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On the second-order regularity of solutions to widely singular or degenerate elliptic equations

Pasquale Ambrosio[1] · **Antonio Giuseppe Grimaldi**[2] · **Antonia Passarelli di Napoli**[3]

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Abstract

We consider local weak solutions to PDEs of the type

$$-\operatorname{div}\left((|Du|-\lambda)_{+}^{p-1}\frac{Du}{|Du|}\right)=f \quad \text{in } \Omega,$$

where $1<p<\infty$, Ω is an open subset of $\mathbb{R}^n$ for $n\geq 2$, λ is a positive constant and $(\,\cdot\,)_+$ stands for the positive part. Equations of this form are widely degenerate for $p\geq 2$ and widely singular for $1<p<2$. We establish higher differentiability results for a suitable nonlinear function of the gradient *Du* of the local weak solutions, assuming that *f* belongs to the local Besov space $B^{(p-2)/p}_{p',1,loc}(\Omega)$ when $p>2$, and that $f\in L^{\frac{np}{n(p-1)+2-p}}_{loc}(\Omega)$ if $1<p\leq 2$. The conditions on the datum *f* are essentially sharp. As a consequence, we obtain the local higher integrability of *Du* under the same minimal assumptions on *f*. For $\lambda=0$, our results give back those contained in Clop et al. (Bull Math Sci 13(12):2350008, 2023) and Irving and Koch (Adv Nonlinear Anal 12(1):20230110, 2023).

Keywords Degenerate elliptic equations · Singular elliptic equations · Sharp second-order regularity

Mathematics Subject Classification 35J70 · 35J75 · 35J92 · 49K20

✉ Pasquale Ambrosio
pasquale.ambrosio@unibo.it

Antonio Giuseppe Grimaldi
antoniogiuseppe.grimaldi@collaboratore.uniparthenope.it

Antonia Passarelli di Napoli
antpassa@unina.it

1 Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy

2 Dipartimento di Ingegneria, Università degli Studi di Napoli "Parthenope", Centro Direzionale Isola C4, 80143 Napoli, Italy

3 Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Via Cintia, 80126 Napoli, Italy

1 Introduction

In this paper, we are interested in the local weak solutions to the strongly singular or degenerate elliptic equation

$$-\operatorname{div}\left((|Du|-\lambda)_+^{p-1}\frac{Du}{|Du|}\right)=f\quad\text{in }\Omega,\tag{1.1}$$

where $1<p<\infty$, Ω is an open subset of $\mathbb{R}^n$ $(n\geq 2)$, $\lambda>0$ is a fixed parameter and $(\,\cdot\,)_+$ stands for the positive part. The peculiarity of equation (1.1) is that it is uniformly elliptic only outside the ball centered at the origin with radius λ, where its principal part behaves asymptotically as the classical p-Laplace operator. Therefore, the study of such an equation fits into the wider class of the asymptotically regular problems that have been extensively studied starting from the pioneering paper [9] (see also [13–17, 21–23, 25, 29, 33, 35] for extensions to various other settings).

As our main result, here we establish the local Sobolev regularity of a nonlinear function of the gradient Du of the weak solutions to equation (1.1), by assuming that the datum f belongs to a suitable local Besov space when $p>2$ (see Theorem 1.1 below) and that $f\in L_{loc}^{\frac{np}{n(p-1)+2-p}}(\Omega)$ if $1<p\leq 2$ (see Theorem 1.4). These results, in turn, imply the local higher integrability of Du under the same hypotheses on the function f (cf. Corollary 1.5).

Before specifying in detail the assumption on f in the case $p>2$, we wish to discuss some results already available in the literature. A common aspect of nonlinear elliptic problems with growth rate $p\geq 2$ is that the higher differentiability is proven for a nonlinear function of the gradient that takes into account the growth of the structure function of the equation. Indeed, already for the p-Poisson equation (which is obtained from (1.1) by setting $\lambda=0$), the higher differentiability is established for the function

$$V_p(Du):=\,|Du|^{\frac{p-2}{2}}Du\,,$$

as can be seen in many papers, starting from the pioneering one by Uhlenbeck [39]. In case of widely degenerate problems, this phenomenon persists and higher differentiability results hold true for the function

$$H_{\frac{p}{2}}(Du):=\,(|Du|-\lambda)_+^{p/2}\frac{Du}{|Du|}\tag{1.2}$$

(see [2] and [7, Theorem 4.2]). However, this function does not provide any information about the second-order regularity of the solutions in the set where the equation becomes degenerate. Actually, since every λ-Lipschitz function is a solution of the homogeneous elliptic equation

$$\operatorname{div}\left((|Du|-\lambda)_+^{p-1}\frac{Du}{|Du|}\right)=0\,,$$

no more than Lipschitz regularity can be expected for the solutions: in this regard, see [4, 6, 7, 27, 34].

In this paper, the nonlinear function of the gradient that gains higher weak differentiability needs to be chosen with two main features: on the one hand, as in the case of the function in (1.2), it has to vanish in the region $\{|Du| \leq \lambda\}$; on the other hand, it has to compensate for the loss of uniform ellipticity of equation (1.1) as $|Du| \to \lambda^+$. Indeed, defining the function $H_{p-1} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$H_{p-1}(\xi) := \begin{cases} (|\xi| - \lambda)_+^{p-1} \frac{\xi}{|\xi|} & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0, \end{cases}$$

and denoting, in accordance with [18, 32], the ellipticity ratio of equation (1.1) by

$$\mathcal{R}(\xi) := \frac{\text{the highest eigenvalue of } D_\xi H_{p-1}(\xi)}{\text{the lowest eigenvalue of } D_\xi H_{p-1}(\xi)} \approx \frac{|\xi|}{|\xi| - \lambda}, \qquad \text{for } |\xi| > \lambda, \tag{1.3}$$

one sees that this quotient clearly blows up as $|\xi| \to \lambda^+$, unless $\lambda = 0$ (cf. Lemma 4.1 below).

To provide context for the main aim of this paper, we would like to point out the paper [28], where Irving and Koch obtained some differentiability results for relaxed minimizers of vectorial convex functionals with non-standard growth of the type

$$\int_\Omega [F(x, Du) - f \cdot u]\, dx.$$

In particular, for the weak solutions $u \in W^{1,p}(\Omega)$ of the p-Poisson equation, they proved that

$$V_p(Du) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^n) \tag{1.4}$$

if the function f belongs to the Besov space $B^{\frac{p-2}{p}}_{p',1}(\Omega)$, with $p > 2$. Their assumption on f is essentially sharp, in the sense that the above result is false if

$$f \in B^s_{p',1}(\Omega) \quad \text{with} \quad s < (p-2)/p\,.$$

Indeed, Brasco and Santambrogio [8, Section 5] showed with an explicit example that condition (1.4) may not hold if f belongs to a fractional Sobolev space $W^{\sigma,p'}_{loc}(\mathbb{R}^n)$ with $0 < \sigma < (p-2)/p$, which is continuously embedded into $B^s_{p',1,loc}(\mathbb{R}^n)$ whenever $s \in (0, \sigma)$ (see Lemma 3.12 below).

This work is in the spirit of the ones mentioned above. Indeed, our main aim here is to find the assumptions to impose on the datum f *in the scale of local Besov or Lebesgue spaces* to obtain the $W^{1,2}$-regularity of a nonlinear function of the gradient of weak solutions to the widely degenerate or singular equation (1.1). In order to state our main results, we introduce the function

$$\mathcal{G}_\lambda(t) := \int_0^t \frac{\omega^{\frac{p}{2}+\frac{1}{p-1}}}{(\omega+\lambda)^{1+\frac{1}{p-1}}}\, d\omega \qquad \text{for } t \geq 0. \tag{1.5}$$

Moreover, for $\xi \in \mathbb{R}^n$ we define the following vector-valued function:

$$\mathcal{V}_\lambda(\xi) := \begin{cases} \mathcal{G}_\lambda((|\xi|-\lambda)_+)\, \frac{\xi}{|\xi|} & \text{if } |\xi| > \lambda, \\ 0 & \text{if } |\xi| \leq \lambda. \end{cases} \tag{1.6}$$

Notice that, for $\lambda = 0$, we have

$$\mathcal{V}_0(\xi) = \frac{2}{p}\, V_p(\xi) := \frac{2}{p}\, |\xi|^{\frac{p-2}{2}} \xi. \tag{1.7}$$

At this point, our first result reads as follows.

Theorem 1.1 *Let $n \geq 2$, $p > 2$, $\lambda \geq 0$ and $f \in B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)$, where $p' = p/(p-1)$ is the conjugate exponent of p. Moreover, let $u \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of equation* (1.1). *Then*

$$\mathcal{V}_\lambda(Du) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^n)\,.$$

Furthermore, for every pair of concentric balls $B_r \subset B_R \Subset \Omega$ we have

$$\begin{aligned} &\int_{B_{r/4}} |D\mathcal{V}_\lambda(Du)|^2\, dx \\ &\leq \left(C + \frac{C}{r^2}\right) \left[1 + \lambda^p + \|Du\|^p_{L^p(B_R)} + \|f\|^{p'}_{L^{p'}(B_R)}\right] + C\, \|f\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_R)} \end{aligned} \tag{1.8}$$

for a positive constant C depending only on n, p and R.

Remark 1.2 Looking at (1.7), one can easily understand that Theorem 1.1 extends the result proved in [28] to a class of widely degenerate elliptic equations with standard growth, under a sharp assumption on the order of differentiation of *f*.

The proof of the previous theorem is achieved combining an *a priori* estimate for the solution of a suitable approximating problem with a comparison estimate. In establishing the *a priori* estimate, we first need to identify a suitable function of the gradient that vanishes in the degeneracy set, for which the second-order *a priori* estimate holds true. Next, we need to estimate the right-hand side in terms of the derivatives of such function, without assuming any Sobolev regularity for the datum *f*. This is done by virtue of the following implication

$$|g|^{\frac{p-2}{2}} g \in W^{1,2}_{loc}(\Omega) \;\Rightarrow\; g \in B^{\frac{2}{p}}_{p,\infty,loc}(\Omega), \tag{1.9}$$

which allows us to use the duality of Besov spaces, provided that one imposes a suitable Besov regularity on the right-hand side *f*. This approach has been inspired by [8], where the authors use for the first time a duality-based inequality in the setting of fractional Sobolev spaces, but limiting themselves to the p-Poisson equation. Finally, we use a comparison argument to transfer the higher differentiability of the approximating solutions to the solution of equation (1.1).

Remark 1.3 We do not know whether Theorem 1.1 is still true when we weaken the regularity of *f* to $B^{(p-2)/p}_{p',q,loc}(\Omega)$ for some $q > 1$, also in the case $\lambda = 0$. In this regard we point out the paper [20], where the authors apparently prove that, for the p-Poisson equation, assertion (1.4) holds under the weaker assumption $f \in W^{(p-2)/p,p'}_{loc}(\Omega)$. However, we believe that they made a mistake at the beginning of [20, page 373] in applying [20, Formula (4)].

We would like to mention that in [10, 11] the authors proved that the assumption $f \in L^2$ is sufficient to prove the $W^{1,2}$-regularity of $|Du|^{p-2}\,Du$, which is of course a different function of the gradient. At the moment, we do not know whether the analogous result can be obtained for the solutions of widely degenerate equations.

Now we turn our attention to the sub-quadratic case, i.e. when $1 < p \le 2$. It is well known that, already for the less degenerate case of the p-Poisson equation, the higher differentiability of the solutions can be achieved without assuming any differentiability on the datum *f*. This different behaviour can be easily explained by observing that, if $1 < p \le 2$,

$$|g|^{\frac{p-2}{2}} g \in W^{1,2}_{loc}(\Omega) \;\Rightarrow\; g \in W^{1,p}_{loc}(\Omega),$$

which of course does not hold true for $p > 2$ (compare with (1.9)). Therefore, the right-hand side can be estimated without assuming any differentiability for *f* (neither of integer nor of fractional order), but only a suitable degree of integrability. The sharp assumption on *f* in the scale of Lebesgue spaces has been recently found in [12].

Here, we prove that a result analogous to [12, Theorem 1.1] holds true also when dealing with solutions of widely singular equations. More precisely, we establish the following result.

Theorem 1.4 *Let* $n \ge 2$, $1 < p \le 2$, $\lambda \ge 0$ *and* $f \in L^{\frac{np}{n(p-1)+2-p}}_{loc}(\Omega)$. *Moreover, let* $u \in W^{1,p}_{loc}(\Omega)$ *be a local weak solution of equation* (1.1). *Then*

$$\mathcal{V}_\lambda(Du) \;\in\; W^{1,2}_{loc}(\Omega, \mathbb{R}^n)\,.$$

Furthermore, for every pair of concentric balls $B_r \subset B_R \Subset \Omega$ *we have*

$$\begin{aligned}\int_{B_{r/4}} |D\mathcal{V}_\lambda(Du)|^2\,dx \\ \le \frac{C}{r^2}\left[1 + \lambda^p + \|Du\|^p_{L^p(B_R)} + \|f\|^{p'}_{L^{\frac{np}{n(p-1)+2-p}}(B_R)}\right] + C\,\|f\|^{p'}_{L^{\frac{np}{n(p-1)+2-p}}(B_R)}\end{aligned} \tag{1.10}$$

for a positive constant C depending only on n, p and R.

As an easy consequence of the higher differentiability results in Theorems 1.1 and 1.4, since the gradient of the solution is bounded in the region $\{|Du| \leq \lambda\}$ and $\mathcal{G}_\lambda(t) \approx t^{p/2}$ for large values of *t* (see Lemma 2.8 below), we are able to establish the following higher integrability result for the gradient of local weak solutions of (1.1):

Corollary 1.5 *Under the assumptions of Theorem 1.1 or Theorem 1.4, we have*

$$Du \in L^q_{loc}(\Omega, \mathbb{R}^n),$$

where

$$q = \begin{cases} any value in [1, \infty) & if \;\; n = 2, \\ \frac{np}{n-2} & if \;\; n \geq 3. \end{cases}$$

Before describing the structure of this paper, we observe that, if we interpret the inverse of the ratio in (1.3) as a weight, the ellipticity bounds of the matrix $D_\xi H_{p-1}(Du)$ can be expressed as

$$\frac{c(p)}{\mathcal{R}(Du)} \, (|Du| - \lambda)_+^{p-2} \, |\zeta|^2 \leq \langle D_\xi H_{p-1}(Du) \, \zeta, \zeta \rangle \leq \, c(p) \, (|Du| - \lambda)_+^{p-2} \, |\zeta|^2,$$

for every $\zeta \in \mathbb{R}^n$ (see Lemma 4.1 with $\varepsilon = 0$). Known results for solutions to non-uniformly elliptic problems rely on integrability properties of $\mathcal{R}(Du)$, which, however, are not available in the present setting. Our choice of the function $\mathcal{V}_\lambda$ implicitly incorporates the aforementioned weight.

The paper is organized as follows. Section 2 is devoted to the preliminaries: after a list of some classic notations and some essentials estimates, we recall the basic properties of the difference quotients of Sobolev functions. Section 3 is entirely devoted to the definitions and properties of Besov spaces that will be useful to prove our results. In Sect. 4, we establish some *a priori* estimates that will be needed to demonstrate Theorems 1.1 and 1.4, whose proofs are contained in Sects. 5 and 6, respectively.

2 Preliminaries

2.1 Notation and essential definitions

In this paper we shall denote by *C* or *c* a general positive constant that may vary on different occasions. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norm we use on $\mathbb{R}^k$, $k \in \mathbb{N}$, will be the standard Euclidean one and it will be denoted by $|\cdot|$. In particular, for the vectors $\xi, \eta \in \mathbb{R}^k$, we write $\langle \xi, \eta \rangle$ for the usual inner product and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm.

In what follows, $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ will denote the *n*-dimensional open ball centered at x_0 with radius *r*. We shall sometimes omit the dependence on the center when all balls occurring in a proof are concentric. Unless otherwise stated, different balls in the same context will have the same center.

For further needs, we now define the auxiliary function $H_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ by

$$H_\gamma(\xi) := \begin{cases} (|\xi| - \lambda)_+^\gamma \frac{\xi}{|\xi|} & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0, \end{cases} \tag{2.1}$$

where $\lambda \geq 0$ and $\gamma > 0$ are parameters. We conclude this first part of the preliminaries by recalling the following definition.

Definition 2.1 Let $\lambda \geq 0$. A function $u \in W^{1,p}_{loc}(\Omega)$ is a *local weak solution* of equation (1.1) if and only if, for any test function $\varphi \in C_0^\infty(\Omega)$, the following integral identity holds:

$$\int_\Omega \langle H_{p-1}(Du), D\varphi \rangle \, dx \; = \; \int_\Omega f \varphi \, dx.$$

2.2 Algebraic inequalities

In this section, we gather some relevant algebraic inequalities that will be needed later on. The first result follows from an elementary computation.

Lemma 2.2 *For $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$, we have*

$$\left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right| \leq \frac{2}{|\eta|} \, |\xi - \eta|.$$

We now recall the following estimate, whose proof can be found in [30, chapter 12].

Lemma 2.3 *Let $p \in (2, \infty)$ and $k \in \mathbb{N}$. Then, for every $\xi, \eta \in \mathbb{R}^k$ we get*

$$|\xi - \eta|^p \leq \; C \left| |\xi|^{\frac{p-2}{2}} \xi - |\eta|^{\frac{p-2}{2}} \eta \right|^2$$

for a constant $C \equiv C(p) > 0$.

Combining [1, Lemma 2.2] with [25, Formula (2.4)], we obtain the following

Lemma 2.4 *Let $1 < p < \infty$. There exists a constant $c \equiv c(n, p) > 0$ such that*

$$c^{-1} (|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \leq \frac{\left| |\xi|^{\frac{p-2}{2}} \xi - |\eta|^{\frac{p-2}{2}} \eta \right|^2}{|\xi - \eta|^2} \leq c \, (|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}$$

for every $\xi, \eta \in \mathbb{R}^n$ with $\xi \neq \eta$.

For the function H_{p-1} defined by (2.1) with $\gamma = p - 1$, we record the following estimates, which can be obtained by suitably modifying the proofs of [7, Lemma 4.1] (for the case $p \geq 2$), [2, Lemma 2.5] (for the case $1 < p < 2$) and [4, Lemma 2.8].

Lemma 2.5 *Let $p \in (1, \infty)$ and $\lambda \geq 0$. Then, there exists a constant $c \equiv c(n, p) > 0$ such that*

$$\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle \geq c\, |H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta)|^2, \tag{2.2}$$

for every $\xi, \eta \in \mathbb{R}^n$. Moreover, if $|\eta| > \lambda > 0$ we have

$$\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle \geq \frac{\min\{1, p-1\}}{2^{p+1}} \frac{(|\eta| - \lambda)^p}{|\eta|\,(|\xi| + |\eta|)} |\xi - \eta|^2.$$

The next result concerns the function $\mathcal{G}_\lambda$ defined by (1.5).

Lemma 2.6 *Let $p \in (1, \infty)$ and $\lambda \geq 0$. Then*

$$\mathcal{G}_\lambda(t) \leq \frac{2}{p}\, t^{\frac{p}{2}} \left(\frac{t}{t + \lambda} \right)^{\frac{p}{p-1}}$$

for every $t > 0$.

Proof Since the function

$$K(\omega) := \left(\frac{\omega}{\omega + \lambda} \right)^{\frac{p}{p-1}}, \qquad \omega > 0,$$

is non-decreasing, for every $t > 0$ we have

$$\mathcal{G}_\lambda(t) = \int_0^t K(\omega)\, \omega^{\frac{p}{2} - 1}\, d\omega \leq \left(\frac{t}{t + \lambda} \right)^{\frac{p}{p-1}} \int_0^t \omega^{\frac{p}{2} - 1}\, d\omega = \frac{2}{p}\, t^{\frac{p}{2}} \left(\frac{t}{t + \lambda} \right)^{\frac{p}{p-1}}.$$

□

The next lemma relates the function $\mathcal{V}_\lambda(\xi)$ with $H_{p-1}(\xi)$.

Lemma 2.7 *Let $p \in (1, \infty)$ and $\lambda \geq 0$. Then, there exists a constant $C \equiv C(n, p) > 0$ such that*

$$|\mathcal{V}_\lambda(\xi) - \mathcal{V}_\lambda(\eta)|^2 \leq C\, \langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle \tag{2.3}$$

for every $\xi, \eta \in \mathbb{R}^n$.

Proof For $\lambda = 0$, estimate (2.3) boils down to (2.2). Therefore, from now on we shall assume that $\lambda > 0$. We first note that inequality (2.3) is trivially satisfied when $|\xi|, |\eta| \leq \lambda$. If $|\eta| \leq \lambda < |\xi|$, using the definitions (1.6), (1.5), (2.1) and Lemma 2.5, we obtain

$$|\mathcal{V}_\lambda(\xi) - \mathcal{V}_\lambda(\eta)|^2 = [\mathcal{G}_\lambda(|\xi| - \lambda)]^2 \leq \left(\int_0^{|\xi|-\lambda} \omega^{\frac{p}{2}-1}\, d\omega \right)^2 = \frac{4}{p^2}\,(|\xi| - \lambda)^p$$
$$= \frac{4}{p^2}\,|H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta)|^2 \leq c(n,p)\,\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle.$$

Now let $|\xi|, |\eta| > \lambda$. Without loss of generality, we may assume that $|\eta| \geq |\xi| > \lambda$. This implies

$$|\eta|^2 = \frac{|\eta|\,(|\eta| + |\eta|)}{2} \geq \frac{|\eta|\,(|\xi| + |\eta|)}{2}. \tag{2.4}$$

Moreover, we have

$$|\mathcal{V}_\lambda(\xi) - \mathcal{V}_\lambda(\eta)| = \left| \mathcal{V}_\lambda(\xi) - \mathcal{G}_\lambda(|\eta| - \lambda)\,\frac{\xi}{|\xi|} + \mathcal{G}_\lambda(|\eta| - \lambda)\,\frac{\xi}{|\xi|} - \mathcal{V}_\lambda(\eta) \right|$$
$$\leq |\mathcal{G}_\lambda(|\xi| - \lambda) - \mathcal{G}_\lambda(|\eta| - \lambda)| + \mathcal{G}_\lambda(|\eta| - \lambda) \left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right|$$
$$\leq \int_{|\xi|-\lambda}^{|\eta|-\lambda} \frac{\omega^{\frac{p}{2}+\frac{1}{p-1}}}{(\omega + \lambda)^{1+\frac{1}{p-1}}}\, d\omega + \frac{2}{|\eta|}\,|\xi - \eta| \int_0^{|\eta|-\lambda} \frac{\omega^{\frac{p}{2}+\frac{1}{p-1}}}{(\omega + \lambda)^{1+\frac{1}{p-1}}}\, d\omega$$
$$\leq \int_{|\xi|-\lambda}^{|\eta|-\lambda} \omega^{\frac{p}{2}-1}\, d\omega + \frac{2}{|\eta|}\,|\xi - \eta| \int_0^{|\eta|-\lambda} \omega^{\frac{p}{2}-1}\, d\omega$$
$$= \frac{2}{p}\, \big| |H_{\frac{p}{2}}(\eta)| - |H_{\frac{p}{2}}(\xi)| \big| + 4\,\frac{(|\eta| - \lambda)^{\frac{p}{2}}}{p\,|\eta|}\,|\xi - \eta|$$
$$\leq \frac{2}{p}\,|H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta)| + 4\,\frac{(|\eta| - \lambda)^{\frac{p}{2}}}{p\,|\eta|}\,|\xi - \eta|,$$

where, in the third line, we have used Lemma 2.2 and the fact that $\mathcal{G}_\lambda$ is an increasing function. Now, applying Young's inequality, estimate (2.4) and Lemma 2.5, we obtain

$$|\mathcal{V}_\lambda(\xi) - \mathcal{V}_\lambda(\eta)|^2 \leq \frac{8}{p^2}\,|H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta)|^2 + 32\,\frac{(|\eta| - \lambda)^p}{p^2\,|\eta|^2}\,|\xi - \eta|^2$$
$$\leq \frac{8}{p^2}\,|H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta)|^2 + \frac{64}{p^2}\,\frac{(|\eta| - \lambda)^p}{|\eta|\,(|\xi| + |\eta|)}\,|\xi - \eta|^2$$
$$\leq C(n,p)\,\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle.$$

This completes the proof. □

We conclude this section with the proof of the following lemma, on which the conclusion of Corollary 1.5 is based.

Lemma 2.8 *Let $p \in (1, \infty)$ and $\lambda > 0$. Then, there exist two positive constants $c \equiv c(p)$ and $\tilde{c} \equiv \tilde{c}(p)$ such that*

$$c\,(t+\lambda)^{p/2} - \tilde{c}\,\lambda^{p/2} \le \mathcal{G}_\lambda(t) \le \frac{2}{p}\,t^{p/2}$$

for all $t \geq 0$.

Proof From the very definition of the function $\mathcal{G}_\lambda$, we easily get the upper bound

$$\mathcal{G}_\lambda(t) \le \int_0^t \omega^{\frac{p-2}{2}}\,d\omega = \frac{2}{p}\,t^{p/2} \qquad \text{for all } t \ge 0.$$

For the derivation of the lower bound, we write the integral that defines $\mathcal{G}_\lambda(t)$ as follows:

$$\int_0^t \frac{\omega^{\frac{p}{2}+\frac{1}{p-1}}}{(\omega+\lambda)^{1+\frac{1}{p-1}}}\,d\omega \;=\; \int_0^t \frac{(\omega+\lambda-\lambda)^{\frac{p}{2}+\frac{1}{p-1}}}{(\omega+\lambda)^{1+\frac{1}{p-1}}}\,d\omega\,. \tag{2.5}$$

Now we recall that for every $\gamma > 0$ it holds

$$2^{-\gamma}a^\gamma - b^\gamma \le (a-b)^\gamma, \quad \forall\, a \ge b \ge 0. \tag{2.6}$$

Using inequality (2.6) with $\gamma = \frac{p}{2} + \frac{1}{p-1}$, $a = \omega + \lambda$ and $b = \lambda$, we find that

$$\begin{aligned}\int_0^t \frac{(\omega+\lambda-\lambda)^{\frac{p}{2}+\frac{1}{p-1}}}{(\omega+\lambda)^{1+\frac{1}{p-1}}}\,d\omega &\ge 2^{\frac{1}{1-p}-\frac{p}{2}}\int_0^t (\omega+\lambda)^{\frac{p}{2}-1}\,d\omega \;-\; \lambda^{\frac{p}{2}+\frac{1}{p-1}}\int_0^t \frac{1}{(\omega+\lambda)^{1+\frac{1}{p-1}}}\,d\omega\\ &= c(p)\,(t+\lambda)^{\frac{p}{2}} - c(p)\,\lambda^{\frac{p}{2}} + (p-1)\,\lambda^{\frac{p}{2}+\frac{1}{p-1}}\left[(\omega+\lambda)^{\frac{1}{1-p}}\right]_0^t\\ &\ge c(p)\,(t+\lambda)^{\frac{p}{2}} - c(p)\,\lambda^{\frac{p}{2}} - (p-1)\,\lambda^{\frac{p}{2}}\,.\end{aligned} \tag{2.7}$$

Joining (2.5) and (2.7), we obtain the asserted lower bound. ☐

2.3 Difference quotients

We recall here the definition and some elementary properties of the difference quotients that will be useful in the following (see, for example, [26]).

Definition 2.9 For every vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^k$ the *finite difference operator* in the direction x_j is defined by

$$\tau_{j,h}F(x) = F(x + he_j) - F(x),$$

where $h \in \mathbb{R}$, e_j is the unit vector in the direction x_j and $j \in \{1, \ldots, n\}$.

The *difference quotient* of F with respect to x_j is defined for $h \in \mathbb{R} \setminus \{0\}$ by

$$\Delta_{j,h}F(x) = \frac{\tau_{j,h}F(x)}{h}\,.$$

When no confusion can arise, we shall omit the index j and simply write τ_h or Δ_h instead of $\tau_{j,h}$ or $\Delta_{j,h}$, respectively.

Proposition 2.10 *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $F \in W^{1,q}(\Omega)$, with $q \geq 1$. Moreover, let $G : \Omega \to \mathbb{R}$ be a measurable function and consider the set*

$$\Omega_{|h|} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > |h|\}.$$

Then:

(i) $\Delta_h F \in W^{1,q}\left(\Omega_{|h|}\right)$ *and* $\partial_i(\Delta_h F) = \Delta_h(\partial_i F)$ *for every* $i \in \{1, \dots, n\}$.

(ii) *If at least one of the functions F or G has support contained in $\Omega_{|h|}$, then*

$$\int_\Omega F\, \Delta_h G\, dx = -\int_\Omega G\, \Delta_{-h} F\, dx.$$

(iii) *We have*

$$\Delta_h(FG)(x) = F(x + he_j)\Delta_h G(x) + G(x)\Delta_h F(x).$$

The next result about the finite difference operator is a kind of integral version of the Lagrange Theorem and its proof can be found in [26, Lemma 8.1].

Lemma 2.11 *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < q < +\infty$ and $F \in L^q(B_R, \mathbb{R}^k)$ is such that $DF \in L^q(B_R, \mathbb{R}^{k\times n})$, then*

$$\int_{B_\rho} |\tau_h F(x)|^q\, dx \leq c^q(n)\, |h|^q \int_{B_R} |DF(x)|^q\, dx.$$

Moreover

$$\int_{B_\rho} |F(x + he_j)|^q\, dx \leq \int_{B_R} |F(x)|^q\, dx.$$

Finally, we recall the following fundamental result, whose proof can be found in [26, Lemma 8.2]:

Lemma 2.12 *Let $F : \mathbb{R}^n \to \mathbb{R}^k$, $F \in L^q(B_R, \mathbb{R}^k)$ with $1 < q < +\infty$. Suppose that there exist $\rho \in (0, R)$ and a constant $M > 0$ such that*

$$\sum_{j=1}^n \int_{B_\rho} |\tau_{j,h} F(x)|^q\, dx \leq M^q\, |h|^q$$

for every $h \in \mathbb{R}$ with $|h| < \frac{R-\rho}{2}$. Then $F \in W^{1,q}(B_\rho, \mathbb{R}^k)$. Moreover

$$\|DF\|_{L^q(B_\rho)} \leq M$$

and

$$\Delta_{j,h} F \to \partial_j F \qquad in\ L^q_{loc}(B_R, \mathbb{R}^k) \ \ as\ h \to 0,$$

for each $j \in \{1, \dots, n\}$.

3 Besov spaces

Here we recall some essential facts on the Besov spaces involved in this paper (see, for example, [37] and [38]).

We denote by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ the Schwartz space and the space of tempered distributions on $\mathbb{R}^n$, respectively. If $v \in \mathcal{S}(\mathbb{R}^n)$, then

$$\hat{v}(\xi) = (\mathcal{F}v)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\,\langle x,\xi\rangle}\, v(x)\, dx, \qquad \xi \in \mathbb{R}^n, \tag{3.1}$$

denotes the Fourier transform of *v*. As usual, $\mathcal{F}^{-1}v$ and $v^\vee$ stand for the inverse Fourier transform, given by the right-hand side of (3.1) with *i* in place of $-i$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way.

Now, let $\Gamma(\mathbb{R}^n)$ be the collection of all sequences $\varphi = \{\varphi_j\}_{j=0}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$\begin{cases} \ \ \operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\} \\ \operatorname{supp}\varphi_j \subset \{x \in \mathbb{R}^n : 2^{j-1} \le |x| \le 2^{j+1}\} \quad \text{if } j \in \mathbb{N}, \end{cases}$$

for every multi-index β there exists a positive number c_β such that

$$2^{j|\beta|}\, |D^\beta \varphi_j(x)| \le c_\beta\,, \qquad \forall\, j \in \mathbb{N}_0,\ \forall\, x \in \mathbb{R}^n$$

and

$$\sum_{j=0}^\infty \varphi_j(x) = 1\,, \qquad \forall\, x \in \mathbb{R}^n.$$

Then, it is well known that $\Gamma(\mathbb{R}^n)$ is not empty (see [37][Section 2, Remark 1]). Moreover, if $\{\varphi_j\}_{j=0}^\infty \in \Gamma(\mathbb{R}^n)$, the entire analytic functions $(\varphi_j\, \hat{v})^\vee(x)$ make sense pointwise in $\mathbb{R}^n$ for any $v \in \mathcal{S}'(\mathbb{R}^n)$. Therefore, the following definition makes sense:

Definition 3.1 Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ and $\varphi = \{\varphi_j\}_{j=0}^\infty \in \Gamma(\mathbb{R}^n)$. We define the *Besov space* $B^s_{p,q}(\mathbb{R}^n)$ as the set of all $v \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|v\|_{B^s_{p,q}(\mathbb{R}^n)} := \left(\sum_{j=0}^\infty 2^{jsq}\, \|(\varphi_j\, \hat{v})^\vee\|^q_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{q}} < +\infty \qquad \text{if } q < \infty, \tag{3.2}$$

and

$$\|v\|_{B^s_{p,q}(\mathbb{R}^n)} := \sup_{j\,\in\,\mathbb{N}_0} 2^{js}\,\|(\varphi_j\,\hat{v})^\vee\|_{L^p(\mathbb{R}^n)} < +\infty \qquad \text{if } q = \infty. \tag{3.3}$$

Remark 3.2 The space $B^s_{p,q}(\mathbb{R}^n)$ defined above is a Banach space with respect to the norm $\|\cdot\|_{B^s_{p,q}(\mathbb{R}^n)}$. Obviously, this norm depends on the chosen sequence $\varphi \in \Gamma(\mathbb{R}^n)$, but this is not the case for the spaces $B^s_{p,q}(\mathbb{R}^n)$ themselves, in the sense that two different choices for the sequence φ give rise to equivalent norms (see [37, Sections 2.3.2 and 2.3.3]). This justifies our omission of the dependence on φ in the left-hand side of (3.2)–(3.3) and in the sequel.

The norms of the *classical Besov spaces* $B^s_{p,q}(\mathbb{R}^n)$ with $s \in (0,1)$, $1 \le p < \infty$ and $1 \le q \le \infty$ can be characterized via differences of the functions involved, cf. [37, Section 2.5.12, Theorem 1]. More precisely, for $h \in \mathbb{R}^n$ and a measurable function $v : \mathbb{R}^n \to \mathbb{R}^k$, let us define

$$\delta_h v(x) := v(x+h) - v(x).$$

Then we have the equivalence

$$\|v\|_{B^s_{p,q}(\mathbb{R}^n)} \approx \|v\|_{L^p(\mathbb{R}^n)} + [v]_{B^s_{p,q}(\mathbb{R}^n)}, \tag{3.4}$$

where

$$[v]_{B^s_{p,q}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|\delta_h v(x)|^p}{|h|^{sp}}\,dx\right)^{\frac{q}{p}}\frac{dh}{|h|^n}\right)^{\frac{1}{q}}, \qquad \text{if } 1 \le q < \infty, \tag{3.5}$$

and

$$[v]_{B^s_{p,\infty}(\mathbb{R}^n)} := \sup_{h\,\in\,\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|\delta_h v(x)|^p}{|h|^{sp}}\,dx\right)^{\frac{1}{p}}. \tag{3.6}$$

In (3.5), if one simply integrates for $|h| < r$ for a fixed $r > 0$, then an equivalent norm is obtained, since

$$\left(\int_{\{|h|\,\ge\,r\}}\left(\int_{\mathbb{R}^n}\frac{|\delta_h v(x)|^p}{|h|^{sp}}\,dx\right)^{\frac{q}{p}}\frac{dh}{|h|^n}\right)^{\frac{1}{q}} \le c(n,s,p,q,r)\,\|v\|_{L^p(\mathbb{R}^n)}.$$

Similarly, in (3.6) one can simply take the supremum over $|h| \le r$ and obtain an equivalent norm. By construction, $B^s_{p,q}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$.

Let Ω be an arbitrary open set in $\mathbb{R}^n$. As usual, $\mathcal{D}(\Omega) = C^\infty_0(\Omega)$ stands for the space of all infinitely differentiable functions in $\mathbb{R}^n$ with compact support in Ω. Let $\mathcal{D}'(\Omega)$ be the dual space of all distributions in Ω and let $g \in \mathcal{S}'(\mathbb{R}^n)$. Then we denote by $g|_\Omega$ its restriction to Ω, i.e.

$$g|_\Omega \in \mathcal{D}'(\Omega): \quad (g|_\Omega)(\phi) = g(\phi) \quad \text{ for } \phi \in \mathcal{D}(\Omega).$$

Definition 3.3 Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Then

$$B^s_{p,q}(\Omega) := \left\{ v \in \mathcal{D}'(\Omega) : \, v = g|_\Omega \;\text{ for some } g \in B^s_{p,q}(\mathbb{R}^n) \right\}$$

and

$$\|v\|_{B^s_{p,q}(\Omega)} := \inf \, \|g\|_{B^s_{p,q}(\mathbb{R}^n)} \, ,$$

where the infimum is taken over all $g \in B^s_{p,q}(\mathbb{R}^n)$ such that $g|_\Omega = v$.
If Ω is a bounded C^∞-domain in $\mathbb{R}^n$, then the *restriction operator*

$$\mathrm{re}_\Omega : \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\Omega), \quad \mathrm{re}_\Omega(v) = v|_\Omega$$

generates a linear and bounded map from $B^s_{p,q}(\mathbb{R}^n)$ onto $B^s_{p,q}(\Omega)$. Furthermore, the spaces $B^s_{p,q}(\Omega)$ satisfy the so-called *extension property*, as ensured by the next theorem.

Theorem 3.4 *Let $s \in \mathbb{R}$, let $1 \leq p, q \leq \infty$ and let Ω be a bounded C^∞-domain in $\mathbb{R}^n$. Then, there exists a linear and bounded extension operator* $\mathrm{ext}_\Omega : B^s_{p,q}(\Omega) \hookrightarrow B^s_{p,q}(\mathbb{R}^n)$ *such that* $\mathrm{re}_\Omega \circ \mathrm{ext}_\Omega = \mathrm{id}$, *where* id *is the identity in $B^s_{p,q}(\Omega)$.*

We refer to [38, Theorem 2.82] and [37, Theorem 3.3.4] for a proof of the previous theorem.

If $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is a dense subset of $B^s_{p,q}(\mathbb{R}^n)$ (cf. [37, Theorem 2.3.3]). Consequently, in that case, a continuous linear functional on $B^s_{p,q}(\mathbb{R}^n)$ can be interpreted in the usual way as an element of $\mathcal{S}'(\mathbb{R}^n)$. More precisely, $g \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the dual space $(B^s_{p,q}(\mathbb{R}^n))'$ of the space $B^s_{p,q}(\mathbb{R}^n)$ if and only if there exists a positive number c such that

$$|g(\phi)| \leq \, c \, \|\phi\|_{B^s_{p,q}(\mathbb{R}^n)} \qquad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n) \, .$$

We endow $(B^s_{p,q}(\mathbb{R}^n))'$ with the natural dual norm, defined by

$$\|g\|_{(B^s_{p,q}(\mathbb{R}^n))'} = \sup \left\{ |g(\phi)| : \phi \in \mathcal{S}(\mathbb{R}^n) \;\text{ and }\; \|\phi\|_{B^s_{p,q}(\mathbb{R}^n)} \leq 1 \right\}, \qquad g \in (B^s_{p,q}(\mathbb{R}^n))'.$$

Now we recall the following duality formula, which has to be meant as an isomorphism of normed spaces (see [37, Theorem 2.11.2]).

Theorem 3.5 *Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q < \infty$. Then*

$$(B^s_{p,q}(\mathbb{R}^n))' = B^{-s}_{p',q'}(\mathbb{R}^n) \, ,$$

where $p' = \infty$ if $p = 1$ (similarly for q').

Remark 3.6 The restrictions $p < \infty$ and $q < \infty$ in Theorem 3.5 are natural, since, if either $p = \infty$ or $q = \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is not dense in $B^s_{p,q}(\mathbb{R}^n)$, and the density of $\mathcal{S}(\mathbb{R}^n)$ in $B^s_{p,q}(\mathbb{R}^n)$ is the basis of our interpretation of the dual space $(B^s_{p,q}(\mathbb{R}^n))'$.

For our purposes, we now give the following definition.

Definition 3.7 For $s \in \mathbb{R}$, $1 \le p \le \infty$ and $1 \le q \le \infty$, we define $\mathring{B}^s_{p,q}(\mathbb{R}^n)$ as the completion of $\mathcal{S}(\mathbb{R}^n)$ in $B^s_{p,q}(\mathbb{R}^n)$ with respect to the norm

$$v \mapsto \|v\|_{B^s_{p,q}(\mathbb{R}^n)} .$$

Of course, only the limit cases $\max\{p, q\} = \infty$ are of interest. We shall denote by $(\mathring{B}^s_{p,q}(\mathbb{R}^n))'$ the topological dual of $\mathring{B}^s_{p,q}(\mathbb{R}^n)$, which is endowed with the natural dual norm

$$\|g\|_{(\mathring{B}^s_{p,q}(\mathbb{R}^n))'} = \sup\left\{|g(\phi)| : \phi \in \mathcal{S}(\mathbb{R}^n) \text{ and } \|\phi\|_{B^s_{p,q}(\mathbb{R}^n)} \le 1\right\}, \qquad g \in (\mathring{B}^s_{p,q}(\mathbb{R}^n))'.$$

The following duality result can be found in [37, Section 2.11.2, Remark 2] (see also [36, pages 121 and 122]).

Theorem 3.8 *Let $s \in \mathbb{R}$, $1 \le p \le \infty$ and $1 \le q \le \infty$. Then*

$$(\mathring{B}^s_{p,q}(\mathbb{R}^n))' = B^{-s}_{p',q'}(\mathbb{R}^n),$$

where $p' = 1$ if $p = \infty$ (similarly for q').

The next result is a key ingredient for the proof of Theorem 1.1 and its proof can be found in [37, Section 3.3.5].

Theorem 3.9 *Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Moreover, assume that Ω is a bounded C^∞-domain in $\mathbb{R}^n$. Then, for every $v \in B^s_{p,q}(\Omega)$ and every $j \in \{1, \dots, n\}$ we have*

$$\|\partial_j v\|_{B^{s-1}_{p,q}(\Omega)} \le c\,\|v\|_{B^s_{p,q}(\Omega)}$$

for a positive constant c which is independent of v.

We can also define local Besov spaces as follows. Given a domain $\Omega \subset \mathbb{R}^n$, we say that a function *v* belongs to $B^s_{p,q,loc}(\Omega)$ if $\phi\, v \in B^s_{p,q}(\mathbb{R}^n)$ whenever $\phi \in C^\infty_0(\Omega)$.

Definition 3.10 Let $\Omega \subseteq \mathbb{R}^n$ be an open set. For any $s \in (0, 1)$ and for any $q \in [1, +\infty)$, we define the *fractional Sobolev space* $W^{s,q}(\Omega, \mathbb{R}^k)$ as follows:

$$W^{s,q}(\Omega, \mathbb{R}^k) := \left\{ v \in L^q(\Omega, \mathbb{R}^k) : \frac{|v(x) - v(y)|}{|x - y|^{\frac{n}{q} + s}} \in L^q(\Omega \times \Omega) \right\},$$

i.e. an intermerdiate Banach space between $L^q(\Omega, \mathbb{R}^k)$ and $W^{1,q}(\Omega, \mathbb{R}^k)$, endowed with the norm

$$\|v\|_{W^{s,q}(\Omega)} := \|v\|_{L^q(\Omega)} + [v]_{W^{s,q}(\Omega)} ,$$

where the term

$$[v]_{W^{s,q}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|v(x)-v(y)|^q}{|x-y|^{n+sq}}\, dx\, dy\right)^{\frac{1}{q}} \tag{3.7}$$

is the so-called *Gagliardo seminorm* of *v*.

Remark 3.11 For every $s \in (0,1)$ and every $q \in [1,\infty)$, we have $B^s_{q,q}(\mathbb{R}^n) = W^{s,q}(\mathbb{R}^n)$. In fact, using the change of variable $y = x + h$ in (3.7) with $\Omega = \mathbb{R}^n$, one gets the seminorm (3.5) with $p = q$.

We conclude this section with the following embedding result, whose proof can be obtained by combining [37, Section 2.2.2, Remark 3] with [37, Section 2.3.2, Proposition 2(ii)].

Lemma 3.12 *Let $s \in (0,1)$ and $q \geq 1$. Then, for every $\sigma \in (0, 1-s)$ we have the continuous embedding $W^{s+\sigma,q}_{loc}(\mathbb{R}^n) \hookrightarrow B^s_{q,1,loc}(\mathbb{R}^n)$.*

4 Estimates for a regularized problem

The aim of this section is to establish some uniform estimates for the gradient of the weak solutions of a family of suitable approximating problems. More precisely, let $\lambda \geq 0$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of (1.1), for some $p > 1$. Fix an open ball $B_R \Subset \Omega$ and assume without loss of generality that $R \leq 1$. For $\varepsilon \in (0,1]$, we consider the problem

$$\begin{cases} -\operatorname{div}(DG_\varepsilon(Du_\varepsilon)) = f_\varepsilon & \text{in } B_R, \\ u_\varepsilon = u & \text{on } \partial B_R, \end{cases} \tag{4.1}$$

where:

- $G_\varepsilon(z) := \frac{1}{p}(|z| - \lambda)^p_+ + \frac{\varepsilon}{p}(1+|z|^2)^{\frac{p}{2}}$, for every $z \in \mathbb{R}^n$;

- $f_\varepsilon := f * \phi_\varepsilon$ and $\{\phi_\varepsilon\}_{\varepsilon>0}$ is a family of standard compactly supported C^∞ mollifiers. Observe that

$$D_z G_\varepsilon(z) = H_{p-1}(z) + \varepsilon\,(1+|z|^2)^{\frac{p-2}{2}}\, z. \tag{4.2}$$

Now we set, for $s > \lambda$,

$$\boldsymbol{\lambda}(s) := \begin{cases} \dfrac{(s-\lambda)^{p-1}}{s} & \text{if } p > 2, \\ (p-1)\,\dfrac{(s-\lambda)^{p-1}}{s} & \text{if } 1 < p \leq 2, \end{cases} \tag{4.3}$$

and

$$\mathbf{\Lambda}(s) := \begin{cases} (p-1)\,(s-\lambda)^{p-2} & \text{if } p > 2, \\ (s-\lambda)^{p-2} & \text{if } 1 < p \le 2, \end{cases} \tag{4.4}$$

and $\boldsymbol{\lambda}(s) = 0 = \mathbf{\Lambda}(s)$ for $0 \le s \le \lambda$. These definitions prove to be useful in the formulation of the next lemma, whose proof follows from [4, Lemma 2.7] (see also [5]) together with standard estimates for the p-Laplace operator.

Lemma 4.1 *Let $\varepsilon \in [0, 1]$ and $z \in \mathbb{R}^n \setminus \{0\}$. Then, for every $\zeta \in \mathbb{R}^n$ we have*

$$\left[\varepsilon\, c_0\, (1+|z|^2)^{\frac{p-2}{2}} + \boldsymbol{\lambda}(|z|)\right] |\zeta|^2 \le \langle D^2 G_\varepsilon(z)\, \zeta, \zeta\rangle \le \left[\varepsilon\, c_1\, (1+|z|^2)^{\frac{p-2}{2}} + \mathbf{\Lambda}(|z|)\right] |\zeta|^2,$$

where $c_0 = \min\{1, p-1\}$ and $c_1 = \max\{1, p-1\}$.

Proof Actually, setting for $t > 0$

$$h(t) = \frac{(t-\lambda)_+^{p-1}}{t} \qquad \text{and} \qquad a(t) = (1+t^2)^{\frac{p-2}{2}},$$

we can write

$$D_z G_\varepsilon(z) = \Big[h(|z|) + \varepsilon a(|z|)\Big]\, z$$

and we can easily calculate

$$D^2 G_\varepsilon(z) = \Big[h(|z|) + \varepsilon a(|z|)\Big]\mathbb{I} + \Big[h'(|z|) + \varepsilon a'(|z|)\Big]\frac{z \otimes z}{|z|}\,.$$

Thus we get

$$\langle D^2 G_\varepsilon(z)\, \eta, \zeta\rangle = \Big[h(|z|) + \varepsilon a(|z|)\Big]\langle \eta, \zeta\rangle + \Big[h'(|z|) + \varepsilon a'(|z|)\Big] \sum_{i,j=1}^{n} \frac{z_i\, \eta_i\, z_j\, \zeta_j}{|z|}\,,$$

for any $\eta, \zeta \in \mathbb{R}^n$. By the Cauchy-Schwarz inequality, we have

$$0 \le \sum_{i,j=1}^{n} \frac{z_i\, \zeta_i\, z_j\, \zeta_j}{|z|} \le |z|\, |\zeta|^2. \tag{4.5}$$

At this point, if $h'(|z|) + \varepsilon a'(|z|) \ge 0$ (which occurs when $p \ge 2$), from the lower bound in (4.5) we immediately obtain

$$\langle D^2 G_\varepsilon(z)\, \zeta, \zeta\rangle \ge \big[h(|z|) + \varepsilon a(|z|)\big]|\zeta|^2 = \frac{(|z|-\lambda)_+^{p-1}}{|z|}\, |\zeta|^2 + \varepsilon(1+|z|^2)^{\frac{p-2}{2}}|\zeta|^2.$$

On the other hand, using the upper bound in (4.5), for $p \geq 2$ we deduce

$$\langle D^2 G_\varepsilon(z)\,\zeta, \zeta\rangle \leq (p-1)(|z|-\lambda)_+^{p-2}\,|\zeta|^2 + \varepsilon(p-1)(1+|z|^2)^{\frac{p-2}{2}}\,|\zeta|^2,$$

where we have also used that

$$h(t) + t h'(t) = (p-1)(t-\lambda)_+^{p-2} \tag{4.6}$$

and

$$a(t) + t a'(t) \leq (p-1)(1+t^2)^{\frac{p-2}{2}} \qquad \text{when } p \geq 2.$$

Otherwise, if $h'(|z|) + \varepsilon a'(|z|) < 0$ (which may happen only when $1 < p < 2$), we easily get

$$\langle D^2 G_\varepsilon(z)\,\zeta, \zeta\rangle \leq \big[h(|z|) + \varepsilon a(|z|)\big]\,|\zeta|^2.$$

For the derivation of the lower bound, we use the right inequality in (4.5) to deduce that

$$\begin{aligned}\langle D^2 G_\varepsilon(z)\zeta, \zeta\rangle \geq& \big[h(|z|) + \varepsilon a(|z|)\big]|\zeta|^2 + \big[h'(|z|) + \varepsilon a'(|z|)\big]\,|z|\,|\zeta|^2 \\ =& \big[h(|z|) + h'(|z|)|z|\big]|\zeta|^2 + \big[a(|z|) + \varepsilon a'(|z|)|z|\big]\,|\zeta|^2 \\ \geq& (p-1)(|z|-\lambda)_+^{p-2}|\zeta|^2 + \varepsilon(p-1)(1+|z|^2)^{\frac{p-2}{2}}\,|\zeta|^2,\end{aligned}$$

where, in the last line, we have used (4.6) and the fact that

$$a(t) + t a'(t) \geq (p-1)(1+t^2)^{\frac{p-2}{2}} \qquad \text{when } 1 < p < 2.$$

This proves the claim. □

In what follows, $u_\varepsilon \in u + W_0^{1,p}(B_R)$ will be the unique weak solution to (4.1). By standard elliptic regularity [26, Chapter 8], we know that $(1+|Du_\varepsilon|^2)^{\frac{p-2}{4}} Du_\varepsilon \in W^{1,2}_{loc}(B_R)$ and therefore $u_\varepsilon \in W^{2,2}_{loc}(B_R)$. Moreover, as usual, we shall denote by p^* the Sobolev conjugate exponent of p, defined as

$$p^* := \begin{cases} \frac{np}{n-p} & \text{if } p < n, \\ \text{any value in } (p,\infty) & \text{if } p \geq n, \end{cases}$$

and denote by $(p^*)'$ its Hölder conjugate exponent.

The proofs of Theorems 1.1 and 1.4 are crucially based on the following results.

Proposition 4.2 *(Uniform energy estimate) With the notation and under the assumptions above, if $f \in L^{(p^*)'}(B_R)$, there exist two positive constants $\varepsilon_0 \leq 1$ and $C \equiv C(n,p)$ such that*

$$\int_{B_R} |Du_\varepsilon|^p \, dx \le C \left[1 + \lambda^p + \|Du\|^p_{L^p(B_R)} + \|f\|^{p'}_{L^{(p^*)'}(B_R)} \right] \tag{4.7}$$

for all $\varepsilon \in (0, \varepsilon_0]$.

Proof We insert in the weak formulation of (4.1)

$$\int_{B_R} \langle DG_\varepsilon(Du_\varepsilon), D\varphi \rangle \, dx = \int_{B_R} f_\varepsilon \, \varphi \, dx \qquad \text{for every } \varphi \in W_0^{1,p}(B_R),$$

the test function $\varphi = u_\varepsilon - u$. Recalling (4.2), this gives

$$\begin{aligned} &\int_{B_R} \langle H_{p-1}(Du_\varepsilon) + \varepsilon \, (1 + |Du_\varepsilon|^2)^{\frac{p-2}{2}} \, Du_\varepsilon, Du_\varepsilon \rangle \, dx \\ &\quad = \int_{B_R} \langle H_{p-1}(Du_\varepsilon) + \varepsilon \, (1 + |Du_\varepsilon|^2)^{\frac{p-2}{2}} \, Du_\varepsilon, Du \rangle \, dx + \int_{B_R} f_\varepsilon \, (u_\varepsilon - u) \, dx. \end{aligned} \tag{4.8}$$

Since

$$\langle H_{p-1}(z) + \varepsilon \, (1 + |z|^2)^{\frac{p-2}{2}} \, z, z \rangle \ge (|z| - \lambda)_+^p + \varepsilon \, (1 + |z|^2)^{\frac{p-2}{2}} |z|^2 \ge (|z| - \lambda)_+^p$$

for every $z \in \mathbb{R}^n$, we can estimate the integrals in (4.8), thus obtaining

$$\begin{aligned} &\int_{B_R} (|Du_\varepsilon| - \lambda)_+^p \, dx \\ &\quad \le \int_{B_R} |Du_\varepsilon|^{p-1} \, |Du| \, dx + \varepsilon \int_{B_R} (1 + |Du_\varepsilon|^2)^{\frac{p-1}{2}} \, |Du| \, dx + \|f_\varepsilon\|_{L^{(p^*)'}(B_R)} \, \|u_\varepsilon - u\|_{L^{p^*}(B_R)} \\ &\quad \le \left(1 + 2^{\frac{p-1}{2}} \right) \int_{B_R} |Du_\varepsilon|^{p-1} \, |Du| \, dx + 2^{\frac{p-1}{2}} \int_{B_R} |Du| \, dx \\ &\qquad + c(n,p) \, \|f_\varepsilon\|_{L^{(p^*)'}(B_R)} \, \|Du_\varepsilon - Du\|_{L^p(B_R)}, \end{aligned}$$

where we have used Hölder's and Sobolev inequalities and the fact that $\varepsilon, R \le 1$. Now, applying Young's inequality with $\sigma > 0$, we arrive at

$$\begin{aligned} &\int_{B_R} (|Du_\varepsilon| - \lambda)_+^p \, dx \\ &\quad \le \sigma \int_{B_R} |Du_\varepsilon|^p \, dx + c(n,p,\sigma) \int_{B_R} |Du|^p \, dx + c(n,p,\sigma) \left[1 + \|f_\varepsilon\|^{p'}_{L^{(p^*)'}(B_R)} \right], \end{aligned}$$

where we have used again that $R \le 1$. Since

$$f_\varepsilon \to f \qquad \text{strongly in } L^{(p^*)'}(B_R) \quad \text{as } \varepsilon \to 0^+, \tag{4.9}$$

there exists a positive number $\varepsilon_0 \le 1$ such that

$$\|f_\varepsilon\|_{L^{(p^*)'}(B_R)} \le 1 + \|f\|_{L^{(p^*)'}(B_R)} \qquad \text{for all } \varepsilon \in (0, \varepsilon_0].$$

Then, for $\varepsilon \in (0, \varepsilon_0]$, we have

$$
\begin{aligned}
&\int_{B_R} (|Du_\varepsilon| - \lambda)_+^p \, dx \\
&\quad \le \sigma \int_{B_R} |Du_\varepsilon|^p \, dx + c\,\|Du\|^p_{L^p(B_R)} + c\left[1 + \|f\|^{p'}_{L^{(p^*)'}(B_R)}\right] \\
&\quad \le \sigma \int_{B_R} [\lambda + (|Du_\varepsilon| - \lambda)_+]^p \, dx + c\,\|Du\|^p_{L^p(B_R)} + c\left[1 + \|f\|^{p'}_{L^{(p^*)'}(B_R)}\right] \\
&\quad \le 2^{p-1}\sigma \int_{B_R} (|Du_\varepsilon| - \lambda)_+^p \, dx + c\,\lambda^p + c\,\|Du\|^p_{L^p(B_R)} + c\left[1 + \|f\|^{p'}_{L^{(p^*)'}(B_R)}\right],
\end{aligned}
\tag{4.10}
$$

where $c \equiv c(n, p, \sigma) > 0$. Choosing $\sigma = \frac{1}{2^p}$ and absorbing the first term on the right-hand side of (4.10) into the left-hand side, we obtain

$$
\int_{B_R} (|Du_\varepsilon| - \lambda)_+^p \, dx \le C\left[1 + \lambda^p + \|Du\|^p_{L^p(B_R)} + \|f\|^{p'}_{L^{(p^*)'}(B_R)}\right]
$$

for some finite positive constant C depending on n and p, but not on ε. This estimate is sufficient to ensure the validity of the assertion. □

Proposition 4.3 *(Comparison estimate) With the notation and under the assumptions above, if $f \in L^{(p^*)'}(B_R)$, there exists a positive constant C depending only on n and p such that the estimate*

$$
\begin{aligned}
&\int_{B_R} |\mathcal{V}_\lambda(Du_\varepsilon) - \mathcal{V}_\lambda(Du)|^2 \, dx \\
&\quad \le C\,\|f_\varepsilon - f\|_{L^{(p^*)'}(B_R)} \left[1 + \lambda + \|Du\|_{L^p(B_R)} + \|f\|^{\frac{1}{p-1}}_{L^{(p^*)'}(B_R)}\right] \\
&\qquad + C\,\varepsilon\left[1 + \lambda^p + \|Du\|^p_{L^p(B_R)} + \|f\|^{p'}_{L^{(p^*)'}(B_R)}\right]
\end{aligned}
\tag{4.11}
$$

holds for every $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is the constant from Proposition 4.2.

Proof We proceed by testing equations (1.1) and $(4.1)_1$ with the map $\varphi = u_\varepsilon - u$. Thus we find

$$
\begin{aligned}
&\int_{B_R} \langle H_{p-1}(Du_\varepsilon) - H_{p-1}(Du), Du_\varepsilon - Du\rangle \, dx + \varepsilon \int_{B_R} \langle (1 + |Du_\varepsilon|^2)^{\frac{p-2}{2}} Du_\varepsilon, Du_\varepsilon - Du\rangle \, dx \\
&\quad = \int_{B_R} (f_\varepsilon - f)(u_\varepsilon - u) \, dx.
\end{aligned}
\tag{4.12}
$$

Using Lemma 2.7, the Cauchy-Schwarz inequality as well as Hölder's and Young's inequalities, from (4.12) we obtain

$$
\begin{aligned}
C \int_{B_R} & |\mathcal{V}_\lambda(Du_\varepsilon) - \mathcal{V}_\lambda(Du)|^2 \, dx \, + \, \varepsilon \int_{B_R} (1+|Du_\varepsilon|^2)^{\frac{p-2}{2}} \, |Du_\varepsilon|^2 \, dx \\
&\le \|f_\varepsilon - f\|_{L^{(p^*)'}(B_R)} \, \|u_\varepsilon - u\|_{L^{p^*}(B_R)} \, + \, \varepsilon \int_{B_R} (1+|Du_\varepsilon|^2)^{\frac{p-2}{2}} \, |Du_\varepsilon| \, |Du| \, dx \\
&\le \|f_\varepsilon - f\|_{L^{(p^*)'}(B_R)} \, \|u_\varepsilon - u\|_{L^{p^*}(B_R)} \, + \, \varepsilon \int_{B_R} (1+|Du_\varepsilon|^2)^{\frac{p-1}{2}} \, |Du| \, dx \\
&\le \|f_\varepsilon - f\|_{L^{(p^*)'}(B_R)} \, \|u_\varepsilon - u\|_{L^{p^*}(B_R)} \, + \, \frac{\varepsilon}{p'} \int_{B_R} (1+|Du_\varepsilon|^2)^{\frac{p}{2}} \, dx \, + \, \frac{\varepsilon}{p} \, \|Du\|^p_{L^p(B_R)} \, ,
\end{aligned}
$$

where C is a positive constant depending only on n and p. Now, let us consider the same $\varepsilon_0 \in (0,1]$ as in Proposition 4.2 and let $\varepsilon \in (0, \varepsilon_0]$. Then, applying Sobolev's and Minkowski's inequalities, we get

$$
\begin{aligned}
\int_{B_R} & |\mathcal{V}_\lambda(Du_\varepsilon) - \mathcal{V}_\lambda(Du)|^2 \, dx \\
&\le C \, \|f_\varepsilon - f\|_{L^{(p^*)'}(B_R)} \, \|Du_\varepsilon - Du\|_{L^p(B_R)} \, + \, \frac{C\,\varepsilon}{p'} \int_{B_R} (1+|Du_\varepsilon|^2)^{\frac{p}{2}} \, dx \, + \, \frac{C\,\varepsilon}{p} \, \|Du\|^p_{L^p(B_R)} \\
&\le C \, \|f_\varepsilon - f\|_{L^{(p^*)'}(B_R)} \, \big(\|Du_\varepsilon\|_{L^p(B_R)} \, + \, \|Du\|_{L^p(B_R)} \big) \\
&\quad + \, \frac{C\,\varepsilon}{p'} \int_{B_R} (1+|Du_\varepsilon|^2)^{\frac{p}{2}} \, dx \, + \, \frac{C\,\varepsilon}{p} \, \|Du\|^p_{L^p(B_R)} \\
&\le C \, \|f_\varepsilon - f\|_{L^{(p^*)'}(B_R)} \left[1 + \lambda + \, \|Du\|_{L^p(B_R)} \, + \, \|f\|^{\frac{1}{p-1}}_{L^{(p^*)'}(B_R)} \right] \\
&\quad + C\,\varepsilon \left[1 + \lambda^p \, + \, \|Du\|^p_{L^p(B_R)} \, + \, \|f\|^{p'}_{L^{(p^*)'}(B_R)} \right] ,
\end{aligned}
$$

where, in the last two lines, we have used inequality (4.7). This concludes the proof. □

5 Proof of Theorem 1.1

In this section we prove Theorem 1.1, by dividing the proof into two steps. First, we shall derive a suitable uniform *a priori* estimate for the weak solutions u_ε of the regularized problems (4.1). Then, we conclude with a standard comparison argument (see e.g. [3, 19, 24]) which, combined with the estimates from Propositions 4.2, 4.3 and 5.1, yields the local Sobolev regularity of the function $\mathcal{V}_\lambda(Du)$. We begin with the following result.

Proposition 5.1 *(Uniform Sobolev estimate) Under the assumptions of Theorem 1.1 and with the notation above, there exists a positive number $\varepsilon_1 \le 1$ such that, for every $\varepsilon \in (0, \varepsilon_1]$ and every pair of concentric balls $B_{r/2} \subset B_r \subset B_R$, we have*

$$
\begin{aligned}
\int_{B_{r/2}} & |D\mathcal{V}_\lambda(Du_\varepsilon)|^2 \, dx \\
&\le \left(C + \frac{C}{r^2} \right) \left[1 + \lambda^p + \|Du\|^p_{L^p(B_R)} \, + \|f\|^{p'}_{L^{p'}(B_R)} \right] + \, C \, \|f\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_R)}
\end{aligned}
\tag{5.1}
$$

for a positive constant C depending only on n and p.

Proof Let us first assume that $\lambda > 0$. Differentiating the equation in (4.1) with respect to x_j for some $j \in \{1, \dots, n\}$ and then integrating by parts, we obtain

$$\int_{B_R} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D\varphi\rangle\, dx = \int_{B_R} (\partial_j f_\varepsilon)\, \varphi\, dx, \qquad \forall\, \varphi \in W_0^{1,p}(B_R). \tag{5.2}$$

Let $\eta \in C_0^\infty(B_r)$ be a standard cut-off function such that

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ on } \overline{B}_{r/2}, \quad \|D\eta\|_\infty \le \frac{\tilde{c}}{r}, \tag{5.3}$$

and choose

$$\varphi = \eta^2\, (\partial_j u_\varepsilon)\, \Phi\big((|Du_\varepsilon| - \lambda)_+\big),$$

where $\Phi : [0, \infty) \to [0, \infty)$ is an increasing, locally Lipschitz continuous function, such that Φ and Φ' are bounded on $[0, \infty)$, $\Phi(0) = 0$ and

$$\Phi'(t)\, t \le c_\Phi\, \Phi(t) \tag{5.4}$$

for a suitable constant $c_\Phi > 0$. Using the above choice of φ as a test function in (5.2), we get

$$\begin{aligned}
&\int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon)\rangle\, \eta^2\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right)\, dx \\
&+ \int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D[(|Du_\varepsilon| - \lambda)_+]\rangle\, \eta^2\, (\partial_j u_\varepsilon)\, \Phi'\left((|Du_\varepsilon| - \lambda)_+\right)\, dx \\
&\quad = -2 \int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D\eta\rangle\, \eta\, (\partial_j u_\varepsilon)\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right)\, dx \\
&\qquad + \int_{B_r} (\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\, \eta^2\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right)\, dx.
\end{aligned} \tag{5.5}$$

As for the first term on the right-hand side of (5.5), we have

$$\begin{aligned}
&-2 \int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D\eta\rangle\, \eta\, (\partial_j u_\varepsilon)\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right)\, dx \\
&\quad \le 2 \int_{B_r} \sqrt{\langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon)\rangle}\, \sqrt{\langle D^2 G_\varepsilon(Du_\varepsilon)\, D\eta, D\eta\rangle}\, \eta\, |\partial_j u_\varepsilon|\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right)\, dx \\
&\quad \le \frac{1}{2} \int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon)\rangle\, \eta^2\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right)\, dx \\
&\qquad + 2 \int_{B_r} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D\eta, D\eta\rangle\, |\partial_j u_\varepsilon|^2\, \Phi\left((|Du_\varepsilon| - \lambda)_+\right)\, dx,
\end{aligned} \tag{5.6}$$

where we have used Cauchy-Schwarz and Young's inequalities. Joining (5.5) and (5.6), we get

$$\begin{aligned}
&\int_{B_r} \langle D^2G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon)\rangle\, \eta^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx \\
&+\int_{B_r} \langle D^2G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D[(|Du_\varepsilon|-\lambda)_+]\rangle\, \eta^2\, (\partial_j u_\varepsilon)\, \Phi'\left((|Du_\varepsilon|-\lambda)_+\right)\, dx \\
&\quad\le \frac{1}{2}\int_{B_r} \langle D^2G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon)\rangle\, \eta^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx \\
&\qquad+\ 2\int_{B_r} \langle D^2G_\varepsilon(Du_\varepsilon)\, D\eta, D\eta\rangle\, |\partial_j u_\varepsilon|^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx \\
&\qquad+\int_{B_r} (\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\, \eta^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx.
\end{aligned} \tag{5.7}$$

Reabsorbing the first integral in the right-hand side of (5.7) by the left-hand side and summing the resulting inequalities with respect to *j* from 1 to *n*, we obtain

$$I_1 + I_2 \ \le\ I_3 + I_4, \tag{5.8}$$

where

$$\begin{aligned}
I_1 &:= \int_{B_r} \sum_{j=1}^{n} \langle D^2G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D(\partial_j u_\varepsilon)\rangle\, \eta^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx, \\
I_2 &:= 2\int_{B_r} \sum_{j=1}^{n} \langle D^2G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D[(|Du_\varepsilon|-\lambda)_+]\rangle\, \eta^2\, (\partial_j u_\varepsilon)\, \Phi'\left((|Du_\varepsilon|-\lambda)_+\right)\, dx, \\
I_3 &:= 4\int_{B_r} \sum_{j=1}^{n} \langle D^2G_\varepsilon(Du_\varepsilon)\, D\eta, D\eta\rangle\, |\partial_j u_\varepsilon|^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx, \\
I_4 &:= 2\int_{B_r} \sum_{j=1}^{n} (\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\, \eta^2\, \Phi\left((|Du_\varepsilon|-\lambda)_+\right)\, dx.
\end{aligned}$$

We now prove that I_2 is non-negative, thus we can drop it in the following. Recalling the definitions (4.3) and (4.4), for $|Du_\varepsilon| > \lambda$ we have

$$
\begin{aligned}
&\sum_{j=1}^{n} \langle D^2 G_\varepsilon(Du_\varepsilon)\, D(\partial_j u_\varepsilon), D[(|Du_\varepsilon| - \lambda)_+]\rangle\, (\partial_j u_\varepsilon) \\
&\quad = \left[\frac{\boldsymbol{\Lambda}(|Du_\varepsilon|)}{|Du_\varepsilon|^2} - \frac{\boldsymbol{\lambda}(|Du_\varepsilon|)}{|Du_\varepsilon|^2} + \varepsilon\,(p-2)\,(1+|Du_\varepsilon|^2)^{\frac{p-4}{2}} \right] \\
&\qquad \cdot \sum_{i,\,j,\,k\,=\,1}^{n} (\partial_j u_\varepsilon)(\partial_i u_\varepsilon)(\partial_k u_\varepsilon)(\partial_{ij}^2 u_\varepsilon)\, \partial_k[(|Du_\varepsilon| - \lambda)_+] \\
&\qquad + \left[\boldsymbol{\lambda}(|Du_\varepsilon|) + \varepsilon\,(1+|Du_\varepsilon|^2)^{\frac{p-2}{2}} \right] \sum_{i,\,j\,=\,1}^{n} (\partial_j u_\varepsilon)(\partial_{ij}^2 u_\varepsilon)\, \partial_i[(|Du_\varepsilon| - \lambda)_+] \qquad (5.9) \\
&\quad = \left[\frac{\boldsymbol{\Lambda}(|Du_\varepsilon|)}{|Du_\varepsilon|} - \frac{\boldsymbol{\lambda}(|Du_\varepsilon|)}{|Du_\varepsilon|} + \varepsilon\,(p-2)\,(1+|Du_\varepsilon|^2)^{\frac{p-4}{2}}\, |Du_\varepsilon| \right] \\
&\qquad \cdot \left[\sum_{k=1}^{n} (\partial_k u_\varepsilon)\, \partial_k[(|Du_\varepsilon| - \lambda)_+] \right]^2 \\
&\qquad + \left[\boldsymbol{\lambda}(|Du_\varepsilon|)|Du_\varepsilon| + \varepsilon\,(1+|Du_\varepsilon|^2)^{\frac{p-2}{2}}\, |Du_\varepsilon| \right] \left| D[(|Du_\varepsilon| - \lambda)_+] \right|^2,
\end{aligned}
$$

where we have used the fact that

$$
\partial_k[(|Du_\varepsilon| - \lambda)_+] \,=\, \partial_k(|Du_\varepsilon|) \,=\, \frac{1}{|Du_\varepsilon|} \sum_{j=1}^{n} (\partial_j u_\varepsilon)(\partial_{kj}^2 u_\varepsilon) \qquad \text{when } |Du_\varepsilon| > \lambda.
$$

Thus, coming back to the estimate of I_2, from (5.9) we deduce

$$
\begin{aligned}
I_2 \,\geq\, 2 \int_{B_r} \eta^2\, \Phi'\left((|Du_\varepsilon| - \lambda)_+\right) \Bigg\{ &\left[\frac{\boldsymbol{\Lambda}(|Du_\varepsilon|)}{|Du_\varepsilon|} - \frac{\boldsymbol{\lambda}(|Du_\varepsilon|)}{|Du_\varepsilon|} \right] \cdot \left[\sum_{k=1}^{n} (\partial_k u_\varepsilon)\, \partial_k[(|Du_\varepsilon| - \lambda)_+] \right]^2 \\
&+ \boldsymbol{\lambda}(|Du_\varepsilon|)|Du_\varepsilon| \left| D[(|Du_\varepsilon| - \lambda)_+] \right|^2 \Bigg\}\, dx.
\end{aligned}
$$

Now, arguing as in the proof of [31, Lemma 4.1], for $|Du_\varepsilon| > \lambda$ we have

$$
\left[\sum_{k=1}^{n} (\partial_k u_\varepsilon)\, \partial_k[(|Du_\varepsilon| - \lambda)_+] \right]^2 \leq |Du_\varepsilon|^2 \left| D[(|Du_\varepsilon| - \lambda)_+] \right|^2. \tag{5.10}
$$

This implies

$$
I_2 \,\geq\, 2 \int_{B_r} \eta^2\, \Phi'\left((|Du_\varepsilon| - \lambda)_+\right) \frac{\boldsymbol{\Lambda}(|Du_\varepsilon|)}{|Du_\varepsilon|} \left[\sum_{k=1}^{n} (\partial_k u_\varepsilon)\, \partial_k[(|Du_\varepsilon| - \lambda)_+] \right]^2 dx \,\geq\, 0,
$$

where we have used the fact that $\Phi'\left((|Du_\varepsilon|-\lambda)_+\right)\geq 0$. Thus, inequality (5.8) boils down to

$$I_1 \leq I_3 + I_4. \tag{5.11}$$

Now we choose

$$\Phi(t) := \left(\frac{t}{t+\lambda}\right)^{1+\frac{2}{p-1}} \qquad \text{for } t \geq 0, \tag{5.12}$$

and therefore

$$\Phi'(t) = \frac{p+1}{p-1}\cdot\frac{\lambda\, t^{\frac{2}{p-1}}}{(t+\lambda)^{2+\frac{2}{p-1}}}\,.$$

Clearly, the function Φ in (5.12) satisfies (5.4) with $c_\Phi = \frac{p+1}{p-1}$. At this stage, we proceed by estimating separately the integrals in (5.11).

ESTIMATE OF I_1
Applying Lemma 4.1, we get

$$I_1 \geq \int_{B_r} \boldsymbol{\lambda}(|Du_\varepsilon|)\,|D^2u_\varepsilon|^2\,\eta^2\,\Phi\left((|Du_\varepsilon|-\lambda)_+\right)\,dx. \tag{5.14}$$

ESTIMATE OF I_3
Using Lemma 4.1, (5.3) and the fact that $\Phi \leq 1$, we infer

$$I_3 \leq \frac{c(p)}{r^2}\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx. \tag{5.15}$$

ESTIMATE OF I_4
By Theorem 3.4, there exists a linear and bounded extension operator

$$\mathrm{ext}_{B_r} : B^{-2/p}_{p',1}(B_r) \hookrightarrow B^{-2/p}_{p',1}(\mathbb{R}^n)$$

such that $\mathrm{re}_{B_r}\circ\mathrm{ext}_{B_r} = \mathrm{id}$, where re_{B_r} is the restriction operator defined in Sect. 3 and the symbol id denotes the identity in $B^{-2/p}_{p',1}(B_r)$. Since $\partial_j f_\varepsilon = \mathrm{ext}_{B_r}(\partial_j f_\varepsilon)$ almost everywhere in B_r, we have

$$\int_{B_r}(\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx = \int_{B_r}\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\cdot(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx.$$

At this point, we need to estimate the integral containing $\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)$. To this aim, we argue as in [8, Proposition 3.2]. By definition of dual norm, we get

$$
\begin{aligned}
&\left|\int_{B_r} \mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\cdot(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx\right| \\
&\quad \le \|\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'}\,\|(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\|_{B^{2/p}_{p,\infty}(\mathbb{R}^n)} \\
&\quad = \|\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'} \\
&\qquad \cdot\Big(\|(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\|_{L^p(\mathbb{R}^n)}+\big[(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\big]_{B^{2/p}_{p,\infty}(\mathbb{R}^n)}\Big),
\end{aligned}
\tag{5.16}
$$

where, in the last line, we have used the equivalence (3.4). By the properties of η and the fact that $\Phi \le 1$, we have

$$
\|(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\|_{L^p(\mathbb{R}^n)} \;\le\; \|Du_\varepsilon\|_{L^p(B_r)}. \tag{5.17}
$$

Moreover, using Theorem 3.8, we obtain

$$
\|\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'} \;\le\; c\,\|\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\|_{B^{-2/p}_{p',1}(\mathbb{R}^n)},
$$

for some positive constant $c \equiv c(n,p)$. Combining the above inequality and the boundedness of the operator ext_{B_r} yields

$$
\|\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'} \;\le\; c\,\|\partial_j f_\varepsilon\|_{B^{-2/p}_{p',1}(B_r)}.
$$

Furthermore, applying Theorem 3.9, we find that

$$
\|\partial_j f_\varepsilon\|_{B^{-2/p}_{p',1}(B_r)} \;\le\; c\,\|f_\varepsilon\|_{B^{\frac{p-2}{p}}_{p',1}(B_r)}.
$$

Combining the preceding inequalities, we infer

$$
\|\mathrm{ext}_{B_r}(\partial_j f_\varepsilon)\|_{(\mathring{B}^{2/p}_{p,\infty}(\mathbb{R}^n))'} \;\le\; c\,\|f_\varepsilon\|_{B^{\frac{p-2}{p}}_{p',1}(B_r)}, \tag{5.18}
$$

for a positive constant c depending only on n and p. Now it remains to estimate the term

$$
\big[(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\big]_{B^{2/p}_{p,\infty}(\mathbb{R}^n)} := \sup_{|h|>0}\left(\int_{\mathbb{R}^n}\frac{\big|\delta_h\big((\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\big)\big|^p}{|h|^2}\,dx\right)^{\frac{1}{p}}.
$$

By applying Lemma 2.3, we deduce

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\frac{\big|\delta_h\big((\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\big)\big|^p}{|h|^2}\,dx \\
&\quad \le \frac{c}{|h|^2}\int_{\mathbb{R}^n}\left|\delta_h\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,\eta^p\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 dx \\
&\quad \le c(n,p)\int_{\mathbb{R}^n}\left|D\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,\eta^p\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 dx,
\end{aligned}
$$

where, in the last line, we have used the first statement in Lemma 2.11. By the properties of η at (5.3) and the boundedness of Φ, one can easily obtain

$$\left[(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\right]^p_{B^{2/p}_{p,\infty}(\mathbb{R}^n)} \le c\int_{B_r}\left|D\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2\eta^2\,dx + \frac{c}{r^2}\int_{B_r}|Du_\varepsilon|^p\,dx\,, \tag{5.19}$$

where $c\equiv c(n,p)>0$. Now, a straightforward computation reveals that, for every $k\in\{1,\ldots,n\}$, we have

$$\begin{aligned}\partial_k&\left[|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right]\\ &=\frac{p}{2}\,|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial^2_{kj}u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\\ &\quad+\frac{p}{2}\,|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p-2}{2}}\,\Phi'((|Du_\varepsilon|-\lambda)_+)\,\frac{\langle Du_\varepsilon,\partial_k Du_\varepsilon\rangle}{|Du_\varepsilon|}\,.\end{aligned}$$

This yields

$$\left|D\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 \le c(n,p)\,(\mathrm{A}_1+\mathrm{A}_2), \tag{5.20}$$

where we set

$$\mathrm{A}_1 := |Du_\varepsilon|^{p-2}\,|D^2u_\varepsilon|^2\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^p$$

and

$$\mathrm{A}_2 := |Du_\varepsilon|^p\,|D^2u_\varepsilon|^2\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{p-2}\,[\Phi'((|Du_\varepsilon|-\lambda)_+)]^2.$$

We now estimate A_1 and A_2 in the set where $|Du_\varepsilon|>\lambda$, since both A_1 and A_2 vanish in the set $\{|Du_\varepsilon|\le\lambda\}$. Note that, for $|Du_\varepsilon|>\lambda$, we have

$$\begin{aligned}\mathrm{A}_1 &= \boldsymbol{\lambda}(|Du_\varepsilon|)\,\Phi(|Du_\varepsilon|-\lambda)\,|D^2u_\varepsilon|^2\,\frac{[\Phi(|Du_\varepsilon|-\lambda)]^{p-1}}{\boldsymbol{\lambda}(|Du_\varepsilon|)}\,|Du_\varepsilon|^{p-2}\\ &= \boldsymbol{\lambda}(|Du_\varepsilon|)\,\Phi(|Du_\varepsilon|-\lambda)\,|D^2u_\varepsilon|^2\left[\frac{\Phi(|Du_\varepsilon|-\lambda)}{|Du_\varepsilon|-\lambda}\,|Du_\varepsilon|\right]^{p-1}\end{aligned}$$

and

$$\begin{aligned}\mathrm{A}_2 &= \boldsymbol{\lambda}(|Du_\varepsilon|)\,\Phi(|Du_\varepsilon|-\lambda)\,|D^2u_\varepsilon|^2\left[\frac{\Phi(|Du_\varepsilon|-\lambda)}{|Du_\varepsilon|-\lambda}\,|Du_\varepsilon|\right]^{p-1}\left[\frac{\Phi'(|Du_\varepsilon|-\lambda)}{\Phi(|Du_\varepsilon|-\lambda)}\,|Du_\varepsilon|\right]^2\\ &= \mathrm{A}_1\left[\frac{\Phi'(|Du_\varepsilon|-\lambda)}{\Phi(|Du_\varepsilon|-\lambda)}\,|Du_\varepsilon|\right]^2.\end{aligned}$$

Recalling the definition of Φ in (5.12), we find that

$$\left[\frac{\Phi(|Du_\varepsilon|-\lambda)}{|Du_\varepsilon|-\lambda}\,|Du_\varepsilon|\right]^{p-1} = \left(\frac{|Du_\varepsilon|-\lambda}{|Du_\varepsilon|}\right)^2 \le 1$$

and, moreover,

$$\left[\frac{\Phi'(|Du_\varepsilon|-\lambda)}{\Phi(|Du_\varepsilon|-\lambda)}\,|Du_\varepsilon|\right]^2 = \left(\frac{p+1}{p-1}\right)^2 \frac{\lambda^2}{(|Du_\varepsilon|-\lambda)^2}.$$

Therefore, combining the four previous estimates, for $|Du_\varepsilon| > \lambda$ we get

$$\mathrm{A}_1 \le c(p)\,\boldsymbol{\lambda}(|Du_\varepsilon|)\,\Phi(|Du_\varepsilon|-\lambda)\,|D^2u_\varepsilon|^2 \tag{5.21}$$

and

$$\begin{aligned} \mathrm{A}_2 &= c(p)\,\boldsymbol{\lambda}(|Du_\varepsilon|)\,\Phi(|Du_\varepsilon|-\lambda)\,|D^2u_\varepsilon|^2\,\frac{(|Du_\varepsilon|-\lambda)^2}{|Du_\varepsilon|^2}\cdot\frac{\lambda^2}{(|Du_\varepsilon|-\lambda)^2} \\ &\le c(p)\,\boldsymbol{\lambda}(|Du_\varepsilon|)\,\Phi(|Du_\varepsilon|-\lambda)\,|D^2u_\varepsilon|^2. \end{aligned} \tag{5.22}$$

Joining estimates (5.19)–(5.22), we obtain

$$\begin{aligned} \left[(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\right]^p_{B^{2/p}_{p,\infty}(\mathbb{R}^n)} &\le c\int_{B_r}\boldsymbol{\lambda}(|Du_\varepsilon|)\,\Phi((|Du_\varepsilon|-\lambda)_+)\,|D^2u_\varepsilon|^2\,\eta^2\,dx \\ &\quad + \frac{c}{r^2}\int_{B_r}|Du_\varepsilon|^p\,dx\,, \end{aligned}$$

where $c \equiv c(n,p) > 0$. Combining the previous inequality, (5.17) and (5.18) with (5.16), and recalling the definition of I_4, we get

$$\begin{aligned} I_4 &\le 2\sum_{j=1}^n\left|\int_{B_r}(\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx\right| \\ &\le c\,\|f_\varepsilon\|_{B^{\frac{p-2}{p}}_{p',1}(B_r)}\left[\left(\int_{B_r}\boldsymbol{\lambda}(|Du_\varepsilon|)\,|D^2u_\varepsilon|^2\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx\right)^{\frac{1}{p}} + \left(1+\frac{1}{r^{2/p}}\right)\|Du_\varepsilon\|_{L^p(B_r)}\right], \end{aligned} \tag{5.23}$$

for a constant $c \equiv c(n,p) > 0$.

Now, inserting estimates (5.14), (5.15) and (5.23) in (5.11), we obtain

$$\begin{aligned} &\int_{B_r}\boldsymbol{\lambda}(|Du_\varepsilon|)\,|D^2u_\varepsilon|^2\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx \\ &\quad\le c\left(\int_{B_r}\boldsymbol{\lambda}(|Du_\varepsilon|)\,|D^2u_\varepsilon|^2\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx\right)^{\frac{1}{p}}\,\|f_\varepsilon\|_{B^{\frac{p-2}{p}}_{p',1}(B_r)} \\ &\qquad + \left(c+\frac{c}{r^{2/p}}\right)\|Du_\varepsilon\|_{L^p(B_r)}\,\|f_\varepsilon\|_{B^{\frac{p-2}{p}}_{p',1}(B_r)} + \frac{c}{r^2}\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx\,, \end{aligned}$$

where $c \equiv c(n,p) > 0$. Applying Young's inequality to the first two terms on the right-hand side of the previous estimate, we get

$$\begin{aligned} &\int_{B_r} \boldsymbol{\lambda}(|Du_\varepsilon|)\,|D^2u_\varepsilon|^2\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx \\ &\quad\le \left(C+\frac{C}{r^2}\right)\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx \,+\, C\,\|f_\varepsilon\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_r)}, \end{aligned} \tag{5.24}$$

for some constant $C \equiv C(n,p) > 0$.

At this point, recalling the definition of $\mathcal{V}_\lambda$ in (1.5)–(1.6), a straightforward computation reveals that, for every $j \in \{1,\ldots,n\}$, we have

$$\begin{aligned} \partial_j\mathcal{V}_\lambda(Du_\varepsilon) \;=\; &\frac{(|Du_\varepsilon|-\lambda)_+^{\frac{p}{2}+\frac{1}{p-1}}}{|Du_\varepsilon|^2\,[\lambda+(|Du_\varepsilon|-\lambda)_+]^{1+\frac{1}{p-1}}}\,\langle Du_\varepsilon,\partial_j Du_\varepsilon\rangle\,Du_\varepsilon \\ &+\,\mathcal{G}_\lambda((|Du_\varepsilon|-\lambda)_+)\left[\frac{\partial_j Du_\varepsilon}{|Du_\varepsilon|}\,-\,\frac{\langle Du_\varepsilon,\partial_j Du_\varepsilon\rangle}{|Du_\varepsilon|^3}\,Du_\varepsilon\right] \end{aligned}$$

if $|Du_\varepsilon| > \lambda$, and $\partial_j\mathcal{V}_\lambda(Du_\varepsilon) = 0$ otherwise. In the set $\{|Du_\varepsilon| > \lambda\}$, this yields

$$|D\mathcal{V}_\lambda(Du_\varepsilon)|^2 \le \mathrm{B}_1 + \mathrm{B}_2, \tag{5.25}$$

where we define

$$\mathrm{B}_1 := 2\,\frac{(|Du_\varepsilon|-\lambda)_+^{p+\frac{2}{p-1}}\,|D^2u_\varepsilon|^2}{[\lambda+(|Du_\varepsilon|-\lambda)_+]^{2+\frac{2}{p-1}}}$$

and

$$\mathrm{B}_2 := 8\,\frac{[\mathcal{G}_\lambda((|Du_\varepsilon|-\lambda)_+)]^2\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|^2}\,.$$

We now estimate B_1 and B_2 separately in the set where $|Du_\varepsilon| > \lambda$, since both B_1 and B_2 vanish for $0 < |Du_\varepsilon| \le \lambda$. Recalling the definitions (4.3) and (5.12), we immediately have

$$\begin{aligned} \mathrm{B}_1 \;&=\; 2\,\frac{(|Du_\varepsilon|-\lambda)^{p-1}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|}\left(\frac{|Du_\varepsilon|-\lambda}{|Du_\varepsilon|}\right)^{1+\frac{2}{p-1}} \\ &=\; 2\,\boldsymbol{\lambda}(|Du_\varepsilon|)\,|D^2u_\varepsilon|^2\,\Phi(|Du_\varepsilon|-\lambda)\,. \end{aligned} \tag{5.26}$$

As for B_2, by Lemma 2.6 we obtain

$$
\begin{aligned}
\mathrm{B}_2 \;&\le\; \frac{32}{p^2}\,\frac{(|Du_\varepsilon|-\lambda)^{p+\frac{2p}{p-1}}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|^{2+\frac{2p}{p-1}}}\\
&=\frac{32}{p^2}\,\frac{(|Du_\varepsilon|-\lambda)^{p-1}\,|D^2u_\varepsilon|^2}{|Du_\varepsilon|}\left(\frac{|Du_\varepsilon|-\lambda}{|Du_\varepsilon|}\right)^{1+\frac{2p}{p-1}}\\
&=\frac{32}{p^2}\,\boldsymbol{\lambda}(|Du_\varepsilon|)\,|D^2u_\varepsilon|^2\,\Phi(|Du_\varepsilon|-\lambda)\,.
\end{aligned}
\tag{5.27}
$$

Joining estimates (5.25)–(5.27), we then find

$$
\int_{B_r}|D\mathcal{V}_\lambda(Du_\varepsilon)|^2\,\eta^2\,dx\;\le\;c(p)\int_{B_r}\boldsymbol{\lambda}(|Du_\varepsilon|)\,|D^2u_\varepsilon|^2\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx, \tag{5.28}
$$

which combined with (5.24), gives

$$
\int_{B_r}|D\mathcal{V}_\lambda(Du_\varepsilon)|^2\,\eta^2\,dx\;\le\;\left(C+\frac{C}{r^2}\right)\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx\;+\;C\,\|f_\varepsilon\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_r)}\,.
$$

Let us now consider the same $\varepsilon_0\in(0,1]$ as in Proposition 4.2 and let $\varepsilon\in(0,\varepsilon_0]$. Then, recalling that $\eta\equiv 1$ on $\overline{B}_{r/2}$ and applying estimate (4.7), we obtain

$$
\int_{B_{r/2}}|D\mathcal{V}_\lambda(Du_\varepsilon)|^2\,dx\le\left(C+\frac{C}{r^2}\right)\left[1+\lambda^p+\|Du\|^p_{L^p(B_R)}+\|f\|^{p'}_{L^{(p^*)'}(B_R)}\right]+C\,\|f_\varepsilon\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_r)},
$$

where we have used the fact that $r<R\le 1$. Since $(p^*)'<p'$, using Hölder's inequality, from the above estimate we get

$$
\int_{B_{r/2}}|D\mathcal{V}_\lambda(Du_\varepsilon)|^2\,dx\le\left(C+\frac{C}{r^2}\right)\left[1+\lambda^p+\|Du\|^p_{L^p(B_R)}+\|f\|^{p'}_{L^{p'}(B_R)}\right]+C\,\|f_\varepsilon\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_r)}\,.
$$

Furthermore, there exists a positive number $\varepsilon_1\le\varepsilon_0$ such that

$$
\|f_\varepsilon\|_{B^{\frac{p-2}{p}}_{p',1}(B_r)}\;\le\;\|f\|_{B^{\frac{p-2}{p}}_{p',1}(B_R)}<+\infty\qquad\text{for every }\varepsilon\in(0,\varepsilon_1].
$$

Combining the last two estimates for $\varepsilon\in(0,\varepsilon_1]$, we conclude the proof in the case $\lambda>0$.

Finally, when $\lambda=0$ the above proof can be greatly simplified, as we can choose $\Phi\equiv 1$ and we have

$$
\mathcal{V}_0(Du_\varepsilon)=\frac{2}{p}\,H_{\frac{p}{2}}(Du_\varepsilon)=\frac{2}{p}\,|Du_\varepsilon|^{\frac{p-2}{2}}Du_\varepsilon\,.
$$

In this regard, we leave the details to the reader. □

Combining Lemma 2.11 with estimate (5.1), we obtain the following

Corollary 5.2 *Let $\varepsilon_1 \in (0, 1]$ be the constant from Proposition 5.1. Under the assumptions of Theorem 1.1 and with the notation above, for every pair of concentric balls $B_{r/4} \subset B_r \subset B_R$ we have*

$$\begin{aligned}&\int_{B_{r/4}} |\tau_{j,h}\mathcal{V}_\lambda(Du_\varepsilon)|^2\,dx \\ &\quad\le \frac{C\,|h|^2\,(1+r^2)}{r^2}\left[1+\lambda^p+\|Du\|^p_{L^p(B_R)}+\|f\|^{p'}_{L^{p'}(B_R)}\right]+C\,|h|^2\,\|f\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_R)},\end{aligned}\tag{5.29}$$

for every $j \in \{1,\ldots,n\}$, for every $h \in \mathbb{R}$ such that $|h| < \frac{r}{8}$, for every $\varepsilon \in (0,\varepsilon_1]$ and a positive constant C depending only on n and p.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Consider the same $\varepsilon_1 \in (0, 1]$ as in Proposition 5.1 and let $\varepsilon \in (0, \varepsilon_1]$. Moreover, let u_ε be the unique energy solution of the Dirichlet problem (4.1). Now we fix three concentric balls $B_{r/4}$, $B_{r/2}$ and B_r, with $B_r \subset B_R \Subset \Omega$, $R \le 1$, and use the finite difference operator $\tau_{j,h}$ defined in Sect. 2.3, for increments $h \in \mathbb{R}\setminus\{0\}$ such that $|h| < \frac{r}{8}$. In what follows, we will denote by C a positive constant which neither depends on ε nor on h. In order to obtain an estimate for the finite difference $\tau_{j,h}\mathcal{V}_\lambda(Du)$, we use the following comparison argument:

$$\begin{aligned}&\int_{B_{r/4}} |\tau_{j,h}\mathcal{V}_\lambda(Du)|^2\,dx \\ &\quad\le 4\int_{B_{r/4}} |\tau_{j,h}\mathcal{V}_\lambda(Du_\varepsilon)|^2\,dx + 4\int_{B_{r/4}} |\mathcal{V}_\lambda(Du_\varepsilon)-\mathcal{V}_\lambda(Du)|^2\,dx \\ &\qquad + 4\int_{B_{r/4}} |\mathcal{V}_\lambda(Du_\varepsilon(x+he_j))-\mathcal{V}_\lambda(Du(x+he_j))|^2\,dx \\ &\quad\le 4\int_{B_{r/4}} |\tau_{j,h}\mathcal{V}_\lambda(Du_\varepsilon)|^2\,dx + 8\int_{B_R} |\mathcal{V}_\lambda(Du_\varepsilon)-\mathcal{V}_\lambda(Du)|^2\,dx,\end{aligned}$$

where we have used the second statement in Lemma 2.11. Combining the previous estimate with (5.29) and (4.11), for every $j \in \{1,\ldots,n\}$ we get

$$\begin{aligned}&\int_{B_{r/4}} |\tau_{j,h}\mathcal{V}_\lambda(Du)|^2\,dx \\ &\quad\le \frac{C\,|h|^2\,(1+r^2)}{r^2}\left[1+\lambda^p+\|Du\|^p_{L^p(B_R)}+\|f\|^{p'}_{L^{p'}(B_R)}\right]+C\,|h|^2\,\|f\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_R)} \\ &\qquad + C\,\|f_\varepsilon - f\|_{L^{(p^*)'}(B_R)}\left[1+\lambda+\|Du\|_{L^p(B_R)}+\|f\|^{\frac{1}{p-1}}_{L^{(p^*)'}(B_R)}\right] \\ &\qquad + C\,\varepsilon\left[1+\lambda^p+\|Du\|^p_{L^p(B_R)}+\|f\|^{p'}_{L^{(p^*)'}(B_R)}\right],\end{aligned}\tag{5.30}$$

which holds for every sufficiently small $h \in \mathbb{R} \setminus \{0\}$ and a constant $C \equiv C(n,p) > 0$. Therefore, recalling (4.9) and letting $\varepsilon \searrow 0$ in (5.30), we obtain

$$\begin{aligned}&\int_{B_{r/4}} |\Delta_{j,h}\mathcal{V}_\lambda(Du)|^2\,dx\\&\quad\le \left(C+\frac{C}{r^2}\right)\left[1+\lambda^p+\|Du\|^p_{L^p(B_R)}+\|f\|^{p'}_{L^{p'}(B_R)}\right]+C\,\|f\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_R)}.\end{aligned}$$

Since the above inequality holds for every $j \in \{1,\dots,n\}$ and every sufficiently small $h \neq 0$, by Lemma 2.12 we may conclude that $\mathcal{V}_\lambda(Du) \in W^{1,2}_{loc}(\Omega,\mathbb{R}^n)$. Moreover, letting $h \to 0$ in the previous inequality, we obtain estimate (1.8) for every ball $B_R \Subset \Omega$ with $R \le 1$. The validity of (1.8) for arbitrary balls follows from a standard covering argument. □

6 Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. Actually, here we limit ourselves to deriving the *a priori* estimates for $1 < p \le 2$, since inequality (1.10) can be obtained using the same arguments presented in Sect. 5. In what follows, we shall keep the notations used for the proof of Proposition 5.1.

Proof of Theorem 1.4 Let us first assume that $\lambda > 0$. Arguing as in the proof of Proposition 5.1, we define the integrals $I_1 - I_4$ exactly as in (5.8). We need to treat differently only the integrals I_2 and I_4, in which the new assumptions $1 < p \le 2$ and $f \in L^{\frac{np}{n(p-1)+2-p}}_{loc}(\Omega)$ are involved. Under these new hypotheses and for $|Du_\varepsilon| > \lambda$, equality (5.9) is replaced by

$$\begin{aligned}&\sum_{j=1}^{n}\langle D^2G_\varepsilon(Du_\varepsilon)\,D(\partial_j u_\varepsilon), D[(|Du_\varepsilon|-\lambda)_+]\rangle\,(\partial_j u_\varepsilon)\\&\quad=\left[(p-1)\,\frac{\mathbf{\Lambda}(|Du_\varepsilon|)}{|Du_\varepsilon|}-\frac{\boldsymbol{\lambda}(|Du_\varepsilon|)}{(p-1)|Du_\varepsilon|}+\varepsilon\,(p-2)\,(1+|Du_\varepsilon|^2)^{\frac{p-4}{2}}\,|Du_\varepsilon|\right]\\&\qquad\cdot\left[\sum_{k=1}^{n}(\partial_k u_\varepsilon)\,\partial_k[(|Du_\varepsilon|-\lambda)_+]\right]^2\\&\qquad+\left[\frac{\boldsymbol{\lambda}(|Du_\varepsilon|)\,|Du_\varepsilon|}{p-1}+\varepsilon\,(1+|Du_\varepsilon|^2)^{\frac{p-2}{2}}\,|Du_\varepsilon|\right]\left|D[(|Du_\varepsilon|-\lambda)_+]\right|^2,\end{aligned}\tag{6.1}$$

where we have used the definitions (4.3) and (4.4) again. It comes out that I_2 is non-negative, as in the super-quadratic case. Indeed, estimates (6.1) and (5.10) lead us to

$$
\begin{aligned}
I_2 \,\geq\, & 2\int_{B_r} \eta^2\, \Phi'((|Du_\varepsilon|-\lambda)_+) \\
& \cdot \Bigg\{ \left[(p-1)\,\frac{\mathbf{\Lambda}(|Du_\varepsilon|)}{|Du_\varepsilon|} - \frac{\boldsymbol{\lambda}(|Du_\varepsilon|)}{(p-1)|Du_\varepsilon|} + \varepsilon(p-2)(1+|Du_\varepsilon|^2)^{\frac{p-4}{2}}|Du_\varepsilon| \right] \\
& \cdot \left[\sum_{k=1}^{n} (\partial_k u_\varepsilon)\,\partial_k[(|Du_\varepsilon|-\lambda)_+] \right]^2 + \frac{\boldsymbol{\lambda}(|Du_\varepsilon|)\,|Du_\varepsilon|}{p-1}\left|D[(|Du_\varepsilon|-\lambda)_+]\right|^2 \\
& + \varepsilon\,(1+|Du_\varepsilon|^2)^{\frac{p-4}{2}}\,|Du_\varepsilon|^3\left|D[(|Du_\varepsilon|-\lambda)_+]\right|^2 \Bigg\}\,dx \\
\geq\, & 2\int_{B_r} \eta^2\, \Phi'((|Du_\varepsilon|-\lambda)_+) \left[(p-1)\,\frac{\mathbf{\Lambda}(|Du_\varepsilon|)}{|Du_\varepsilon|} + \varepsilon(p-1)(1+|Du_\varepsilon|^2)^{\frac{p-4}{2}}|Du_\varepsilon| \right] \\
& \cdot \left[\sum_{k=1}^{n} (\partial_k u_\varepsilon)\,\partial_k[(|Du_\varepsilon|-\lambda)_+] \right]^2 dx \,\geq\, 0\,,
\end{aligned}
$$

where the function Φ is defined in (5.12). Then, using (5.3), (5.12), Lemma 4.1 and the fact that $I_2 \geq 0$, from (5.8) we now obtain

$$
\begin{aligned}
& \int_{B_r} \boldsymbol{\lambda}(|Du_\varepsilon|)\,|D^2 u_\varepsilon|^2\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx \\
& \quad \leq \frac{c(p)}{r^2}\int_{B_r} (1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx + \frac{c(p)}{r^2}\int_{B_r} \mathbf{\Lambda}(|Du_\varepsilon|)\,|Du_\varepsilon|^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx \\
& \qquad + c(p)\sum_{j=1}^{n}\int_{B_r} (\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx \\
& \quad \leq \frac{c(p)}{r^2}\int_{B_r} (1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx + c(p)\sum_{j=1}^{n}\int_{B_r} (\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx\,.
\end{aligned} \tag{6.2}
$$

At this point, we integrate by parts and then apply Hölder's inequality in the second integral on right-hand side of (6.2). This gives

$$
\begin{aligned}
& \left| \int_{B_r} (\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx \right| \\
& \quad \leq \|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\,\|\partial_j[(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)]\|_{L^{\frac{np}{n-2+p}}(\mathbb{R}^n)}.
\end{aligned} \tag{6.3}
$$

From now on, we will only deal with the case $n \geq 3$ and $1 < p < 2$, since the remaining cases imply that

$$
\frac{np}{n(p-1)+2-p} = \frac{np}{n-2+p} = 2
$$

and can be addressed by suitably modifying the arguments used in Sect. 5. Note that in the case $p = n = 2$, we have $(p^*)' < p' = 2$, and therefore we can continue to argue as in the proof of Proposition 5.1.

For ease of notation, we now set

$$Z(x) := \eta^2(x) \cdot (\partial_j u_\varepsilon(x)) \cdot \Phi((|Du_\varepsilon| - \lambda)_+) \, .$$

Since $\frac{np}{n-2+p} < 2$ for $n \geq 3$ and $1 < p < 2$, an application of Hölder's inequality and Lemma 2.4 yield

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\tau_{j,h} Z(x)|^{\frac{np}{n-2+p}} \, dx \\
&= \int_{\mathbb{R}^n} |\tau_{j,h} Z(x)|^{\frac{np}{n-2+p}} \left(|Z(x+he_j)|^2 + |Z(x)|^2\right)^{\frac{(p-2)}{4}\frac{np}{n-2+p}} \left(|Z(x+he_j)|^2 + |Z(x)|^2\right)^{\frac{(2-p)}{4}\frac{np}{n-2+p}} dx \\
&\leq c(n,p) \left(\int_{\mathbb{R}^n} |\tau_{j,h} Z(x)|^2 \left(|Z(x+he_j)|^2 + |Z(x)|^2\right)^{\frac{p-2}{2}} dx\right)^{\frac{np}{2(n-2+p)}} \left(\int_{\mathbb{R}^n} |Z(x)|^{\frac{np}{n-2}} \, dx\right)^{\frac{(n-2)(2-p)}{2(n-2+p)}} \\
&\leq c(n,p) \left(\int_{\mathbb{R}^n} \left|\tau_{j,h} \left(|Z(x)|^{\frac{p-2}{2}} Z(x)\right)\right|^2 dx\right)^{\frac{np}{2(n-2+p)}} \left(\int_{\mathbb{R}^n} |Z(x)|^{\frac{np}{n-2}} \, dx\right)^{\frac{(n-2)(2-p)}{2(n-2+p)}}
\end{aligned}$$

for every $h \in \mathbb{R} \setminus \{0\}$. Dividing both sides by $|h|^{\frac{np}{n-2+p}}$ and letting $h \to 0$, by virtue of Lemma 2.12 we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\partial_j Z(x)|^{\frac{np}{n-2+p}} \, dx \\
&\leq c(n,p) \left(\int_{\mathbb{R}^n} \left|\partial_j \left(|Z(x)|^{\frac{p-2}{2}} Z(x)\right)\right|^2 dx\right)^{\frac{np}{2(n-2+p)}} \left(\int_{\mathbb{R}^n} |Z(x)|^{\frac{np}{n-2}} \, dx\right)^{\frac{(n-2)(2-p)}{2(n-2+p)}} \\
&\leq c(n,p) \left(\int_{\mathbb{R}^n} \left|D \left(|Z(x)|^{\frac{p-2}{2}} Z(x)\right)\right|^2 dx\right)^{\frac{np}{2(n-2+p)}} \left(\int_{\mathbb{R}^n} |Z(x)|^{\frac{np}{n-2}} \, dx\right)^{\frac{(n-2)(2-p)}{2(n-2+p)}} \\
&= c(n,p) \left(\int_{\mathbb{R}^n} \left|D \left(|Z(x)|^{\frac{p-2}{2}} Z(x)\right)\right|^2 dx\right)^{\frac{np}{2(n-2+p)}} \left(\int_{\mathbb{R}^n} \left||Z(x)|^{\frac{p-2}{2}} Z(x)\right|^{\frac{2n}{n-2}} dx\right)^{\frac{(n-2)(2-p)}{2(n-2+p)}} \qquad (6.4) \\
&\leq c(n,p) \left(\int_{\mathbb{R}^n} \left|D \left(|Z(x)|^{\frac{p-2}{2}} Z(x)\right)\right|^2 dx\right)^{\frac{np}{2(n-2+p)}} \left(\int_{\mathbb{R}^n} \left|D \left(|Z(x)|^{\frac{p-2}{2}} Z(x)\right)\right|^2 dx\right)^{\frac{n(2-p)}{2(n-2+p)}} \\
&= c(n,p) \left(\int_{\mathbb{R}^n} \left|D \left(|Z(x)|^{\frac{p-2}{2}} Z(x)\right)\right|^2 dx\right)^{\frac{n}{n-2+p}} .
\end{aligned}$$

Recalling the definition of Z, calculating the gradient in the right-hand side of (6.4), using the properties of η and recalling that $\Phi \leq 1$, we get

$$
\begin{aligned}
&\| \partial_j[\eta^2(\partial_j u_\varepsilon)\,\Phi((|Du_\varepsilon|-\lambda)_+)]\|_{L^{\frac{np}{n-2+p}}(\mathbb{R}^n)} \\
&\quad\le c\left(\int_{\mathbb{R}^n}\left|D\left(\eta^p|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 dx\right)^{\frac{1}{p}} \\
&\quad\le c\left(\int_{B_r}\eta^{2p}\left|D\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 dx\right)^{\frac{1}{p}} \\
&\qquad + c\left(\int_{B_r}\eta^{2p-2}\,|D\eta|^2\,|Du_\varepsilon|^p\,dx\right)^{\frac{1}{p}}.
\end{aligned}
\tag{6.5}
$$

Inserting (6.5) into (6.3) and using the properties of η, we obtain

$$
\begin{aligned}
&\left|\int_{B_r}(\partial_j f_\varepsilon)(\partial_j u_\varepsilon)\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx\right| \\
&\quad\le c\,\|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\left(\int_{B_r}\eta^2\left|D\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 dx\right)^{\frac{1}{p}} \\
&\qquad + \frac{c}{r^{2/p}}\,\|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\left(\int_{B_r}|Du_\varepsilon|^p\,dx\right)^{\frac{1}{p}}.
\end{aligned}
\tag{6.6}
$$

Now, combining (6.6) with (6.2), we have

$$
\begin{aligned}
&\int_{B_r}\boldsymbol{\lambda}(|Du_\varepsilon|)\,|D^2u_\varepsilon|^2\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx \\
&\quad\le \frac{c}{r^2}\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx + \frac{c}{r^{2/p}}\,\|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\,\|Du_\varepsilon\|_{L^p(B_r)} \\
&\qquad + c\,\|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\sum_{j=1}^{n}\left(\int_{B_r}\eta^2\left|D\left(|\partial_j u_\varepsilon|^{\frac{p-2}{2}}(\partial_j u_\varepsilon)\,[\Phi((|Du_\varepsilon|-\lambda)_+)]^{\frac{p}{2}}\right)\right|^2 dx\right)^{\frac{1}{p}}.
\end{aligned}
$$

The last integral can be estimated using (5.20), (5.21) and (5.22). Thus we infer

$$
\begin{aligned}
&\int_{B_r}\boldsymbol{\lambda}(|Du_\varepsilon|)\,|D^2u_\varepsilon|^2\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx \\
&\quad\le \frac{c}{r^2}\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx + \frac{c}{r^{2/p}}\,\|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\,\|Du_\varepsilon\|_{L^p(B_r)} \\
&\qquad + c\,\|f_\varepsilon\|_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}\left(\int_{B_r}\boldsymbol{\lambda}(|Du_\varepsilon|)\,|D^2u_\varepsilon|^2\,\eta^2\,\Phi((|Du_\varepsilon|-\lambda)_+)\,dx\right)^{\frac{1}{p}},
\end{aligned}
$$

where $c\equiv c(n,p)>0$. Applying Young's inequality to reabsorb the last integral by the left-hand side, and then using inequality (5.28), we derive

$$
\int_{B_r}|D\mathcal{V}_\lambda(Du_\varepsilon)|^2\,\eta^2\,dx \le \frac{c}{r^2}\int_{B_r}(1+|Du_\varepsilon|^2)^{\frac{p}{2}}\,dx + c\,\|f_\varepsilon\|^{p'}_{L^{\frac{np}{n(p-1)+2-p}}(B_r)}.
$$

The desired conclusion follows by arguing as in the proofs of Proposition 5.1 and Theorem 1.1, observing that

$$(p^*)' = \frac{np}{np - n + p} < \frac{np}{n(p-1) + 2 - p} \qquad \text{for every } p > 1.$$

Finally, when $\lambda = 0$ the above proof can be greatly simplified, as we can choose $\Phi \equiv 1$ and we have

$$\mathcal{V}_0(Du_\varepsilon) = \frac{2}{p} |Du_\varepsilon|^{\frac{p-2}{2}} Du_\varepsilon \, .$$

We leave the details to the reader. □

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