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Hair and entropy for slowly rotating quantum black holes

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Abstract We study the quantum hair associated with coherent states describing slowly rotating black holes and show how it can be naturally related with the Bekenstein–Hawking entropy and with 1-loop quantum corrections of the metric for the (effectively) non-rotating case. We also estimate corrections induced by such quantum hair to the temperature of the Hawking radiation through the tunnelling method.

1 Introduction

The breakthrough in gravitational wave astronomy from the LIGO and Virgo collaboration [1–3] has opened up a new observational window, allowing us to directly learn more about black holes. As solutions to the Einstein equations, these spacetimes contain singularities which might just signal the breakdown of classical physics in the strong field regime. In recent years, several ways of describing quantum aspects of black holes have been proposed in the literature. Some approaches, like the corpuscular picture [4–7], assume that the geometry should only emerge at suitable (macroscopic) scales from the underlying (microscopic) quantum field theory of gravitons [8–10]. Bekenstein's conjecture for the horizon area quantisation [11,12] then naturally follows for the occupation number of gravitons is proportional to the square of the ADM mass M [13] in units of the Planck mass m_{p}.[1]

An improved description of nonuniform geometries can be obtained by employing coherent states of gravitons [14,15], which then leads to necessary departures from the classical Schwarzschild metric [16] (and thermodynamics [17,18]). In particular, the central singularity of the Schwarzschild black hole is replaced by an integrable singularity [19] without Cauchy horizons. The coherent state is built for a scalar field whose expectation value effectively describes the geometry emerging from the (longitudinal or temporal) polarisation of the graviton in the linearised theory. A similar analysis for electrically charged spherically symmetric black holes was then shown to remove both the central singularity and the Cauchy horizon [20].

The majority of black holes in nature are very likely to spin, which motivates investigating quantum descriptions of black holes with non-vanishing specific angular momentum $a = J/M$ [21]. A complete description of axisymmetric Kerr black holes [22] remains beyond our scope, but this (conceptually and phenomenologically) important issue can be addressed for slow rotation by considering coherent states of gravitons similarly to the spherically symmetric case. In particular, we will focus on the quantum description of the approximate Kerr metric for $|a| \ll G_{\mathrm{N}} M$, which can be written as [23]

$$\mathrm{d}s^2 \simeq -(1+2V)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+2V} - \frac{4\,G_{\mathrm{N}}\,M\,a}{r}\,\sin^2\theta\,\mathrm{d}t\,\mathrm{d}\phi + r^2\,\mathrm{d}\Omega^2 \tag{1.1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. In the above, the metric function

$$V = V_M + W_a \tag{1.2}$$

where

$$V_M = -\frac{G_{\mathrm{N}}\,M}{r} \tag{1.3}$$

corresponds to the Schwarzschild metric [24] for $a = 0$, and

$$W_a = \frac{a^2}{2\,r^2}. \tag{1.4}$$

[1] We shall often use units with $c = 1$, $G_{\mathrm{N}} = \ell_{\mathrm{p}}/m_{\mathrm{p}}$ and $\hbar = \ell_{\mathrm{p}}\,m_{\mathrm{p}}$, where ℓ_{p} and m_{p} denote the Planck length and mass, respectively.

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In the above stationary geometry, the possible event horizon is a sphere located at $r = r_{\rm H}$ defined as the largest (real) solution of $1+2\,V = 0$. We shall find that the very existence of a quantum coherent state again requires departures from the classical geometry (at least) near the (would-be) classical central singularity. This induces the presence of "quantum hair",[2] which we will further connect with the Bekenstein–Hawking entropy [11], the Hawking evaporation [31], and 1-loop quantum corrections to the metric obtained in the weak-field approximation (see [32] and references therein for earlier works). Since the quantum corrected geometry obtained from coherent states is not perturbative (in the ratio $G_{\rm N}\,M/r$ or Planck constant), the latter result extends, and provides an independent support for, perturbative calculations.

In Sect. 2, we first review the classical solutions of the Klein–Gordon equation and show how coherent states of a massless scalar field on a reference flat spacetime associated to the vacuum can be used to reproduce a black hole geometry with small angular momentum; Sect. 3 is devoted to studying the quantum hair of such coherent state black holes, whose existence implies that information about the interior state is present outside the horizon; the relation with the Bekenstein–Hawking entropy is derived in Sect. 4, where corrections to the Hawking temperature are also estimated using the semiclassical tunnelling methods; Sect. 5 contains concluding remarks.

2 Coherent quantum states for slowly rotating geometry

Like in Ref. [16], the quantum vacuum is here assumed to correspond to a spacetime devoid of matter and gravitational excitations. Any classical metric should then emerge from a suitable (highly excited) quantum state. A standard approach for recovering classical behaviours employs coherent states, which is generically motivated by their property of minimising the quantum uncertainty, and is further supported by studies of electrodynamics [15,33], linearised gravity [34], and the de Sitter spacetime [35–37].

In particular, we want to reproduce the slowly rotating stationary geometry (1.1) as the full general relativistic extension of the Newtonian potential. The latter can be derived from the longitudinal mode of gravitons in the linearised theory and, like for the static case of Ref. [16], we assume that this feature is preserved in the stationary limit of full general relativity. We therefore try and obtain the complete metric function (1.2) as the expectation value of an effective free massless scalar field $\sqrt{G_{\rm N}}\,\Phi = V$ satisfying the Klein–Gordon equation

$$\Box\Phi = 0. \tag{2.1}$$

It is convenient to employ spherical coordinates in which a complete (normalised) set of positive frequency solutions is given by

$$u_{\omega\ell m} = \frac{e^{-i\,\omega\,t}}{\sqrt{2\,\omega}}\,j_\ell(\omega\,r)\,Y_{\ell m}(\theta,\varphi) \tag{2.2}$$

where j_ℓ are spherical Bessel functions of the first kind, and

$$Y_\ell^m = (-1)^m\sqrt{\frac{(2\,\ell+1)(\ell-m)!}{4\,\pi\,(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{i\,m\,\varphi} \tag{2.3}$$

are spherical harmonics of degree ℓ and order m, P_ℓ^m being associated Legendre polynomials. We recall that these solutions are orthonormal,[3]

$$(u_{\omega\ell m}|u_{\omega'\ell' m'}) = \frac{\pi}{2\,\omega^2}\,\delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{mm'} \qquad \left(u_{\omega\ell m}|u^*_{\omega'\ell' m'}\right) = 0 \tag{2.4}$$

in the Klein–Gordon scalar product

$$(f_1|f_2) = i\int \mathrm{d}^3x\,\left(f_1^*\,\partial_t f_2 - f_2\,\partial_t f_1^*\right). \tag{2.5}$$

The quantum theory is built by mapping the field Φ into an operator expanded in terms of the normal modes (2.2),

$$\hat\Phi = \sum_\ell\sum_{m=-\ell}^{\ell}\frac{2}{\pi}\int_0^\infty \omega^2\,\mathrm{d}\omega\,\sqrt{\hbar}\Big[u_{\omega\ell m}\,\hat a_{\ell m}(\omega) + u^*_{\omega\ell m}\,\hat a^\dagger_{\ell m}(\omega)\Big]. \tag{2.6}$$

Likewise, its conjugate momentum reads

$$\hat\Pi = i\sum_\ell\sum_{m=-\ell}^{\ell}\frac{2}{\pi}\int_0^\infty \omega^3\,\mathrm{d}\omega\,\sqrt{\hbar}\Big[u_{\omega\ell m}\,\hat a_{\ell m}(\omega) - u^*_{\omega\ell m}\,\hat a^\dagger_{\ell m}(\omega)\Big]. \tag{2.7}$$

These operators satisfy the equal-time commutation relations,

$$\Big[\hat\Phi(t,r,\theta,\varphi),\hat\Pi(t,r',\theta',\varphi')\Big] = i\,\hbar\,\frac{\delta(r-r')}{r^2}\,\frac{\delta(\theta-\theta')}{\sin\theta}\,\delta(\varphi-\varphi') \tag{2.8}$$

provided the creation and annihilation operators obey the commutation rules

$$\Big[\hat a_{\ell m}(\omega),\hat a^\dagger_{\ell' m'}(\omega')\Big] = \frac{\pi}{2\,\omega^2}\,\delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{mm'}. \tag{2.9}$$

[2] The concept of quantum hair has been explored through different quantum gravity frameworks, see e.g. Refs. [25–30].

[3] See Appendix A for more details about the notation.

The vacuum state is first defined by $\hat{a}_{\ell m}(\omega)\,|0\rangle = 0$ for all allowed values of ω, ℓ and m, and a basis for the Fock space is constructed by the usual action of creation operators.

2.1 Semiclassical metric function

We seek a quantum state of Φ which effectively reproduces (as closely as possible) the expected slow-rotation limit of the Kerr geometry (1.1), that is

$$\sqrt{G_{\rm N}}\,\langle V|\,\hat{\Phi}(t,r,\theta,\varphi)\,|V\rangle \simeq V(r). \tag{2.10}$$

We can build $|V\rangle$ as a superposition of coherent states satisfying

$$\hat{a}_{\ell m}(\omega)\,|g_{\ell m}(\omega)\rangle = g_{\ell m}(\omega)\,e^{i\,\gamma_{\ell m}(\omega)}\,|g_{\ell m}(\omega)\rangle \tag{2.11}$$

where $g_{\ell m} = g^*_{\ell m}$ and $\gamma_{\ell m} = \gamma^*_{\ell m}$, so that

$$\begin{aligned}\sqrt{G_{\rm N}}\,\langle V|\,\hat{\Phi}\,|V\rangle &= \ell_{\rm p}\sum_{\ell}\sum_{m=-\ell}^{\ell}\frac{2}{\pi}\\ &\times\int_0^\infty \omega^2\,{\rm d}\omega\; j_\ell(\omega\,r)\,\frac{(-1)^m}{\sqrt{2\,\omega}}\sqrt{\frac{(2\,\ell+1)(\ell-m)!}{4\,\pi\,(\ell+m)!}}\\ &\times 2\,\cos(\omega\,t-\gamma_{\ell m}+m\,\varphi)\,P_\ell^m(\cos\theta)\,g_{\ell m}(\omega).\end{aligned} \tag{2.12}$$

Since the Kerr metric is stationary and axially symmetric, we impose that the phases $\gamma_{\ell m} \simeq \omega\,t + m\,\varphi$. Indeed, one could argue that recovering exact spacetime symmetries with such a limiting procedure reflects the fact that no perfect isometries exist in nature [16].

The coefficients $g_{\ell m}$ can be determined by expanding the metric field V on the spatial part of the normal modes (2.2),

$$\begin{aligned}V(r,\theta) &= \sum_{\ell}\sum_{m=-\ell}^{\ell}\frac{2}{\pi}\int_0^\infty \omega^2\,{\rm d}\omega\; j_\ell(\omega r)\,(-1)^m\\ &\times\sqrt{\frac{(2\,\ell+1)\,(\ell-m)!}{4\,\pi\,(\ell+m)!}}\,P_\ell^m(\cos\theta)\,\tilde{V}_{\ell m}(\omega).\end{aligned} \tag{2.13}$$

By comparing the expansions (2.12) and (2.13), we obtain

$$g_{\ell m} = \sqrt{\frac{\omega}{2}}\,\frac{\tilde{V}_{\ell m}(\omega)}{\ell_{\rm p}}. \tag{2.14}$$

The coherent state finally reads

$$\begin{aligned}|V\rangle &= \prod_{\ell}\prod_{m=-\ell}^{\ell} e^{-N_{\ell m}/2}\\ &\times\exp\left\{\frac{2}{\pi}\int_0^\infty \omega^2\,{\rm d}\omega\; g_{\ell m}(\omega)\,\hat{a}^\dagger_{\ell m}(\omega)\right\}|0\rangle\end{aligned} \tag{2.15}$$

where

$$N_{\ell m} = \frac{2}{\pi}\int_0^\infty \omega^2\,{\rm d}\omega\,|g_{\ell m}(\omega)|^2 \tag{2.16}$$

is the occupation number for the state $|g_{\ell m}(\omega)\rangle$. We note in particular that $N_V = \sum_{\ell m} N_{\ell m}$ measures the "distance" of $|V\rangle$ from the vacuum $|0\rangle$ in the Fock space and should be finite [16].

2.2 Schwarzschild geometry

For zero angular momentum, hence $a = W_a = 0$, the metric function (1.3) is obtained from

$$\tilde{V}_{00} = -\frac{2\,\sqrt{\pi}}{\omega^2}\,G_{\rm N}\,M \tag{2.17}$$

so that the only contributions to the coherent state $|V_M\rangle$ are given by the eigenvalues [16]

$$g_{00} = -\sqrt{\frac{2\,\pi}{\omega^3}}\,\frac{M}{m_{\rm p}} \tag{2.18}$$

yielding the total occupation number

$$N_M = N_{00} = 4\,\frac{M^2}{m_{\rm p}^2}\int_0^\infty\frac{{\rm d}\omega}{\omega}. \tag{2.19}$$

The number N_M diverges logarithmically both in the infrared (IR) and in the ultraviolet (UV). In particular, the UV divergence arises from demanding a Schwarzschild geometry for all values of $r > 0$ and can be formally regularised by introducing a cut-off $\omega_{\rm UV} \sim 1/R_{\rm s}$, where $R_{\rm s}$ can be interpreted as the finite radius of a regular matter source [38].

Note in fact that the static geometry we are reconstructing from the coherent state should be completely determined by the energy-momentum of the matter source in general relativity, like the Newtonian potential is fully determined by the energy density in the linearised theory. The UV cut-off is therefore just a mathematically simple way of accounting for the fact that the very existence of a proper quantum state $|V_M\rangle$ requires the coefficients $g_{00} = g_{00}(\omega)$ to depart from their purely classical expression (2.18) for $\omega \to \infty$. This departure from the classical expression would in turn be related with the actual state of matter in the interior of the black hole. Since we aim at a general analysis of the geometry, we just demand that $R_{\rm s} \lesssim R_{\rm H} = 2\,G_{\rm N}\,M$ for a (quantum) black hole [16] and do not investigate the connection between the geometry and possible matter sources any further here (see Refs. [39–42]). Likewise, we introduce a IR cut-off $\omega_{\rm IR} = 1/R_\infty$ to account for the necessarily finite lifetime $\tau \sim R_\infty$ of the system, and finally write

$$N_M = 4\,\frac{M^2}{m_{\rm p}^2}\ln\left(\frac{R_\infty}{R_{\rm s}}\right). \tag{2.20}$$

The coherent state $|V_M\rangle$ so defined corresponds to a quantum-corrected metric function

$$V_{\mathrm{q}M} \simeq \sqrt{G_{\mathrm{N}}}\,\langle V_M|\,\hat{\Phi}\,|V_M\rangle = \frac{1}{\pi^{3/2}} \int_{\omega_{\mathrm{IR}}}^{\omega_{\mathrm{UV}}} \omega^2\,\mathrm{d}\omega\; j_0(\omega r)\,\tilde{V}_{00}(\omega)$$
$$\simeq -\frac{2\,G_{\mathrm{N}}\,M}{\pi\,r} \int_{R_\infty^{-1}}^{R_{\mathrm{s}}^{-1}} \mathrm{d}\omega\, \frac{\sin(\omega\,r)}{\omega}$$
$$\simeq -\frac{G_{\mathrm{N}}\,M}{r}\left\{1-\left[1-\frac{2}{\pi}\,\mathrm{Si}\left(\frac{r}{R_{\mathrm{s}}}\right)\right]\right\} \tag{2.21}$$

where we let $\omega_{\mathrm{IR}} = 1/R_\infty \to 0$ and Si denotes the sine integral function. The emerging quantum-corrected geometry is correspondingly given by [4]

$$\mathrm{d}s^2 \simeq -\left(1+2\,V_{\mathrm{q}M}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1+2\,V_{\mathrm{q}M}} + r^2\,\mathrm{d}\Omega^2 \tag{2.22}$$

which was already analysed in Ref. [16], where further details can be found.

2.3 Slowly rotating black hole

The classical metric (1.1) is characterised by an angular momentum of modulus $\hbar \ll J = |a|\,M \ll G_{\mathrm{N}}\,M^2$ oriented along the axis of symmetry, so that $J^z = J$ for $a > 0$, and by the metric function W_a in Eq. (1.4). We can now show that a quantum state that reproduces such a metric, as closely as possible, like we discussed in the previous Sect. 2.2, can be obtained by linearly combining the coherent state $|V_M\rangle$ of the Schwarzschild geometry with a suitable coherent state $|W_a\rangle$.

The normal modes (2.2) are eigenfunctions of the angular momentum operators $\hat{L}^2$ and $\hat{L}_z$ (in Minkowski spacetime) with eigenvalues $\hbar^2\,\ell\,(\ell+1)$ and $\hbar\,m$, respectively. The expectation values of the angular momentum operators on the coherent state $|g_{\ell m}(\omega)\rangle$ are therefore given by (see Appendix B)

$$J_{\ell m} = \langle g_{\ell m}(\omega)|\,\sqrt{\hat{L}^2}\,|g_{\ell m}(\omega)\rangle$$
$$= \hbar\,\sqrt{\ell\,(\ell+1)}\,|g_{\ell m}(\omega)|^2 \tag{2.23}$$

and

$$J^z_{\ell m} = \langle g_{\ell m}(\omega)|\,\hat{L}_z\,|g_{\ell m}(\omega)\rangle$$
$$= \hbar\,m\,|g_{\ell m}(\omega)|^2\,. \tag{2.24}$$

[4] For $R_{\mathrm{s}} \to 0$, the term in square brackets in Eq. (2.21) vanishes at any $r > 0$ and the Schwarzschild metric is formally recovered.

The total angular momentum for a superposition $|W\rangle$ of states $|g_{\ell m}(\omega)\rangle$ can be obtained as

$$J \equiv \langle W|\,\sqrt{\hat{L}^2}\,|W\rangle = \sum_{\ell>0}\sum_{m=-\ell}^{\ell} \frac{2}{\pi}\int_0^\infty \omega^2\,\mathrm{d}\omega\; J_{\ell m}(\omega)$$
$$= \sum_{\ell>0} \hbar\,\sqrt{\ell\,(\ell+1)} \sum_{m=-\ell}^{\ell} N_{\ell m}\,. \tag{2.25}$$

Likewise,

$$J^z \equiv \langle W|\,\hat{L}_z\,|W\rangle = \sum_{\ell>0}\sum_{m=-\ell}^{\ell} \frac{2}{\pi}\int_0^\infty \omega^2\,\mathrm{d}\omega\; J^z_{\ell m}(\omega)$$
$$= \sum_{\ell>0}\sum_{m=-\ell}^{\ell} \hbar\,m\,N_{\ell m}\,. \tag{2.26}$$

Let us next consider coherent states defined by the eigenvalues

$$g_{\ell m} = C_{\ell m}\,\frac{\sqrt{2\,\pi}\,\ell_{\mathrm{p}}^\alpha\,M}{\omega^{3/2-\alpha}\,m_{\mathrm{p}}} \tag{2.27}$$

where $C_{\ell m}$ are numerical coefficients that do not depend on ω and $\ell \geq 1$. The corresponding occupation numbers (2.16) are given by

$$N_{\ell m} \simeq \begin{cases} C_{\ell m}^2\,N_M & \text{for } \alpha = 0 \\ 4\,C_{\ell m}^2\,\dfrac{M^2}{m_{\mathrm{p}}^2}\left[\left(\dfrac{\ell_{\mathrm{p}}}{R_{\mathrm{s}}}\right)^{2\alpha} - \left(\dfrac{\ell_{\mathrm{p}}}{R_\infty}\right)^{2\alpha}\right] & \text{for } \alpha \neq 0 \end{cases} \tag{2.28}$$

where $N_M \sim M^2/m_{\mathrm{p}}^2$ is given in Eq. (2.20). Note that the IR limit $R_\infty \to \infty$ is regular only for $\alpha > 0$, for which $N_{\ell m} \ll N_M$ if $R_{\mathrm{s}} \gg \ell_{\mathrm{p}}$. In this case, we can further approximate

$$N_{\ell m} \simeq 4\,C_{\ell m}^2\,\frac{M^2}{m_{\mathrm{p}}^2}\left(\frac{\ell_{\mathrm{p}}}{R_{\mathrm{s}}}\right)^{2\alpha} \sim C_{\ell m}^2 \tag{2.29}$$

where we considered $R_{\mathrm{s}} \sim R_{\mathrm{H}}$ for a black hole. [5] Moreover, the modification (2.13) to the metric function is given by

$$W_{\ell m} \simeq \ell_{\mathrm{p}}\,\frac{2}{\pi}$$
$$\times \int_{\omega_{\mathrm{IR}}}^{\omega_{\mathrm{UV}}} \omega^2\,\mathrm{d}\omega\; j_\ell(\omega\,r)\,\frac{(-1)^m}{\sqrt{2\,\omega}}\sqrt{\frac{(2\,\ell+1)\,(\ell-m)!}{\pi\,(\ell+m)!}}$$
$$P_\ell^m(\cos\theta)\,g_{\ell m}(\omega)$$
$$\simeq \frac{G_{\mathrm{N}}\,M}{r}\left(\frac{\ell_{\mathrm{p}}}{r}\right)^\alpha\left[C_{\ell m}\,(-1)^m\sqrt{\frac{(2\,\ell+1)\,(\ell-m)!}{(\ell+m)!}}\right.$$
$$\left.P_\ell^m(\cos\theta)\,\frac{2}{\pi}\int_0^{r/R_{\mathrm{s}}} z^\alpha\,\mathrm{d}z\; j_\ell(z)\right] \tag{2.30}$$

[5] All numerical factors can be included in $C_{\ell m}$.

where the integral in square brackets can be expressed in terms of regularised hypergeometric functions [see Eq. (A.11)]. We then see that the leading terms in the correction (2.30) are of the classical form $W_a \sim r^{-2}$ in Eq. (1.4) if $\alpha = 1$.

Finally, the contribution to the angular momentum satisfies the classicality conditions

$$\hbar \ll J_{\ell m} \simeq \hbar \sqrt{\ell\,(\ell+1)}\, N_{\ell m} \simeq \hbar\, m\, N_{\ell m} \simeq J^z_{\ell m} \tag{2.31}$$

provided $m \simeq \ell$ and $\ell\, N_{\ell m} \sim \ell\, C^2_{\ell m} \gg 1$. We can build a coherent state $|W_a\rangle$ that reproduces the geometry (1.1) by including different coherent states (2.27) with $\alpha = 1$ and angular momentum numbers satisfying these conditions. The rotation coefficient will then be given by

$$\frac{m_\mathrm{p}}{M} \ll \frac{a}{G_\mathrm{N}\, M} \sim \frac{m_\mathrm{p}^2}{M^2} \sum_{\ell=1}^{\ell_\mathrm{c}} \sqrt{\ell\,(\ell+1)}\, N_{\ell\ell}$$
$$\sim \frac{1}{N_M} \sum_{\ell=1}^{\ell_\mathrm{c}} \sqrt{\ell\,(\ell+1)}\, N_{\ell\ell} \lesssim \delta_J \ll 1 \tag{2.32}$$

where we introduced a parameter $\delta_J > 0$ to define the slow rotation regime in terms of a maximum value of ℓ, denoted by ℓ_c.

The inclusion of states $|W\rangle$ like the above will give rise to quantum-corrected geometries

$$\mathrm{d}s^2 \simeq -\left(1 + 2\, V_{\mathrm{q}M} + 2\, W_{\mathrm{q}a}\right) \mathrm{d}t^2$$
$$+ \frac{\mathrm{d}r^2}{1 + 2\, V_{\mathrm{q}M} + 2\, W_{\mathrm{q}a}} - \frac{4\, G_\mathrm{N}\, J}{r} \sin^2\theta\, \mathrm{d}t\, \mathrm{d}\phi + r^2\, \mathrm{d}\Omega^2 \tag{2.33}$$

where $V_{\mathrm{q}M}$ is given in Eq. (2.21), $W_{\mathrm{q}a} \simeq W_{\ell m}$ in Eq. (2.30) and J in Eq. (2.25). These geometries do not entail a weak-field approximation but are perturbative in the angular momentum contributions, that is in J and $W_{\mathrm{q}a}$.

3 Quantum hair

Black hole solutions in general relativity are determined only by the total mass, angular momentum, and electric charge (if present). These uniqueness theorems [43] strongly limit the information about the internal state of a black hole that can be obtained by outside observers. However, the situation changes when we consider the quantum description of black holes given by coherent states already for the spherically symmetric case of Sect. 2.2. In fact, the coherent states from which the geometry emerges as a mean field effect cannot accommodate for perfect Schwarzschild spacetimes [16], but they instead depend on the internal structure of the matter sources (classically) hidden inside the horizon, as we recalled in Sect. 2.2.

The classical case of slow rotation was considered in Sect. 2.3, where we assumed that the quantum states of the geometry only include specific coherent states (2.27) with $\alpha = 1$ satisfying the relations in Eq. (2.31) for the angular momentum. However, the possibility that other states contribute can only be limited from the condition of recovering the classical metric (1.1) within the experimental bounds. Their presence, on the other hand, will constitute a further example of quantum hair [25–30], with departures from the classical geometry.

Instead of attempting a general analysis, we shall only consider states that violate one of the classicality conditions defined in Sect. 2.3 at a time. In particular, we will study (a) quantum contributions with $J^z \simeq J$ but $\alpha > 1$ inducing departures from V_M smaller than W_a at large r in Sect. 3.1 and (b) modes with $\alpha = 1$ and $a > 0$ given by Eq. (2.32) but such that $|J^z| \ll J$ in Sect. 3.2.

3.1 Quantum metric corrections

An explicit example of a coherent state which satisfies the classical conditions $J^z \simeq J$ for the angular momentum but leads to a geometry with terms that fall off at $r \gg R_\mathrm{H} = 2\, G_\mathrm{N}\, M$ faster than W_a in Eq. (1.4) is built from

$$g_{\bar\ell\bar\ell} = C_{\bar\ell}\, \frac{\sqrt{2\,\pi}\, \ell_\mathrm{p}^\alpha\, M}{\omega^{3/2-\alpha}\, m_\mathrm{p}} \tag{3.1}$$

where $\alpha > 1$ and $\bar\ell$ is a fixed integer value. The hairy geometry can now be obtained from

$$W_{\bar\ell\bar\ell} \simeq \frac{\ell_\mathrm{p}}{\pi^2} \int\limits_{\omega_\mathrm{IR}}^{\omega_\mathrm{UV}} \omega^{3/2}\, \mathrm{d}\omega\, j_{\bar\ell}(\omega\, r)\, \frac{2\,\bar\ell+1}{2^{\bar\ell-1/2}\, \bar\ell!}\, (\sin\theta)^{\bar\ell}\, g_{\bar\ell\bar\ell}(\omega) \tag{3.2}$$

where we used Eq. (A.8). We thus find

$$W_{\bar\ell\bar\ell} \simeq C_{\bar\ell}\, \frac{\ell_\mathrm{p}^\alpha\, G_\mathrm{N}\, M}{\pi^{3/2}}\, \frac{2\,\bar\ell+1}{2^{\bar\ell-1}\, \bar\ell!}\, (\sin\theta)^{\bar\ell} \int\limits_{\omega_\mathrm{IR}}^{\omega_\mathrm{UV}} \omega^\alpha\, \mathrm{d}\omega\, j_{\bar\ell}(\omega\, r)$$
$$\simeq C_{\bar\ell}\, \frac{G_\mathrm{N}\, M}{\pi^{3/2}\, r} \left(\frac{\ell_\mathrm{p}}{r}\right)^\alpha \frac{2\,\bar\ell+1}{2^{\bar\ell-1}\, \bar\ell!}\, (\sin\theta)^{\bar\ell} \int_0^{r/R_\mathrm{s}} z^\alpha\, \mathrm{d}z\, j_{\bar\ell}(z)$$
$$\sim \frac{G_\mathrm{N}\, M}{r} \left(\frac{\ell_\mathrm{p}}{r}\right)^\alpha \tag{3.3}$$

where the integral is given in Eq. (A.11).

We can in particular estimate the correction on the (unperturbed) Schwarzschild horizon at $r = R_\mathrm{H}$,

$$W_{\bar\ell\bar\ell}(R_\mathrm{H}) \sim \left(\frac{m_\mathrm{p}}{M}\right)^\alpha (\sin\theta)^{\bar\ell}. \tag{3.4}$$

Such corrections with different $\bar\ell$ represent purely axial perturbations on the horizon, with vanishingly small amplitude for macroscopically large black holes of mass $M \gg m_\mathrm{p}$ provided $\alpha \gtrsim 1$.

3.2 Quantum angular momentum

States that lead to metric functions of the classical form W_a in Eq. (1.4) with a given by Eq. (2.32) but satisfy

$$J^z \sim \sum_{\ell=1}^{\ell_c} \sum_{m=-\ell}^{\ell} m\, N_{\ell m} \simeq 0 \tag{3.5}$$

can be simply obtained by assuming $|g_{\ell m}| = |g_{\ell -m}|$ so that $N_{\ell m} = N_{\ell -m}$. As an example, we consider

$$g_{\bar\ell\bar\ell} = g_{\bar\ell -\bar\ell} = C_{\bar\ell} \sqrt{\frac{2\pi}{\omega}}\, \frac{M}{m_p} \tag{3.6}$$

where $\bar\ell$ is again a fixed integer value and $\bar\ell\, C_{\bar\ell}^2$ is of the correct size to yield a rotation parameter $a > 0$ satisfying the bounds in Eq. (2.32). The metric correction is now given by

$$W_{\bar\ell\bar\ell} \simeq \frac{\ell_p}{\pi^2} \int_{\omega_{IR}}^{\omega_{UV}} \omega^2\, d\omega\, j_{\bar\ell}(\omega\, r)\, \frac{(-1)^{\bar\ell}+1}{\sqrt{2\,\omega}}\, \frac{2\,\bar\ell+1}{2^{\bar\ell-1}\,\bar\ell!} \times (\sin\theta)^{\bar\ell}\, g_{\bar\ell\bar\ell}(\omega) \tag{3.7}$$

where we used the known relation (A.9). For $\bar\ell$ odd the above expression vanishes, whereas for $\bar\ell$ even we find twice the value in Eq. (3.3) with $\alpha = 1$, that is

$$W_{\bar\ell\bar\ell} \sim \frac{\ell_p\, G_N\, M}{r^2}. \tag{3.8}$$

From the above few examples, it should be clear that one can engineer many different axially symmetric configurations, all of which differ from the (slowly-rotating) Kerr geometry only by terms of order $(\ell_p/r)^\alpha$ for $\alpha \geq 1$. Of course, this ambiguity would be removed by computing the coherent state generated by a given matter source, which is however supposedly hidden behind the horizon. Moreover, we remark that such terms would result in a (slight) shift in the position r_H of the event horizon with respect to the classical value R_H.

4 Entropy and evaporation

In the previous sections, for simplicity, we have modelled the dependence of the geometry from the internal structure of the black hole by introducing cut-offs $\omega_{IR} \sim 1/R_\infty$ and $\omega_{UV} \sim 1/R_s$ in momentum space and allowing for contributions of angular momentum that have no classical counterpart. Were we able to test the gravitational field with sufficient accuracy, for instance from the motion of test particles and light in the outer region to the horizon, we could remove these uncertainties and gather information about the interior of the black hole.

4.1 Bekenstein–Hawking entropy

A common way to measure our ignorance about the actual state of a system is provided by the thermodynamic entropy, which is obtained by counting the possible microstates corresponding to a given macroscopic configuration. For a Schwarzschild black hole, the Bekenstein–Hawking entropy [11]

$$S_{BH} = \frac{\pi\, R_H^2}{\ell_p^2} = \frac{4\,\pi\, M^2}{m_p^2} \tag{4.1}$$

can be obtained [17] by supplementing a pure coherent state of the Schwarzschild geometry (2.18) with the Planckian distribution of Hawking quanta at the temperature [31]

$$T_H = \frac{m_p^2}{8\,\pi\, M}. \tag{4.2}$$

Given a black hole of mass M, instead of one pure coherent state, we could consider all possible states giving rise to (practically) indistinguishable semiclassical geometries with the mass M. We can employ the total occupation number (2.20) [6] of the corresponding coherent state to estimate the total number of microstates available to build such configurations as

$$\mathcal{N}_M \sim \sum_{n=0}^{N_M} \binom{N_M}{n} = \sum_{n=0}^{N_M} \frac{N_M!}{(N_M-n)!\, n!} = 2^{N_M}. \tag{4.3}$$

The thermodynamic entropy is thus

$$S_M \propto \ln(\mathcal{N}_M) \sim \left(\frac{M}{m_p}\right)^2 \tag{4.4}$$

which is clearly proportional to the Bekenstein–Hawking entropy (4.1). One can therefore envisage that the coherent states giving rise to Schwarzschild black hole geometries contain the precursors (or proxies) of the Hawking particles, like in the original corpuscular picture [4–7].

4.2 Entropy and angular momentum

We can also estimate the number of quantum states with angular momentum corresponding to geometric configurations that cannot be observationally distinguished from a non-rotating Schwarzschild black hole. For this purpose, we can consider again a maximum angular momentum parameter $\delta_J \ll 1$ such that configurations with

$$\frac{J}{M\, R_H} \simeq \frac{a}{G_N\, M} \lesssim \delta_J \tag{4.5}$$

cannot be distinguished from the coherent state reproducing the quantum-corrected Schwarzschild geometry (2.21).

[6] We just mention that the same quantisation law is obtained for the ground state of a dust ball [40,41].

Furthermore, we shall include in this count only those contributions of the form in Eq. (2.27),

$$g_{\ell m} \sim C_{\ell m} \frac{\omega^\alpha\, \ell_{\rm p}^\alpha\, M}{\omega^{3/2}\, m_{\rm p}} \tag{4.6}$$

that violate both of the classicality conditions considered in Sects. 3.1 and 3.2, that is $\alpha \gtrsim 1$ and $0 \le |m| \ll \ell$.

In particular, the contribution of modes with $m \simeq 0$ to the angular momentum (2.25) is approximately given by

$$\frac{a_\ell}{G_{\rm N}\, M} \simeq \ell\, \frac{m_{\rm p}^2}{M^2}\, N_{\ell 0} \sim \ell \left(\frac{\ell_{\rm p}}{R_{\rm s}}\right)^{2\alpha} \tag{4.7}$$

in which we assumed $N_{\ell 0} \sim C_{\ell 0}^2 \sim 1$ and $\ell \gg 1$. Imposing the constraint (4.5) on the total angular momentum,

$$\sum_{\ell=1}^{\ell_{\rm c}} \frac{a_\ell}{G_{\rm N}\, M} \sim \ell_{\rm c}^2 \left(\frac{\ell_{\rm p}}{R_{\rm s}}\right)^{2\alpha} \lesssim \delta_J \tag{4.8}$$

yields

$$\ell_{\rm c} \lesssim \left(\frac{R_{\rm s}}{\ell_{\rm p}}\right)^{\alpha} \sqrt{\delta_J} \sim \left(\frac{M}{m_{\rm p}}\right)^{\alpha} \sqrt{\delta_J} \tag{4.9}$$

where we again set $R_{\rm s} \sim R_{\rm H}$ for a black hole. Upon allowing for the inclusion of modes $|g_{\ell 0}\rangle$ with $1 \le \ell \le \ell_{\rm c}$, we can estimate the degeneracy of the quantum black hole given by the total number of possible combinations in angular momentum, that is

$$\mathcal{N}_{\rm c} = \sum_{\ell=0}^{\ell_{\rm c}} \binom{\ell_{\rm c}}{\ell} = \sum_{\ell=0}^{\ell_{\rm c}} \frac{\ell_{\rm c}!}{(\ell_{\rm c}-\ell)!\,\ell!} = 2^{\ell_{\rm c}}. \tag{4.10}$$

The corresponding thermodynamic entropy,

$$S \propto \ln(\mathcal{N}_{\rm c}) \sim \left(\frac{M}{m_{\rm p}}\right)^{\alpha} \sqrt{\delta_J} \tag{4.11}$$

is also proportional to the Bekenstein–Hawking entropy (4.1) for $\alpha = 2$.

It is then interesting to notice that the metric corrections for $\alpha = 2$ are of the same order in $G_{\rm N}$, $\ell_{\rm p}$ and $1/r$ as those obtained from 1-loop corrections to the Schwarzschild metric [32], that is

$$\begin{aligned} W_{{\rm q}a} &\simeq \sum_{\ell=1}^{\ell_{\rm c}} W_{\ell 0} \simeq \frac{G_{\rm N}\, M}{r} \left(\frac{\ell_{\rm p}}{r}\right)^2 \\ &\quad \times \sum_{\ell=1}^{\ell_{\rm c}} \left[C_{\ell 0} \sqrt{2\,\ell+1}\, P_\ell(\cos\theta)\, \frac{2}{\pi} \int_0^{r/R_{\rm s}} z^2\, {\rm d}z\, j_\ell(z) \right] \\ &\sim \frac{G_{\rm N}\, M}{r} \left(\frac{\ell_{\rm p}}{r}\right)^2 \end{aligned} \tag{4.12}$$

where $P_\ell = P_\ell^0$ are Legendre polynomials. The corresponding quantum corrected Schwarzschild geometry is now given by

$$\begin{aligned} {\rm d}s^2 &\simeq -\left(1 + 2\, V_{{\rm q}M} + 2\, W_{{\rm q}a}\right) {\rm d}t^2 \\ &\quad + \frac{{\rm d}r^2}{1 + 2\, V_{{\rm q}M} + 2\, W_{{\rm q}a}} + r^2\, {\rm d}\Omega^2 \end{aligned} \tag{4.13}$$

where $V_{{\rm q}M}$ is the metric function in Eq. (2.21). It is important to remark that $W_{{\rm q}a}$ represents a perturbation over the full quantum-corrected geometry (2.22) and is not restricted to the weak-field approximation employed to perform 1-loop corrections.

Finally, we can check that the condition (4.9) guarantees that the horizon does not shift significantly from the unperturbed Schwarzschild radius. In fact, for $\alpha = 2$ and neglecting the effect of $V_{{\rm q}M}$, we can write the metric function

$$V \simeq V_M + \epsilon\, \frac{\ell_{\rm p}^2\, G_{\rm N}\, M}{r^3} \tag{4.14}$$

where $\epsilon \sim \sqrt{\delta_J}$ now contains all the parameters (and angular dependence) shown in the first line of Eq. (4.12). The largest solution to $V = -1/2$ is then given by

$$r_{\rm H} \simeq 2\, G_{\rm N}\, M - \epsilon\, \ell_{\rm p} \tag{4.15}$$

which represents a negligible correction to $R_{\rm H} = 2\, G_{\rm N}\, M$. Given the fast fall-off of the metric correction in Eq. (4.12), one could interpret these perturbations as being "confined" about the horizon $R_{\rm H}$, like in the membrane approach [44] and in derivation of the entropy (4.1) based on conformal symmetry [45].

4.3 Hawking radiation

The Hawking evaporation has been studied with several methods since its discovery [31]. In particular, semiclassical approaches describe this effect as particles that tunnel out from within the event horizon on classically forbidden paths [46–55]. We will employ the WKB approach to compute corrections to the Hawking temperature for slowly rotating black holes described by the quantum-corrected Schwarzschild metric (2.21) with the metric modification in Eq. (4.12) that we showed can contribute to the Bekenstein–Hawking entropy.

We start by noting that, replacing the WKB ansatz

$$\Phi \simeq \exp\left[\frac{i}{\hbar}\, S(t, r, \theta, \phi)\right] \tag{4.16}$$

in the Klein–Gordon Eq. (2.1) at leading order in $\hbar$, yields the Hamilton-Jacobi equation

$$g^{\mu\nu}\, \partial_\mu S\, \partial_\nu S \simeq 0. \tag{4.17}$$

Solutions can be written in the form

$$S = -E\, t + \mathcal{W}(r) + \mathcal{J}(\theta, \phi) + K \tag{4.18}$$

where E represents the energy of the emitted boson and K is a complex constant that will be fixed later. The ratio E/M regulates the magnitude of the backreaction of the emission on the black hole, which can alter the thermal nature of the Hawking radiation [46]. We only consider large black holes with mass $M \gg m_{\rm p}$, hence this effect can be neglected for $E \ll M$.

Given the inverse of the metric (1.1), Eq. (4.17) can be written as

$$(1+2\,V)\left[\left(\frac{\partial \mathcal{W}}{\partial r}\right)^2+\frac{r^2}{w\,\sin\theta}\left(\frac{\partial \mathcal{J}}{\partial \phi}\right)^2\right] +\frac{1}{r^2}\left(\frac{\partial \mathcal{J}}{\partial \theta}\right)^2+\frac{4\,a\,G_{\rm N}\,M\,r}{w}\,E\,\frac{\partial \mathcal{J}}{\partial \phi}\simeq\frac{r^4}{w}E^2 \tag{4.19}$$

where we used the form (4.18) for S and defined

$$w\equiv r^4\,(1+2\,V)+4\,a^2\,G_{\rm N}^2\,M^2\,\sin^2\theta. \tag{4.20}$$

For Hawking particles in a quantum corrected Schwarzschild geometry, we can just consider purely radial trajectories [52], along which $\mathcal{J}$ is constant, and further approximate $w\simeq r^4\,(1+2\,V)$. In this case, Eq. (4.19) is solved by

$$\mathcal{W}_\pm\simeq\pm E\int^r\frac{\mathrm{d}\bar{r}}{1+2\,V} \tag{4.21}$$

with + (−) for outgoing (ingoing) particles.

Imaginary terms in the action S correspond to the Boltzmann factor for emission and absorption across the event horizon. Such terms can only arise due to the pole at $r=r_{\rm H}\simeq R_{\rm H}$, where $1+2\,V=0$, and from the imaginary part of K in Eq. (4.18), resulting in the probabilities

$$P_\pm\propto\exp\left[-\frac{2}{\hbar}\left(\Im\mathcal{W}_\pm+\Im K\right)\right] \tag{4.22}$$

where $\Im$ denotes the imaginary part. Assuming that ingoing particles necessarily cross the event horizon, that is $P_-\simeq 1$, one must set $\Im K=-\Im\mathcal{W}_-$. Since $\mathcal{W}_+=-\mathcal{W}_-$, the probability of a particle tunnelling out then reads

$$P_+\simeq\exp\left(-\frac{4}{\hbar}\,\Im\mathcal{W}_+\right). \tag{4.23}$$

The integral (4.21) around the pole at $r\simeq R_{\rm H}$ with the Feynman prescription for the propagator [52] yields

$$\Im\mathcal{W}_+\simeq\lim_{r\to R_{\rm H}}\frac{\pi\,E}{2\,\hbar\,V'(r)} \tag{4.24}$$

where $V'=\partial_r V$. Finally,

$$P_+\simeq\exp\left[-\frac{2\,\pi\,E}{V'(R_{\rm H})}\right] \tag{4.25}$$

which implies that the temperature must be given by

$$T_M\simeq\frac{\hbar}{2\,\pi}\,V'(R_{\rm H})=\frac{\hbar}{2\,\pi}\left[V'_{{\rm q}M}(R_{\rm H})+W'_{{\rm q}a}(R_{\rm H})\right]. \tag{4.26}$$

This expression with the metric function (2.21) and the contribution (4.12) with $\alpha=2$ for the case of Sect. 4.1 gives

$$T_M\simeq T_{\rm H}\,\frac{2}{\pi}\left[\mathrm{Si}\left(\frac{R_{\rm H}}{R_s}\right)-\sin\left(\frac{R_{\rm H}}{R_s}\right) -\frac{3\,m_{\rm p}^2}{4\,\pi\,M^2}\sum_{\ell=1}^{\ell_c}C_{\ell 0}\,\sqrt{2\,\ell+1}\,P_\ell(\cos\theta)\int_0^{R_{\rm H}/R_s}z^2\,\mathrm{d}z\;j_\ell(z)\right] \tag{4.27}$$

where $T_{\rm H}$ is the standard Hawking temperature (4.2), which is therefore recovered asymptotically for $M\gg m_{\rm p}$ and $R_{\rm s}\ll R_{\rm H}$.

Using the metric function in Eq. (4.14), one analogously finds

$$T_M\simeq T_{\rm H}\left(1-\frac{3\,\epsilon\,m_{\rm p}^2}{4\,M^2}\right). \tag{4.28}$$

On equating the two corrections of order $m_{\rm p}^2/M^2$, we obtain

$$\epsilon\simeq\frac{1}{\sqrt{\pi^3}}\sum_{\ell=1}^{\ell_c}C_{\ell 0}\,\frac{\sqrt{2\,\ell+1}\,\Gamma(\ell/2+3/2)\,P_\ell(\cos\theta)}{2^\ell\,\Gamma\,(\ell+3/2)\,\Gamma(\ell/2+5/2)}\,{}_1F_2 \times\left(\frac{\ell+3}{2},\ell+\frac{3}{2},\frac{\ell+5}{2},-\frac{1}{4}\right) \tag{4.29}$$

where we used Eq. (A.11) with $\alpha=2$.

5 Conclusions and outlook

In this work, the semiclassical metric function reproducing a Kerr geometry in the slow-rotation regime was shown to arise from suitable highly-excited coherent states, thus generalising previous results obtained for spherically symmetric geometries [16,20,37]. Quantum hair naturally emerges in this context, since the existence of the quantum coherent state does not allow for any possible IR and UV divergences in general.

An additional source of quantum hair was then identified in angular momentum modes that do not satisfy the conditions for giving rise to a classical rotating geometry described in Sect. 2.3. Such modes were further associated with the Bekenstein–Hawking entropy of Schwarzschild black holes and are therefore expected to play the role of precursors of the Hawking radiation, at least for very massive black holes. The Hawking evaporation was then studied with the Hamilton-Jacobi method, from which modes representing quantum hair in the geometry were related to metric corrections of the form that one expects from 1-loop quantum corrections in the weak-field approximation [32].

There are different directions along which the present results could be improved and developed. First of all, results regarding the Hawking evaporation can be straightforwardly generalised to massive bosons and fermions [53–55]. One could furthermore study other black hole solutions that can emerge from coherent quantum states and eventually attempt at a quantum description of black holes with arbitrary angular momentum [21].

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A Normalisations and conventions

We summarise here the convention we use in the main text. Projections on the spatial part of the normal modes (2.2) are defined as

$$\tilde{f}_{\ell m}(\omega) = \int_{-1}^{+1} \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\infty} r^2\, \mathrm{d}r \times j_\ell(\omega\, r) \left[Y_\ell^m(\theta,\varphi)\right]^* f(r,\theta,\varphi). \tag{A.1}$$

The orthonormality relations (2.4) then follow from the orthonormality of spherical Bessel functions,

$$\int_0^{\infty} r^2\, \mathrm{d}r\, j_\ell(\omega\, r)\, j_{\ell'}(\omega'\, r) = \frac{\pi}{2\,\omega^2}\, \delta(\omega-\omega')\, \delta_{\ell\ell'} \tag{A.2}$$

as well as the orthonormality of spherical harmonics,

$$\int_{-1}^{+1} \mathrm{d}\cos\theta \int_0^{2\pi} \mathrm{d}\varphi\, Y_\ell^m(\theta,\varphi) \left[Y_{\ell'}^{m'}(\theta,\varphi)\right]^* = \delta_{\ell\ell'}\, \delta_{mm'}. \tag{A.3}$$

The commutation relations (2.8) and (2.9) follow from the completeness relations

$$\frac{2}{\pi} \int_0^{\infty} \omega^2\, \mathrm{d}\omega\, j_\ell(\omega\, r)\, j_\ell(\omega\, r') = \frac{\delta(r-r')}{r^2} \tag{A.4}$$

and

$$\sum_\ell \sum_{m=-\ell}^{\ell} Y_\ell^m(\theta,\varphi) \left[Y_\ell^m(\theta',\varphi')\right]^* = \frac{\delta(\theta-\theta')}{\sin\theta}\, \delta(\varphi-\varphi'). \tag{A.5}$$

Other useful properties of spherical harmonics are given by

$$\left[Y_\ell^m\right]^* = (-1)^m\, Y_\ell^{-m} \tag{A.6}$$

and

$$P_\ell^{-m} = (-1)^m \frac{(\ell-m)!}{(\ell+m)!}\, P_\ell^m. \tag{A.7}$$

From

$$P_\ell^\ell = \frac{(-1)^\ell}{2^\ell\, \ell!} \sqrt{\frac{(2\,\ell+1)!}{4\,\pi}}\, (\sin\theta)^\ell \tag{A.8}$$

we then obtain

$$P_\ell^{-\ell} = \frac{1}{2^\ell\, \ell!\, (2\,\ell)!} \sqrt{\frac{(2\,\ell+1)!}{4\,\pi}}\, (\sin\theta)^\ell. \tag{A.9}$$

In all of the above expressions, the Kronecker delta is defined by $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. The Dirac delta is defined by

$$\int \mathrm{d}z\, \delta(z-z_0)\, f(z) = f(z_0) \tag{A.10}$$

where integration is assumed on the natural domain of the variable z.

Relevant integrals of the spherical Bessel functions are given by

$$\int_0^x z^\alpha\, \mathrm{d}z\, j_\ell(z) = \frac{\sqrt{\pi}}{2^{\ell+2}} \frac{\Gamma((1+\alpha+\ell)/2)}{\Gamma(3/2+\ell)\,\Gamma((3+\alpha+\ell)/2)} \times {}_1F_2((1+\alpha+\ell)/2, \ell +3/2, (3+\alpha+\ell)/2, -x^2/4) \tag{A.11}$$

where ${}_1F_2$ is the generalised hypergeometric function. In particular, for $\alpha = 1$, we have

$$\int_0^x z\, \mathrm{d}z\, j_\ell(z) = \frac{\sqrt{\pi}}{2^{\ell+2}} \frac{\Gamma(1+\ell/2)}{\Gamma(3/2+\ell)\,\Gamma(2+\ell/2)}\\ {}_1F_2(1+\ell/2, \ell+3/2, 2+\ell/2, -x^2/4). \tag{A.12}$$

B Angular momentum

The normal modes (2.2) are eigenfunctions of the angular momentum, that is

$$\hat{L}^2\, u_{\omega\ell m} = \hbar^2\, \ell\,(\ell+1)\, u_{\omega\ell m} \quad \text{and}\\ \hat{L}_z\, u_{\omega\ell m} = \hbar\, m\, u_{\omega\ell m}. \tag{B.1}$$

It then follows that

$$\hat{L}^2\, |1_{\ell m}(\omega)\rangle = \hbar^2\, \ell\,(\ell+1)\, |1_{\ell m}(\omega)\rangle \quad \text{and}\\ \hat{L}_z\, |1_{\ell m}(\omega)\rangle = \hbar\, m\, |1_{\ell m}(\omega)\rangle \tag{B.2}$$

where $|1_{\ell m}(\omega)\rangle = \hat{a}^\dagger_{\ell m}(\omega)\,|0\rangle$. We can also write the first relation as defining the operator

$$\sqrt{\hat{L}^2}\, |1_{\ell m}(\omega)\rangle = \hbar\, \sqrt{\ell\,(\ell+1)}\, |1_{\ell m}(\omega)\rangle\,. \tag{B.3}$$

Likewise, we have

$$\sqrt{\hat{L}^2}\, |n_{\ell m}(\omega)\rangle = \hbar\, \sqrt{\ell\,(\ell+1)}\, n_{\ell m}\, |n_{\ell m}(\omega)\rangle \quad \text{and}\\ \hat{L}_z\, |n_{\ell m}(\omega)\rangle = \hbar\, m\, n_{\ell m}\, |n_{\ell m}(\omega)\rangle \tag{B.4}$$

where $|n_{\ell m}(\omega)\rangle = (n!)^{-1/2} \left[\hat{a}^\dagger_{\ell m}(\omega)\right]^n |0\rangle$ (with $n = n_{\ell m}$ for brevity).

Let us consider a coherent state of fixed ω (which we omit for simplicity), ℓ and m,

$$\begin{aligned} |g_{\ell m}\rangle &= e^{-N_{\ell m}/2} \exp\left\{g_{\ell m}\, \hat{a}^\dagger_{\ell m}\right\} |0\rangle \\ &= e^{-N_{\ell m}/2} \sum_n \frac{\left(g_{\ell m}\, \hat{a}^\dagger_{\ell m}\right)^n}{n!} |0\rangle \\ &= e^{-N_{\ell m}/2} \sum_n \frac{g_{\ell m}^n}{\sqrt{n!}} |n_{\ell m}\rangle\,. \end{aligned} \tag{B.5}$$

From $\langle\, n_{\ell m} \mid n'_{\ell m}\,\rangle = \delta_{nn'}$, the normalisation

$$\langle\, g_{\ell m} \mid g_{\ell m}\,\rangle = e^{-N_{\ell m}} \sum_n \frac{g_{\ell m}^{2\,n}}{n!} = 1 \tag{B.6}$$

implies $N_{\ell m} = g_{\ell m}^2$. From Eq. (B.4), we then find

$$\begin{aligned} \langle g_{\ell m}|\, \sqrt{\hat{L}^2}\, |g_{\ell m}\rangle &= e^{-g_{\ell m}^2} \sum_{n,s} \frac{g_{\ell m}^s}{\sqrt{s!}} \frac{g_{\ell m}^n}{\sqrt{n!}} \langle s_{\ell m}|\, \sqrt{\hat{L}^2}\, |n_{\ell m}\rangle \\ &= e^{-g_{\ell m}^2} \sum_{n_{\ell m}} \frac{g_{\ell m}^{2\,n_{\ell m}}}{n_{\ell m}!} \hbar\, \sqrt{\ell\,(\ell+1)}\, n_{\ell m} \\ &= e^{-g_{\ell m}^2}\, \hbar\, \sqrt{\ell\,(\ell+1)} \sum_{n_{\ell m}} \frac{g_{\ell m}^{2\,n_{\ell m}}}{(n_{\ell m}-1)!} \\ &= \hbar\, \sqrt{\ell\,(\ell+1)}\, g_{\ell m}^2\, e^{-g_{\ell m}^2} \sum_n \frac{g_{\ell m}^{2\,n}}{n!} \\ &= \hbar\, \sqrt{\ell\,(\ell+1)}\, N_{\ell m} \end{aligned} \tag{B.7}$$

which is Eq. (2.23) with $N_{\ell m} = g_{\ell m}^2(\omega)$. Likewise,

$$\langle g_{\ell m}|\, \hat{L}_z\, |g_{\ell m}\rangle = \hbar\, m\, N_{\ell m} \tag{B.8}$$

which is Eq. (2.24).

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