time-to-maturity (between 150 and 365 days), the *B-GARCH-DIR* yields a pricing error that is almost one-half the pricing error of the *GARCH-HN*. Therefore, the use of a GARCH asset pricing model with stochastic interest rates provides a high degree of flexibility, which is crucial to obtaining a better pricing performance.

The out-of-sample results, which are reported in Table 6, confirm the in-sample analysis. In fact, the *B-GARCH-DIR* is superior to both the *GARCH-HN* and *GARCH-C* models for all the considered levels of moneyness and maturity.

We conclude that, also when pricing options, the *B-GARCH-DIR* is more suitable than the other considered benchmarks, and that taking into account a stochastic interest rate is essential for an accurate description of the volatility surface implied by S&P500 options.

Table 4
Joint maximum likelihood estimation results. Normalized values are divided by the respective joint log-likelihood from *B-GARCH-DIR*. The estimation period spans from January 01, 1975 to December 31, 2022 (12 522 daily observations), and the time series of the options spans from January 14, 2016 to January 14, 2023. The standard errors reported in parenthesis are computed by inverting the negative Hessian matrix evaluated at the optimum parameter values.

	B-GARCH-DIR	*GARCH-HN*	*GARCH-C*
Interest rate			
$\bar{r}$	−0.000	(−)	(−)
	(0.004)		
β_r	0.000	(−)	(−)
	(0.001)		
λ_v	0.000	(−)	(−)
	(0.028)		
ω_v	0.001**	(−)	(−)
	(0.000)		
α_v	0.001**	(−)	(−)
	(0.000)		
β_v	0.914**	(−)	(−)
	(0.338)		
ψ_v	-0.213**	(−)	(−)
	(0.016)		
S&P500 log-return			
λ_h	−0.031*	−0.003	−0.000
	(0.009)	(0.037)	(0.036)
ω_h	-0.002**	-0.001**	(−)
	(0.001)	(0.000)	
α_h	0.008**	0.006**	0.008**
	(0.001)	(0.001)	(0.001)
β_h	0.908**	0.781**	0.750**
	(0.015)	(0.006)	(0.004)
γ_h	2.707**	1.090**	2.803**
	(0.241)	(0.178)	(0.324)
κ_h	0.049**	(−)	(−)
	(0.001)		
ω_q	(−)	(−)	0.001**
			(0.001)
α_q	(−)	(−)	0.011**
			(0.001)
β_q	(−)	(−)	0.998**
			(0.005)
γ_q	(−)	(−)	0.778**
			(0.308)
Joint log-likelihood			
Normalized	1.00	1.235	1.067

** Denote statistical significance at the 0.01 levels, respectively.
* Denote statistical significance at the 0.05 levels, respectively.

Table 5
In-sample implied volatility root mean squared error in percentage points (*IVRMSE*(%)). The options' time series spans from January 14, 2016 to January 14, 2020.

Panel A: IVRMSE(%) overall			
	B-GARCH-DIR	*GARCH*	*GARCH-C*
	11.35	16.99	12.52
Panel B: IVRMSE(%) sorted by moneyness			
Moneyness	*B-GARCH-DIR*	*GARCH*	*GARCH-C*
$0.8 \leq S_t/K \leq 0.9$	5.63	7.83	5.92
$0.9 < S_t/K \leq 1.02$	7.07	11.26	7.54
$1.02 < S_t/K \leq 1.2$	9.66	15.28	10.88
Panel C: IVRMSE(%) sorted by day-to-maturity (DTM)			
Maturity	*B-GARCH-DIR*	*GARCH*	*GARCH-C*
14 ≤*DTM*≤ 50	21.09	25.67	21.71
50 <*DTM*≤ 150	12.87	18.73	13.03
150 <*DTM*≤ 365	6.13	11.34	7.57

6. Conclusions

We have introduced a novel bivariate GARCH process to model the interaction between the volatility of stock returns and the short-term interest rate. The proposed framework uses a linear specification for the drift of the interest rate and a nonlinear asymmetric GARCH for its conditional variance. Accordingly, the conditional variance of the asset returns is specified by an augmented version of the nonlinear GARCH, which combines the innovations that drive both the returns and the interest rate.

An empirical analysis is carried out on the S&P500 and the 3-month Treasury Bill daily time series. The results obtained reveal that our model provides both a better in-sample fitting and a higher predictive performance than the *GARCH-HN* and *GARCH-C*.

Furthermore, we derive analytical backward recursions to compute the moment generating function, which allow for fast and accurate derivative pricing. We present an option pricing application in which, by considering a rich sample of Call and Put S&P500 option values, we show that our novel approach yields significantly more accurate in-sample and out-of-sample implied volatility surfaces that the GARCH model of Heston and Nandi (2000). Therefore, the methodology proposed in the present paper turns out to be an efficient, tractable and easy to implement approach for managing asset price risk and valuing options.

Table 6
Out-of-sample implied volatility root mean squared error in percentage points (*IVRMSE*(%)). The options' time series spans from January 14, 2020 to January 14, 2023.

Panel A: IVRMSE(%) overall			
	B-GARCH-DIR	*GARCH-HN*	*GARCH-C*
	20.87	26.17	22.06
Panel B: IVRMSE(%) sorted by moneyness			
Moneyness	*B-GARCH-DIR*	*GARCH-HN*	*GARCH-C*
$0.8 \leq S_t/K \leq 0.9$	5.01	8.45	5.81
$0.9 < S_t/K \leq 1.02$	6.63	10.55	7.02
$1.02 < S_t/K \leq 1.2$	13.99	20.82	15.31
Panel C: IVRMSE(%) sorted by day-to-maturity (DTM)			
Maturity	*B-GARCH-DIR*	*GARCH-HN*	*GARCH-C*
14 ≤*DTM*≤ 50	21.09	24.94	21.81
50 <*DTM*≤ 150	23.64	30.03	24.84
150 <*DTM*≤ 365	16.24	20.74	18.01

Appendix

Proof of Proposition 1. We guess that the moment generating function takes the log-linear form in (7). Therefore, by the law of iterated expectation, we have

$$f(t,T,\phi,\xi) = \mathbb{E}_t\left[\exp\left\{\phi x_{t+1}+A_{t+1}+B_{t+1}h_{t+2}+C_{t+1}r_{t+2}+D_{t+1}v_{t+2}+\xi r_{t+1}\right\}\right].$$

By considering Eqs. (1), (2), (3), and(4), we obtain

$$\mathbb{E}_t[\exp\{\phi x_{t+1} + A_{t+1} + B_{t+1}h_{t+2} + C_{t+1}r_{t+2} + D_{t+1}v_{t+2} + \xi r_{t+1}\}]$$
$$= \mathbb{E}_t \left[\exp\left\{\begin{array}{c} \phi(x_t + r_{t+1} + \lambda_h h_{t+1} + \sqrt{h_{t+1}}Z_{t+1}) + A_{t+1} \\ +B_{t+1}\left(\omega_h + \beta_h h_{t+1} + \alpha_h\left(Z_{t+1} - \gamma_h\sqrt{h_{t+1}}\right)^2 + \kappa_h\left(\epsilon_{t+1} - \psi_v\sqrt{v_{t+1}}\right)^2\right) \\ +C_{t+1}\left(\bar{r} + (\beta_r + 1)r_{t+1} + \lambda_v v_{t+1} + \sqrt{v_{t+1}}\epsilon_{t+1}\right) \\ +D_{t+1}\left(\omega_v + \beta_v v_{t+1} + \alpha_v\left(\epsilon_{t+1} - \psi_v\sqrt{v_{t+1}}\right)^2\right) + \xi r_{t+1} \end{array}\right\}\right].$$

By simple algebra, we have

$$\mathbb{E}_t[\exp\{\phi x_{t+1} + A_{t+1} + B_{t+1}h_{t+2} + C_{t+1}r_{t+2} + D_{t+1}v_{t+2} + \xi r_{t+1}\}] \tag{23}$$
$$=\mathbb{E}_t \left[\exp\left\{\begin{array}{c} \phi x_t + A_{t+1} + B_{t+1}\omega_h + C_{t+1}\bar{r} + D_{t+1}\omega_v \\ +B_{t+1}\alpha_h\left[Z_{t+1}^2 + \gamma_h^2 h_{t+1} - \left(2\gamma_h - \frac{\phi}{B_{t+1}\alpha_h}\right)\sqrt{h_{t+1}}Z_{t+1}\right] \\ +\left(B_{t+1}\beta_h + \phi\lambda_h\right)h_{t+1} + \left(C_{t+1}(\beta_r + 1) + \phi + \xi\right)r_{t+1} + (C_{t+1}\lambda_v + D_{t+1}\beta_v)v_{t+1} \\ +\left(B_{t+1}\kappa_h + D_{t+1}\alpha_v\right)\left[\epsilon_{t+1}^2 + \psi_v^2 v_{t+1} - \left(2\psi_v - \frac{C_{t+1}}{B_{t+1}\kappa_h + D_{t+1}\alpha_v}\right)\sqrt{v_{t+1}}\epsilon_{t+1}\right] \end{array}\right\}\right].$$

$$
\begin{aligned}
&\mathbb{E}_t[\exp\{\phi x_{t+1} + A_{t+1} + B_{t+1}h_{t+2} + C_{t+1}r_{t+2} + D_{t+1}v_{t+2} + \xi r_{t+1}\}] \qquad (25)\\
&= \exp\left\{\begin{array}{c}
\phi x_t + A_{t+1} + B_{t+1}\omega_h + C_{t+1}\bar{r} + D_{t+1}\omega_v - \frac{1}{2}\ln(1-2B_{t+1}\alpha_h) - \frac{1}{2}\ln[1-2(B_{t+1}\kappa_h + D_{t+1}\alpha_v)]\\
+\left(B_{t+1}\beta_h + \phi\lambda_h + \phi\gamma_h - \frac{\phi^2}{4B_{t+1}\alpha_h} + \frac{B_{t+1}\alpha_h(\gamma_h - \frac{\phi}{2B_{t+1}\alpha_h})^2}{1-2B_{t+1}\alpha_h}\right)h_{t+1}\\
+\left(C_{t+1}(\beta_r+1) + \phi + \xi\right)r_{t+1}\\
\left(C_{t+1}\lambda_v + D_{t+1}\beta_v + \psi_v C_{t+1} - \frac{C_{t+1}^2}{4(B_{t+1}\kappa_h + D_{t+1}\alpha_v)} + \frac{(B_{t+1}\kappa_h + D_{t+1}\alpha_v)(\psi_v - \frac{C_{t+1}}{2(B_{t+1}\kappa_h + D_{t+1}\alpha_v)})^2}{1-2(B_{t+1}\kappa_h + D_{t+1}\alpha_v)}\right)v_{t+1}
\end{array}\right\},
\end{aligned}
$$

Box I.

Now, by completing the square in (23), we achieve

$$
\begin{aligned}
&\mathbb{E}_t[\exp\{\phi x_{t+1} + A_{t+1} + B_{t+1}h_{t+2} + C_{t+1}r_{t+2} + D_{t+1}v_{t+2} + \xi r_{t+1}\}] \qquad (24)\\
&= \mathbb{E}_t\left[\exp\left\{\begin{array}{c}
\phi x_t + A_{t+1} + B_{t+1}\omega_h + C_{t+1}\bar{r} + D_{t+1}\omega_v\\
+B_{t+1}\alpha_h\left[Z_{t+1} - \left(\gamma_h - \frac{\phi}{2B_{t+1}\alpha_h}\right)\sqrt{h_{t+1}}\right]^2\\
+\left(B_{t+1}\beta_h + \phi\lambda_h + \phi\gamma_h - \frac{\phi^2}{4B_{t+1}\alpha_h}\right)h_{t+1}\\
+\left(C_{t+1}(\beta_r+1) + \phi + \xi\right)r_{t+1}\\
\left(B_{t+1}\kappa_h + D_{t+1}\alpha_v\right)\left[\epsilon_{t+1} - \left(\psi_v - \frac{C_{t+1}}{2(B_{t+1}\kappa_h + D_{t+1}\alpha_v)}\right)\sqrt{v_{t+1}}\right]^2\\
\left(C_{t+1}\lambda_v + D_{t+1}\beta_v + \psi_v C_{t+1} - \frac{C_{t+1}^2}{4(B_{t+1}\kappa_h + D_{t+1}\alpha_v)}\right)v_{t+1}
\end{array}\right\}\right].
\end{aligned}
$$

Since Z_t and ϵ_t are independent and normally distributed $\forall t \in \mathbb{Z}$, we can simplify (24) by using the well-known result

$$
\mathbb{E}[\exp\{a(w+b)^2\}] = \exp\left\{-\frac{1}{2}\ln(1-2a) + \frac{ab^2}{1-2a}\right\},
$$

where $w \sim \mathcal{N}(0,1)$.

Therefore Eq. (25) (see Box I), and, finally, by equating homogeneous terms of (7) and (25), the relations in (8)–(11) are obtained, which concludes the proof. □

Proof of Proposition 2. The option price is the expected discounted value of the option payoff:

$$
C(S_t,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-Y_{t,T}}\max(e^{x_T}-K,0)p(x_T,Y_{t,T}|x_t,r_t)\mathrm{d}Y_{t,T}\mathrm{d}x_T. \quad (26)
$$

Let us define the bivariate density function

$$
p_t^{\star\star}(x_T,Y_{t,T}) = \frac{e^{-Y_{t,T}}p(x_T,Y_{t,T}|x_t,r_t)}{\mathbb{E}_t[\exp\{-Y_{t,T}\}]}, \quad (27)
$$

and its marginal

$$
p_t^{\star\star}(x_T) = \int_{-\infty}^{+\infty} p_t^{\star\star}(x_T,Y_{t,T})\mathrm{d}Y_{t,T}. \quad (28)
$$

Eq. (26) can be rewritten as follows:

$$
C(S_t,t) = \mathbb{E}_t[\exp\{-Y_{t,T}\}] \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\max(e^{x_T}-K,0)\frac{e^{-Y_{t,T}}p(x_T,Y_{t,T}|x_t,r_t)}{\mathbb{E}_t[\exp\{-Y_{t,T}\}]}\mathrm{d}Y_{t,T}\mathrm{d}x_T.
$$

By using (27) and (28), and by noticing that $\mathbb{E}_t[\exp\{-Y_{t,T}\}] = f(t,T,0,-1)$, Eq. (12) is readily obtained. □

Proof of Proposition 3. Using (13), we have

$$
\int_{-\infty}^{+\infty} e^{\phi x_T}p_t^{\star\star}(x_T)\mathrm{d}x_T = \int_{-\infty}^{+\infty} e^{\phi x_T}\int_{-\infty}^{+\infty}\frac{e^{-Y_{t,T}}p(x_T,Y_{t,T}|x_t,r_t)}{f(t,T,0,-1)}\mathrm{d}Y_{t,T}\mathrm{d}x_T = \frac{\mathbb{E}_t[e^{\phi x_T - Y_{t,T}}]}{f(t,T,0,-1)} = \frac{f(t,T,\phi,-1)}{f(t,T,0,-1)}. \quad \square
$$

Proof of Proposition 4. Eq. (26) can be rewritten as follows

$$
\begin{aligned}
C(S_t,t) =& f(t,T,1,-1)\int_{\ln K}^{+\infty}\int_{-\infty}^{+\infty} p_t^{\star}(x_T,Y_{t,T})\mathrm{d}Y_{t,T}\mathrm{d}x_T\\
&- Kf(t,T,0,-1)\int_{\ln K}^{+\infty}\int_{-\infty}^{+\infty} p_t^{\star\star}(x_T,Y_{t,T})\mathrm{d}Y_{t,T}\mathrm{d}x_T, \quad (29)
\end{aligned}
$$

where

$$
p_t^{\star}(x_T,Y_{t,T}) = \frac{e^{-Y_{t,T}}e^{x_T}p(x_T,Y_{t,T}|x_t,r_t)}{f(t,T,1,-1)}, \quad (30)
$$

and $p_t^{\star\star}(x_T,Y_{t,T})$ is given by (27).

The characteristic function associated to $p(x_T,Y_{t,T}|x_t,r_t)$ is $f(t,T,i\phi,i\xi)$. Then, from (27) and (30) it is easy to see that the characteristic function associated to $p_t^{\star}(x_T,Y_{t,T})$ is $f(t,T,i\phi+1,i\xi-1)/f(t,T,1,-1)$, whereas the characteristic function associated to $p_t^{\star\star}(x_T,Y_{t,T})$ is $f(t,T,i\phi,i\xi-1)/f(t,T,0,-1)$, because

$$
\begin{aligned}
&\int_{\ln K}^{+\infty}\int_{-\infty}^{+\infty} p_t^{\star}(x_T,Y_{t,T})\mathrm{d}Y_{t,T}\mathrm{d}x_T\\
&= \frac{1}{f(t,T,1,-1)}\int_{\ln K}^{+\infty}\int_{-\infty}^{+\infty} e^{(\phi+1)x_T+(\xi-1)Y_{t,T}}p_t(x_T,Y_{t,T}|x_t,r_t)\mathrm{d}Y_{t,T}\mathrm{d}x_T,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\ln K}^{+\infty}\int_{-\infty}^{+\infty} p_t^{\star\star}(X_{t,T},Y_{t,T})\mathrm{d}Y_{t,T}\mathrm{d}x_T\\
&= \frac{1}{f(t,T,0,-1)}\int_{\ln K}^{+\infty}\int_{-\infty}^{+\infty} e^{\phi x_T+(\xi-1)Y_{t,T}}p(x_T,Y_{t,T}|x_t,r_t)\mathrm{d}Y_{t,T}\mathrm{d}x_T.
\end{aligned}
$$

Since $f(t,T,\phi,\xi) = \mathbb{E}_t[\exp\{\phi x_T + \xi Y_{t,T}\}]$ and $f(t,T,\phi,\xi-1) = \mathbb{E}_t[\exp\{\phi x_T + (\xi-1)Y_{t,T}\}]$, we can marginalize the double integral in (29) by fixing $\xi = 1$, so that

$$
\begin{aligned}
C(S_t,t) =& f(t,T,1,-1)\int_{\ln K}^{+\infty} p_t^{\star}(x_T)\mathrm{d}x_T\\
&- Kf(t,T,0,-1)\int_{\ln K}^{+\infty} p_t^{\star\star}(x_T)\mathrm{d}x_T, \quad (31)
\end{aligned}
$$

where

$$
p_t^{\star}(x_T) = \int_{-\infty}^{+\infty} p_t^{\star}(x_T,Y_{t,T})\mathrm{d}Y_{t,T},
$$

and $p_t^{\star\star}(x_T)$ is given by (28).

Then, by using the well-known result of Gil-Pelaez (1951), we have

$$
\int_{\ln K}^{+\infty} p_t^{\star}(x_T)\mathrm{d}x_T = \frac{1}{2} - \frac{1}{2\pi}\int_0^{+\infty}\frac{K^{i\phi}f(t,T,-i\phi+1,-1) - K^{-i\phi}f(t,T,i\phi+1,-1)}{i\phi f(t,T,1,-1)}\mathrm{d}\phi, \quad (32)
$$

$$\int_{\ln K}^{+\infty} p_t^{\star\star}(x_T)\mathrm{d}x_T = \frac{1}{2} - \frac{1}{2\pi}$$

$$\int_0^{+\infty} \frac{K^{i\phi} f(t,T,-i\phi,-1) - K^{-i\phi} f(t,T,i\phi,-1)}{i\phi f(t,T,0,-1)}\mathrm{d}\phi. \tag{33}$$

Finally, by substituting (32) and (33) into (31) we obtain (16). Relation (17) can be derived by using an analogous procedure, which is omitted to save space. □

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