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The Frobenius characteristic of the Orlik-Terao algebra of type A

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We provide a new virtual description of the symmetric group action on the cohomology of ordered configuration space on SU_2 up to translations. We use this formula to prove the Moseley-Proudfoot-Young conjecture. As a consequence we obtain the graded Frobenius character of the Orlik-Terao algebra of type A_n.

1 Introduction

The Orlik-Terao algebra OT_n is the subalgebra of rational functions on $\mathbb{C}^n$ generated by $\frac{1}{x_i - x_j}$ for all $i \neq j$. It has been intensively studied in [Ter02, PS06, ST09, Ber10, Sch11, DGT14, Le14, Liu16, EPW16, MPY17, MMPR21]. Only recently, has an attempt to describe the symmetric group action on OT_n been made by Moseley, Proudfoot, and Young [MPY17]. They provided a recursive algorithm for computing the graded Frobenius character of the OT_n. That algorithm is based on a surprising relation between the Orlik-Terao algebra and the intersection cohomology ring M_n of a certain hypertoric variety constructed from the root system of type A_n [BP09, MP15].

Computation of M_n using the aforementioned algorithm has suggested the following conjecture. Let D_n be the cohomology algebra of the configuration spaces of n ordered points in SU_2 up to translations.

Conjecture 1.1 ([MPY17, Conjecture 2.10])**.** For each n, there exists an isomorphism of graded S_n-representations $D_n \simeq M_n$. □

It has been verified for $n \leq 10$ in [MPY17] and for $n \leq 22$ in [MMPR21].

The algebra D_n has an independent interest, indeed each graded piece is the Whitehouse lift of Eulerian S_n-representation up to a sign ($D_n^k = \mathrm{sgn}_n \otimes F_n^{(n-1-k)}$ see [GS87, Han90, Whi97, ER19]). The Eulerian representations appear also in the study of the free Lie algebra [Reu93]. These representations are used to decompose the Hochshild Cohomology and Cyclic Cohomology in simpler pieces [Whi97]. Moreover, D_n appears in the Hochschild-Pirashvili homology of a wedge of circles and in the weight-zero compactly supported cohomology of $\mathcal{M}_{2,n}$ [GH22].

Some tentatives to prove the Moseley-Proudfoot-Young conjecture failed for two reason: firstly the only known formula describing D_n is

$$C_n = (V_{(n)} \oplus qV_{(n-1,1)}) \otimes D_n,$$

where V_λ is the Schur representation and C_n is the cohomology of the configuration space of $\mathbb{R}^3$. Although there is an explicit formula for C_n involving plethysm (Theorem 2.6), inverting the Kronecker (tensor) product is very difficult. The second issue is that the recursive formula of [MPY17] for M_n is complicate and involves plethysm, Kronecker product and the character of C_n.

We overcome the first problem providing a new virtual formula for the graded Frobenius character of D_n (Theorem 3.1) by using the Cohen–Taylor-Totaro-Kříz spectral sequence [CT78, Tot96, Kri94]. Instead of working on the recursive formula [MPY17, Theorem 3.2], we use the isomorphism of graded S_n-representations

$$OT_n \simeq M_n \otimes R_n$$

provided in [PS06, Proposition 7], where R_n is the symmetric algebra on $V_{n-1,1}$. Then we virtually invert R_n (Theorem 4.2) with respect the Kronecker product and we prove the conjecture by induction on n (Theorem 4.7) relying on a certain subspace T_n of OT_n (Theorem 4.6). Finally, we obtain an explicit formula for the character of OT_n (Theorem 4.8) and the generating functions for the characters of D_n and of OT_n (Theorem 4.11).

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2 Definitions

We introduce the main objects of study and some notations. The Orlik-Terao algebra was introduced in [Ter02] and its Artinian reduction in [OT94]. In type A_{n-1} the definitions specialize as follows.

Definition 2.1. The Orlik-Terao algebra of type A_{n-1} is the ring $OT_n = \mathbb{Q}[e_{ij}]/I_n^{OT}$ generated by e_{ij} for distinct $i, j \in [n]$ and relations I_n^{OT} given by:

- $e_{ij} + e_{ji} = 0$ for all i, j distinct,
- $e_{ij}e_{jk} + e_{jk}e_{ki} + e_{ki}e_{ij} = 0$ for all i, j, k distinct.

□

Definition 2.2. Let $C_n^\bullet := H^{2\bullet}(\mathrm{Conf}_n(\mathbb{R}^3); \mathbb{Q})$ be the cohomology algebra of the ordered configuration space of n points in $\mathbb{R}^3$. □

The ring C_n can be presented as quotient of OT_n by the equations

- $e_{ij}^2 = 0$ for all i, j distinct.

The above presentation was proved for the first time in [Coh76].

Definition 2.3. Let $D_n^\bullet := H^{2\bullet}(\mathrm{Conf}_n(SU_2)/SU_2; \mathbb{Q})$ be the cohomology algebra of the ordered configuration space of n points in SU_2 up to translations. □

The algebra D_n can be presented as $\mathbb{Q}[e_{ij}]/I_n^D$ generated by e_{ij} for distinct $i, j \in [n]$ and relations I_n^D given by:

- $e_{ij} + e_{ji} = 0$ for all i, j distinct,
- $(e_{ij} + e_{jk} + e_{ki})^2 = 0$ for all i, j, k distinct,
- $\sum_{j \neq i} e_{ij} = 0$ for all $i \in [n]$.

This presentation is due to Matherne, Miyata, Proudfoot, and Ramos [MMPR21, Theorem A4].

Definition 2.4. Let $M_n = OT_n/I_n^M$ be the quotient of the Orlik-Terao algebra by the relations:

- $\sum_{j \neq i} e_{ij} = 0$ for all $i \in [n]$.

□

The algebra M_n was originally defined in a geometric way in [BP09, Corollary 4.5] (see also [MMPR21, Theorem A.6]).

Theorem 2.5. The algebra $M_n^\bullet$ is isomorphic to $\mathrm{IH}^{2\bullet}(X_n; \mathbb{Q})$, the intersection cohomology of a hypertoric variety X_n associated with the root system of the Lie algebra $\mathfrak{sl}_n$. □

We use the standard notation for symmetric polynomial: let h_λ, e_λ, s_λ, p_λ for $\lambda \vdash n$ a partition of n be the complete homogeneous, elementary, Schur, and power sum symmetric polynomials, respectively. Given a graded S_n-representation V we consider the graded Frobenius character $\mathrm{ch}_V(q)$, frequently will omit the dependence on q. As an example if V_λ is the irreducible Schur representation in degree zero, then $\mathrm{ch}_{V_\lambda} = s_\lambda$.

We denote the *plethysm* of symmetric functions f, g by $f[g]$. For W a representation of S_j we denote $\widetilde{W} = W^{\boxtimes m}$ the representation of the wreath product $S_j \wr S_m = (S_j)^{\times m} \rtimes S_m$, where $S_j^{\times m}$ acts coordinatewise and S_m by permuting the coordinates. Let V be a representation of S_m and $V \otimes \widetilde{W}$ be the representation of $S_j \wr S_m$ where $S_j^{\times m}$ acts only on $\widetilde{W}$ and S_m on both factors. The group $S_j \wr S_m$ is naturally a subgroup of S_{jm}, the main property of the plethysm is

$$\mathrm{ch}_{\mathrm{Ind}_{S_j \wr S_m}^{S_{jm}} V \otimes \widetilde{W}} = \mathrm{ch}_V[\mathrm{ch}_W].$$

Let Lie_n be the submodule of the multilinear part of the free Lie algebra on n generators. As S_n representation $Lie_n = \mathrm{Ind}_{Z_n}^{S_n} \zeta_n$ where Z_n is the cyclic group generated by an n-cycle in S_n and ζ_n is a primitive root of the unity. We denote by l_j its character, cf. Remark 4.10 for an explicit description. The following result is due to Sundaram and Welker [SW97, Theorem 4.4(iii)], see also [HR15, Theorem 2.7].

Proposition 2.6. The graded character of C_n is

$$\mathrm{ch}_{C_n} = \sum_{\lambda \vdash n} q^{n-\ell(\lambda)} \prod_{j \geq 1} h_{m_j}[l_j],$$

where $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ in the exponential notation and $\ell(\lambda) = \sum_j m_j$ is the number of blocks. □

Finally, we define $R_n = S^\bullet V_{(n-1,1)}$ and $\Lambda_n = \Lambda^\bullet V_{(n-1,1)}$ be the symmetric (resp. alternating) algebra on the standard representation of S_n. We regard $V_{(n-1,1)}$ in degree one, hence $\mathrm{ch}_{\Lambda_n} = \sum_{i=0}^{n-1} q^i s_{n-i,1^i}$. See Remark 4.10 for an expression of ch_{R_n} in term of Schur polynomials.

3 Graded Frobenius characteristic of D_n

In this section we provide a virtual formula for ch_{D_n} that will be used in the proof of Theorem 4.7. We denote by ch'_V the expression $\mathrm{ch}_V(-q)$ for V a graded S_n-representation. Let P_n be the S_n-representation by permutations, i.e. $P_n = V_{(n-1,1)} \oplus V_{(n)}$. For a partition $\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n}) \vdash n$ let S_λ be the subgroup of S_n stabilizing λ, i.e. $S_\lambda = \prod_{j\geq 1} S_j \wr S_{m_j}$.

Theorem 3.1. The graded character of D_n is:

$$\mathrm{ch}_{D_n}(q) = \sum_{\lambda\vdash n} \frac{q^{n-\ell(\lambda)}}{1-q} \prod_{j\geq 1} \mathrm{ch}'_{\Lambda^\bullet(P_{m_j})}[l_j]. \tag{1}$$

□

Proof. We consider the Cohen–Taylor-Totaro-Křiz spectral sequence $E_\bullet(SU_2, n)$ [CT78, Tot96, Kri94] that converge to $H^\bullet(\mathrm{Conf}_n(SU_2))$. In our case since SU_2 is 3-dimensional and has nonzero cohomology only in degree 0 and 3, we have that $E_2^{p,q} = 0$ if $3 \nmid p$ and $2 \nmid q$. The S_n-representation on the second page is described in [AAB14, Theorem 3.15]:

$$E_2^{3p,2q}(SU_2, n) = \bigoplus_{\substack{\lambda\vdash n \\ \ell(\lambda)=n-q}} \mathrm{Ind}_{S_\lambda}^{S_n} \left(\boxtimes_j (\mathrm{Ind}_{Z_j}^{S_j} \zeta_j)^{\boxtimes m_j} \otimes \mathrm{Res}_{W_\lambda}^{S_{\ell(\lambda)}} \Lambda^p P_{\ell(\lambda)} \right). \tag{2}$$

Since $\mathrm{Res}_{W_\lambda}^{S_{\ell(\lambda)}} P_{\ell(\lambda)} = \oplus_{j\geq 1} P_{m_j}$ we have

$$\mathrm{ch}_{E_2}(s,t) = \sum_{\lambda\vdash n} t^{2(n-\ell(\lambda))} \prod_{j\geq 1} \mathrm{ch}_{\Lambda^\bullet P_{m_j}}(s^3)[l_j]. \tag{3}$$

Topologically $SU_2 \simeq S^3$ is a formal orientable manifold, the only nonzero differential of $E_\bullet(SU_2, n)$ is d_3 as observed in [Pet20, §1.10] and in [Get99, Section 2]. The differential d_3 is compatible with the S_n-action by the functoriality property of the spectral sequence. It follows

$$\mathrm{ch}_{E_2}(-q^2, q^3) = \mathrm{ch}_{E_\infty}(-q^2, q^3), \tag{4}$$

because this is the right evaluation that simplifies the coimage of d_3 with its image.

Consider the map $f\colon (\mathbb{R}^3)^{n-1} \to (SU_2)^n$ defined by $(x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, e)$ where e is the identity of SU_2 and $\mathbb{R}^3$ is identified with $SU_2 \setminus \{e\}$. The map f restricts to the subspaces $\mathrm{Conf}_{n-1}(\mathbb{R}^3) \to \mathrm{Conf}_n(SU_2)$ and the restricted map has a retraction defined by

$$(g_1, g_2, \dots, g_n) \mapsto (g_n^{-1} g_1, g_n^{-1} g_2, \dots, g_n^{-1} g_{n-1}).$$

This implies that $E_\bullet(\mathbb{R}^3, n-1)$ is a direct addendum of $E_\bullet(SU_2, n)$. Notice that $\mathrm{Conf}_{n-1}(\mathbb{R}^3) \times SU_2 \simeq \mathrm{Conf}_n(SU_2)$ via the map $((x_1, \dots, x_{n-1}), g) \mapsto g \cdot f(\underline{x})$, hence $E_\infty(SU_2, n) = E_\infty(\mathbb{R}^3, n-1) \otimes H^\bullet(SU_2)$ as graded vector spaces. Since $E_2(\mathbb{R}^3)$ is supported on the column $p = 0$, so is $E_\infty(\mathbb{R}^3)$. Therefore $E_\infty(SU_2)$ is supported only on the column $p = 0$ and $p = 3$, indeed the even cohomology of $\mathrm{Conf}_n(SU_2)$ is supported in degrees $(0, 2q)$ and the odd one in degrees $(3, 2q)$. So

$$\mathrm{ch}_{E_\infty}(s,t) = \mathrm{ch}_{H^{\mathrm{even}}(\mathrm{Conf}_n(SU_2))}(t) + s^3 t^{-3}\, \mathrm{ch}_{H^{\mathrm{odd}}(\mathrm{Conf}_n(SU_2))}(t).$$

Let $\pi\colon \mathrm{Conf}_n(SU_2) \to \mathrm{Conf}_n(SU_2)/SU_2$ be the natural projection, it is a S_n-equivariant fiber bundle. The Leray-Hirsch theorem for rational cohomology asserts that $H(\mathrm{Conf}_n(SU_2); \mathbb{Q})$ is a free $H(\mathrm{Conf}_n(SU_2)/SU_2; \mathbb{Q})$-module with basis given by $1, \omega$ for any nonzero $\omega \in H^3(\mathrm{Conf}_n(SU_2))$. The module structure is given by π^* so it is S_n-equivariant. We observe that S_n acts trivially on $H^0(\mathrm{Conf}_n(SU_2))$ and on $H^3(\mathrm{Conf}_n(SU_2))$, because the latter is a 1-dimensional quotient of $E_2^{3,0}(SU_2) \cong P_n$. Therefore

$$\mathrm{ch}_{H^{\mathrm{even}}(\mathrm{Conf}_n(SU_2))}(t) = \mathrm{ch}_{H(\mathrm{Conf}_n(SU_2)/SU_2)}(t) = \mathrm{ch}_{D_n}(t^2),$$
$$\mathrm{ch}_{H^{\mathrm{odd}}(\mathrm{Conf}_n(SU_2))}(t) = t^3\, \mathrm{ch}_{H(\mathrm{Conf}_n(SU_2)/SU_2)}(t) = t^3\, \mathrm{ch}_{D_n}(t^2).$$

We have $\mathrm{ch}_{E_\infty}(s,t) = (1 + s^3)\, \mathrm{ch}_{D_n}(t^2)$ and together with eq. (3) and (4) they imply

$$(1 - q^6)\, \mathrm{ch}_{D_n}(q^6) = \sum_{\lambda\vdash n} q^{6(n-\ell(\lambda))} \prod_{j\geq 1} \mathrm{ch}_{\Lambda^\bullet P_{m_j}}(-q^6)[l_j].$$

That is our claim. ■

Remark 3.2. The formula (1) has $(1-q)$ in the denominator and seems to be an infinite series. However it can be written as a polynomial in q of degree $n-1$:

$$\mathrm{ch}_{D_n}(q) = \sum_{\lambda \vdash n} q^{n-\ell(\lambda)}(1-q)^{c_\lambda - 1} \prod_{j \geq 1} \mathrm{ch}'_{\Lambda_{m_j}}[l_j],$$

where $c_\lambda = |\{j \mid m_j \neq 0\}|$. Furthermore, since the left hand side is a polynomial in q of degree $n-2$, the coefficient of q^{n-1} in the right hand side must be zero. □

4 Proof of the MPY conjecture

Now we prove the conjecture and provide a new formula for the character of the Orlik-Terao algebra. The Kronecker product of two symmetric function $f * g$ is the linear extension of the tensor product for representation, i.e. $\mathrm{ch}_{V\otimes W} = \mathrm{ch}_V * \mathrm{ch}_W$.

Theorem 4.1 ([PS06, Proposition 7])**.** For each n the equation

$$\mathrm{ch}_{OT_n} = \mathrm{ch}_{M_n} * \mathrm{ch}_{R_n}$$

holds. □

Lemma 4.2. Let V be any representation of the symmetric group S_n. We have:

$$\mathrm{ch}_{S^\bullet V} * \mathrm{ch}'_{\Lambda^\bullet V} = s_n.$$

□

Proof. The Koszul complex for the ring $S^\bullet V$ is a free resolution of $\mathbb{Q} = S^\bullet V/(V)$. The bigraded character of the Koszul complex is $\mathrm{ch}_{S^\bullet V}(s) * \mathrm{ch}_{\Lambda^\bullet V}(t)$, hence by exactness we have $\mathrm{ch}_{S^\bullet V}(q) * \mathrm{ch}_{\Lambda^\bullet V}(-q) = s_n$. ■

It follows that ch_{R_n} is invertible with respect to the Kronecker product, whose inverse is ch'_{Λ_n}.

Lemma 4.3. Let g be a symmetric function of degree j and m a positive integer. We have

$$\mathrm{ch}'_{\Lambda^\bullet P_m}[g] = h_m[(1-q)g].$$

□

Proof. Using the identity $h_{n-k}e_k = s_{n-k,1^k} + s_{n-k+1,1^{k-1}}$ we obtain

$$\mathrm{ch}'_{\Lambda^\bullet P_n} = (1-q)\sum_{k=0}^{n-1}(-q)^k s_{n-k,1^k} = \sum_{k=0}^{n}(-q)^k h_{n-k}e_k.$$

Recall the subtraction formula (see for example in [LR11, §3.3])

$$h_m[f-g] = \sum_{i=0}^{m}(-1)^k h_{m-k}[f]e_k[g],$$

we obtain

$$\begin{aligned} h_m[(1-q)g] &= \sum_{k=0}^{m}(-1)^k h_{m-k}[g]e_k[qg] \\ &= \sum_{k=0}^{m}(-q)^k (h_{m-k}e_k)[g] \\ &= \mathrm{ch}'_{\Lambda^\bullet P_m}[g]. \blacksquare \end{aligned}$$

Using the Lemma above we can rewrite the character of D_n as follow.

Corollary 4.4. The graded character of D_n is

$$\mathrm{ch}_{D_n}(q) = \frac{1}{1-q}\sum_{\lambda\vdash n}\prod_{j\geq 1} h_{m_j}[q^{j-1}(1-q)l_j]. \tag{5}$$

□

Proof. It follows from Theorem 3.1 and Theorem 4.3. ■

Lemma 4.5. Let $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ be a partition of n and g_j, f_{m_j} be symmetric functions of degree j and m_j respectively. We have:

$$\mathrm{ch}'_{\Lambda^\bullet P_n} * \prod_{j\geq 1} f_{m_j}[g_j] = \prod_{j\geq 1} f_{m_j}[g_j * \mathrm{ch}'_{\Lambda^\bullet P_j}].$$

□

Proof. Firstly observe that

$$\mathrm{Res}^{S_n}_{\prod_{j\geq 1} S_{jm_j}} P_n = \bigoplus_{j\geq 1} P_{jm_j},$$

and so

$$\mathrm{Res}^{S_n}_{\prod_{j\geq 1} S_{jm_j}} \Lambda^\bullet P_n = \bigotimes_{j\geq 1} \Lambda^\bullet P_{jm_j}.$$

Using the projection formula (sometimes called Frobenius reciprocity) we obtain:

$$\mathrm{ch}'_{\Lambda^\bullet P_n} * \prod_{j\geq 1} f_{m_j}[g_j] = \prod_{j\geq 1} \mathrm{ch}'_{\Lambda^\bullet P_{jm_j}} * f_{m_j}[g_j].$$

Thus it is enough to show

$$\mathrm{ch}'_{\Lambda^\bullet P_{jm}} * f[g] = f[g * \mathrm{ch}'_{\Lambda^\bullet P_j}].$$

This last equality is linear and multiplicative in the entry f: the linearity is trivial and the multiplicativity follow from the argument above

$$\begin{aligned}\mathrm{ch}'_{\Lambda^\bullet P_{jm}} * (f_1 f_2)[g] &= \mathrm{ch}'_{\Lambda^\bullet P_{jm}} * (f_1[g] f_2[g]) \\ &= (\mathrm{ch}'_{\Lambda^\bullet P_{jm_1}} * f_1[g])(\mathrm{ch}'_{\Lambda^\bullet P_{jm_2}} * f_2[g]).\end{aligned}$$

Therefore we may assume $f = p_m$. Again $\mathrm{ch}'_{\Lambda^\bullet P_{jm}} * p_m[g] = p_m[g * \mathrm{ch}'_{\Lambda^\bullet P_j}]$ is linear and multiplicative in the entry g and so we reduce to the case $g = p_j$.

It remains to prove that $\mathrm{ch}'_{\Lambda^\bullet P_{jm}} * p_{jm} = p_m[p_j * \mathrm{ch}'_{\Lambda^\bullet P_j}]$. Since $(p_\lambda)_\lambda$ are orthogonal idempotent with respect to the Kronecker product

$$\mathrm{ch}'_{\Lambda^\bullet P_n} * p_n = \chi'_{\Lambda^\bullet P_n}(c_n) p_n$$

where $\chi'_V(\sigma)$ is the graded character of $\sigma \in S_n$ with q replaced by $-q$ and $c_n \in S_n$ be an n-cycle. It is easy to see that

$$\chi'_{\Lambda^\bullet P_n}(c_n) = 1 + (-1)^{n-1}(-q)^n = 1 - q^n$$

on the canonical base of $\Lambda^\bullet P_n$: let $(v_i)_i$ the standard base of P_n, the product of some v_j is invariant for c_n if and only if each generator appears a fixed number of times (i.e. 0 or 1 times). Finally the equalities

$$\begin{aligned}p_m[p_j * \mathrm{ch}'_{\Lambda^\bullet P_j}] &= p_m[(1-q^j)p_j] \\ &= (1-q^{jm})p_{jm} \\ &= \mathrm{ch}'_{\Lambda^\bullet P_{jm}} * p_{jm}\end{aligned}$$

conclude the proof. ■

For each monomial $m = \prod_k e_{i_k,j_k} \in \mathbb{Q}[e_{i,j}]$ we define the *support* of m as the finest set partition $B(m) \vdash [n]$ such that for all k i_k and j_k belong to the same block of $B(m)$. We also define the *type* of m as the partition $\lambda(m) \vdash n$ collecting the size of blocks of $B(m)$. Notice that the relations defining OT_n (Theorem 2.1) are sum of monomials with the same support, hence the notion of support and type are well defined in OT_n. Moreover, monomials with different supports are linearly independent.

For $B \vdash [n]$ a set partition let $T_B \subset OT_n$ be the vector space generated by all monomials m such that $B(m) = B$. For $S \subseteq [n]$ we define $T_S = T_B$ where B is the finest set partition of $[n]$ with a block equal to S. Given two monomials m, m' such that $mm' \neq 0$ in OT_n, we have that $B(mm')$ is the finest set partition coarsening both $B(m)$ and $B(m')$, hence

$$T_B \cong \bigotimes_{i=1}^{l} T_{B_i}$$

where we denote by B_i the blocks of $B = \{B_1, B_2, \dots, B_l\}$.

Consider a partition $\lambda \vdash n$, let T_λ be the vector space generated by all monomials of type λ. Choose a set partition $B_\lambda \vdash [n]$ whose blocks B_i are of length λ_i and let S_{B_λ} be the subgroup of S_n stabilizing B_λ, if $\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n})$ then $S_{B_\lambda} \cong \prod_{j\geq 1} S_j \wr S_{m_j}$. We have

$$T_\lambda \cong \operatorname{Ind}_{S_{B_\lambda}}^{S_n} T_{B_\lambda}$$

as representation of S_n, where S_{B_i} acts on the factor T_{B_i} of $T_{B_\lambda} = \otimes_{i=1}^{|B|} T_{B_i}$ and S_{m_j} permutes the m_j factors of size j. For the sake of notation we set $T_n = T_{(n)}$.

Lemma 4.6. We have

$$\mathrm{ch}_{OT_n} = \sum_{\lambda \vdash n} \prod_{j \geq 1} h_{m_j}[\mathrm{ch}_{T_j}].$$

□

Proof. The Orlik-Terao algebra decomposes

$$\begin{aligned} OT_n &= \bigoplus_{B \vdash [n]} T_B \\ &= \bigoplus_{B \vdash [n]} \bigotimes_{i=1}^{|B|} T_{B_i} \\ &= \bigoplus_{\lambda \vdash n} \operatorname{Ind}_{S_{B_\lambda}}^{S_n} \bigotimes_{i=1}^{\ell(\lambda)} T_{B_i} \\ &= \bigoplus_{\lambda \vdash n} \operatorname{Ind}_{\prod_j S_{jm_j}}^{S_n} \left(\bigotimes_{j \geq 1} \operatorname{Ind}_{S_j \wr S_{m_j}}^{S_{jm_j}} \widetilde{T_j} \right) \end{aligned}$$

as S_n-representation. Taking the character we obtain the claimed relation. ■

Theorem 4.7. We have

$$\mathrm{ch}_{D_n} = \mathrm{ch}_{M_n}$$

and

$$\mathrm{ch}_{T_n} = q^{n-1} l_n * \mathrm{ch}_{R_n}.$$

□

Proof. We prove both equality by induction on n. The base case $n = 1$ is trivial. For the inductive step we consider:

$$\begin{aligned} \mathrm{ch}_{M_n} &= \mathrm{ch}_{OT_n} * \mathrm{ch}'_{\Lambda_n} \\ &= \frac{1}{(1-q)} \sum_{\lambda \vdash n} \prod_{j \geq 1} h_{m_j}[\mathrm{ch}_{T_j} * \mathrm{ch}'_{\Lambda^\bullet P_j}] \\ &= \mathrm{ch}_{T_n} * \mathrm{ch}'_{\Lambda_n} + \frac{1}{(1-q)} \sum_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} \prod_{j \geq 1} h_{m_j}[q^{j-1}(1-q) l_j]. \end{aligned}$$

The first equality follows from Theorem 4.1 and Lemma 4.2. The second one follows from Lemma 4.6 and Lemma 4.5 together with the identity $\mathrm{ch}'_{\Lambda \bullet P_j} = (1-q)\,\mathrm{ch}'_{\Lambda_j}$. The last one follows from the inductive hypothesis and Theorem 4.2. We have proven the identity

$$\mathrm{ch}_{M_n} - \mathrm{ch}_{T_n} * \mathrm{ch}'_{\Lambda_n} = \frac{1}{(1-q)} \sum_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} \prod_{j \geq 1} h_{m_j}[q^{j-1}(1-q)l_j] = \mathrm{ch}_{D_n} - q^{n-1} l_n,$$

where the last equality is given by Theorem 4.4. Since ch_{D_n} and ch_{M_n} has degree less than $n-1$ and $\mathrm{ch}_{T_n} * \mathrm{ch}'_{\Lambda_n}$ bigger than $n-2$, $\mathrm{ch}_{M_n} = \mathrm{ch}_{D_n}$ and $\mathrm{ch}_{T_n} * \mathrm{ch}'_{\Lambda_n} = q^{n-1} l_n$ hold. Therefore $\mathrm{ch}_{T_n} = q^{n-1} l_n * \mathrm{ch}_{R_n}$. ■

Corollary 4.8. We obtain the character of OT_n:

$$\mathrm{ch}_{OT_n} = \sum_{\lambda \vdash n} q^{n-\ell(\lambda)} \prod_{j \geq 1} h_{m_j}[l_j * \mathrm{ch}_{R_j}]. \tag{6}$$

□

Proof. It follows from Theorem 4.7 and Theorem 4.6. ■

An important object for the proof of Theorem 4.7 is the R_n-module T_n. It is a submodule of the free module OT_n and its Frobenius character is equal to the one of the free module $R_n \otimes_{\mathbb{Q}} T_n^{n-1}$. This observations lead to the following conjecture:

Conjecture 4.9. The R_n-module T_n is free. □

Remark 4.10. The formula (6) is completely explicit because ch_{R_j} and l_j are known. Indeed

$$\mathrm{ch}_{R_n} = (1-q) \sum_{\lambda \vdash n} s_\lambda(1, q, q^2, \ldots) s_\lambda = (1-q) h_n \left[\frac{X}{1-q} \right]$$

by [Pro03, Section 5.6] or [Sta99, Exercise 7.73] where $X = h_1$. Moreover,

$$l_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{\frac{n}{d}},$$

by [Reu93, Theorem 8.3], l_n is known as the Lyndon symmetric function or as Gessel-Reutenauer symmetric function [GR93]. □

Let Exp be the plethystic exponential defined by

$$\mathrm{Exp}(f) := \exp\left(\sum_{k \geq 1} \frac{p_k[f]}{k} \right) = \sum_{k \geq 0} h_k[f],$$

see [LR11, Section 5.3] for the equivalence between the two formulas. We denote by Log the inverse of Exp and we define the symmetric functions

$$L = \sum_{n \geq 1} q^{n-1} t^n l_n = -\frac{\mathrm{Log}(1-qtX)}{q}.$$

Corollary 4.11. The generating functions for ch_D and ch_{OT} are:

$$\sum_{n \geq 1} \mathrm{ch}_{D_n}(q) t^n = \frac{1}{1-q} (\mathrm{Exp}((1-q)L) - 1), \tag{7}$$

$$\sum_{n \geq 1} \mathrm{ch}_{OT_n}(q) t^n = \mathrm{Exp}\left((1-q)L * \mathrm{Exp}\left(\frac{X}{1-q} \right) \right) - 1. \tag{8}$$

□

Proof. Let f be a symmetric function and call f_j be the homogeneous part of degree j. Assume that f has zero constant term, i.e. $f = \sum_{j\geq 1} f_j$, then

$$\begin{aligned}\mathrm{Exp}(f) &= \prod_{j\geq 1} \mathrm{Exp}(f_j)\\ &= \prod_{j\geq 1}\sum_{m\geq 0} h_m[f_j]\\ &= \sum_{\lambda}\prod_{j\geq 1} h_{m_j}[f_j],\end{aligned}$$

where the sum is taken over all partitions $\lambda = (1^{m_1}, 2^{m_2}, \dots)$. The corollary follows by taking $f = (1-q)L$ and $f = (1-q)L * \mathrm{Exp}((1-q)^{-1}X)$. ■

Formulas of this paper are checked and implemented in SageMath [Sage]. The code is available at

`https://github.com/paga92/character_OT`.

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