

1 **SUPPLEMENTARY MATERIALS: CONSTRUCTION AND**
2 **EVALUATION OF PYTHAGOREAN HODOGRAPH CURVES IN**
3 **EXPONENTIAL-POLYNOMIAL SPACES ***

4 LUCIA ROMANI[†] AND ALBERTO VISCARDI[‡]

5 **SM1. Proof of Proposition 3.3.**

Proof. (a) is a consequence of the fact that $\phi_{i,1}^\omega(t) \geq 0$ for all $t \in [0, 1]$ and $\sum_{i=0}^3 \phi_{i,1}^\omega(t) = 1$ for all $t \in [0, 1]$;
(b) is due to the fact that $\phi_{i,1}^\omega(t) = \phi_{3-i,1}^\omega(1-t)$ for all $t \in [0, 1]$, $i = 0, \dots, 3$;
(c) follows from the fact that

$$\begin{aligned} \frac{d}{dt} \phi_{0,1}^\omega(t) &= -\frac{\varphi_{0,1}^\omega(t)}{\int_0^1 \varphi_{0,1}^\omega(x) dx}, \\ \frac{d}{dt} \phi_{i,1}^\omega(t) &= \frac{\varphi_{i-1,1}^\omega(t)}{\int_0^1 \varphi_{i-1,1}^\omega(x) dx} - \frac{\varphi_{i,1}^\omega(t)}{\int_0^1 \varphi_{i,1}^\omega(x) dx} \quad i = 1, 2 \\ \frac{d}{dt} \phi_{3,1}^\omega(t) &= \frac{\varphi_{2,1}^\omega(t)}{\int_0^1 \varphi_{2,1}^\omega(x) dx}; \end{aligned}$$

6 (d) is a consequence of (c). □

7 **SM2. Proof of Proposition 3.4.**

Proof. By substituting (3.8) into (2.5) we obtain

$$\mathbf{r}'(t) = \mathbf{A}_0 \mathbf{iA}_0^* (\psi_{0,1}^\omega(t))^2 + (\mathbf{A}_0 \mathbf{iA}_1^* + \mathbf{A}_1 \mathbf{iA}_0^*) \psi_{0,1}^\omega(t) \psi_{1,1}^\omega(t) + \mathbf{A}_1 \mathbf{iA}_1^* (\psi_{1,1}^\omega(t))^2$$

8 and thus, in light of (3.3),

9 (SM2.1) $\mathbf{r}'(t) = \mathbf{A}_0 \mathbf{iA}_0^* \varphi_{0,1}^\omega(t) + (\mathbf{A}_0 \mathbf{iA}_1^* + \mathbf{A}_1 \mathbf{iA}_0^*) \frac{1}{2} c_1(\omega) \varphi_{1,1}^\omega(t) + \mathbf{A}_1 \mathbf{iA}_1^* \varphi_{2,1}^\omega(t).$

10 Since $\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{r}'(x) dx$, by integrating the expression in (SM2.1) exploiting the
11 formulae in (3.5), we obtain the Bézier-like form of $\mathbf{r}(t)$ with control points in (3.10).□

12 **SM3. Proof of Proposition 3.6.**

Proof. Since $\sigma(t) = \mathbf{A}(t) \mathbf{A}^*(t)$, in light of (3.8) we can write

$$\sigma(t) = |\mathbf{A}_0|^2 (\psi_{0,1}^\omega(t))^2 + (\mathbf{A}_1 \mathbf{A}_0^* + \mathbf{A}_0 \mathbf{A}_1^*) \psi_{0,1}^\omega(t) \psi_{1,1}^\omega(t) + |\mathbf{A}_1|^2 (\psi_{1,1}^\omega(t))^2.$$

13 Then, exploiting (3.3), the claimed result is obtained. □

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[†]AM² - Dipartimento di Matematica, Alma Mater Studiorum Università di Bologna, Bologna, Italy (lucia.romani@unibo.it)

[‡]Dipartimento di Matematica “Giuseppe Peano”, Università di Torino, Torino, Italy (alberto.viscardi@unito.it)

14 **SM4. Proof of Proposition 3.7.**

Proof. Since

$$s(t) = \int_0^t \sigma(x) dx = \int_0^t \sum_{i=0}^2 \sigma_i \varphi_{i,1}^\omega(x) dx = \sum_{i=0}^2 \sigma_i \int_0^t \varphi_{i,1}^\omega(x) dx,$$

15 then, recalling formulae (3.5) we arrive at

$$16 \quad s(t) = \sigma_2 c_2(\omega) \phi_{3,1}^\omega(t) + \sigma_1 \frac{c_3(\omega)}{c_1(\omega)} \sum_{i=2}^3 \phi_{i,1}^\omega(t) + \sigma_0 c_2(\omega) \sum_{i=1}^3 \phi_{i,1}^\omega(t)$$

17 and, by collecting the coefficients of each basis function $\phi_{i,1}^\omega(t)$, $i = 0, \dots, 3$, we get
18 the claimed result. \square

19 **SM5. Proof of Proposition 4.3.**

Proof. (a) is a consequence of the fact that $\phi_{i,2}^\omega(t) \geq 0$ for all $t \in [0, 1]$ and $\sum_{i=0}^5 \phi_{i,2}^\omega(t) = 1$ for all $t \in [0, 1]$;
(b) is due to the fact that $\phi_{i,2}^\omega(t) = \phi_{5-i,2}^\omega(1-t)$ for all $t \in [0, 1]$, $i = 0, \dots, 5$;
(c) follows from the fact that

$$\begin{aligned} \frac{d}{dt} \phi_{0,2}^\omega(t) &= -\frac{\varphi_{0,2}^\omega(t)}{\int_0^1 \varphi_{0,2}^\omega(x) dx}, \\ \frac{d}{dt} \phi_{i,2}^\omega(t) &= \frac{\varphi_{i-1,2}^\omega(t)}{\int_0^1 \varphi_{i-1,2}^\omega(x) dx} - \frac{\varphi_{i,2}^\omega(t)}{\int_0^1 \varphi_{i,2}^\omega(x) dx}, \quad i = 1, \dots, 4 \\ \frac{d}{dt} \phi_{5,2}^\omega(t) &= \frac{\varphi_{4,2}^\omega(t)}{\int_0^1 \varphi_{4,2}^\omega(x) dx}; \end{aligned}$$

20 (d) is a consequence of (c). \square

21 **SM6. Proof of Proposition 4.4.**

22 *Proof.* By substituting (4.6) into (2.5) we obtain

$$\begin{aligned} 24 \quad \mathbf{r}'(t) &= \sum_{j=0}^2 \mathbf{A}_j \mathbf{iA}_j^* (\psi_{j,2}^\omega(t))^2 + (\mathbf{A}_0 \mathbf{iA}_1^* + \mathbf{A}_1 \mathbf{iA}_0^*) \psi_{0,2}^\omega(t) \psi_{1,2}^\omega(t) \\ 25 \quad &+ (\mathbf{A}_1 \mathbf{iA}_2^* + \mathbf{A}_2 \mathbf{iA}_1^*) \psi_{1,2}^\omega(t) \psi_{2,2}^\omega(t) + (\mathbf{A}_0 \mathbf{iA}_2^* + \mathbf{A}_2 \mathbf{iA}_0^*) \psi_{0,2}^\omega(t) \psi_{2,2}^\omega(t) \end{aligned}$$

27 and thus, in light of (4.1),

$$\begin{aligned} 29 \quad (\text{SM6.1}) \quad \mathbf{r}'(t) &= \mathbf{A}_0 \mathbf{iA}_0^* \varphi_{0,2}^\omega(t) + (\mathbf{A}_0 \mathbf{iA}_1^* + \mathbf{A}_1 \mathbf{iA}_0^*) \frac{1}{2} \varphi_{1,2}^\omega(t) \\ 30 \quad &+ \left(\mathbf{A}_1 \mathbf{iA}_1^* q_0(\omega) + (\mathbf{A}_0 \mathbf{iA}_2^* + \mathbf{A}_2 \mathbf{iA}_0^*) \frac{1}{2} q_1(\omega) \right) \varphi_{2,2}^\omega(t) \\ 31 \quad &+ (\mathbf{A}_1 \mathbf{iA}_2^* + \mathbf{A}_2 \mathbf{iA}_1^*) \frac{1}{2} \varphi_{3,2}^\omega(t) + \mathbf{A}_2 \mathbf{iA}_2^* \varphi_{4,2}^\omega(t). \end{aligned}$$

33 Since $\mathbf{r}(t) = \mathbf{r}_0 + \int_0^t \mathbf{r}'(x) dx$, by integrating the expression in (SM6.1) exploiting the
34 formulae in (4.3), we obtain the Bézier-like form of $\mathbf{r}(t)$ with control points in (4.8). \square

35 **SM7. Proof of Proposition 4.6.**

 36 *Proof.* Since $\sigma(t) = \mathbf{A}(t)\mathbf{A}^*(t)$, in light of (4.6) we can write

37

$$\begin{aligned}
 38 \quad \sigma(t) &= \sum_{j=0}^2 |\mathbf{A}_j|^2 (\psi_{j,2}^\omega(t))^2 + (\mathbf{A}_1\mathbf{A}_0^* + \mathbf{A}_0\mathbf{A}_1^*) \psi_{0,2}^\omega(t)\psi_{1,2}^\omega(t) \\
 39 &\quad + (\mathbf{A}_1\mathbf{A}_2^* + \mathbf{A}_2\mathbf{A}_1^*) \psi_{1,2}^\omega(t)\psi_{2,2}^\omega(t) + (\mathbf{A}_2\mathbf{A}_0^* + \mathbf{A}_0\mathbf{A}_2^*) \psi_{0,2}^\omega(t)\psi_{2,2}^\omega(t).
 \end{aligned}$$

 41 Then, exploiting (4.1), we obtain the claimed result. \square

 42 **SM8. Proof of Proposition 4.7.**
Proof. Since

$$s(t) = \int_0^t \sigma(x)dx = \int_0^t \sum_{i=0}^4 \sigma_i \varphi_{i,2}^\omega(x)dx = \sum_{i=0}^4 \sigma_i \int_0^t \varphi_{i,2}^\omega(x)dx,$$

43 recalling formulae (4.3), we then arrive at

44

$$\begin{aligned}
 45 \quad s(t) &= \sigma_0 q_2(\omega) \sum_{i=1}^5 \phi_{i,2}^\omega(t) + \sigma_1 q_3(\omega) \sum_{i=2}^5 \phi_{i,2}^\omega(t) + \sigma_2 \frac{q_4(\omega)}{q_1(\omega)} \sum_{i=3}^5 \phi_{i,2}^\omega(t) \\
 46 &\quad + \sigma_3 q_3(\omega) \sum_{i=4}^5 \phi_{i,2}^\omega(t) + \sigma_4 q_2(\omega) \phi_{5,2}^\omega(t).
 \end{aligned}$$

47

 48 By collecting the coefficients of each basis function $\phi_{i,2}^\omega(t)$, $i = 0, \dots, 5$, we get the

 49 claimed result. \square

 50 **SM9. 5th order Taylor expansions at $\omega = 0$ of $\{\phi_{j,m}^\omega(t)\}_{j=0}^{2m+1}$, $m \in \{1, 2\}$.**

 51 For $\{\phi_{j,1}^\omega(t)\}_{j=0}^3$,

$$T_{0,1}^\omega(t) = T_{3,1}^\omega(1-t), \quad T_{1,1}^\omega(t) = T_{2,1}^\omega(1-t),$$

$$52 \quad T_{2,1}^\omega(t) = 3t^2(1-t) \frac{(30t^4 - 40t^3 + 23t^2 - 12t - 3)\omega^4 + 420(3t^2 - 2t + 1)\omega^2 + 25200}{25200},$$

$$T_{3,1}^\omega(t) = t^3 \frac{(10t^4 - 21t^2 + 11)\omega^4 + 420(t^2 - 1)\omega^2 + 8400}{8400},$$

 53 and, for $\{\phi_{j,2}^\omega(t)\}_{j=0}^5$,

$$T_{0,2}^\omega(t) = T_{5,2}^\omega(1-t), \quad T_{1,2}^\omega(t) = T_{4,2}^\omega(1-t), \quad T_{2,2}^\omega(t) = T_{3,2}^\omega(1-t),$$

$$T_{3,2}^\omega(t) = 10t^3(1-t)^2 \frac{(245t^4 - 392t^3 + 253t^2 - 82t + 3)\omega^4 + 420(10t^2 - 8t + 3)\omega^2 + 35280}{35280},$$

54

$$T_{4,2}^\omega(t) = 5t^4(1-t) \frac{(245t^4 - 196t^3 - 96t^2 + 44t - 1)\omega^4 + 840(5t^2 - 2t - 1)\omega^2 + 35280}{35280},$$

$$T_{5,2}^\omega(t) = t^5 \frac{(49t^4 - 100t^2 + 51)\omega^4 + 840(t^2 - 1)\omega^2 + 7056}{7056}.$$

 55 **SM10. Stable expressions of $\{\phi_{j,m}^\omega(t)\}_{j=0}^{2m+1}$, $m \in \{1, 2\}$, for large ω .**

 56 For $m = 1$,

57

$$\phi_{0,1}^\omega(t) = \phi_{3,1}^\omega(1-t), \quad \phi_{1,1}^\omega(t) = \phi_{2,1}^\omega(1-t),$$

58

$$59 \quad \phi_{2,1}^\omega(t) = \frac{N\phi_{2,1}^\omega(t)}{D\phi_{2,1}^\omega}, \quad \phi_{3,1}^\omega(t) = \frac{e^{-2\omega t} + 2\omega t e^{-\omega t} - 1}{e^{-2\omega} + 2\omega e^{-\omega} - 1} e^{\omega(t-1)},$$

60 where

61

$$62 \quad N\phi_{2,1}^\omega(t) = \left(\frac{1}{\omega} + t\right) e^{-3\omega} - \frac{1}{\omega} e^{\omega(t-3)} - \left(\frac{1}{\omega} + 1\right) e^{-\omega(t+2)} - \left(\frac{1}{\omega} + 3t - 2\right) e^{-2\omega}$$

63

$$64 \quad + \left(\frac{2}{\omega} - 1\right) e^{\omega(t-2)} + \left(\frac{2}{\omega} + 1\right) e^{-\omega(t+1)}$$

65

$$66 \quad - \left(\frac{1}{\omega} - 3t + 2\right) e^{-\omega} - \left(\frac{1}{\omega} - 1\right) e^{\omega(t-1)} - \frac{1}{\omega} e^{-t\omega} + \frac{1}{\omega} - t,$$

67

68

69

$$70 \quad D\phi_{2,1}^\omega = \left(\frac{2}{\omega} + 1\right) e^{-3\omega} - \left(\frac{2}{\omega} - 5 - 2\omega\right) e^{-2\omega} - \left(\frac{2}{\omega} + 5 - 2\omega\right) e^{-\omega} + \frac{2}{\omega} - 1.$$

71

72 For $m = 2$,

$$73 \quad \phi_{0,2}^\omega(t) = \phi_{5,2}^\omega(1-t), \quad \phi_{1,2}^\omega(t) = \phi_{4,2}^\omega(1-t), \quad \phi_{2,2}^\omega(t) = \phi_{3,2}^\omega(1-t),$$

74

$$75 \quad \phi_{3,2}^\omega(t) = \frac{N\phi_{3,2}^\omega(t)}{D\phi_{3,2}^\omega}, \quad \phi_{4,2}^\omega(t) = \frac{N\phi_{4,2}^\omega(t)}{D\phi_{4,2}^\omega} e^{\omega(t-1)},$$

76

$$77 \quad \phi_{5,2}^\omega(t) = \frac{e^{-4\omega t} - 8e^{-3\omega t} - 12\omega t e^{-2\omega t} + 8e^{-\omega t} - 1}{e^{-4\omega} - 8e^{-3\omega} - 12\omega e^{-2\omega} + 8e^{-\omega} - 1} e^{2\omega(t-1)},$$

78 where

79

$$80 \quad N\phi_{3,2}^\omega(t) = \left(\frac{3}{\omega} + 2t\right) e^{-5\omega} - \frac{4}{\omega} e^{\omega(t-5)} + \frac{1}{\omega} e^{\omega(2t-5)} - 4\left(\frac{2}{\omega} + 1\right) e^{-\omega(t+4)}$$

81

$$82 \quad + \left(\frac{9}{\omega} - 2(5t-6)\right) e^{-4\omega} - 4\left(\frac{1}{\omega} + 3\right) e^{\omega(t-4)} + \left(\frac{3}{\omega} + 4\right) e^{2\omega(t-2)}$$

83

$$84 \quad + \left(\frac{5}{\omega} + 2\right) e^{-\omega(2t+3)} + 4\left(\frac{1}{\omega} - 1\right) e^{-\omega(t+3)} - 4\left(\frac{3}{\omega} - 5t\right) e^{-3\omega}$$

85

$$86 \quad + 4\left(\frac{3}{\omega} + 1\right) e^{\omega(t-3)} - \left(\frac{9}{\omega} + 2\right) e^{\omega(2t-3)} - \left(\frac{9}{\omega} - 2\right) e^{-2\omega(t+1)} + 4\left(\frac{3}{\omega} - 1\right) e^{-\omega(t+2)}$$

87

$$88 \quad - 4\left(\frac{3}{\omega} + 5t\right) e^{-2\omega} + 4\left(\frac{1}{\omega} + 1\right) e^{\omega(t-2)} + \left(\frac{5}{\omega} - 2\right) e^{2\omega(t-1)}$$

89

$$90 \quad + \left(\frac{3}{\omega} - 4\right) e^{-\omega(2t+1)} - 4\left(\frac{1}{\omega} - 3\right) e^{-\omega(t+1)} + \left(\frac{9}{\omega} + 2(5t-6)\right) e^{-\omega}$$

91

$$92 \quad - 4\left(\frac{2}{\omega} - 1\right) e^{\omega(t-1)} + \frac{1}{\omega} e^{-2\omega t} - \frac{4}{\omega} e^{-\omega t} + \frac{3}{\omega} - 2t,$$

93

94

$$95 \quad D\phi_{3,2}^\omega = 2\left[\left(\frac{3}{\omega} + 1\right) e^{-5\omega} + \left(\frac{27}{\omega} + 31 + 6\omega\right) e^{-4\omega} - 2\left(\frac{15}{\omega} - 23 - 15\omega\right) e^{-3\omega}\right.$$

96

$$97 \quad \left. - 2\left(\frac{15}{\omega} + 23 - 15\omega\right) e^{-2\omega} + \left(\frac{27}{\omega} - 31 + 6\omega\right) e^{-\omega} + \frac{3}{\omega} - 1\right],$$

98

99

$$100 \quad N\phi_{4,2}^\omega(t) = 2e^{-\omega(2t+5)} + 3(1+2\omega t)e^{-\omega(t+5)} - 6e^{-5\omega} + e^{\omega(t-5)} - 2e^{-\omega(3t+4)}$$

101

$$102 \quad - 2e^{-2\omega(t+2)} - 3(9+10\omega t)e^{-\omega(t+4)} + 38e^{-4\omega} - 7e^{\omega(t-4)}$$

103

$$104 \quad - (1+6\omega)e^{-3\omega(t+1)} + 24(1+\omega)e^{-\omega(2t+3)} + 12(2+\omega(5t-3))e^{-\omega(t+3)}$$

105

$$106 \quad - 8(7-3\omega)e^{-3\omega} + 3(3-2\omega)e^{\omega(t-3)} + 3(3+2\omega)e^{-\omega(3t+2)} - 8(7+3\omega)e^{-2\omega(t+1)}$$

107

$$108 \quad + 12(2-\omega(5t-3))e^{-\omega(t+2)} + 24(1-\omega)e^{-2\omega} - (1-6\omega)e^{\omega(t-2)}$$

109

$$110 \quad - 7e^{-\omega(3t+1)} + 38e^{-\omega(2t+1)} - 3(9-10\omega t)e^{-\omega(t+1)} - 2e^{-\omega}$$

111

$$112 \quad - 2e^{\omega(t-1)} + e^{-3\omega t} - 6e^{-2\omega t} + 3(1-2\omega t)e^{-\omega t} + 2,$$

113

$$114 \quad D\phi_{4,2}^\omega = e^{-7\omega} + (1+6\omega)e^{-6\omega} - 27(3+2\omega)e^{-5\omega} + (79-156\omega-72\omega^2)e^{-4\omega}$$

115

$$116 \quad + (79+156\omega-72\omega^2)e^{-3\omega} - 27(3-2\omega)e^{-2\omega} + (1-6\omega)e^{-\omega} + 1.$$

115 **SM11. Stable expressions of $\{\tau_{j,m}^\omega(t)\}_{j=0}^{2m}$, $m \in \{1, 2\}$, for large ω .**
 116 For $m = 1$,

$$117 \quad \tau_{0,1}^\omega(t) = \frac{e^{\omega(t-2)} - 2(t-1)\omega e^{-\omega} - e^{-t\omega}}{(t-1)^2(e^{-2\omega} + 2\omega e^{-\omega} - 1)}, \quad \tau_{1,1}^\omega(t) = \frac{N\tau_{1,1}^\omega(t)}{D\tau_{1,1}^\omega(t)},$$

118

$$119 \quad \tau_{2,1}^\omega(t) = \frac{t^2 e^{-2\omega} - e^{-\omega(t+1)} + 2t(t-1)\omega e^{-\omega} + e^{\omega(t-1)} - t^2}{t^2(e^{-2\omega} + 2\omega e^{-\omega} - 1)},$$

120 where

121

$$122 \quad N\tau_{1,1}^\omega(t) = \left(\left(\frac{2}{\omega} + 1 \right) t^2 - \left(\frac{4}{\omega} + 1 \right) t + \frac{1}{\omega} \right) e^{-2\omega} - \frac{1}{\omega} e^{\omega(t-1)}$$

123

$$124 \quad - \frac{1}{\omega} e^{\omega(t-2)} + 2t(t-1)e^{-\omega} + \frac{1}{\omega} e^{-t\omega}$$

125

$$126 \quad + \frac{1}{\omega} e^{-\omega(t+1)} - \left(\frac{2}{\omega} - 1 \right) t^2 + \left(\frac{4}{\omega} - 1 \right) t - \frac{1}{\omega},$$

127

128

$$129 \quad D\tau_{1,1}^\omega(t) = 2t(t-1)(e^{-\omega} + 1) \left(\left(\frac{2}{\omega} + 1 \right) e^{-\omega} - \frac{2}{\omega} + 1 \right).$$

130 For $m = 2$, $\tau_{j,2}^\omega(t) = N\tau_{j,2}^\omega(t)/D\tau_{j,2}^\omega(t)$, $j = 0, \dots, 4$, where

$$131 \quad N\tau_{0,2}^\omega(t) = -e^{2\omega(t-2)} + 8e^{\omega(t-3)} - 12(t-1)\omega e^{-2\omega} - 8e^{-\omega(t+1)} + e^{-2t\omega},$$

132

$$133 \quad D\tau_{0,2}^\omega(t) = (t-1)^4 (-e^{-4\omega} + 8e^{-3\omega} + 12\omega e^{-2\omega} - 8e^{-\omega} + 1),$$

134

$$135 \quad N\tau_{1,2}^\omega(t) = (t-1)^4 e^{-3\omega} - 2e^{\omega(t-3)} + e^{\omega(2t-3)}$$

$$136 \quad + 3(3t^4 - 12t^3 + 18t^2 - 12t + 2 + 2t\omega(t^3 - 4t^2 + 6t - 3)) e^{-2\omega}$$

$$137 \quad - 6e^{\omega(t-2)} + 6e^{-\omega(t+1)}$$

$$138 \quad - 3(3t^4 - 12t^3 + 18t^2 - 12t + 2 - 2t\omega(t^3 - 4t^2 + 6t - 3)) e^{-\omega}$$

$$139 \quad - e^{-2t\omega} + 2e^{-t\omega} - (t-1)^4,$$

140

$$141 \quad D\tau_{1,2}^\omega(t) = 4t(t-1)^3 (e^{-3\omega} + 3(2\omega + 3)e^{-2\omega} + 3(2\omega - 3)e^{-\omega} - 1),$$

142

143

144

$$145 \quad N\tau_{2,2}^\omega(t) = - \left(\frac{3(6t^4 - 16t^3 + 12t^2 - 1)}{\omega} + 2t(3t^3 - 8t^2 + 6t - 1) \right) e^{-2\omega}$$

146

$$147 \quad + \frac{4}{\omega} e^{\omega(t-2)} - \frac{1}{\omega} e^{2\omega(t-1)} - \frac{4}{\omega} e^{-\omega(t+1)}$$

148

$$149 \quad + 8t(3t^3 - 8t^2 + 6t - 1) e^{-\omega}$$

150

$$151 \quad + \frac{4}{\omega} e^{\omega(t-1)} + \frac{1}{\omega} e^{-2t\omega} - \frac{4}{\omega} e^{-t\omega}$$

152

$$153 \quad - \frac{3(6t^4 - 16t^3 + 12t^2 - 1)}{\omega} + 2t(3t^3 - 8t^2 + 6t - 1),$$

154

155

$$156 \quad D\tau_{2,2}^\omega(t) = 12t^2(t-1)^2 \left(\left(\frac{3}{\omega} + 1 \right) e^{-2\omega} + 4e^{-\omega} - \frac{3}{\omega} + 1 \right),$$

157

$$158 \quad N\tau_{3,2}^\omega(t) = t^3(3t-4)e^{-3\omega} + 2e^{-\omega(t+2)}$$

159

$$+ 3(2t\omega - 8t^3\omega + 6t^4\omega - 12t^3 + 9t^4 + 1)e^{-2\omega}$$

160

$$- 6e^{\omega(t-2)} + e^{2\omega(t-1)} - e^{-\omega(2t+1)} + 6e^{-\omega(t+1)}$$

161

$$+ 3(2t\omega - 8t^3\omega + 6t^4\omega + 12t^3 - 9t^4 - 1)e^{-\omega}$$

162

$$- 2e^{\omega(t-1)} - t^3(3t-4),$$

164

$$165 \quad D\tau_{3,2}^\omega(t) = 4t^3(t-1)(e^{-3\omega} + 3(2\omega+3)e^{-2\omega} + 3(2\omega-3)e^{-\omega} - 1),$$

166

$$167 \quad N\tau_{4,2}^\omega(t) = -t^4e^{-4\omega} + 8t^4e^{-3\omega} + e^{-2\omega(t+1)} - 8e^{-\omega(t+2)}$$

168

$$+ 12t(t^3-1)\omega e^{-2\omega} + 8e^{\omega(t-2)} - e^{2\omega(t-1)} - 8t^4e^{-\omega} + t^4,$$

170

$$171 \quad D\tau_{4,2}^\omega(t) = t^4(-e^{-4\omega} + 8e^{-3\omega} + 12\omega e^{-2\omega} - 8e^{-\omega} + 1).$$