# Alma Mater Studiorum Università di Bologna
## Archivio istituzionale della ricerca

The Frobenius Characteristic of the Orlik-Terao Algebra of Type A

This is the final peer-reviewed author's accepted manuscript (postprint) of the following publication:

*Published:*

(Article begins on next page)

# The Frobenius characteristic of the Orlik-Terao algebra of type A

## Roberto Pagaria[1]

[1]Università di Bologna, Dipartimento di Matematica, Piazza di Porta San Donato 5 – 40126 Bologna
Italy

*Correspondence to be sent to: roberto.pagaria@unibo.it*

We provide a new virtual description of the symmetric group action on the cohomology of ordered configuration space on $SU_2$ up to translations. We use this formula to prove the Moseley-Proudfoot-Young conjecture. As a consequence we obtain the graded Frobenius character of the Orlik-Terao algebra of type $A_n$.

## 1  Introduction

The Orlik-Terao algebra $OT_n$ is the subalgebra of rational functions on $\mathbb{C}^n$ generated by $\frac{1}{x_i-x_j}$ for all $i \neq j$. It has been intensively studied in [Ter02, PS06, ST09, Ber10, Sch11, DGT14, Le14, Liu16, EPW16, MPY17, MMPR21]. Only recently, has an attempt to describe the symmetric group action on $OT_n$ been made by Moseley, Proudfoot, and Young [MPY17]. They provided a recursive algorithm for computing the graded Frobenius character of the $OT_n$. That algorithm is based on a surprising relation between the Orlik-Terao algebra and the intersection cohomology ring $M_n$ of a certain hypertoric variety constructed from the root system of type $A_n$ [BP09, MP15].

Computation of $M_n$ using the aforementioned algorithm has suggested the following conjecture. Let $D_n$ be the cohomology algebra of the configuration spaces of $n$ ordered points in $SU_2$ up to translations.

**Conjecture 1.1** ([MPY17, Conjecture 2.10])**.** For each $n$, there exists an isomorphism of graded $S_n$-representations $D_n \simeq M_n$. $\qquad\square$

It has been verified for $n \leq 10$ in [MPY17] and for $n \leq 22$ in [MMPR21].

The algebra $D_n$ has an independent interest, indeed each graded piece is the Whitehouse lift of Eulerian $S_n$-representation up to a sign $(D_n^k = \mathrm{sgn}_n \otimes F_n^{(n-1-k)}$ see [GS87, Han90, Whi97, ER19]). The Eulerian representations appear also in the study of the free Lie algebra [Reu93]. These representations are used to decompose the Hochshild Cohomology and Cyclic Cohomology in simpler pieces [Whi97]. Moreover, $D_n$ appears in the Hochschild-Pirashvili homology of a wedge of circles and in the weight-zero compactly supported cohomology of $\mathcal{M}_{2,n}$ [GH22].

Some tentatives to prove the Moseley-Proudfoot-Young conjecture failed for two reason: firstly the only known formula describing $D_n$ is

$$C_n = (V_{(n)} \oplus qV_{(n-1,1)}) \otimes D_n,$$

where $V_\lambda$ is the Schur representation and $C_n$ is the cohomology of the configuration space of $\mathbb{R}^3$. Although there is an explicit formula for $C_n$ involving plethysm (Theorem 2.6), inverting the Kronecker (tensor) product is very difficult. The second issue is that the recursive formula of [MPY17] for $M_n$ is complicate and involves plethysm, Kronecker product and the character of $C_n$.

We overcome the first problem providing a new virtual formula for the graded Frobenius character of $D_n$ (Theorem 3.1) by using the Cohen–Taylor-Totaro-Křiz spectral sequence [CT78, Tot96, Kri94]. Instead of working on the recursive formula [MPY17, Theorem 3.2], we use the isomorphism of graded $S_n$-representations

$$OT_n \simeq M_n \otimes R_n$$

provided in [PS06, Proposition 7], where $R_n$ is the symmetric algebra on $V_{n-1,1}$. Then we virtually invert $R_n$ (Theorem 4.2) with respect the Kronecker product and we prove the conjecture by induction on $n$ (Theorem 4.7) relying on a certain subspace $T_n$ of $OT_n$ (Theorem 4.6). Finally, we obtain an explicit formula for the character of $OT_n$ (Theorem 4.8) and the generating functions for the characters of $D_n$ and of $OT_n$ (Theorem 4.11).

## 2    Definitions

We introduce the main objects of study and some notations. The Orlik-Terao algebra was introduced in [Ter02] and its Artinian reduction in [OT94]. In type $A_{n-1}$ the definitions specialize as follows.

**Definition 2.1.** The Orlik-Terao algebra of type $A_{n-1}$ is the ring $OT_n = \mathbb{Q}[e_{ij}]/I_n^{OT}$ generated by $e_{ij}$ for distinct $i,j \in [n]$ and relations $I_n^{OT}$ given by:

- $e_{ij} + e_{ji} = 0$ for all $i,j$ distinct,
- $e_{ij}e_{jk} + e_{jk}e_{ki} + e_{ki}e_{ij} = 0$ for all $i,j,k$ distinct.

□

**Definition 2.2.** Let $C_n^\bullet := H^{2\bullet}(\mathrm{Conf}_n(\mathbb{R}^3); \mathbb{Q})$ be the cohomology algebra of the ordered configuration space of $n$ points in $\mathbb{R}^3$. □

The ring $C_n$ can be presented as quotient of $OT_n$ by the equations

- $e_{ij}^2 = 0$ for all $i,j$ distinct.

The above presentation was proved for the first time in [Coh76].

**Definition 2.3.** Let $D_n^\bullet := H^{2\bullet}(\mathrm{Conf}_n(SU_2)/SU_2; \mathbb{Q})$ be the cohomology algebra of the ordered configuration space of $n$ points in $SU_2$ up to translations. □

The algebra $D_n$ can be presented as $\mathbb{Q}[e_{ij}]/I_n^D$ generated by $e_{ij}$ for distinct $i,j \in [n]$ and relations $I_n^D$ given by:

- $e_{ij} + e_{ji} = 0$ for all $i,j$ distinct,
- $(e_{ij} + e_{jk} + e_{ki})^2 = 0$ for all $i,j,k$ distinct,
- $\sum_{j \neq i} e_{ij} = 0$ for all $i \in [n]$.

This presentation is due to Matherne, Miyata, Proudfoot, and Ramos [MMPR21, Theorem A4].

**Definition 2.4.** Let $M_n = OT_n/I_n^M$ be the quotient of the Orlik-Terao algebra by the relations:

- $\sum_{j \neq i} e_{ij} = 0$ for all $i \in [n]$.

□

The algebra $M_n$ was originally defined in a geometric way in [BP09, Corollary 4.5] (see also [MMPR21, Theorem A.6]).

**Theorem 2.5.** The algebra $M_n^\bullet$ is isomorphic to $\mathrm{IH}^{2\bullet}(X_n; \mathbb{Q})$, the intersection cohomology of a hypertoric variety $X_n$ associated with the root system of the Lie algebra $\mathfrak{sl}_n$. □

We use the standard notation for symmetric polynomial: let $h_\lambda, e_\lambda, s_\lambda, p_\lambda$ for $\lambda \vdash n$ a partition of $n$ be the complete homogeneous, elementary, Schur, and power sum symmetric polynomials, respectively. Given a graded $S_n$-representation $V$ we consider the graded Frobenius character $\mathrm{ch}_V(q)$, frequently will omit the dependence on $q$. As an example if $V_\lambda$ is the irreducible Schur representation in degree zero, then $\mathrm{ch}_{V_\lambda} = s_\lambda$.

We denote the *plethysm* of symmetric functions $f, g$ by $f[g]$. For $W$ a representation of $S_j$ we denote $\widetilde{W} = W^{\boxtimes m}$ the representation of the wreath product $S_j \wr S_m = (S_j)^{\times m} \rtimes S_m$, where $S_j^{\times m}$ acts coordinatewise and $S_m$ by permuting the coordinates. Let $V$ be a representation of $S_m$ and $V \otimes \widetilde{W}$ be the representation of $S_j \wr S_m$ where $S_j^{\times m}$ acts only on $\widetilde{W}$ and $S_m$ on both factors. The group $S_j \wr S_m$ is naturally a subgroup of $S_{jm}$, the main property of the plethysm is

$$\mathrm{ch}_{\mathrm{Ind}_{S_j \wr S_m}^{S_{jm}} V \otimes \widetilde{W}} = \mathrm{ch}_V[\mathrm{ch}_W].$$

Let $Lie_n$ be the submodule of the multilinear part of the free Lie algebra on $n$ generators. As $S_n$ representation $Lie_n = \mathrm{Ind}_{Z_n}^{S_n} \zeta_n$ where $Z_n$ is the cyclic group generated by an $n$-cycle in $S_n$ and $\zeta_n$ is a primitive root of the unity. We denote by $l_j$ its character, cf. Remark 4.10 for an explicit description. The following result is due to Sundaram and Welker [SW97, Theorem 4.4(iii)], see also [HR15, Theorem 2.7].

**Proposition 2.6.** The graded character of $C_n$ is

$$\mathrm{ch}_{C_n} = \sum_{\lambda \vdash n} q^{n - \ell(\lambda)} \prod_{j \geq 1} h_{m_j}[l_j],$$

where $\lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n})$ in the exponential notation and $\ell(\lambda) = \sum_j m_j$ is the number of blocks. □

Finally, we define $R_n = S^\bullet V_{(n-1,1)}$ and $\Lambda_n = \Lambda^\bullet V_{(n-1,1)}$ be the symmetric (resp. alternating) algebra on the standard representation of $S_n$. We regard $V_{(n-1,1)}$ in degree one, hence $\mathrm{ch}_{\Lambda_n} = \sum_{i=0}^{n-1} q^i s_{n-i,1^i}$. See Remark 4.10 for an expression of $\mathrm{ch}_{R_n}$ in term of Schur polynomials.

## 3  Graded Frobenius characteristic of $D_n$

In this section we provide a virtual formula for $\mathrm{ch}_{D_n}$ that will be used in the proof of Theorem 4.7. We denote by $\mathrm{ch}'_V$ the expression $\mathrm{ch}_V(-q)$ for $V$ a graded $S_n$-representation. Let $P_n$ be the $S_n$-representation by permutations, i.e. $P_n = V_{(n-1,1)} \oplus V_{(n)}$. For a partition $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n}) \vdash n$ let $S_\lambda$ be the subgroup of $S_n$ stabilizing $\lambda$, i.e. $S_\lambda = \prod_{j \geq 1} S_j \wr S_{m_j}$.

**Theorem 3.1.** The graded character of $D_n$ is:

$$\mathrm{ch}_{D_n}(q) = \sum_{\lambda \vdash n} \frac{q^{n-\ell(\lambda)}}{1-q} \prod_{j \geq 1} \mathrm{ch}'_{\Lambda^\bullet(P_{m_j})}[l_j]. \tag{1}$$

$\square$

**Proof**. We consider the Cohen–Taylor-Totaro-Křiz spectral sequence $E_\bullet(SU_2, n)$ [CT78, Tot96, Kri94] that converge to $H^\bullet(\mathrm{Conf}_n(SU_2))$. In our case since $SU_2$ is 3-dimensional and has nonzero cohomology only in degree 0 and 3, we have that $E_2^{p,q} = 0$ if $3 \nmid p$ and $2 \nmid q$. The $S_n$-representation on the second page is described in [AAB14, Theorem 3.15]:

$$E_2^{3p,2q}(SU_2, n) = \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) = n - q}} \mathrm{Ind}_{S_\lambda}^{S_n} \left( \boxtimes_j (\mathrm{Ind}_{Z_j}^{S_j} \zeta_j)^{\boxtimes m_j} \otimes \mathrm{Res}_{W_\lambda}^{S_{\ell(\lambda)}} \Lambda^p P_{\ell(\lambda)} \right). \tag{2}$$

Since $\mathrm{Res}_{W_\lambda}^{S_{\ell(\lambda)}} P_{\ell(\lambda)} = \oplus_{j \geq 1} P_{m_j}$ we have

$$\mathrm{ch}_{E_2}(s, t) = \sum_{\lambda \vdash n} t^{2(n-\ell(\lambda))} \prod_{j \geq 1} \mathrm{ch}_{\Lambda^\bullet P_{m_j}}(s^3)[l_j]. \tag{3}$$

Topologically $SU_2 \simeq S^3$ is a formal orientable manifold, the only nonzero differential of $E_\bullet(SU_2, n)$ is $\mathrm{d}_3$ as observed in [Pet20, §1.10] and in [Get99, Section 2]. The differential $\mathrm{d}_3$ is compatible with the $S_n$-action by the functoriality property of the spectral sequence. It follows

$$\mathrm{ch}_{E_2}(-q^2, q^3) = \mathrm{ch}_{E_\infty}(-q^2, q^3), \tag{4}$$

because this is the right evaluation that simplifies the coimage of $\mathrm{d}_3$ with its image.

Consider the map $f \colon (\mathbb{R}^3)^{n-1} \to (SU_2)^n$ defined by $(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, e)$ where $e$ is the identity of $SU_2$ and $\mathbb{R}^3$ is identified with $SU_2 \setminus \{e\}$. The map $f$ restricts to the subspaces $\mathrm{Conf}_{n-1}(\mathbb{R}^3) \to \mathrm{Conf}_n(SU_2)$ and the restricted map has a retraction defined by

$$(g_1, g_2, \ldots, g_n) \mapsto (g_n^{-1} g_1, g_n^{-1} g_2, \ldots, g_n^{-1} g_{n-1}).$$

This implies that $E_\bullet(\mathbb{R}^3, n-1)$ is a direct addendum of $E_\bullet(SU_2, n)$. Notice that $\mathrm{Conf}_{n-1}(\mathbb{R}^3) \times SU_2 \simeq \mathrm{Conf}_n(SU_2)$ via the map $((x_1, \ldots, x_{n-1}), g) \mapsto g \cdot f(\underline{x})$, hence $E_\infty(SU_2, n) = E_\infty(\mathbb{R}^3, n-1) \otimes H^\bullet(SU_2)$ as graded vector spaces. Since $E_2(\mathbb{R}^3)$ is supported on the column $p = 0$, so is $E_\infty(\mathbb{R}^3)$. Therefore $E_\infty(SU_2)$ is supported only on the column $p = 0$ and $p = 3$, indeed the even cohomology of $\mathrm{Conf}_n(SU_2)$ is supported in degrees $(0, 2q)$ and the odd one in degrees $(3, 2q)$. So

$$\mathrm{ch}_{E_\infty}(s, t) = \mathrm{ch}_{H^{\mathrm{even}}(\mathrm{Conf}_n(SU_2))}(t) + s^3 t^{-3} \mathrm{ch}_{H^{\mathrm{odd}}(\mathrm{Conf}_n(SU_2))}(t).$$

Let $\pi \colon \mathrm{Conf}_n(SU_2) \to \mathrm{Conf}_n(SU_2)/SU_2$ be the natural projection, it is a $S_n$-equivariant fiber bundle. The Leray-Hirsch theorem for rational cohomology asserts that $H(\mathrm{Conf}_n(SU_2); \mathbb{Q})$ is a free $H(\mathrm{Conf}_n(SU_2)/SU_2; \mathbb{Q})$-module with basis given by $1, \omega$ for any nonzero $\omega \in H^3(\mathrm{Conf}_n(SU_2))$. The module structure is given by $\pi^*$ so it is $S_n$-equivariant. We observe that $S_n$ acts trivially on $H^0(\mathrm{Conf}_n(SU_2))$ and on $H^3(\mathrm{Conf}_n(SU_2))$, because the latter is a 1-dimensional quotient of $E_2^{3,0}(SU_2) \cong P_n$. Therefore

$$\mathrm{ch}_{H^{\mathrm{even}}(\mathrm{Conf}_n(SU_2))}(t) = \mathrm{ch}_{H(\mathrm{Conf}_n(SU_2)/SU_2)}(t) = \mathrm{ch}_{D_n}(t^2),$$

$$\mathrm{ch}_{H^{\mathrm{odd}}(\mathrm{Conf}_n(SU_2))}(t) = t^3 \mathrm{ch}_{H(\mathrm{Conf}_n(SU_2)/SU_2)}(t) = t^3 \mathrm{ch}_{D_n}(t^2).$$

We have $\mathrm{ch}_{E_\infty}(s, t) = (1 + s^3) \mathrm{ch}_{D_n}(t^2)$ and together with eq. (3) and (4) they imply

$$(1 - q^6) \mathrm{ch}_{D_n}(q^6) = \sum_{\lambda \vdash n} q^{6(n-\ell(\lambda))} \prod_{j \geq 1} \mathrm{ch}_{\Lambda^\bullet P_{m_j}}(-q^6)[l_j].$$

That is our claim. $\blacksquare$

**Remark 3.2.** The formula (1) has $(1 - q)$ in the denominator and seems to be an infinite series. However it can be written as a polynomial in $q$ of degree $n - 1$:

$$\mathrm{ch}_{D_n}(q) = \sum_{\lambda \vdash n} q^{n - \ell(\lambda)} (1 - q)^{c_\lambda - 1} \prod_{j \geq 1} \mathrm{ch}'_{\Lambda_{m_j}}[l_j],$$

where $c_\lambda = |\{j \mid m_j \neq 0\}|$. Furthermore, since the left hand side is a polynomial in $q$ of degree $n - 2$, the coefficient of $q^{n-1}$ in the right hand side must be zero. □

## 4  Proof of the MPY conjecture

Now we prove the conjecture and provide a new formula for the character of the Orlik-Terao algebra. The Kronecker product of two symmetric function $f * g$ is the linear extension of the tensor product for representation, i.e. $\mathrm{ch}_{V \otimes W} = \mathrm{ch}_V * \mathrm{ch}_W$.

**Theorem 4.1** ([PS06, Proposition 7]). For each $n$ the equation

$$\mathrm{ch}_{OT_n} = \mathrm{ch}_{M_n} * \mathrm{ch}_{R_n}$$

holds. □

**Lemma 4.2.** Let $V$ be any representation of the symmetric group $S_n$. We have:

$$\mathrm{ch}_{S^\bullet V} * \mathrm{ch}'_{\Lambda^\bullet V} = s_n.$$

□

**Proof**. The Koszul complex for the ring $S^\bullet V$ is a free resolution of $\mathbb{Q} = S^\bullet V/(V)$. The bigraded character of the Koszul complex is $\mathrm{ch}_{S^\bullet V}(s) * \mathrm{ch}_{\Lambda^\bullet V}(t)$, hence by exactness we have $\mathrm{ch}_{S^\bullet V}(q) * \mathrm{ch}_{\Lambda^\bullet V}(-q) = s_n$. ∎

It follows that $\mathrm{ch}_{R_n}$ is invertible with respect to the Kronecker product, whose inverse is $\mathrm{ch}'_{\Lambda_n}$.

**Lemma 4.3.** Let $g$ be a symmetric function of degree $j$ and $m$ a positive integer. We have

$$\mathrm{ch}'_{\Lambda^\bullet P_m}[g] = h_m[(1 - q)g].$$

□

**Proof**. Using the identity $h_{n-k} e_k = s_{n-k,1^k} + s_{n-k+1,1^{k-1}}$ we obtain

$$\mathrm{ch}'_{\Lambda^\bullet P_n} = (1 - q) \sum_{k=0}^{n-1} (-q)^k s_{n-k,1^k} = \sum_{k=0}^{n} (-q)^k h_{n-k} e_k.$$

Recall the subtraction formula (see for example in [LR11, §3.3])

$$h_m[f - g] = \sum_{i=0}^{m} (-1)^k h_{m-k}[f] e_k[g],$$

we obtain

$$h_m[(1 - q)g] = \sum_{k=0}^{m} (-1)^k h_{m-k}[g] e_k[qg]$$

$$= \sum_{k=0}^{m} (-q)^k (h_{m-k} e_k)[g]$$

$$= \mathrm{ch}'_{\Lambda^\bullet P_m}[g]. \blacksquare$$

Using the Lemma above we can rewrite the character of $D_n$ as follow.

**Corollary 4.4.** The graded character of $D_n$ is

$$\mathrm{ch}_{D_n}(q) = \frac{1}{1-q} \sum_{\lambda \vdash n} \prod_{j \geq 1} h_{m_j}[q^{j-1}(1-q)l_j]. \tag{5}$$

□

**Proof**. It follows from Theorem 3.1 and Theorem 4.3. ∎

**Lemma 4.5.** Let $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ be a partition of $n$ and $g_j, f_{m_j}$ be symmetric functions of degree $j$ and $m_j$ respectively. We have:

$$\mathrm{ch}'_{\Lambda^\bullet P_n} * \prod_{j \geq 1} f_{m_j}[g_j] = \prod_{j \geq 1} f_{m_j}[g_j * \mathrm{ch}'_{\Lambda^\bullet P_j}].$$

□

**Proof**. Firstly observe that

$$\mathrm{Res}^{S_n}_{\prod_{j \geq 1} S_{jm_j}} P_n = \bigoplus_{j \geq 1} P_{jm_j},$$

and so

$$\mathrm{Res}^{S_n}_{\prod_{j \geq 1} S_{jm_j}} \Lambda^\bullet P_n = \bigotimes_{j \geq 1} \Lambda^\bullet P_{jm_j}.$$

Using the projection formula (sometimes called Frobenius reciprocity) we obtain:

$$\mathrm{ch}'_{\Lambda^\bullet P_n} * \prod_{j \geq 1} f_{m_j}[g_j] = \prod_{j \geq 1} \mathrm{ch}'_{\Lambda^\bullet P_{jm_j}} * f_{m_j}[g_j].$$

Thus it is enough to show

$$\mathrm{ch}'_{\Lambda^\bullet P_{jm}} * f[g] = f[g * \mathrm{ch}'_{\Lambda^\bullet P_j}].$$

This last equality is linear and multiplicative in the entry $f$: the linearity is trivial and the multiplicativity follow from the argument above

$$\mathrm{ch}'_{\Lambda^\bullet P_{jm}} * (f_1 f_2)[g] = \mathrm{ch}'_{\Lambda^\bullet P_{jm}} * (f_1[g] f_2[g])$$
$$= (\mathrm{ch}'_{\Lambda^\bullet P_{jm_1}} * f_1[g])(\mathrm{ch}'_{\Lambda^\bullet P_{jm_2}} * f_2[g]).$$

Therefore we may assume $f = p_m$. Again $\mathrm{ch}'_{\Lambda^\bullet P_{jm}} * p_m[g] = p_m[g * \mathrm{ch}'_{\Lambda^\bullet P_j}]$ is linear and multiplicative in the entry $g$ and so we reduce to the case $g = p_j$.

It remains to prove that $\mathrm{ch}'_{\Lambda^\bullet P_{jm}} * p_{jm} = p_m[p_j * \mathrm{ch}'_{\Lambda^\bullet P_j}]$. Since $(p_\lambda)_\lambda$ are orthogonal idempotent with respect to the Kronecker product

$$\mathrm{ch}'_{\Lambda^\bullet P_n} * p_n = \chi'_{\Lambda^\bullet P_n}(c_n) p_n$$

where $\chi'_V(\sigma)$ is the graded character of $\sigma \in S_n$ with $q$ replaced by $-q$ and $c_n \in S_n$ be an $n$-cycle. It is easy to see that

$$\chi'_{\Lambda^\bullet P_n}(c_n) = 1 + (-1)^{n-1}(-q)^n = 1 - q^n$$

on the canonical base of $\Lambda^\bullet P_n$: let $(v_i)_i$ the standard base of $P_n$, the product of some $v_j$ is invariant for $c_n$ if and only if each generator appears a fixed number of times (i.e. 0 or 1 times). Finally the equalities

$$p_m[p_j * \mathrm{ch}'_{\Lambda^\bullet P_j}] = p_m[(1 - q^j) p_j]$$
$$= (1 - q^{jm}) p_{jm}$$
$$= \mathrm{ch}'_{\Lambda^\bullet P_{jm}} * p_{jm}$$

conclude the proof. ∎

For each monomial $m = \prod_k e_{i_k, j_k} \in \mathbb{Q}[e_{i,j}]$ we define the *support* of $m$ as the finest set partition $B(m) \vdash [n]$ such that for all $k$ $i_k$ and $j_k$ belong to the same block of $B(m)$. We also define the *type* of $m$ as the partition $\lambda(m) \vdash n$ collecting the size of blocks of $B(m)$. Notice that the relations defining $OT_n$ (Theorem 2.1) are sum of monomials with the same support, hence the notion of support and type are well defined in $OT_n$. Moreover, monomials with different supports are linearly independent.

For $B \vdash [n]$ a set partition let $T_B \subset OT_n$ be the vector space generated by all monomials $m$ such that $B(m) = B$. For $S \subseteq [n]$ we define $T_S = T_B$ where $B$ is the finest set partition of $[n]$ with a block equal to $S$. Given two monomials $m, m'$ such that $mm' \neq 0$ in $OT_n$, we have that $B(mm')$ is the finest set partition coarsening both $B(m)$ and $B(m')$, hence

$$T_B \cong \bigotimes_{i=1}^{l} T_{B_i}$$

where we denote by $B_i$ the blocks of $B = \{B_1, B_2, \ldots, B_l\}$.

Consider a partition $\lambda \vdash n$, let $T_\lambda$ be the vector space generated by all monomials of type $\lambda$. Choose a set partition $B_\lambda \vdash [n]$ whose blocks $B_i$ are of length $\lambda_i$ and let $S_{B_\lambda}$ be the subgroup of $S_n$ stabilizing $B_\lambda$, if $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ then $S_{B_\lambda} \cong \prod_{j \geq 1} S_j \wr S_{m_j}$. We have

$$T_\lambda \cong \mathrm{Ind}_{S_{B_\lambda}}^{S_n} T_{B_\lambda}$$

as representation of $S_n$, where $S_{B_i}$ acts on the factor $T_{B_i}$ of $T_{B_\lambda} = \otimes_{i=1}^{|B|} T_{B_i}$ and $S_{m_j}$ permutes the $m_j$ factors of size $j$. For the sake of notation we set $T_n = T_{(n)}$.

**Lemma 4.6.** We have

$$\mathrm{ch}_{OT_n} = \sum_{\lambda \vdash n} \prod_{j \geq 1} h_{m_j}[\mathrm{ch}_{T_j}].$$

$\square$

**Proof.** The Orlik-Terao algebra decomposes

$$OT_n = \bigoplus_{B \vdash [n]} T_B$$

$$= \bigoplus_{B \vdash [n]} \bigotimes_{i=1}^{|B|} T_{B_i}$$

$$= \bigoplus_{\lambda \vdash n} \mathrm{Ind}_{S_{B_\lambda}}^{S_n} \bigotimes_{i=1}^{\ell(\lambda)} T_{B_i}$$

$$= \bigoplus_{\lambda \vdash n} \mathrm{Ind}_{\prod_j S_{jm_j}}^{S_n} \left( \bigotimes_{j \geq 1} \mathrm{Ind}_{S_j \wr S_{m_j}}^{S_{jm_j}} \widetilde{T}_j \right)$$

as $S_n$-representation. Taking the character we obtain the claimed relation. ∎

**Theorem 4.7.** We have

$$\mathrm{ch}_{D_n} = \mathrm{ch}_{M_n}$$

and

$$\mathrm{ch}_{T_n} = q^{n-1} l_n * \mathrm{ch}_{R_n}.$$

$\square$

**Proof.** We prove both equality by induction on $n$. The base case $n = 1$ is trivial. For the inductive step we consider:

$$\mathrm{ch}_{M_n} = \mathrm{ch}_{OT_n} * \mathrm{ch}'_{\Lambda_n}$$

$$= \frac{1}{(1-q)} \sum_{\lambda \vdash n} \prod_{j \geq 1} h_{m_j}[\mathrm{ch}_{T_j} * \mathrm{ch}'_{\Lambda \bullet P_j}]$$

$$= \mathrm{ch}_{T_n} * \mathrm{ch}'_{\Lambda_n} + \frac{1}{(1-q)} \sum_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} \prod_{j \geq 1} h_{m_j}[q^{j-1}(1-q)l_j].$$

The first equality follows from Theorem 4.1 and Lemma 4.2. The second one follows from Lemma 4.6 and Lemma 4.5 together with the identity $\mathrm{ch}'_{\Lambda \bullet P_j} = (1 - q) \mathrm{ch}'_{\Lambda_j}$. The last one follows from the inductive hypothesis and Theorem 4.2. We have proven the identity

$$\mathrm{ch}_{M_n} - \mathrm{ch}_{T_n} * \mathrm{ch}'_{\Lambda_n} = \frac{1}{(1-q)} \sum_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} \prod_{j \geq 1} h_{m_j}[q^{j-1}(1-q)l_j] = \mathrm{ch}_{D_n} - q^{n-1}l_n,$$

where the last equality is given by Theorem 4.4. Since $\mathrm{ch}_{D_n}$ and $\mathrm{ch}_{M_n}$ has degree less than $n - 1$ and $\mathrm{ch}_{T_n} * \mathrm{ch}'_{\Lambda_n}$ bigger than $n - 2$, $\mathrm{ch}_{M_n} = \mathrm{ch}_{D_n}$ and $\mathrm{ch}_{T_n} * \mathrm{ch}'_{\Lambda_n} = q^{n-1}l_n$ hold. Therefore $\mathrm{ch}_{T_n} = q^{n-1}l_n * \mathrm{ch}_{R_n}$. ∎

**Corollary 4.8.** We obtain the character of $OT_n$:

$$\mathrm{ch}_{OT_n} = \sum_{\lambda \vdash n} q^{n-\ell(\lambda)} \prod_{j \geq 1} h_{m_j}[l_j * \mathrm{ch}_{R_j}]. \tag{6}$$

□

**Proof.** It follows from Theorem 4.7 and Theorem 4.6. ∎

An important object for the proof of Theorem 4.7 is the $R_n$-module $T_n$. It is a submodule of the free module $OT_n$ and its Frobenius character is equal to the one of the free module $R_n \otimes_{\mathbb{Q}} T_n^{n-1}$. This observations lead to the following conjecture:

**Conjecture 4.9.** The $R_n$-module $T_n$ is free. □

**Remark 4.10.** The formula (6) is completely explicit because $\mathrm{ch}_{R_j}$ and $l_j$ are known. Indeed

$$\mathrm{ch}_{R_n} = (1 - q) \sum_{\lambda \vdash n} s_\lambda(1, q, q^2, \ldots) s_\lambda = (1 - q) h_n \left[ \frac{X}{1 - q} \right]$$

by [Pro03, Section 5.6] or [Sta99, Exercise 7.73] where $X = h_1$. Moreover,

$$l_n = \frac{1}{n} \sum_{d \mid n} \mu(d) p_d^{\frac{n}{d}},$$

by [Reu93, Theorem 8.3], $l_n$ is known as the Lyndon symmetric function or as Gessel-Reutenauer symmetric function [GR93]. □

Let Exp be the plethystic exponential defined by

$$\mathrm{Exp}(f) := \exp \left( \sum_{k \geq 1} \frac{p_k[f]}{k} \right) = \sum_{k \geq 0} h_k[f],$$

see [LR11, Section 5.3] for the equivalence between the two formulas. We denote by Log the inverse of Exp and we define the symmetric functions

$$L = \sum_{n \geq 1} q^{n-1} t^n l_n = -\frac{\mathrm{Log}(1 - qtX)}{q}.$$

**Corollary 4.11.** The generating functions for $\mathrm{ch}_D$ and $\mathrm{ch}_{OT}$ are:

$$\sum_{n \geq 1} \mathrm{ch}_{D_n}(q) t^n = \frac{1}{1-q} (\mathrm{Exp}((1 - q)L) - 1), \tag{7}$$

$$\sum_{n \geq 1} \mathrm{ch}_{OT_n}(q) t^n = \mathrm{Exp} \left( (1 - q)L * \mathrm{Exp} \left( \frac{X}{1 - q} \right) \right) - 1. \tag{8}$$

□

**Proof**. Let $f$ be a symmetric function and call $f_j$ be the homogeneous part of degree $j$. Assume that $f$ has zero constant term, i.e. $f = \sum_{j \geq 1} f_j$, then

$$\mathrm{Exp}(f) = \prod_{j \geq 1} \mathrm{Exp}(f_j)$$
$$= \prod_{j \geq 1} \sum_{m \geq 0} h_m[f_j]$$
$$= \sum_{\lambda} \prod_{j \geq 1} h_{m_j}[f_j],$$

where the sum is taken over all partitions $\lambda = (1^{m_1}, 2^{m_2}, \dots)$. The corollary follows by taking $f = (1-q)L$ and $f = (1-q)L * \mathrm{Exp}((1-q)^{-1}X)$. ∎

Formulas of this paper are checked and implemented in SageMath [Sage]. The code is available at

https://github.com/paga92/character_OT.

## Funding

## Acknowledgements

## References

[AAB14]  Samia Ashraf, Haniya Azam, and Barbu Berceanu, *Representation theory for the Križ model*, Algebr. Geom. Topol. **14** (2014), no. 1, 57–90. MR 3158753

[Ber10]  Andrew Berget, *Products of linear forms and Tutte polynomials*, European J. Combin. **31** (2010), no. 7, 1924–1935. MR 2673030

[BP09]  Tom Braden and Nicholas Proudfoot, *The hypertoric intersection cohomology ring*, Invent. Math. **177** (2009), no. 2, 337–379. MR 2511745

[Coh76]  Frederick R. Cohen, *The homology of $C_{n+1}$-spaces, $n \geq 0$*, The homology of iterated loop spaces, Lecture Notes in Mathematics, Vol. 533, Springer-Verlag, Berlin-New York, 1976, pp. 207 – 351. MR 0436146

[CT78]  F. R. Cohen and L. R. Taylor, *Computations of Gel′fand-Fuks cohomology, the cohomology of function spaces, and the cohomology of configuration spaces*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), I, Lecture Notes in Math., vol. 657, Springer, Berlin, 1978, pp. 106–143. MR 513543

[DGT14]  Graham Denham, Mehdi Garrousian, and Ştefan O. Tohăneanu, *Modular decomposition of the Orlik-Terao algebra*, Ann. Comb. **18** (2014), no. 2, 289–312. MR 3206154

[EPW16]  Ben Elias, Nicholas Proudfoot, and Max Wakefield, *The Kazhdan-Lusztig polynomial of a matroid*, Adv. Math. **299** (2016), 36–70. MR 3519463

[ER19]  Nicholas Early and Victor Reiner, *On configuration spaces and Whitehouse's lifts of the Eulerian representations*, J. Pure Appl. Algebra **223** (2019), no. 10, 4524–4535. MR 3958103

[Get99]  Ezra Getzler, *The homology groups of some two-step nilpotent lie algebras associated to symplectic vector spaces*, 1999.

[GH22]   Nir Gadish and Louis Hainaut, *Configuration spaces on a wedge of spheres and Hochschild-Pirashvili homology*, February 2022.

[GR93]   Ira M. Gessel and Christophe Reutenauer, *Counting permutations with given cycle structure and descent set*, J. Combin. Theory Ser. A **64** (1993), no. 2, 189–215. MR 1245159

[GS87]   Murray Gerstenhaber and S. D. Schack, *A Hodge-type decomposition for commutative algebra cohomology*, J. Pure Appl. Algebra **48** (1987), no. 3, 229–247. MR 917209

[Han90]   Phil Hanlon, *The action of $S_n$ on the components of the Hodge decomposition of Hochschild homology*, Michigan Math. J. **37** (1990), no. 1, 105–124. MR 1042517

[HR15]   Patricia Hersh and Victor Reiner, *Representation stability for cohomology of configuration spaces in $\mathbf{R}^d$*, 2015.

[Kri94]   Igor Kř
íž, *On the rational homotopy type of configuration spaces*, Ann. of Math. (2) **139** (1994), no. 2, 227–237. MR 1274092

[Le14]   Dinh Van Le, *On the Gorensteinness of broken circuit complexes and Orlik-Terao ideals*, J. Combin. Theory Ser. A **123** (2014), 169–185. MR 3157806

[Liu16]   Ricky Ini Liu, *On the commutative quotient of Fomin-Kirillov algebras*, European J. Combin. **54** (2016), 65–75. MR 3459053

[LR11]   Nicholas A. Loehr and Jeffrey B. Remmel, *A computational and combinatorial exposé of plethystic calculus*, J. Algebraic Combin. **33** (2011), no. 2, 163–198. MR 2765321

[MMPR21]   Jacob P. Matherne, Dane Miyata, Nicholas Proudfoot, and Eric Ramos, *Equivariant log concavity and representation stability*, 2021.

[MP15]   Michael McBreen and Nicholas Proudfoot, *Intersection cohomology and quantum cohomology of conical symplectic resolutions*, Algebr. Geom. **2** (2015), no. 5, 623–641. MR 3421784

[MPY17]   Daniel Moseley, Nicholas Proudfoot, and Ben Young, *The Orlik-Terao algebra and the cohomology of configuration space*, Exp. Math. **26** (2017), no. 3, 373–380. MR 3642114

[OT94]   Peter Orlik and Hiroaki Terao, *Commutative algebras for arrangements*, Nagoya Math. J. **134** (1994), 65–73. MR 1280653

[Pet20]   Dan Petersen, *Cohomology of generalized configuration spaces*, Compos. Math. **156** (2020), no. 2, 251–298. MR 4045070

[Pro03]   Claudio Procesi, *Symmetric group*, `http://garsia.math.yorku.ca/ghana03/chapt2.pdf`, 2003, [Online].

[PS06]   Nicholas Proudfoot and David Speyer, *A broken circuit ring*, Beiträge Algebra Geom. **47** (2006), no. 1, 161–166. MR 2246531

[Reu93]   Christophe Reutenauer, *Free Lie algebras*, London Mathematical Society Monographs. New Series, vol. 7, The Clarendon Press, Oxford University Press, New York, 1993, Oxford Science Publications. MR 1231799

[Sage]   The Sage Developers, William Stein, David Joyner, David Kohel, John Cremona, and Burçin Eröcal, *Sagemath, version 9.0*, 2020.

[Sch11]   Hal Schenck, *Resonance varieties via blowups of $\mathbb{P}^2$ and scrolls*, Int. Math. Res. Not. IMRN (2011), no. 20, 4756–4778. MR 2844937

[ST09]   Hal Schenck and Ştefan O. Tohăneanu, *The Orlik-Terao algebra and 2-formality*, Math. Res. Lett. **16** (2009), no. 1, 171–182. MR 2480571

[Sta99]   Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR 1676282

[SW97]    Sheila Sundaram and Volkmar Welker, *Group actions on arrangements of linear subspaces and applications to configuration spaces*, Trans. Amer. Math. Soc. **349** (1997), no. 4, 1389–1420. MR 1340186

[Ter02]    Hiroaki Terao, *Algebras generated by reciprocals of linear forms*, J. Algebra **250** (2002), no. 2, 549–558. MR 1899865

[Tot96]    Burt Totaro, *Configuration spaces of algebraic varieties*, Topology **35** (1996), no. 4, 1057–1067. MR 1404924

[Whi97]    Sarah Whitehouse, *The Eulerian representations of $\Sigma_n$ as restrictions of representations of $\Sigma_{n+1}$*, J. Pure Appl. Algebra **115** (1997), no. 3, 309–320. MR 1431838