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Published Version:
Capelli-Deruyts bitableaux and the classical Capelli generators of the center of the enveloping algebra U(gl(n)) / Andrea Brini. - In: COMMUNICATIONS IN ALGEBRA. - ISSN 0092-7872. - STAMPA. - Early Access:(2023), pp. 1-37. [10.1080/00927872.2023.2197072]

Availability:
This version is available at: https://hdl.handle.net/11585/921594 since: 2023-03-30

Published:
DOI: http://doi.org/10.1080/00927872.2023.2197072

(Article begins on next page)

Capelli-Deruyts bitableaux and the classical Capelli generators of the center of the enveloping algebra $\mathbf{U}(gl(n))$

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Abstract

In this paper, we consider a special class of Capelli bitableaux, namely the Capell bitableaux of the form $\mathbf{K}^\lambda = [Der^*_\lambda|Der_\lambda] \in \mathbf{U}(gl(n))$. The main results we prove are the hook coefficient lemma and the expansion theorem. Capelli-Deruyts bitableaux $\mathbf{K}^p_n$ of rectangular shape are of particular interest since they are central elements in the enveloping algebra $\mathbf{U}(gl(n))$. The expansion theorem implies that the central element $\mathbf{K}^p_n$ is explicitely described as a polynomial in the classical Capelli central elements $\mathbf{H}^{(j)}_n$. The hook coefficient lemma implies that the Capelli-Deruyts bitableaux $\mathbf{K}^p_n$ are (canonically) expressed as the products of column determinants.

Keyword: Capelli bitableaux; Capelli-Deruyts bitableaux; Capelli column determinants; central elements in $\mathbf{U}(gl(n))$; Lie superalgebras.

AMSC: 17B10, 05E10, 17B35

Contents

1 Introduction

The study of the center $\boldsymbol{\zeta}(n)$ of the enveloping algebra $\mathbf{U}(gl(n))$ of the general linear Lie algebra $gl(n, \mathbb{C})$, and the study of the algebra $\Lambda^*(n)$ of shifted symmetric polynomials have noble and rather independent origins and motivations. The theme of central elements in $\mathbf{U}(gl(n))$ is a standard one in the general theory of Lie algebras, see e.g. [18]. It is an old and actual one, since it is an offspring of the celebrated Capelli identity (see e.g. [11], [14], [21], [22], [36], [41], [42]), relates to its modern generalizations and applications (see e.g. [1], [24], [25], [29], [30], [31], [32], [40]) as well as to the theory of *Yangians* (see, e.g. [27], [28]).

Capelli bitableaux $[S|T]$ and their variants (such as *Young-Capelli bitableaux* and *double Young-Capelli bitableaux*) have been proved to be relevant in the study of the enveloping algebra $\mathbf{U}(gl(n)) = \mathbf{U}(gl(n), \mathbb{C})$ of the general linear Lie algebra and of its center $\zeta(n)$.

To be more specific, the *superalgebraic method of virtual variables* (see, e.g. [4], [5], [6], [7], [8], [9], [10]) allowed us to express remarkable classes of elements in $\mathbf{U}(gl(n))$, namely,

- the class of *Capelli bitableaux* $[S|T] \in \mathbf{U}(gl(n))$
- the class of *Young-Capelli bitableaux* $[S|\boxed{T}] \in \mathbf{U}(gl(n))$
- the class of *double Young-Capelli bitableaux* $[\,\boxed{S \mid T}\,] \in \mathbf{U}(gl(n))$

as the images - with respect to the $Ad_{gl(n)}$-adjoint equivariant Capelli *devirtualization epimorphism* - of simple expressions in an enveloping superalgebra $\mathbf{U}(gl(m_0|m_1 + n))$ (see, e.g [10]).

Capelli (determinantal) bitableaux are generalizations of the famous *column determinant element* in $\mathbf{U}(gl(n))$ introduced by Capelli in 1887 [11] (see, e.g. [9]). Young-Capelli bitableaux were introduced by the present authors several years ago [5], [6], [7] and might be regarded as generalizations of the Capelli column determinant elements in $\mathbf{U}(gl(n))$ as well as of the *Young symmetrizers* of the classical representation theory of symmetric groups (see, e.g. [42]). Double Young-Capelli bitableaux play a crucial role in the study of the center $\boldsymbol{\zeta}(n)$ of the enveloping algebra ([8], [10]).

In plain words, the Young-Capelli bitableau $[S|\boxed{T}]$ is obtained by adding a *column symmetrization* to the Capelli bitableau $[S|T]$ and turn out to be a linear combination of

Capelli bitableaux (see, e.g [10], Proposition 2.13). The double Young-Capelli bitableau $[\,\boxed{S \mid T}\,]$ is obtained by adding a further *row skew-symmetrization* to the Young-Capelli bitableau $[S \| \boxed{T}]$ ([10], Proposition 5.1), turn out to be a linear combination of Young-Capelli bitableaux (see, e.g [10], Proposition 2.14) and, therefore, it is in turn a linear combination of Capelli bitableaux.

Capelli bitableaux are the preimages - with respect to the *Koszul linear* $\mathbf{U}(gl(n))$-*equivariant isomorphism* $\mathcal{K}$ from the enveloping algebra $\mathbf{U}(gl(n))$ to the polynomial algebra $\mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n))$ ([26], [7], [9]) - of the classical *determinant bitableaux* (see, e.g. [19], [17], [16], [20], [4]). Hence, they are ruled by the *straightening laws* and the set of standard Capelli bitableaux is a basis of $\mathbf{U}(gl(n))$.

The set of standard Young-Capelli bitableaux is another relavant basis of $\mathbf{U}(gl(n))$ whose elements act in a nondegenerate orhogonal way on the set of standard right symmetrized bitableaux (the *Gordan-Capelli basis* of $\mathbb{C}[M_{n,n}]$) and this fact leads to explicit complete decompositions of the semisimple $\mathbf{U}(gl(n))$-module $\mathbb{C}[M_{n,n}]$ (see, e.g. [4], [5]).

The linear combinations of double Young-Capelli bitableaux

$$\mathbf{S}_\lambda(n) = \frac{1}{H(\tilde{\lambda})} \sum_S [\,\boxed{S \mid S}\,] \in \mathbf{U}(gl(n)), \tag{1}$$

where the sum is extended to all row (strictly) increasing tableaux S of shape $sh(S) = \widetilde{\lambda} \vdash h$, $\widetilde{\lambda}$ the conjugate shape/partition of λ ([1]), are *central elements* of $\mathbf{U}(gl(n))$.

We called the elements $\mathbf{S}_\lambda(n)$ the *Schur elements*. The Schur elements $\mathbf{S}_\lambda(n)$ are the preimages - with respect to the Harish-Chandra isomorphism - of the elements of the basis of shifted Schur polynomials $s^*_{\lambda|n}$ of the algebra $\Lambda^*(n)$ of shifted symmetric polynomials [38], [33]. Hence, the Schur elements are the same [10] as the *quantum immanants* ([38], [31], [32], [33]) , first presented by Okounkov as traces of *fusion matrices* ([31], [32]) and, recently, described by the present authors as linear combinations (with explicit coefficients) of "diagonal" *Capelli immanants* [8]. Presentation (1) of Schur elements/quantum immanants doesn't involve the irreducible characters of symmetric groups. Furthermore, it is better suited to the study of the eigenvalues on irreducible $gl(n)-$modules and of the duality in the algebra $\boldsymbol{\zeta}(n)$, as well as to the study of the limit $n \to \infty$, via the *Olshanski decomposition* (see, Olshanski [34], [35] and Molev [27], pp. 928 ff.)

In this paper, we consider a special class of Capelli bitableaux, namely the class of *Capelli-Deruyts bitableaux*. These elements are Capelli bitableaux of the form

$$\mathbf{K}^\lambda = [Der^*_\lambda | Der_\lambda] \in \mathbf{U}(gl(n)),$$

[1]Given a partition (shape) $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p) \vdash n$, let $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2 \geq \cdots \geq \widetilde{\lambda}_q) \vdash n$ denote its *conjugate* partition, where $\widetilde{\lambda}_s = \#\{t; \lambda_t \geq s\}$.

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$ is a partition with $\lambda_1 \leq n$, and

– Der_λ is the *Deruyts tableaux* of shape λ, that is the Young tableau of shape λ:

$$Der_\lambda = \begin{bmatrix} 1\ 2\ \ldots\ \ldots\ \ldots\ \lambda_1 \\ 1\ 2\ \ldots\ \ldots\ \lambda_2 \\ \ldots\ \ldots\ \ldots \\ 1\ 2\ \ldots\ \lambda_p \end{bmatrix}$$

– Der_λ^* is the *reverse Deruyts tableaux* of shape λ, that is the Young tableau of shape λ:

$$Der_\lambda^* = \begin{bmatrix} \lambda_1\ \ldots\ \ldots\ \ldots\ 2\ 1 \\ \lambda_2\ \ldots\ \ldots\ 2\ 1 \\ \ldots\ \ldots\ \ldots \\ \lambda_p\ \ldots\ 2\ 1. \end{bmatrix}.$$

Capelli-Deruyts bitableaux arise, in a natural way, as generalizations to arbitrary shapes $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$ of the well-known *Capelli column determinant*[2] elements:

$$\mathbf{H}_n^{(n)} = \mathbf{cdet} \begin{pmatrix} e_{1,1} + (n-1) & e_{1,2} & \ldots & e_{1,n} \\ e_{2,1} & e_{2,2} + (n-2) & \ldots & e_{2,n} \\ \vdots & \vdots & \vdots & \\ e_{n,1} & e_{n,2} & \ldots & e_{n,n} \end{pmatrix} \in \mathbf{U}(gl(n)), \qquad (2)$$

introduced by Alfredo Capelli [11] in the celebrated identities that bear his name (see, e.g. [11], [14], [21], [22], [36], [41], [42], [1], [24], [25], [29], [30], [31], [32], [40]).

The main results we prove are the following:

– **The hook coefficient lemma**: let v_μ be a $gl(n, \mathbb{C})$-highest weight vector of weight $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n)$, with $\mu_i \in \mathbb{N}$ for every $i = 1, 2, \ldots, n$. Then, v_μ is an *eigenvector* of the action of the Capelli-Deruyts bitableau $\mathbf{K}^\lambda$ with *eigenvalue* the (signed) product of *hook numbers* in the Ferrers diagram of the partition μ (Proposition 5).

– **The expansion theorem**: the Capelli-Deruyts bitableau $\mathbf{K}^\lambda \in \mathbf{U}(gl(n))$ expands as a polynomial, with explicit coefficients, in the *Capelli generators*

$$\mathbf{H}_k^{(j)} = \sum_{1 \leq i_1 < \cdots < i_j \leq k} \mathbf{cdet} \begin{pmatrix} e_{i_1,i_1} + (j-1) & e_{i_1,i_2} & \ldots & e_{i_1,i_j} \\ e_{i_2,i_1} & e_{i_2,i_2} + (j-2) & \ldots & e_{i_2,i_j} \\ \vdots & \vdots & \vdots & \\ e_{i_k,i_1} & e_{i_j,i_2} & \ldots & e_{i_j,i_j} \end{pmatrix}$$

[2]The symbol **cdet** denotes the column determinat of a matrix $A = [a_{ij}]$ with noncommutative entries: $\mathbf{cdet}(A) = \sum_\sigma (-1)^{|\sigma|}\ a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$.

of the centers of the enveloping algebras $\mathbf{U}(gl(k))$, $k = 1, 2, \ldots, n$, $j = 1, 2, \ldots, k$ (Theorem 3).

Capelli-Deruyts bitableaux $\mathbf{K_n^p}$ of *rectangular shape* $\lambda = n^p = (n, \overset{p\ times}{n, n, \cdots}, n)$ are of particular interest since they are *central elements* in the enveloping algebra $\mathbf{U}(gl(n))$.

– The expansion theorem implies that the Capelli-Deruyts bitableau $\mathbf{K_n^p}$ (with p rows) equals the product of the Capelli-Deruyts bitableau $\mathbf{K_n^{p-1}}$ (with $p-1$ rows) and the central element

$$\mathbf{C}_n(p-1) = \sum_{j=0}^{n} (-1)^{n-j}(p-1)_{n-j}\ \mathbf{H}_n^{(j)}$$

(see Corollary 1). Hence, by iterating this procedure, the central element $\mathbf{K_n^p}$ is explicitely described as a polynomial in the classical Capelli central elements $\mathbf{H}_n^{(j)}$ (see Corollary 3).

– The hook coefficient lemma implies -via the HarishChandra isomorphism- that the element $\mathbf{C}_n(p)$ also equals the column determinant element

$$\mathbf{H}_n(p) = \mathbf{cdet}\left[e_{h,k} + \delta_{hk}(-p+n-h)\right]_{h,k=1,\ldots,n} \in \mathbf{U}(gl(n)).$$

Notice that

$$\mathbf{H}_n(0) = \ \mathbf{cdet}\begin{pmatrix} e_{1,1}+(n-1) & e_{1,2} & \ldots & e_{1,n} \\ e_{2,1} & e_{2,2}+(n-2) & \ldots & e_{2,n} \\ \vdots & \vdots & \vdots & \\ e_{n,1} & e_{n,2} & \ldots & e_{n,n} \end{pmatrix} = \mathbf{H}_n^{(n)},$$

the classical Capelli column determinant element.

From these facts, the Capelli-Deruyts bitableaux $\mathbf{K_n^p}$ are (canonically) expressed as the products of column determinants:

$$\mathbf{K_n^p} = (-1)^{n\binom{p}{2}}\ \mathbf{H}_n(p-1)\ \cdots\ \mathbf{H}_n(1)\ \mathbf{H}_n(0)$$

(see Corollary 7).

The method of *superalgebraic virtual variables* ([4], [5], [6], [7], [8], [9], [10]) plays a crucial role in the present paper; we provide a short presentation of the method in the Appendix.

2 The classical Capelli identities

The *algebra of algebraic forms* $\mathbf{f}(\underline{x}_1, \ldots, \underline{x}_n)$ *in* n *vector variables* $\underline{x}_i = (\underline{x}_{i1}, \ldots, \underline{x}_{id})$ *of dimension* d is the polynomial algebra in $n \times d$ (commutative) variables:

$$\mathbb{C}[M_{n,d}] = \mathbb{C}[x_{ij}]_{i=1,\ldots,n;j=1,\ldots,d},$$

and $M_{n,d}$ denotes the matrix with n rows and d columns with "generic" entries x_{ij}:

$$M_{n,d} = [x_{ij}]_{i=1,\ldots,n;j=1,\ldots,d} = \begin{bmatrix} x_{11} & \ldots & x_{1d} \\ x_{21} & \ldots & x_{2d} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nd} \end{bmatrix}. \tag{3}$$

The algebra $\mathbb{C}[M_{n,d}]$ is a $\mathbf{U}(gl(n))-$module, with respect to the action:

$$e_{x_j,x_i} \cdot \mathbf{f} = D^l_{x_j,x_i}(\mathbf{f}),$$

for every $\mathbf{f} \in \mathbb{C}[M_{n,d}]$, where, for any $i, j = 1, 2, \ldots, n$, where $D^l_{x_j,x_i}$ is the unique *derivation* of the algebra $\mathbb{C}[M_{n,d}]$ such that

$$D^l_{x_j,x_i}(x_{hk}) = \delta_{ih}\ x_{jk},$$

for every $k = 1, 2. \ldots, d$.

Proposition 1. (**The Capelli identities**, **1887**)

$$\mathbf{H}_n^{(n)}(\mathbf{f}) = \begin{cases} 0 & \text{if } n > d \\ [\underline{x}_1, \ldots, \underline{x}_n]\ \Omega_n(\mathbf{f}) & \text{if } n = d, \end{cases}$$

where $\mathbf{f}(\underline{x}_1, \ldots, \underline{x}_n) \in \mathbb{C}[M_{n,d}]$ *is an* algebraic form *(polynomial) in the* n *vector variables* $\underline{x}_i = (x_{i1}, \ldots, x_{id})$ *of dimension* d*, and, if* $d = n$*,* $[\underline{x}_1, \ldots, \underline{x}_n]$ *is the* bracket

$$[\underline{x}_1, \ldots, \underline{x}_n] = det \begin{bmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & \vdots & \vdots \\ x_{n1} & \ldots & x_{nn} \end{bmatrix},$$

and Ω_n *is the* Cayley Ω-process

$$\Omega_n = det \begin{bmatrix} \frac{\partial}{\partial x_{11}} & \ldots & \frac{\partial}{\partial x_{1n}} \\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x_{n1}} & \ldots & \frac{\partial}{\partial x_{nn}} \end{bmatrix}.$$

□

From [9], we recall that the determinant element $\mathbf{H}_n^{(n)}$ can be written as the (one row) *Capelli-Deruyts bitableau* $[n \ldots 21|12 \ldots n]$ ([5], see also [8], [26]).

Proposition 2. *The element*

$$\mathbf{H}_n^{(n)} = \ \boldsymbol{cdet} \begin{pmatrix} e_{1,1} + (n-1) & e_{1,2} & \ldots & e_{1,n} \\ e_{2,1} & e_{2,2} + (n-2) & \ldots & e_{2,n} \\ \vdots & \vdots & \vdots & \\ e_{n,1} & e_{n,2} & \ldots & e_{n,n} \end{pmatrix} \in \mathbf{U}(gl(n))$$

equals the one row Capelli-Deruyts bitableau *(see, e.g. Subsection 9.6 below)*

$$[n \ldots 21|12 \ldots n] = \mathfrak{p}\left(e_{n,\alpha} \cdots e_{2,\alpha} e_{1,\alpha} \cdot e_{\alpha,1} e_{\alpha,2} \cdots e_{\alpha,n}\right),$$

where $\mathfrak{p}$ *denotes the* Capelli devirtualization epimorphism *(see, e.g. Subsection 9.5 below).*

From eq. (2) and Proposition 2, it follows:

Proposition 3. *We have:*

1. *Let v_μ be a $gl(n, \mathbb{C})$-highest weight vector of weight $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n)$, with $\mu_i \in \mathbb{N}$ for every $i = 1, 2, \ldots, n$. Then v_μ is an* eigenvector *of the action of* $\mathbf{H}_n^{(n)}$ *with* eigenvalue*:*

$$(\mu_1 + n - 1)(\mu_2 + n - 2) \cdots \mu_n.$$

   In symbols,

$$\mathbf{H}_n^{(n)} \cdot v_\mu = ((\mu_1 + n - 1)(\mu_2 + n - 2) \cdots \mu_n) \ \ v_\mu.$$

2. *The element* $\mathbf{H}_n^{(n)}$ *is* central *in the enveloping algebra* $\mathbf{U}(gl(n))$*.*

3 The Capelli-Deruyts bitableaux in $\mathbf{U}(gl(n))$

We generalize the *one row* Capelli bitableau $\mathbf{H}_n^{(n)} = [n \ldots 21|12 \ldots n]$ to arbitrary shapes (partitions)

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p), \qquad \lambda_i \in \mathbb{Z}^+.$$

3.1 Capelli-Deruyts bitableaux $\mathbf{K}^{\lambda}$ of shape λ.

Given a partition(shape) $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$, we recall that the *Deruyts tableaux* of shape λ is the Young tableau

$$Der_{\lambda} = (\underline{\lambda_1}, \underline{\lambda_2}, \ldots, \underline{\lambda_p}) \tag{4}$$

and the *reverse Deruyts tableaux* of shape λ is the Young tableau

$$Der^*_{\lambda} = (\underline{\lambda_1}^*, \underline{\lambda_2}^*, \ldots, \underline{\lambda_p}^*),$$

where

$$\underline{\lambda_i} = 1\ 2\ \cdots\ \lambda_i$$

and

$$\underline{\lambda_i}^* = \lambda_i\ \cdots\ 2\ 1,$$

for every $i = 1, 2, \ldots, p$.

The *Capelli-Deruyts bitableau* $\mathbf{K}^{\lambda}$ is the Capelli bitableau in $\mathbf{U}(gl(n))$, $n \geq \lambda_1$:

$$\mathbf{K}^{\lambda} = [Der^*_{\lambda} | Der_{\lambda}] = \mathfrak{p}\big(e_{Der^*_{\lambda} C_{\lambda}} \cdot e_{C_{\lambda} Der_{\lambda}}\big),$$

where $\mathfrak{p}$ denotes the Capelli devirtualization epimorphism and $e_{Der^*_{\lambda} C_{\lambda}}$, $e_{C_{\lambda} Der_{\lambda}}$ are *bitableax monomials* (see., e.g. Subsection 9.6, eq. (9.6)).

Example 1. *Let* $\lambda = (3, 2, 2)$*. Then*

$$\mathbf{K}^{(3,2,2)} = \left[\begin{array}{c|c} 3\ 2\ 1 & 1\ 2\ 3 \\ 2\ 1 & 1\ 2 \\ 2\ 1 & 1\ 2 \end{array} \right] =$$

$$= \mathfrak{p}\big(e_{3\alpha_1} e_{2\alpha_1} e_{1\alpha_1} e_{2\alpha_2} e_{1\alpha_2} e_{2\alpha_3} e_{1\alpha_3} \cdot e_{\alpha_1 1} e_{\alpha_1 2} e_{\alpha_1 3} e_{\alpha_2 1} e_{\alpha_2 2} e_{\alpha_3 1} e_{\alpha_3 2}\big) \in \mathbf{U}(gl(n)), \quad n \geq 3,$$

where $\alpha_1, \alpha_2, \alpha_3$ *are (arbitrary, distinct)* positive virtual symbols.

□

Remark 1. *Given a Young tableau*

$$T = \left[\begin{array}{l} x_{11}\ x_{12}\ \cdots\ \cdots\ \cdots\ x_{1\lambda_1} \\ x_{21}\ x_{22}\ \cdots\ \cdots\ \cdot\cdot\ x_{2\lambda_2} \\ \vdots \\ x_{i1}\ x_{i2}\ \cdots\ \cdots\ \cdot\ x_{i\lambda_i} \\ \vdots \\ x_{p1}\ x_{p2}\ \cdots\ \cdots\ x_{p\lambda_p} \end{array} \right], \ x_{ij} \in X, \tag{5}$$

of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$ *over the set* X *is said to be of* Deruyts type *whenever*

$$\{x_{i1},\ x_{i2}, \ldots,\ x_{i\lambda_i}\} \subseteq \{x_{i-1\ 1},\ x_{i-1\ 2}, \ldots,\ x_{i-1\ \lambda_{i-1}}\},$$

for $i = 2, \ldots, p$.

Clearly, any tableau of Deruyts type (5) can be regarded as a Deruyts tableau (4), by suitably renaming and reordering the entries.

3.2 The Capelli-Deruyts bitableaux $\mathbf{K_n^p}$ of rectangular shape $\lambda = n^p$

Given any positive integer p, we define the *rectangular Capelli/Deruyts bitableau*, with p rows of length $\lambda_1 = \lambda_2 = \cdots = \lambda_p = n$:

$$\mathbf{K_n^p} = \left[\begin{array}{l|l} n\ n-1\ \ldots\ 3\ 2\ 1 & 1\ 2\ 3\ \ldots\ n-1\ n \\ n\ n-1\ \ldots\ 3\ 2\ 1 & 1\ 2\ 3\ \ldots\ n-1\ n \\ \ldots & \ldots \\ & \\ \ldots & \ldots \\ n\ n-1\ \ldots\ 3\ 2\ 1 & 1\ 2\ 3\ \ldots\ n-1\ n \end{array} \right] \in \mathbf{U}(gl(n)).$$

From Proposition 26, we infer:

Proposition 4. *The elements* $\mathbf{K_n^p}$ *are central in* $\mathbf{U}(gl(n))$.

Set, by definition, $\mathbf{K_n^0} = \mathbf{1}$.

4 The hook eigenvalue Theorem for Capelli-Deruyts bitableaux

Any rectangular Capelli-Deruyts bitableau $\mathbf{K_n^p}$ well behaves on $gl(n, \mathbb{C})$-highest weight vectors (compare with Proposition 3, item 1)).

Theorem 1. *(***The hook coefficient lemma***)*

Let v_μ *be a highest weight vector of weight* $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n)$, *with* $\mu_i \in \mathbb{N}$ *for every* $i = 1, 2, \ldots, n$. *Then* v_μ *is an* eigenvector *of the action of* $\mathbf{K_n^p}$ *with* eigenvalue *the (signed) product of* hook numbers *in the Ferrers diagram of the partition* μ*:*

$$(-1)^{\binom{p}{2}n} \left(\prod_{j=1}^{p} (\mu_1 - j + n)(\mu_2 - j + n - 1) \cdots (\mu_n - j + 1) \right).$$

In symbols,

$$\mathbf{K_n^p} \cdot v_\mu = (-1)^{\binom{p}{2}n} \left(\prod_{j=1}^{p} (\mu_1 - j + n)(\mu_2 - j + n - 1) \cdots (\mu_n - j + 1) \right) v_\mu.$$

Theorem 1 generalizes to arbitrary Capelli-Deruyts bitableaux $\mathbf{K}_\lambda$ of shape λ as follows:

Proposition 5. *Let v_μ be a highest weight vector of weight $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n)$, with $\mu_i \in \mathbb{N}$ for every $i = 1, 2, \ldots, n$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_p)$ be a partition(shape). Then*

$$\begin{aligned} \mathbf{K}^\lambda \cdot v_\mu = \ & (-1)^{\lambda_p(\lambda_{p-1}+\cdots+\lambda_1)+\lambda_{p-1}(\lambda_{p-2}+\cdots+\lambda_1)+\cdots+\lambda_2\lambda_1} \times \\ & \times \left(\prod_{i=1}^{p} (\mu_1 - i + \lambda_i)(\mu_2 - i + \lambda_i - 1) \cdots (\mu_{\lambda_i} - i + 1) \right) v_\mu. \end{aligned}$$

5 The factorization Theorem for Capelli-Deruyts bitableaux

Let $J = \{j_1 < j_2 < \cdots < j_k\} \subseteq \underline{n} = \{1, 2; \ldots, n\}$. With a slight abuse of notation, we write $\underline{J}$ for the increasing word $\underline{J} = j_1 j_2 \cdots j_k$ and $\underline{J}^*$ for the decreasing word $\underline{J}^* = j_k \cdots j_2 j_1$.

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$, set $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_p$.

We have

$$\mathbf{K}^\lambda = \left[\begin{array}{c|c} \underline{\lambda_1}^* & \underline{\lambda_1} \\ \underline{\lambda_2}^* & \underline{\lambda_2} \\ \vdots & \vdots \\ \underline{\lambda_p}^* & \underline{\lambda_p} \end{array} \right]$$

and, consistently, we write, for $J \subseteq M$,

$$\left[\begin{array}{c} \mathbf{K}^\lambda \\ J \end{array} \right] = \left[\begin{array}{c|c} \underline{\lambda_1}^* & \underline{\lambda_1} \\ \underline{\lambda_2}^* & \underline{\lambda_2} \\ \vdots & \vdots \\ \underline{\lambda_p}^* & \underline{\lambda_p} \\ \underline{J}^* & \underline{J} \end{array} \right], \quad [J] = [\underline{J}^* | \underline{J}].$$

Theorem 2. (**The row insertion theorem**) *Let $m \leq \lambda_p$. Given $M \subseteq \underline{\lambda_p}$, $|M| = m$, we have*

$$[M^*|M] \ \mathbf{K}^\lambda = \sum_{k=0}^{m} \langle p \rangle_{m-k} \sum_{J;\ J \subseteq M;\ |J|=k} (-1)^{|\lambda|k} \left[\begin{array}{c} \mathbf{K}^\lambda \\ J \end{array} \right],$$

where $\langle p \rangle_j$ *denonotes the* raising factorial

$$\langle p \rangle_j = p(p+1)\cdots(p+j-1).$$

□

Theorem 3. (**The expansion theorem**) *Let* $m \leq \lambda_p$. *Given* $M \subseteq \underline{\lambda_p}$, $|M| = m$, *we have*

$$(-1)^{|\lambda| m} \left[\begin{array}{c} \mathbf{K}^{\lambda} \\ M \end{array} \right] = \sum_{k=0}^{m} (-1)^{m-k} (p)_{m-k} \sum_{J;\ J \subseteq M;\ |J|=k} [\underline{J}^{*} | \underline{J}]\ \mathbf{K}^{\lambda},$$

where $(p)_j$ *denonotes the* falling factorial

$$(p)_j = p(p-1)\cdots(p-j+1).$$

Proof. By Theorem 2,

$$\sum_{k=0}^{m} (-1)^{m-k} (p)_{m-k} \sum_{J;\ J \subseteq\ M;\ |J|=k} [J]\ \mathbf{K}^{\lambda} =$$

$$= \sum_{k=0}^{m} (-1)^{m-k} (p)_{m-k} \sum_{J;\ J \subseteq\ M;\ |J|=k} \sum_{i=0}^{k} \langle p \rangle_{k-i} \sum_{I;\ I \subseteq J;\ |I|=i} (-1)^{|\lambda| i} \left[\begin{array}{c} \mathbf{K}^{\lambda} \\ I \end{array} \right] =$$

$$= \sum_{i=0}^{m} \sum_{k=i}^{m} \sum_{I;\ I \subseteq\ M;\ |I|=i} \Big(\sum_{J;\ M\ \supseteq J\ \supseteq I;\ |J|=k} (-1)^{m-k} (p)_{m-k} \ \langle p \rangle_{k-i} \Big)\ (-1)^{|\lambda| i} \left[\begin{array}{c} \mathbf{K}^{\lambda} \\ I \end{array} \right] =$$

$$= \sum_{i=0}^{m} \sum_{I;\ I \subseteq\ M;\ |I|=i} \Big(\sum_{k=i}^{m} (-1)^{m-k} (p)_{m-k} \ \langle p \rangle_{k-i} \binom{m-i}{k-i} \Big)\ (-1)^{|\lambda| i} \left[\begin{array}{c} \mathbf{K}^{\lambda} \\ I \end{array} \right] =$$

$$= \sum_{i=0}^{m} \sum_{I;\ I \subseteq\ M;\ |I|=i} \Big((m-i)! \sum_{k=i}^{m} (-1)^{m-k} \binom{p}{m-k} \left\langle \begin{array}{c} p \\ k-i \end{array} \right\rangle \Big)\ (-1)^{|\lambda| i} \left[\begin{array}{c} \mathbf{K}^{\lambda} \\ I \end{array} \right] =$$

$$= \sum_{i=0}^{m} \sum_{I;\ I \subseteq\ M;\ |I|=i} \big((m-i)!\ \delta_{m-i,0} \big)\ (-1)^{|\lambda| i} \left[\begin{array}{c} \mathbf{K}^{\lambda} \\ I \end{array} \right] =$$

$$= \sum_{i=0}^{m} \sum_{I;\ I \subseteq\ M;\ |I|=i} \big((m-i)!\ \delta_{m,i} \big)\ (-1)^{|\lambda| i} \left[\begin{array}{c} \mathbf{K}^{\lambda} \\ I \end{array} \right] = (-1)^{|\lambda| m} \left[\begin{array}{c} \mathbf{K}^{\lambda} \\ M \end{array} \right].$$

□

Example 2.

1. *We have*

$$[21|12]\left[\begin{array}{l|l} 3\ 2\ 1 & 1\ 2\ 3 \\ 2\ 1 & 1\ 2 \end{array}\right] = 6\left[\begin{array}{l|l} 3\ 2\ 1 & 1\ 2\ 3 \\ 2\ 1 & 1\ 2 \end{array}\right] + 2\left[\begin{array}{l|l} 3\ 2\ 1 & 1\ 2\ 3 \\ 2\ 1 & 1\ 2 \\ 1 & 1 \end{array}\right]$$
$$+ 2\left[\begin{array}{l|l} 3\ 2\ 1 & 1\ 2\ 3 \\ 2\ 1 & 1\ 2 \\ 2 & 2 \end{array}\right] + \left[\begin{array}{l|l} 3\ 2\ 1 & 1\ 2\ 3 \\ 2\ 1 & 1\ 2 \\ 2\ 1 & 1\ 2 \end{array}\right].$$

2. *We have*

$$\left[\begin{array}{l|l} 3\ 2\ 1 & 1\ 2\ 3 \\ 2\ 1 & 1\ 2 \\ 2\ 1 & 1\ 2 \end{array}\right] = 2\left[\begin{array}{l|l} 3\ 2\ 1 & 1\ 2\ 3 \\ 2\ 1 & 1\ 2 \end{array}\right] - 2\,[1|1]\left[\begin{array}{l|l} 3\ 2\ 1 & 1\ 2\ 3 \\ 2\ 1 & 1\ 2 \end{array}\right]$$
$$- 2\,[2|2]\left[\begin{array}{l|l} 3\ 2\ 1 & 1\ 2\ 3 \\ 2\ 1 & 1\ 2 \end{array}\right] + [2\ 1|1\ 2]\left[\begin{array}{l|l} 3\ 2\ 1 & 1\ 2\ 3 \\ 2\ 1 & 1\ 2 \end{array}\right].$$

6 The center $\zeta(n)$ of $\mathbf{U}(gl(n))$

6.1 The Capelli generators of the center $\zeta(n)$ of $\mathbf{U}(gl(n))$

In the enveloping algebra $\mathbf{U}(gl(n))$, given any increasing k-tuple integers $1 \leq i_1 < \cdots < i_k \leq n$.

We recall that the column determinant

$$\mathbf{cdet}\begin{pmatrix} e_{i_1,i_1} + (k-1) & e_{i_1,i_2} & \dots & e_{i_1,i_k} \\ e_{i_2,i_1} & e_{i_2,i_2} + (k-2) & \dots & e_{i_2,i_k} \\ \vdots & \vdots & \vdots & \\ e_{i_k,i_1} & e_{i_k,i_2} & \dots & e_{i_k,i_k} \end{pmatrix} \in \mathbf{U}(gl(n))$$

equals the *one-row* Capelli-Deruyts bitableau

$$[i_k i_{k-1} \cdots i_1 | i_1 \cdots i_{k-1} i_k] = \mathfrak{p}\left(e_{i_k\alpha} e_{i_{k-1}\alpha} \cdots e_{i_1\alpha} e_{\alpha i_1} \cdots e_{\alpha i_{k-1}} e_{\alpha i_k}\right) \in \mathbf{U}(gl(n))$$

(see, e.g. [9]).

Consider the k-th *Capelli element*

$$\mathbf{H}_n^{(k)} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbf{cdet}\begin{pmatrix} e_{i_1,i_1} + (k-1) & e_{i_1,i_2} & \dots & e_{i_1,i_k} \\ e_{i_2,i_1} & e_{i_2,i_2} + (k-2) & \dots & e_{i_2,i_k} \\ \vdots & \vdots & \vdots & \\ e_{i_k,i_1} & e_{i_k,i_2} & \dots & e_{i_k,i_k} \end{pmatrix}$$

Clearly, we have

$$\mathbf{H}_n^{(k)} = \sum_{1 \le i_1 < \cdots < i_k \le n} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k]. \tag{6}$$

We recall the following fundamental result, proved by Capelli in two papers ([12], [13]) with deceiving titles.

Proposition 6. (**Capelli**, **1893**) *Let* $\zeta(n)$ *denote be* center *of* $\mathbf{U}(gl(n))$*. We have:*

- *The elements* $\mathbf{H}_n^{(k)}$*,* $k = 1, 2, \ldots, n$ *belong to the center* $\zeta(n)$*.*
- *The subalgebra* $\zeta(n)$ *of* $\mathbf{U}(gl(n))$ *is the polynomial algebra*

  $$\zeta(n) \ = \ \mathbb{C}[\mathbf{H}_n^{(1)}, \mathbf{H}_n^{(2)}, \ldots, \mathbf{H}_n^{(n)}],$$

  where

  $$\mathbf{H}_n^{(1)}, \mathbf{H}_n^{(2)}, \ldots, \mathbf{H}_n^{(n)}$$

  is a set of algebraically independent generators of $\zeta(n)$*.*

6.2 The factorization Theorem for rectangular Capelli-Deruyts bitableaux $\mathbf{K_n^p}$

The crucial result in this section is that Capelli-Deruyts bitableaux $\mathbf{K_n^p}$ of *rectangular* shape $\lambda = n^p$ expand into *commutative* polynomials in the Capelli elements $\mathbf{H}_n^{(j)}$, with explicit coefficients.

The next result was announced, without proof, in [3]. By eq. (6), it is a special case of Theorem 3.

Corollary 1. (**ExpansionTheorem**)

Let $p \in \mathbb{N}$ *and set* $\mathbf{H}_n^{(0)} = \mathbf{1}$*, by definition. The following identity in* $\zeta(n)$ *holds:*

$$\mathbf{K_n^p} = (-1)^{n(p-1)} \ \mathbf{C}_n(p-1) \ \mathbf{K_n^{p-1}},$$

where, given $p \in \mathbb{N}$*,*

$$\mathbf{C}_n(p-1) = \sum_{j=0}^{n} \ (-1)^{n-j} (p-1)_{n-j} \ \mathbf{H}_n^{(j)}. \tag{7}$$

where

$$(m)_k = m(m-1) \cdots (m-k+1), \ \ m, k \in \mathbb{N}$$

denotes the falling factorial coefficient.

If $p = 0$, eq. (7) collapses to

$$\mathbf{K}_{\mathbf{n}}^{\mathbf{1}} = \mathbf{H}_n^{(n)} = \mathbf{C}_n(0).$$

Notice that the linear relations (7), for $p = 0, \ldots, n-1$, yield a nonsingular triangular coefficients matrix.

Corollary 2. *The subalgebra $\zeta(n)$ of $\mathbf{U}(gl(n))$ is the polynomial algebra*

$$\zeta(n) \ = \ \mathbb{C}[\mathbf{C}_n(0), \mathbf{C}_n(1), \ldots, \mathbf{C}_n(n-1)],$$

where

$$\mathbf{C}_n(0), \mathbf{C}_n(1), \ldots, \mathbf{C}_n(n-1)$$

is a set of algebraically independent generators of $\zeta(n)$.

Corollary 3. *The rectangular Capelli-Deruyts bitableau $\mathbf{K}_{\mathbf{n}}^{\mathbf{p}}$ equals the* commutative polynomial *in the Capelli generators:*

$$\mathbf{K}_{\mathbf{n}}^{\mathbf{p}} = (-1)^{n\binom{p}{2}} \ \mathbf{C}_n(p-1) \ \cdots \ \mathbf{C}_n(1) \ \mathbf{C}_n(0).$$

Example 3. *Let $n = 3$, $p = 2$. Then*

$$\mathbf{K}_{\mathbf{3}}^{\mathbf{2}} = \left[\begin{array}{c|c} 3 \ 2 \ 1 & 1 \ 2 \ 3 \\ 3 \ 2 \ 1 & 1 \ 2 \ 3 \end{array} \right] = \ - \ \mathbf{C}_3(1) \ \mathbf{C}_3(0) = \left(\mathbf{H}_3^{(2)} - \mathbf{H}_3^{(3)} \right) \mathbf{H}_3^{(3)}.$$

□

6.3 The Harish-Chandra isomorphism and the algebra $\Lambda^*(n)$ of shifted symmetric polynomials

In this subsection we follow A. Okounkov and G. Olshanski [33].

As in the classical context of the algebra $\Lambda(n)$ of symmetric polynomials in n variables $x_1, x_2, \ldots, x_n$, the algebra $\Lambda^*(n)$ of *shifted symmetric polynomials* is an algebra of polynomials $p(x_1, x_2, \ldots, x_n)$ but the ordinary symmetry is replaced by the *shifted symmetry*:

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i+1} - 1, x_i + 1, \ldots, x_n),$$

for $i = 1, 2, \ldots, n-1$.

The *shifted elementary symmetric polynomials* are the elements of $\Lambda^*(n)$

– for every $r \in \mathbb{Z}^+$,

$$\mathbf{e}_k^*(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} (x_{i_1} + k - 1)(x_{i_2} + k - 2) \cdots (x_{i_k}),$$

– $\mathbf{e}_0^*(x_1, x_2, \ldots, x_n) = \mathbf{1}$.

The *Harish-Chandra isomorphism* is the algebra isomorphism

$$\chi_n : \zeta(n) \longrightarrow \Lambda^*(n), \qquad A \mapsto \chi_n(A),$$

$\chi_n(A)$ being the shifted symmetric polynomial such that, for every highest weight module V_μ, the evaluation $\chi_n(A)(\mu_1, \mu_2, \ldots, \mu_n)$ equals the eigenvalue of $A \in \zeta(n)$ in V_μ ([33], Proposition **2**.**1**).

6.4 The Harish-Chandra isomorphism interpretation of Theorem 1 and Theorem 3

Notice that

$$\chi_n(\mathbf{H}_n^{(r)}) = \mathbf{e}_r^*(x_1, x_2, \ldots, x_n) \in \Lambda^*(n),$$

for every $r = 1, 2, \ldots, n$.

Furthermore, from Theorem 1 it follows

Corollary 4.

$$\chi_n(\mathbf{K_n^p}) = (-1)^{\binom{p}{2}n} \left(\prod_{j=1}^{p} (x_1 - j + n)(x_2 - j + n - 1) \cdots (x_n - j - 1) \right).$$

By Corollary 1, we have

$$\chi_n(\mathbf{K_n^{p+1}}) = \ \chi_n(\mathbf{C}_n(p)) \ \chi_n(\mathbf{K_n^p}).$$

and Corollary 4 implies

Proposition 7. *For every $p \in \mathbb{N}$,*

$$\chi_n(\mathbf{C}_n(p)) = (x_1 - p + n - 1)(x_2 - p + n - 2) \cdots (x_n - p).$$

Proposition 8. *The set*

$$\chi_n(\mathbf{C}_n(0)), \ \chi_n(\mathbf{C}_n(1)), \ \ldots \ , \ \chi_n(\mathbf{C}_n(n-1))$$

is a system of algebraically independent generators of the ring $\Lambda^*(n)$ of shifted symmetric polynomials in the variables $x_1, x_2, \ldots, x_n$.

Given $p \in \mathbb{N}$, consider the column determinant

$$\mathbf{H}_n(p) = \mathbf{cdet} \begin{pmatrix} e_{1,1} - p + (n-1) & e_{1,2} & \dots & e_{1,n} \\ e_{2,1} & e_{2,2} - p + (n-2) & \dots & e_{2,n} \\ \vdots & \vdots & \vdots & \\ e_{n,1} & e_{n,2} & \dots & e_{n,n} - p \end{pmatrix}. \qquad (8)$$

We recall a standard result (for an elementary proof see e.g. [41]):

Proposition 9. *For every $p \in \mathbb{N}$, the element*

$$\mathbf{H}_n(p) = \boldsymbol{cdet}\left[e_{h,k} + \delta_{hk}(-p + n - h)\right]_{h,k=1,\dots,n} \in \mathbf{U}(gl(n)).$$

is central. In symbols, $\mathbf{H}_n(p) \in \zeta(n)$.

Equation (8), Proposition 9 and Proposition 7 imply

$$\chi_n(\mathbf{H}_n(p)) = (x_1 - p + n - 1)(x_2 - p + n - 2)\cdots(x_n - p) = \chi_n(\mathbf{C}_n(p)).$$

Hence, we get the well-known identity (see, e.g. [27]):

Corollary 5. *For every $p \in \mathbb{N}$, we have*

$$\begin{aligned} \mathbf{H}_n(p) &= \boldsymbol{cdet}\left[e_{h,k} + \delta_{hk}(-p + n - h)\right]_{h,k=1,\dots,n} \\ &= \sum_{j=0}^{n} (-1)^{n-j}(p)_{n-j}\ \mathbf{H}_n^{(j)} = \mathbf{C}_n(p). \end{aligned}$$

Corollary 6. *The subalgebra $\zeta(n)$ of $\mathbf{U}(gl(n))$ is the polynomial algebra*

$$\zeta(n) \ = \ \mathbb{C}[\mathbf{H}_n(0), \mathbf{H}_n(1), \dots, \mathbf{H}_n(n-1)],$$

where

$$\mathbf{H}_n(0), \mathbf{H}_n(1), \dots, \mathbf{H}_n(n-1)$$

is a set of algebraically independent generators of $\zeta(n)$.

Corollary 7. *The rectangular Capelli-Deruyts bitableau $\mathbf{K_n^p}$ equals the* product of column determinants*:*

$$\mathbf{K_n^p} = (-1)^{n\binom{p}{2}}\ \mathbf{H}_n(p-1)\ \cdots\ \mathbf{H}_n(1)\ \mathbf{H}_n(0).$$

Example 4. *Let $n = 3$, $p = 2$. Then*

$$\mathbf{K}^{\mathbf{2}}_{\mathbf{3}} = \left[\begin{array}{c|c} 3\ 2\ 1 & 1\ 2\ 3 \\ 3\ 2\ 1 & 1\ 2\ 3 \end{array} \right] = -\ \mathbf{H}_3(1)\ \mathbf{H}_3(0) =$$

$$= -\boldsymbol{cdet}\begin{pmatrix} e_{1,1}+1 & e_{1,2} & e_{1,3} \\ e_{2,1} & e_{2,2} & e_{2,3} \\ e_{3,1} & e_{3,2} & e_{3,3}-1 \end{pmatrix}\ \boldsymbol{cdet}\begin{pmatrix} e_{1,1}+2 & e_{1,2} & e_{1,3} \\ e_{2,1} & e_{2,2}+1 & e_{2,3} \\ e_{3,1} & e_{3,2} & e_{3,3} \end{pmatrix}.$$

□

Corollaries 3 and 7 generalize to Capelli-Deruyts bitableaux $\mathbf{K}^\lambda$ of arbitrary shape λ. Theorem 3 implies:

Proposition 10. *Let $n \in \mathbb{Z}$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$, $\lambda_1 \leq n$. Set $\lambda' = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{p-1})$. Then*

$$\mathbf{K}^\lambda = (-1)^{\lambda_p(\lambda_{p-1}+\cdots+\lambda_1)}\ \mathbf{C}_{\lambda_p}(p-1)\ \mathbf{K}^{\lambda'},$$

where

$$\mathbf{C}_{\lambda_p}(p-1) = \sum_{j=0}^{\lambda_p} (-1)^{\lambda_p - j}\ (p-1)_{\lambda_p - j}\ \mathbf{H}^{(j)}_{\lambda_p}.$$

□

Corollary 8. *Let $n \in \mathbb{Z}$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$, $\lambda_1 \leq n$. For $i = 1, 2, \ldots, p$, set*

$$\mathbf{C}_{\lambda_i}(i-1) = \sum_{j=0}^{\lambda_i} (-1)^{\lambda_i - j}\ (i-1)_{\lambda_i - j}\ \mathbf{H}^{(j)}_{\lambda_i}.$$

Then,

1. *The element $\mathbf{C}_{\lambda_i}(i-1)$ is* central *in the enveloping algebra $\mathbf{U}(gl(\lambda_i))$, for $i = 1, 2, \ldots, p$.*

2. *The Capelli-Deruyts bitableau $\mathbf{K}^\lambda$ equals the polynomial in the* Capelli elements *$\mathbf{H}^{(j)}_{\lambda_i}$:*

$$\mathbf{K}^\lambda = (-1)^{\lambda_p(\lambda_{p-1}+\cdots+\lambda_1)+\cdots+\lambda_2\lambda_1}\ \mathbf{C}_{\lambda_p}(p-1)\cdots\mathbf{C}_{\lambda_2}(1)\ \mathbf{C}_{\lambda_1}(0).$$

□

Example 5. *Let $n = 3$, $\lambda = (3,2)$ and let*

$$\mathbf{K}^{(3,2)} = \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \end{array} \right].$$

Then,

$$\mathbf{K}^{(3,2)} = \mathbf{C}_2(1)\ \mathbf{C}_3(0) = \left(\mathbf{H}_2^{(2)} - \mathbf{H}_2^{(1)}\right) \mathbf{H}_3^{(3)}.$$

□

For $i = 1, 2, \ldots, p$, consider the center $\zeta(\lambda_i)$ of $\mathbf{U}(gl(\lambda_i))$ and the Harish-Chandra isomorphisms

$$\chi_{\lambda_i} : \zeta(\lambda_i) \longrightarrow \Lambda^*(\lambda_i).$$

Proposition 5 and Proposition 10 imply:

$$\chi_{\lambda_i}\big(\mathbf{C}_{\lambda_i}(i-1)\big) = (x_1 - i + \lambda_i)(x_2 - i + \lambda_i - 1) \cdots (x_{\lambda_i} - i + 1). \tag{9}$$

Proposition 9 implies that the element

$$\mathbf{H}_{\lambda_i}(i-1) = \mathbf{cdet}\left[e_{h,k} + \delta_{hk}(\lambda_i - i - h + 1)\right]_{h,k=1,\ldots,\lambda_i} \in \mathbf{U}(gl(\lambda_i)).$$

is central in the enveloping algebra $\mathbf{U}(gl(\lambda_i))$. In symbols, $\mathbf{H}_n(p) \in \zeta(\lambda_i)$.

Clearly,

$$\chi_{\lambda_i}\big(\mathbf{H}_{\lambda_i}(i-1)\big) = (x_1 - i + \lambda_i)(x_2 - i + \lambda_i - 1) \cdots (x_{\lambda_i} - i + 1),$$

and, therefore, from eq. (9), we have

Corollary 9. $\mathbf{H}_{\lambda_i}(i-1) = \mathbf{C}_{\lambda_i}(i-1)$.

From Corollary 8, we have

Corollary 10. *The Capelli-Deruyts bitableau $\mathbf{K}_\lambda$ equals the* product of column determinants*:*

$$\mathbf{K}^\lambda = (-1)^{\lambda_p(\lambda_{p-1}+\cdots+\lambda_1)+\cdots+\lambda_2\lambda_1}\ \mathbf{H}_{\lambda_p}(p-1) \cdots \mathbf{H}_{\lambda_2}(1)\ \mathbf{H}_{\lambda_1}(0).$$

Example 6. *We have*

$$\mathbf{K}^{(3,2)} == \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & & 1 & 2 & \end{array} \right] = \mathbf{H}_2(1)\ \mathbf{H}_3(0) =$$

$$= \textbf{\textit{cdet}} \begin{pmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} - 1 \end{pmatrix} \textbf{\textit{cdet}} \begin{pmatrix} e_{1,1} + 2 & e_{1,2} & e_{1,3} \\ e_{2,1} & e_{2,2} + 1 & e_{2,3} \\ e_{3,1} & e_{3,2} & e_{3,3} \end{pmatrix}.$$

6.5 Polynomial identities

Let t be a variable and consider the polynomial

$$\mathbf{H}_n(t) = \mathbf{cdet} \begin{pmatrix} e_{1,1} - t + (n-1) & e_{1,2} & \dots & e_{1,n} \\ e_{2,1} & e_{2,2} - t + (n-2) & \dots & e_{2,n} \\ \vdots & \vdots & \vdots & \\ e_{n,1} & e_{n,2} & \dots & e_{n,n} - t \end{pmatrix} =$$

$$= \mathbf{cdet} \left[e_{i,j} + \delta_{ij}(-t + n - i) \right]_{i,j=1,\dots,n}$$

with coefficients in $\mathbf{U}(gl(n))$.

Corollary 11. *(see, e.g. [41]) In the polynomial algebra $\zeta(n)[t]$, the following identity holds:*

$$\mathbf{H}_n(t) = \sum_{j=0}^{n} (-1)^{n-j} \ \mathbf{H}_n^{(j)} \ (t)_{n-j},$$

where, for every $k \in \mathbb{N}$, $(t)_k = t(t-1)\cdots(t-k+1)$ denotes the $k-th$ falling factorial polynomial.

Corollary 12. *In the polynomial algebra $\Lambda^*(n)[t]$, the following identity holds:*

$$(x_1 - t + n - 1)(x_2 - t + n - 2)\cdots(x_n - t) = \sum_{j=0}^{n} (-1)^{n-j} \ \mathbf{e}_j^*(x_1, x_2, \dots, x_n) \ (t)_{n-j}.$$

Following Molev [28] Chapt. **7** (see also Howe and Umeda [22]), consider the "Capelli determinant"

$$\mathcal{C}_n(s) = \mathbf{cdet} \begin{pmatrix} e_{1,1} + s & e_{1,2} & \dots & e_{1,n} \\ e_{2,1} & e_{2,2} + s - 1 & \dots & e_{2,n} \\ \vdots & \vdots & \vdots & \\ e_{n,1} & e_{n,2} & \dots & e_{n,n} + s - (n-1) \end{pmatrix} =$$

$$= \mathbf{cdet} \left[e_{i,j} + \delta_{ij}(s - i + 1) \right]_{i,j=1,\dots,n},$$

regarded as a polynomial in the variable s.

By the formal (column) Laplace rule, the coefficients $\mathcal{C}_n^{(h)} \in \mathbf{U}(gl(n))$ in the expansion

$$\mathcal{C}_n(s) = s^n + \mathcal{C}_n^{(1)} s^{n-1} + \mathcal{C}_n^{(2)} s^{n-2} + \dots + \mathcal{C}_n^{(n)},$$

are the sums of the minors:

$$\mathcal{C}_n^{(h)} = \sum_{1\leq i_1<i_2<\ldots<i_h\leq n} \mathcal{M}_{i_1,i_2,\ldots,i_h},$$

where $\mathcal{M}_{i_1,i_2,\ldots,i_h}$ denotes the column determinant of the submatrix of the matrix $\mathcal{C}_n(0)$ obtained by selecting the rows and the columns with indices $i_1 < i_2 < \ldots < i_h$.

Since $\mathcal{C}_n(s) = \mathbf{H}_n(-s+(n-1))$, from Proposition 11 it follows:

Corollary 13.

$$\mathcal{C}_n(s) = \sum_{j=0}^{n} (-1)^{n-j}(-s+(n-1))_{n-j}\ \mathbf{H}_n^{(j)}.$$

Corollary 14. *We have:*

- *The elements* $\mathcal{C}_n^{(h)},\ h=1,2,\ldots,n$ *are central and provide a system of algebraically independent generators of* $\zeta(n)$.
- $\chi_n(\mathcal{C}_n^{(h)}) = \bar{\mathbf{e}}_h(x_1,x_2,\ldots,x_n) = \mathbf{e}_h(x_1,x_2-1,\ldots,x_n-(n-1))$, *where* $\mathbf{e}_h$ *denotes the* $h-$*th elementary symmetric polynomial.*

6.6 The shaped Capelli central elements $\mathbf{K}_\lambda(n)$

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$, $\lambda_1 \leq n$, consider the *shaped Capelli elements* (see [9])

$$\mathbf{K}_\lambda(n) = \sum_S \mathfrak{p}\big(e_{S,C^*_\lambda} \cdot e_{C^*_\lambda,S}\big) = \sum_S [S|S] \in \mathbf{U}(gl(n)),$$

where the sum is extended to all row-increasing tableaux S, $sh(S) = \lambda$.

Notice that the elements $\mathbf{K}_\lambda(n)$ are *radically different* from the elements $\mathbf{H}_\lambda(n) = \mathbf{H}_{\lambda_1}(n)\cdots\mathbf{H}_{\lambda_p}(n)$ and are *radically different* from the elements $\mathbf{K}^\lambda$.

Since the adjoint representation acts by derivation, we have

$$ad(e_{ij})\big(\sum_S e_{S,C^*_\lambda} \cdot e_{C^*_\lambda,S}\big) = 0,$$

for every $e_{ij} \in gl(n)$ and, then, from Proposition 26, it follows

Proposition 11. *The elements* $\mathbf{K}_\lambda(n)$ *are central in* $\mathbf{U}(gl(n))$.

Let $\boldsymbol{\zeta}(n)^{(m)}$ be the m-th filtration element of the center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$.

Clearly, $\mathbf{K}_\lambda(n), \mathbf{H}_\lambda(n) \in \boldsymbol{\zeta}(n)^{(m)}$ if and only if $m \geq |\lambda|$.

Proposition 12.

$$\mathbf{K}_\lambda(n) = \pm \mathbf{H}_\lambda(n) + \sum\ c_{\lambda,\mu}\mathbf{F}_\mu(n),$$

where $\mathbf{F}_\mu(n) \in \boldsymbol{\zeta}(n)^{(m)}$ *for some* $m < |\lambda|$.

Proof. Immediate from Corollary 16. □

Therefore, the central elements $\mathbf{K}_\lambda(n)$, $|\lambda| \leq m$ are linearly independent in $\boldsymbol{\zeta}(n)^{(m)}$, and the next result follows at once.

Proposition 13. *The set*

$$\big\{\mathbf{K}_\lambda(n); \lambda_1 \leq n\ \big\}$$

is a linear basis of the center $\boldsymbol{\zeta}(n)$.

Let $\mathcal{K}$ be the *Koszul equivariant isomorphism* [9]

$$\mathcal{K} : \mathbf{U}(gl(n)) \to \mathbb{C}[M_{n,n}],$$
$$\mathcal{K} : [S|S] \mapsto (S|S). \tag{10}$$

Clearly, the Koszul map $\mathcal{K}$ induces, by restriction, an isomorphism from the center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$ to the algebra $\mathbb{C}[M_{n,n}]^{ad_{gl(n)}}$ of $ad_{gl(n)}-$invariants in $\mathbb{C}[M_{n,n}]$.

Consider to the polynomial

$$\begin{aligned}\mathbf{h}_k(n) &= \sum_{1\leq i_1<\cdots<i_k\leq n} (i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k) \\ &= \sum_{1\leq i_1<\cdots<i_k\leq n} \mathbf{det}\begin{pmatrix} (i_1|i_1) & \ldots & (i_1|i_k) \\ \vdots & & \vdots \\ (i_k|i_1) & \ldots & (i_k|i_k)\end{pmatrix} \in \mathbb{C}[M_{n,n}].\end{aligned}$$

Clearly, $\mathbf{h}_k(n) \in \mathbb{C}[M_{n,n}]^{ad_{gl(n)}}$.

Notice that the polynomials $\mathbf{h}_k(n)$'s appear as coefficients (in $\mathbb{C}[M_{n,n}]$) of the characteristic polynomial:

$$P_{M_{n,n}}(t) = det\big(tI - M_{n,n}\big) = t^n + \sum_{i=1}^{n} (-1)^i\ \mathbf{h}_i(n)\ t^{n-i}.$$

From (10), we have

Proposition 14.

$$\mathcal{K}\big(\mathbf{K}_\lambda(n)\big) = (-1)^{\binom{|\lambda|}{2}}\ \mathbf{h}_{\lambda_1}(n)\mathbf{h}_{\lambda_2}(n)\cdots\mathbf{h}_{\lambda_p}(n), \quad |\lambda| = \sum_i\ \lambda_i.$$

Proposition 13 implies (is actually equivalent to) the well-known theorem for the algebra of invariants $\mathbb{C}[M_{n,n}]^{ad_{gl(n)}}$:

Proposition 15.

$$\mathbb{C}[M_{n,n}]^{ad_{gl(n)}} = \mathbb{C}\big[\mathbf{h}_1(n), \mathbf{h}_2(n), \ldots, \mathbf{h}_n(n)\big].$$

Moreover, the $\mathbf{h}_k(n)$*'s are algebraically independent.*

Proposition 15 is usually stated in terms of the algebra $\mathbb{C}[M_{n,n}]^{GL(n)} = \mathbb{C}[M_{n,n}]^{ad_{gl(n)}}$, where $\mathbb{C}[M_{n,n}]^{GL(n)}$ is the subalgebra of invariants with respect to the *conjugation action* of the general linear group $GL(n)$ on $\mathbb{C}[M_{n,n}]$ (see, e.g. [36]).

7 Proof of Theorem 2

7.1 A commutation identity for enveloping algebras of Lie superalgebras

Let $(L = L_0 \oplus L_1, [\ ,\ ])$ be a *Lie superalgebra* over $\mathbb{C}$ (see, e.g. [23], [39]), where $[\ ,\ ]$ denotes the *superbracket* bilinear form.

Given $a \in L$, consider the linear operator T_a from $U(L)$ to itself defined by setting

$$T_a(\mathbf{N}) = a\ \mathbf{N} - (-1)^{|a||\mathbf{N}|}\mathbf{N}\ a,$$

for every $\mathbf{N} \in U(L)$, $\mathbb{Z}_2$-homogeneous of degree $|\mathbf{N}|$.

We recall that T_a is the unique (left) superderivation of $U(L)$, $\mathbb{Z}_2$-homogeneous of degree $|a|$, such that

$$T_a(b) = [a, b],$$

for every $b \in L$.

Furthermore, given $a, b \in L = L_0 \oplus L_1$, from (super) skew-symmetry and the (super) Jacobi identity, it follows:

$$T_a \circ T_b - (-1)^{|a||b|} T_b \circ T_a = T_{[a,b]}.$$

The Lie algebra representation

$$Ad_L : L = L_0 \oplus L_1 \to End_{\mathbb{C}}\big[\mathbf{U}(L)\big] = End_{\mathbb{C}}\big[\mathbf{U}(L)\big]_0 \oplus End_{\mathbb{C}}\big[\mathbf{U}(L)\big]_1$$

$$e_a \mapsto T_a$$

is the *adjoint representation* of $U(L)$ on itself.

Proposition 16.

$$a_{i_1}a_{i_2}\cdots a_{i_m}\omega = \omega a_{i_1}a_{i_2}\cdots a_{i_m}(-1)^{|\omega|(|a_{i_1}|+|a_{i_2}|+\cdots+|a_{i_m}|)}+$$
$$+\sum_{k=1}^{m}\sum_{\sigma(1)<\cdots<\sigma(k);\sigma(k+1)<\cdots<\sigma(m)}\Big((T_{a_{i_{\sigma(1)}}}\dots T_{a_{i_{\sigma(k)}}}(\omega))\ a_{i_{\sigma(k+1)}}\cdots a_{i_{\sigma(m)}}\times$$
$$\times\, sgn(a_{i_{\sigma(1)}}\dots a_{i_{\sigma(k)}};a_{i_{\sigma(k+1)}}\cdots a_{i_{\sigma(m)}})\ (-1)^{|\omega|(|a_{i_{\sigma(k+1)}}|+\cdots+|a_{i_{\sigma(m)}}|)}\Big).$$

Proof. By induction hypotesis,

$$a_{i_1}(a_{i_2}\cdots a_{i_m})\omega = a_{i_1}\omega a_{i_2}\cdots a_{i_m}(-1)^{|\omega|(|a_{i_2}|+\cdots+|a_{i_m}|)}+$$
$$+\,a_{i_1}\sum_{h=2}^{m}\sum_{\tau(2)<\cdots<\tau(h);\tau(h+1)<\cdots<\tau(m)}\Big(T_{a_{i_{\tau(2)}}}\dots T_{a_{i_{\tau(h)}}}(\omega)a_{i_{\tau(h+1)}}\cdots a_{i_{\tau(m)}}\times$$
$$\times\, sgn(a_{i_{\tau(2)}}\cdots a_{i_{\tau(h)}};a_{i_{\tau(h+1)}}\cdots a_{i_{\tau(m)}})(-1)^{|\omega|(|a_{i_{\tau(h+1)}}|+\cdots+\cdots|a_{i_{\tau(m)}}|)}\Big) =$$
$$= \omega a_{i_1}a_{i_2}\cdots a_{i_m}(-1)^{|\omega|(|a_{i_1}|+|a_{i_2}|+\cdots+|a_{i_m}|)} + T_{a_{i_1}}(\omega)a_{i_2}\cdots a_{i_m}(-1)^{|\omega|(|a_{i_2}|+\cdots+|a_{i_m}|)}+$$
$$+\sum_{h=2}^{m}\sum_{\tau(2)<\cdots<\tau(h);\tau(h+1)<\cdots<\tau(m)}\Big(T_{a_{i_1}}T_{a_{\tau(2)}}\cdots T_{a_{\tau(h)}}(\omega)a_{\tau(h+1)}\cdots a_{i_{\tau(m)}}\times$$
$$\times\, sgn(a_{\tau(2)}\cdots a_{\tau(h)};a_{\tau(h+1)}\cdots a_{\tau(m)})(-1)^{|\omega|(|a_{\tau(h+1)}|+\cdots+\cdots|a_{i_{\tau(m)}}|)}+$$
$$+\,T_{a_{\tau(2)}}\cdots T_{a_{\tau(h)}}(\omega)a_{i_1}a_{\tau(h+1)}\cdots a_{i_{\tau(m)}}\times$$
$$(-1)^{|a_{i_1}|(|\omega|+|a_{\tau(2)}|+\cdots+|a_{i_{\tau(m)}}|)}\times sgn(a_{\tau(2)}\cdots a_{\tau(h)};a_{\tau(h+1)}\cdots a_{\tau(m)})(-1)^{|\omega|(|a_{\tau(h+1)}|+\cdots+\cdots|a_{i_{\tau(m)}}|)}\Big),$$

where

$$(-1)^{|a_{i_1}|(|\omega|+|a_{i_{\tau(2)}}|+\cdots+|a_{i_{\tau(m)}}|)+|\omega|(|a_{\tau(h+1)}|+\cdots+|a_{i_{\tau(m)}}|)}\times sgn(a_{i_{\tau(2)}}\cdots a_{i_{\tau(m)}};a_{i_{\tau(h+1)}}\cdots a_{i_{\tau(m)}}) =$$
$$= sgn(a_{i_{\tau(2)}}\cdots a_{i_{\tau(h)}};a_{i_1}a_{i_{\tau(h+1)}}\cdots a_{i_{\tau(m)}})(-1)^{|\omega|(|a_{i_1}|+|a_{i_{\tau(h+1)}}+\cdots+|a_{i_{\tau(m)}}|)}.$$

Then, the assertion follows. □

In the Sweedler notation of the *supersymmetric* superbialgebra $Super(L)$, Theorem 16 can be stated in the following compact form:

Proposition 17. *Let*

$$\alpha = a_{i_1}a_{i_2}\cdots a_{i_m}.$$

Then

$$\alpha\omega = \sum_{(\alpha)} T_{\alpha_{(1)}}(\omega)\alpha_{(2)}(-1)^{|\omega||\alpha_{(2)}|}.$$

Proof. Let
$$\alpha = a_{i_1} a_{i_2} \cdots a_{i_m}.$$
Then, the coproduct (in the Sweedler notation)
$$\Delta(\alpha) = \sum_{(\alpha)} \alpha_{(1)} \otimes \alpha_{(2)}$$
equals
$$\sum_{k=0}^{m} \sum_{\sigma(1)<\cdots<\sigma(k);\sigma(k+1)<\cdots<\sigma(m)} \big(a_{i_{\sigma(1)}} \ldots a_{i_{\sigma(k)}} \otimes a_{i_{\sigma(k+1)}} \cdots a_{i_{\sigma(m)}} \times$$
$$\times sgn(a_{i_{\sigma(1)}} \cdots a_{i_{\sigma(k)}}; a_{i_{\sigma(k+1)}} \cdots a_{i_{\sigma(m)}})\big).$$
□

Furthermore

Lemma 1. *Let $T_\alpha = T_{a_1} T_{a_2} \cdots T_{a_m}$. Then*
$$T_\alpha(\omega_1 \cdot \omega_2) = \sum_{(\alpha)} T_{\alpha_{(1)}}(\omega_1) T_{\alpha_{(2)}}(\omega_2)(-1)^{|\alpha_{(2)}||\omega_1|}.$$

7.2 Some preliminary remarks and definitions

7.2.1 The virtual algebra and the Capelli devirtualization epimorphism

Given a vector space V of dimension n, we will regard it as a subspace of a $\mathbb{Z}_2$−graded vector space $V_0 \oplus V_1$, where $V_1 = V$. The vector spaces V_0 (we assume that $dim(V_0) = m$ is "sufficiently large") is called the *positive virtual (auxiliary) vector space* and V is called the *(negative) proper vector space*.

Let $\mathcal{A}_0 = \{\alpha_1, \ldots, \alpha_{m_0}\}$, $\mathcal{L} = \{1, 2, \ldots, n\}$ denote *fixed bases* of V_0 and $V = V_1$, respectively; therefore $|\alpha_s| = 0 \in \mathbb{Z}_2$, and $|i| = 1 \in \mathbb{Z}_2$.

Let
$$\{e_{a,b}; a, b \in \mathcal{A}_0 \cup \mathcal{L}\}, \qquad |e_{a,b}| = |a| + |b| \in \mathbb{Z}_2$$
be the standard $\mathbb{Z}_2$−homogeneous basis of the Lie superalgebra $gl(m|n)$ provided by the elementary matrices. The elements $e_{a,b} \in gl(m|n)$ are $\mathbb{Z}_2$−homogeneous of $\mathbb{Z}_2$−degree $|e_{a,b}| = |a| + |b|$.

The superbracket of the Lie superalgebra $gl(m|n)$ has the following explicit form:
$$[e_{a,b}, e_{c,d}] = \delta_{bc}\ e_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}\ e_{c,b},$$

$a, b, c, d \in \mathcal{A}_0 \cup \mathcal{L}$.

In the following, the elements of the sets $\mathcal{A}_0, \mathcal{L}$ will be called *positive virtual symbols* and *negative proper symbols*, respectively.

The inclusion $V \subset V_0 \oplus V_1$ induces a natural embedding of the ordinary general linear Lie algebra $gl(n) = gl(0|n)$ of V into the *auxiliary* general linear Lie *superalgebra* $gl(m|n)$ of $V_0 \oplus V_1$ (see, e.g. [23], [39]) and, hence, a natural embedding $\mathbf{U}(gl(n)) \subset \mathbf{U}(gl(m|n))$.

In the following, we will systematically refer to the *Capelli devirtualization epimorphism*

$$\mathfrak{p} : Virt(m, n) \twoheadrightarrow \mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n)),$$

where $Virt(m, n)$ is the *virtual subalgebra* of $\mathbf{U}(gl(m|n))$.

For definitions and details, we refer the reader to Subsection 9.5.

7.2.2 A more readable notation

In the following, we will adopt the more readable notation:

- We write $\{a|b\}$ for the elements $e_{a,b}$ of the standard basis of $gl(m|n)$.
- Given two words $I = i_1\ i_2\ \cdots\ i_p$, $J = j_1\ j_2\ \cdots\ j_p$, with $i_h, j_h \in \mathcal{L}$ and a virtual symbol α, we write

  $$\{J|\alpha\} = \{j_1\ j_2\ \cdots\ j_p|\alpha\}, \quad \{\alpha|I\} = \{\alpha|i_1\ i_2\ \cdots\ i_p\}$$

  in place of

  $$e_{j_1,\alpha} e_{j_2,\alpha} \cdots e_{j_p,\alpha}, \quad e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_p},$$

  respectively.

In this notation, given a pair of Young tableaux

$$S = (w_1, w_2, \ldots, w_p), \quad T = (\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_p), \qquad sh(S) = sh(T) = \lambda,$$

the *Capelli bitableau*

$$[S|T] = \mathfrak{p}\big(e_{SC_\lambda} \cdot e_{C_\lambda T}\big) \in \mathbf{U}(gl(n))$$

is

$$[S|T] = \mathfrak{p}\big(\mathbf{P}_S \cdot \mathbf{P}_T\big),$$

where

$$\mathbf{P}_S = \{w_1|\beta_1\}\{w_2|\beta_2\} \cdots \{w_p|\beta_p\}, \qquad \mathbf{P}_T = \{\beta_1|\overline{w}_1\}\{\beta_2|\overline{w}_2\} \cdots \{\beta_p|\overline{w}_p\}.$$

Furthermore, for the adjoint representation

$$Ad_{gl(m|n)} : gl(m|n) \to End_{\mathbb{C}}\big[\mathbf{U}(gl(m|n))\big]$$

we write

– $T_{i\alpha},\ T_{\alpha i}$ in place of $Te_{i\alpha},\ T_{e_{\alpha i}}$.

– $T_{I\alpha},\ T_{\alpha I}$ in place of $T_{i_1,\alpha}T_{i_2,\alpha}\cdots T_{i_p,\alpha}, \quad T_{\alpha,i_1}T_{\alpha,i_2}\cdots T_{\alpha,i_p}$, respectively.

7.2.3 The coproduct in $\Lambda(V) = \Lambda(\mathcal{L})$, Sweedler notation and *split notation*

Given a word $I = i_1\ i_2\ \ \cdots\ i_m,\ i_t \in \mathcal{L}$ in $\Lambda(V) = \Lambda(\mathcal{L})$, and a natural integer $k, \quad k = 0, 1, \cdots, m$, consider the homogeneous component

$$\Delta_{k,m-k} : \Lambda(\mathcal{L}) \to \Lambda(\mathcal{L})_k \otimes \Lambda(\mathcal{L})_{m-k}$$

of the coproduct

$$\Delta : \Lambda(\mathcal{L}) \to \Lambda(\mathcal{L}) \otimes \Lambda(\mathcal{L}).$$

Given a permutation σ with

$$\sigma(1) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(m),$$

and the two subwords

$$I_{(1)} = i_{\sigma(1)}\ \ \cdots\ i_{\sigma(k)}, \quad I_{(2)} = i_{\sigma(k+1)}\ \ \cdots\ i_{\sigma(m)}$$

we call the pair $(I_{(1)}, I_{(2)})$ a *split* of I of step $(k, m-k)$ of signature $sgn(I; I_{(1)}, I_{(2)}) = sgn(\sigma)$. Clearly, $I = sgn(I; I_{(1)}, I_{(2)})\ I_{(1)}I_{(2)}$.

We denote by $\mathbf{S}(I; k, m-k)$ the set of all splits of I of step $(k, m-k)$.

Then, the coproduct component

$$\Delta_{k,m-k}(I) = \sum_{(I)_{k,m-k}} I_{(1)} \otimes I_{(2)}$$

can be explicitly written as

$$\Delta_{k,m-k}(I) = \sum_{(I_{(1)},I_{(2)})\in\mathbf{S}(I;k,m-k)} sgn(I; I_{(1)}, I_{(2)})\ I_{(1)} \otimes I_{(2)}.$$

7.3 Some lemmas

Consider the Capelli bitableau

$$[S|T] = \mathfrak{p}\big(\mathbf{P}_S \cdot \mathbf{P}_T\big)$$

as in Eq. (7.2.2).

From Proposition 17, we derive the following pair of Lemmas.

Lemma 2. *Let* $I = i_1\ i_2 \cdots\ i_m,\ J = j_1\ j_2\ \cdots\ j_m,\ m \leq \lambda_p$.
Then

$$\{J|\alpha\}\{\alpha|I\}\ \mathbf{P}_S$$

equals

$$\{J|\alpha\}\ \sum_{k=0}^{m}\ \sum_{(I)_{k,m-k}} T_{\alpha\ I_{(1)}}\big(\mathbf{P}_S\big)\{\alpha|I_{(2)}\}(-1)^{|\mathbf{P}_S|(m-k)}.$$

Since

$$\mathfrak{p}\big(\{J|\alpha\}\{\alpha|I\}\ \boldsymbol{P}_S \cdot \boldsymbol{P}_T\big) = [J|I]\ [S|T],$$

Lemma 3. *We have*

$$[J|I]\ [S|T] = \ (-1)^{(|\mathbf{P}_T|+k)(m-k)} \times \\ \times\, \mathfrak{p}\Big(\sum_{k=0}^{m}\ \sum_{(I)_{k,m-k}}\ \sum_{(J)_{k,m-k}} T_{J_{(1)}\alpha} T_{\alpha I_{(1)}}\big(\mathbf{P}_S\big)\ \{J_{(2)}|\alpha\}\ \mathbf{P}_T\ \{\alpha|I_{(2)}\}\Big). \quad (11)$$

Proof. We have

$$\{J|\alpha\}\{\alpha|I\}\ \boldsymbol{P}_S\ \boldsymbol{P}_T = \\ = \{J|\alpha\}\ \sum_{k=0}^{m}\ \sum_{(I)_{k,m-k}} T_{\alpha\ I_{(1)}}\big(\boldsymbol{P}_S\big)\ \{\alpha|I_{(2)}\}\ \boldsymbol{P}_T\ (-1)^{|\boldsymbol{P}_S|(m-k)} = \\ = \sum_{k=0}^{m}\ \sum_{(I)_{k,m-k}} \{J|\alpha\}\ T_{\alpha\ I_{(1)}}\big(\boldsymbol{P}_S\big)\ \{\alpha|I_{(2)}\}\ \boldsymbol{P}_T\ (-1)^{|\boldsymbol{P}_S|(m-k)} = \\ = \sum_{k=0}^{m}\ \sum_{(I)_{k,m-k}} \left(\sum_{h=0}^{m}\ \sum_{(J)_{h,m-h}} T_{J_{(1)}\alpha}\big(T_{\alpha\ I_{(1)}}\big(\boldsymbol{P}_S\big)\big)\{J_{(2)}|\alpha\}\ (-1)^{(|\boldsymbol{P}_S|+h)(m-h)}\right) \times \\ \times \{\alpha|I_{(2)}\}\ \boldsymbol{P}_T\ (-1)^{|\boldsymbol{P}_S|(m-k)}.$$

Now, if $h < k$, then $m - h > m - k$ and, hence,

$$\sum_{(I)_{k,m-k}} \left(\sum_{(J)_{h,m-h}} T_{J_{(1)}\alpha}\big(T_{\alpha\ I_{(1)}}\big(\boldsymbol{P}_S\big)\big)\{J_{(2)}|\alpha\}\ (-1)^{(|\boldsymbol{P}_S|+h)(m-h)} \right) \times$$
$$\times\ \{\alpha|I_{(2)}\}\ \boldsymbol{P}_T\ (-1)^{|\boldsymbol{P}_S|(m-k)}$$

is an *irregular element*, since the $\{J_{(2)}|\alpha\}\{\alpha|I_{(2)}\}$ are irregular monomials; so, its image with respect to the Capelli epimorphism $\mathfrak{p}$ equals zero.

If $h > k$, then,

$$T_{J_{(1)}\alpha}\big(T_{\alpha\ I_{(1)}}\big(\boldsymbol{P}_S\big)\big) = 0.$$

and, hence,

$$\sum_{(I)_{k,m-k}} \left(\sum_{(J)_{h,m-h}} T_{J_{(1)}\alpha}\big(T_{\alpha\ I_{(1)}}\big(\boldsymbol{P}_S\big)\big)\{J_{(2)}|\alpha\}\ (-1)^{(|\boldsymbol{P}_S|+h)(m-h)} \right) \times$$
$$\times\ \{\alpha|I_{(2)}\}\ \boldsymbol{P}_T\ (-1)^{|\boldsymbol{P}_S|(m-k)} = 0.$$

Then,

$$\begin{aligned}
[J|I]\ [S|T] &= (-1)^{(|\boldsymbol{P}_S|+k)(m-k)}(-1)^{|\boldsymbol{P}_S|(m-k)}\times \\
&\quad \times \mathfrak{p}\big(\sum_{k=0}^{m} \sum_{(I)_{k,m-k}} \sum_{(J)_{k,m-k}} T_{J_{(1)}\alpha}T_{\alpha I_{(1)}}\big(\boldsymbol{P}_S\big)\ \{J_{(2)}|\alpha\}\ \{\alpha|I_{(2)}\}\ \boldsymbol{P}_T\big) \\
&= (-1)^{(|\boldsymbol{P}_T|+k)(m-k)}\times \\
&\quad \times \mathfrak{p}\big(\sum_{k=0}^{m} \sum_{(I)_{k,m-k}} \sum_{(J)_{k,m-k}} T_{J_{(1)}\alpha}T_{\alpha I_{(1)}}\big(\boldsymbol{P}_S\big)\ \{J_{(2)}|\alpha\}\ \boldsymbol{P}_T\ \{\alpha|I_{(2)}\}\big).
\end{aligned}$$

□

Corollary 15. *Let* $m \leq \lambda_p$. *Then*

$$[J|I]\ [S|T] = \pm \left[\begin{array}{c|c} S & T \\ J & I \end{array} \right] + \sum c_{m,\lambda}\ \mathbf{G}_{m,\lambda},$$

where

$$[J|I]\ [S|T], \quad \left[\begin{array}{c|c} S & T \\ J & I \end{array} \right] \notin \mathbf{U}(gl(n))^{(n)} \quad \textit{whenever} \quad n < m + |\lambda|,$$

and

$$\mathbf{G}_{m,\lambda} \in \mathbf{U}(gl(n))^{(n)} \quad \textit{for some} \quad n < m + |\lambda|.$$

Corollary 16. *Let $m \leq \lambda_p$. Then*

$$[S|T] = \pm\ [\omega_1|\overline{\omega}_1]\ [\omega_2|\overline{\omega}_2] \cdots [\omega_p|\overline{\omega}_p] + \sum d_\lambda\ \mathbf{F}_\lambda,$$

where

$$[S|T],\ [\omega_1|\overline{\omega}_1]\ [\omega_2|\overline{\omega}_2] \cdots [\omega_p|\overline{\omega}_p] \notin \mathbf{U}(gl(n))^{(n)} \quad whenever \quad n < |\lambda|,$$

and

$$\mathbf{F}_\lambda \in \mathbf{U}(gl(n))^{(n)} \quad for\ some \quad n < |\lambda|.$$

We specialize the previous results to Capelli-Deruyts bitableaux $\mathbf{K}^\lambda$.

Let

$$\mathbf{M}^* = \{\underline{\lambda_1}^*|\beta_1\} \cdots \{\underline{\lambda_p}^*|\beta_p\}, \quad \mathbf{M} = \{\beta_1|\underline{\lambda_1}\} \cdots \{\beta_p|\underline{\lambda_p}\},$$

where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_p)$ and $|\mathbf{M}^*| = |\mathbf{M}| = |\lambda| = \lambda_1 + \cdots + \lambda_p \in \mathbb{Z}_2$.

Given an increasing word $W = h_1\ h_2\ \cdots\ h_p$ on $\mathcal{L} = \{1, 2, \ldots, n\}$, denote by W^* its *reverse* word, that is:

$$W^* = h_p\ \cdots\ h_2\ h_1.$$

Let $I = 1\ 2\ \cdots\ m$, $I^* = m\ m-1\ \cdots\ 1$, $m \leq \lambda_p$.

In this notation

$$\mathbf{K}^\lambda = \mathfrak{p}\big(\mathbf{M}^* \cdot \mathbf{M}\big)$$

and

$$[I^*|I]\ \mathbf{K}^\lambda = \mathfrak{p}\big(\ \{I^*|\alpha\}\{\alpha|I\}\ \mathbf{M}^* \cdot \mathbf{M}\ \big).$$

We apply Lemma 3 to the element $[I^*|I]\ \mathbf{K}^\lambda$. As we shall see, the double sum

$$\sum_{(I^*)_{k,m-k}} \sum_{(I)_{k,m-k}}$$

in eq. (11) reduces to a single sum

$$\sum_{(I)_{k,m-k}}$$

since the only splits $I^*_{(1)},\ I^*_{(2)}$ in $(I^*)_{k,m-k}$ that give rise to nonzero summands are those for

$$I^*_{(1)} = (I_{(1)})^* \quad \text{and} \quad I^*_{(2)} = (I_{(2)})^*,$$

where $(I_{(1)})^*,\ (I_{(2)})^*$ are the reverse words of $I_{(1)}$ and $I_{(2)}$, respectively.

Lemma 4. *The element*

$$[I^*|I]\ \mathbf{K}^\lambda = \mathfrak{p}\big(\{I^*|\alpha\}\{\alpha|I\}\ \mathbf{M}^* \cdot \mathbf{M}\big)$$

equals

$$\sum_{k=0}^{m} \quad (-1)^{(|\boldsymbol{M}|+k)(m-k)} \sum_{(I)_{k,m-k}} \quad \mathfrak{p}\Big(T_{(I_{(1)})^*\alpha}\big(T_{\alpha\ I_{(1)}}(\mathbf{M}^*)\big)\{(I_{(2)})^*|\alpha\}\mathbf{M}\{\alpha|I_{(2)}\}\Big).$$

Proof. From Lemma 3, we have

$$\mathfrak{p}\big(\{I^*|\alpha\}\{\alpha|I\}\ \mathbf{M}^* \cdot \mathbf{M}\big) = \sum_{k=0}^{m} (-1)^{(|\mathbf{M}|+k)(m-k)}$$
$$\Big(\sum_{(I)_{k,m-k}} \sum_{(I^*)_{k,m-k}} \mathfrak{p}\Big(T_{I^*_{(1)}\alpha}\big(T_{\alpha\ I_{(1)}}(\mathbf{M}^*)\big)\ \{(I^*_{(2)}|\alpha\}\ \mathbf{M}\ \{\alpha|I_{(2)}\}\ \Big)\Big).$$

Let $k = 0, 1, \ldots, m$ and examine the element

$$\sum_{(I)_{k,m-k}} \sum_{(I^*)_{k,m-k}} T_{I^*_{(1)}\alpha}\big(T_{\alpha\ I_{(1)}}(\mathbf{M}^*)\big)\{I^*_{(2)}|\alpha\}\ \mathbf{M}\{\alpha|I_{(2)}\} =$$
$$= \sum_{(I)_{k,m-k}} \sum_{(I^*)_{k,m-k}} T_{I^*_{(1)}\alpha}\Big(T_{\alpha\ I_{(1)}}\big(\{\underline{\lambda_1}^*|\beta_1\}\cdots\{\underline{\lambda_p}^*|\beta_p\}\big)\Big)\{(I_{(2)})^*|\alpha\}\ \mathbf{M}\{\alpha|I_{(2)}\}.$$

If $i \in I_{(2)}$, then $i \notin I_{(1)}$. Hence, all the variables

$$\{i|\beta_q\} \qquad q = 1, 2, \ldots, p$$

appear in

$$T_{\alpha, I_{(1)}}\big(\{\underline{\lambda_1}^*|\beta_1\}\cdots\{\underline{\lambda_p}^*|\beta_p\}\big),$$

for every $q = 1, 2, \ldots, p$.

Assume that $i \notin I^*_{(2)}$, then $i \in I^*_{(1)}$. Hence, $\exists\ \underline{q} \in \{1, 2, \ldots, p\}$ such that the variable

$$\{i|\beta_{\underline{q}}\}$$

is *created* by the action of

$$T_{I^*_{(1)}\alpha}$$

on

$$T_{\alpha, I_{(1)}}\big(\{\underline{\lambda_1}^*|\beta_1\}\cdots\{\underline{\lambda_p}^*|\beta_p\}\big) \qquad (*).$$

Then $(*)$ contains two occurrencies of $\{i|\beta_{\underline{q}}\}$ and, hence, *equals zero.* Therefore

$$T_{I^*_{(1)}\alpha}T_{\alpha,I_{(1)}}\big(\{\underline{\lambda_1}^*|\beta_1\}\cdots\{\underline{\lambda_p}^*|\beta_p\}\big)\neq 0$$

implies

$$i\in I_{(2)}\implies i\in I^*_{(2)}.$$

Since $I_{(2)}$ and $I^*_{(2)}$ are words of the same length $m-k$, this implies that the only *not zero* summands - *with respect to the action of the Capelli epimorphism* $\mathfrak{p}$ - in

$$\sum_{(I)_{k,m-k}}\ \sum_{(I^*)_{k,m-k}}\mathfrak{p}\big(T_{I^*_{(1)}\alpha}\big(T_{\alpha\ I_{(1)}}\big(\mathbf{M}^*\big)\big)\{I^*_{(2)}|\alpha\}\ \mathbf{M}\{\alpha|I_{(2)}\}\big)$$

are for $I^*_{(1)}=(I_{(1)})^*$ and $I^*_{(2)}=(I_{(2)})^*$, that is

$$\mathfrak{p}\big(T_{(I_{(1)})^*\alpha}\big(T_{\alpha\ I_{(1)}}\big(\mathbf{M}^*\big)\big)\ \{(I_{(2)})^*|\alpha\}\ \mathbf{M}\{\alpha|I_{(2)}\}\big).$$

□

Let us examine the expression

$$\sum_{(I)_{k.m-k}}(-1)^{k(m-k)}\ T_{(I_{(1)})^*\alpha}\big(T_{\alpha\ I_{(1)}}\big(\mathbf{M}^*\big)\big)\ \{(I_{(2)})^*|\alpha\}\{\alpha|I_{(2)}\}. \tag{12}$$

in the notation of *splits*.

Corollary 17. *The expression (12) equals*

$$\sum_{(A,B)\in S(I;k,m-k)}T_{A^*\alpha}\big(T_{\alpha A}\big(\mathbf{M}^*\big)\big)\ \{B^*|\alpha\}\{\alpha|B\}.$$

Proof. In the notation of *splits*, the expression (12) equals

$$(-1)^{k(m-k)}\sum_{(A,B)\in S(I;k,m-k)}T_{A^*\alpha}\big(T_{\alpha\ A}\big(\boldsymbol{M}^*\big)\big)\ \{B^*|\alpha\}\{\alpha|B\}\times$$
$$sgn(I;A,B)sgn(I^*;A^*,B^*).$$

We have

$$(-1)^{k(m-k)}sgn(I;A,B)sgn(I^*;A^*,B^*)=$$
$$=(-1)^{k(m-k)}(-1)^{k(m-k)}sgn(I;A,B)sgn(I^*;B^*,A^*).$$

But $sgn(I;A,B)sgn(I^*;B^*,A^*)=1$. □

Given $(A,B) \in S(I;k,m-k)$, let $A = a_1a_2\cdots a_k$, $\{a_1 < a_2 < \cdots < a_k\} \subseteq \{1,2,\ldots,m\}$ and recall

$$\mathbf{M}^* = \{\underline{\lambda}_1|\beta_1\}\cdots\{\underline{\lambda_p}^*|\beta_p\};$$

we examine the element

$$T_{A^*\alpha}T_{\alpha\ A}\big(\mathbf{M}^*\big). \tag{13}$$

Lemma 5. *We have*

$$T_{A^*\alpha}T_{\alpha\ A}\big(\mathbf{M}^*\big) = \langle\ p\rangle_k\ \{\underline{\lambda_1}^*|\beta_1\}\cdots\{\underline{\lambda_p}^*|\beta_p\} = \ \langle\ p\rangle_k\ \mathbf{M}^*,$$

where

$$\langle\ p\rangle_k = p(p+1)\cdots(p+k-1)$$

is the raising factorial *coefficient.*

Proof. By skew-symmetry, a simple computation shows that (13) equals

$$\sum_{h_1+\cdots+h_p=k}\ \sum_{(A_1,\ldots,A_p)\in S(A;h_1,\ldots,h_p)} T_{(A_1)^*\alpha}T_{\alpha A_1}\big(\{\underline{\lambda_1}^*|\beta_1\}\big)\cdots T_{(A_p)^*\alpha}T_{\alpha A_p}\big(\{\underline{\lambda_p}^*|\beta_p\}\big). \tag{14}$$

We examine the value of

$$T_{C^*\alpha}T_{\alpha C}\big(\{\underline{q}^*|\beta\}\big)$$

for $C = c_1c_2\cdots c_h$, $\{c_1 < c_2 < \ldots < c_h\} \subseteq \{1,2,\ldots q\}$.
Clearly

$$\{\underline{q}^*|\beta\} = \{\underline{q}|\beta\}(-1)^{\binom{q}{2}},$$

and a simple computation shows that

$$T_{C^*\alpha}T_{\alpha C}\big(\{\underline{q}|\beta\}\big) = h!\ \{\underline{q}|\beta\}.$$

Indeed, we have

$$\begin{aligned} T_{\alpha C}\big(\{\underline{q}|\beta\}\big) &= T_{c_1\alpha}\cdots T_{c_h\alpha}\big(\{1|\beta\}\cdots\{q|\beta\}\big) \\ &= \{1|\beta\}\cdots\widehat{\{c_1|\beta\}}\{\alpha|\beta\}\cdots\widehat{\{c_h|\beta\}}\{\alpha|\beta\}\cdots\{q|\beta\}\ (-1)^{c_h-1+\cdots+c_1-1} \\ &= \{\alpha|\beta\}^h\{1|\beta\}\cdots\widehat{\{c_1|\beta\}}\cdots\widehat{\{c_h|\beta\}}\cdots\{q|\beta\}\ (-1)^{c_h-1+\cdots+c_1-1}; \end{aligned}$$

now,

$$\begin{aligned} T_{C\alpha}T_{\alpha C}\big(\{\underline{q}|\beta\}\big) &= T_{c_h\alpha}\cdots T_{c_1\alpha}\big(\{\alpha|\beta\}^h\{1|\beta\}\cdots\widehat{\{c_1|\beta\}}\cdots\big)(-1)^{c_h-1+\cdots+c_1-1} \\ &= h!\{c_h|\beta\}\cdots\{c_1|\beta\}\cdots\widehat{\{c_1|\beta\}}\cdots\widehat{\{c_h|\beta\}}(-1)^{c_h-1+\cdots+c_1-1} \\ &= h!\{1|\beta\}\cdots\{q|\beta\} = h!\{\underline{q}|\beta\}. \end{aligned}$$

Then,

$$T_{C^*\alpha}T_{\alpha C}\big(\{\underline{q}^*|\beta\}\big) = (-1)^{\binom{q}{2}} T_{C^*\alpha}T_{\alpha C}\big(\{\underline{q}|\beta\}\big) = (-1)^{\binom{q}{2}}\ h!\ \{\underline{q}|\beta\} = h!\ \{\underline{q}^*|\beta\}.$$

Hence, (14) equals

$$\sum_{(h_1,\ldots,h_p);h_1+\cdots+h_p=k}\ \sum_{(A_1,\ldots,A_p)\in S(A;h_1,\ldots,h_p)} h_1!\cdots h_p!\ \big(\{\underline{\lambda_1}^*|\beta_1\}\cdots\{\underline{\lambda_p}^*|\beta_p\}\big) =$$
$$= \sum_{h_1+\cdots+h_p=k} \frac{k!}{h_1!\cdots h_p!}\ h_1!\cdots h_p!\ \big(\{\underline{\lambda_1}^*|\beta_1\}\cdots\{\underline{\lambda_p}^*|\beta_p\}\big)$$

that equals

$$\left\langle \begin{matrix} p \\ k \end{matrix} \right\rangle k!\ \big(\{\underline{\lambda_1}^*|\beta_1\}\cdots\{\underline{\lambda_p}^*|\beta_p\}\big) = \langle\ p\rangle_k\ \{\underline{\lambda_1}^*|\beta_1\}\cdots\{\underline{\lambda_p}^*|\beta_p\}.$$

□

Hence, from Lemma 4 and Lemma 5, we infer:

Proposition 18. *Let* $I = 12\cdots m$, $I^* = m\cdots 21$. *Then*

$$\begin{aligned}[I^*|I]\ \mathbf{K}^\lambda &= \mathfrak{p}\big(\ \{I^*|\alpha\}\{\alpha|I\}\ \boldsymbol{M}^*\cdot\boldsymbol{M}\ \big)\\ &= \mathfrak{p}\big(\{I^*|\alpha\}\{\alpha|I\}\ \{\underline{\lambda_1}^*|\beta_1\}\cdots\{\underline{\lambda_p}^*|\beta_p\}\{\beta_1|\underline{\lambda_1}\}\cdots\{\beta_p|\underline{\lambda_p}\}\big)\end{aligned}$$

equals

$$\sum_{k=0}^{m}(-1)^{|\boldsymbol{M}|(m-k)} \sum_{(A,B)\in S(I;k,m-k)} \langle\ p\rangle_k\ \mathfrak{p}\big(\boldsymbol{M}^*\{B^*|\alpha\}\boldsymbol{M}\{\alpha|B\}\big).$$

7.4 Proof of Theorem 2

Let $m \le \lambda_p$ and $M \subseteq \underline{\lambda_p}$, $|M| = m$, as in Theorem 2.

Recall that $|\mathbf{M}| = |\mathbf{M}^*| = |\lambda| \in \mathbb{Z}_2$, where $|\lambda| = \lambda_1 + \cdots + \lambda_p$.

From Remark 1 and Proposition 18, we have:

$$\begin{aligned}[M^*|M]\,\mathbf{K}^\lambda &= \mathfrak{p}\left(\{M^*|\alpha\}\{\alpha|M\}\mathbf{M}^*\cdot\mathbf{M}\right)\\ &= \sum_{k=0}^{m}\ \langle p\rangle_{m-k}\ (-1)^{|\lambda|k} \sum_{J;\ J\subseteq M;\ |J|=k} \mathfrak{p}\left(\mathbf{M}^*\{J^*|\alpha\}\mathbf{M}\{\alpha|J\}\right)\\ &\stackrel{def}{=} \sum_{k=0}^{m}\ \langle p\rangle_{m-k}\ (-1)^{|\lambda|k} \sum_{J;\ J\subseteq M;\ |J|=k} \left[\begin{matrix}\mathbf{K}^\lambda\\ J\end{matrix}\right].\end{aligned}$$

8 Proof of Theorem 1

Proof. Recall that

$$v_\mu = (Der_{\tilde{\mu}} | Der_{\tilde{\mu}}^P),$$

where $(Der_{\tilde{\mu}} | Der_{\tilde{\mu}}^P)$ is the Young bitableau (see, e.g. Subsection 9.7 below)

$$\left(\begin{array}{l|l} 1 \;\; 2 \;\; \cdots\cdots \;\; \tilde{\mu}_1 & 1 \;\; 2 \;\; \cdots\cdots \;\; \tilde{\mu}_1 \\ & \\ 1 \;\; 2 \;\; \cdots \;\; \tilde{\mu}_2 & 1 \;\; 2 \;\; \cdots \;\; \tilde{\mu}_2 \\ \cdots \; \cdots \; \cdots & \cdots \; \cdots \; \cdots \\ \cdots \; \cdots & \cdots \; \cdots \\ 1 \;\; 2 \;\; \tilde{\mu}_q & 1 \;\; 2 \;\; \tilde{\mu}_q \end{array} \right)$$

in the polynomial algebra $\mathbb{C}[M_{n,d}]$.

Set

$$e_{Der^*_{n^p}, Coder_{n^p}} \;=\; e_{n\alpha_1} \cdots e_{1\alpha_1} \cdots\cdots e_{n\alpha_{p-1}} \cdots e_{1\alpha_{p-1}} e_{n\alpha_p} \cdots e_{1\alpha_p}.$$

Set

$$e_{Coder_{n^p}, Der_{n^p}} \;=\; e_{\alpha_1 1} \cdots e_{\alpha_1 n} \cdots\cdots e_{\alpha_{p-1} 1} \cdots e_{\alpha_{p-1} n} e_{\alpha_p 1} \cdots e_{\alpha_p n}.$$

Since

$$\mathbf{K}_\mathbf{n}^\mathbf{p} = \mathfrak{p}\big(e_{Der^*_{n^p}, Coder_{n^p}} \; e_{Coder_{n^p}, Der_{n^p}}\big),$$

the action of $\mathbf{K}_\mathbf{n}^\mathbf{p}$ on $v_\mu = (Der_{\tilde{\mu}} | Der_{\tilde{\mu}}^P)$ is the same as the action of

$$e_{Der^*_{n^p}, Coder_{n^p}} \; e_{Coder_{n^p}, Der_{n^p}}.$$

We follow [37] (see Proposition 5).

Now, if $\mu_n = 0$, then

$$e_{\alpha_p n} \cdot (Der_{\tilde{\mu}} | Der_{\tilde{\mu}}^P)$$

is zero.

In the following, we limit ourselves to write the left parts of the Young bitableaux involved.

If $\mu_n \geq 1$, then

$$e_{\alpha_p n} \cdot (Der_{\tilde{\mu}} | Der_{\tilde{\mu}}^P)$$

equals

$$(-1)^{n-1}\left(\begin{array}{lllll} 1 & 2 & \cdots & n-1 & \alpha_p \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & \cdots & \\ \cdots & \cdots & \cdots & & \end{array}\right| + \cdots + (-1)^{n(\mu_n-1)+n-1}\left(\begin{array}{lllll} 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & n-1 & \alpha_p \\ 1 & 2 & \cdots & \cdots & \\ \cdots & \cdots & & & \end{array}\right|, \quad (15)$$

by Proposition 30.

A simple sign computation shows that (15) equals

$$(-1)^{n-1}\ \mu_n(-1)^{n-1}\left(\begin{array}{lllll} 1 & 2 & \cdots & n-1 & \alpha_p \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & \cdots & \\ \cdots & \cdots & \cdots & & \end{array}\right|.$$

Now, again by Proposition 30 and simple computation, we have:

$$
e_{\alpha_p n-1} \cdot \left(\begin{array}{lllll} 1 & 2 & \cdots & n-1 & \alpha_p \\ 1 & 2 & \cdots & \cdots & n \\ \cdots & \cdots & \cdots & & \\ \cdots\cdots\cdots & & & & \\ 1\,2 & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & & \end{array} \right| =
$$

$$
= (-1)^{n-2} \left(\begin{array}{lllll} 1 & 2 & \cdots & \alpha_p & \alpha_p \\ 1 & 2 & \cdots & \cdots & n \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & \cdots & n \\ 1\,2 & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & & \end{array} \right| +
$$

$$
+ \sum_{i=2}^{\mu_n} (-1)^{(n-1)+(i-2)n+(n-2)} \left(\begin{array}{lllll} 1 & 2 & \cdots & n-1 & \alpha_p \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & \alpha_p & n \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & \cdots & n \\ 1 & 2 & \cdots & n-1 & \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & n-1 & \\ 1 & 2 & \cdots & & \end{array} \right| +
$$

$$
+ \sum_{i=\mu_n+1}^{\mu_{n-1}} (-1)^{(n-1)+(\mu_n-1)n+(i-\mu_n-1)(n-1)+(n-2)} \left(\begin{array}{lllll} 1 & 2 & \cdots & n-1 & \alpha_p \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & \cdots & n \\ 1 & 2 & \cdots & n-1 & \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & \alpha_p & \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & n-1 & \\ 1 & 2 & \cdots & & \end{array} \right| ,
$$

where the tableaux in the two sums are the tableaux with the second occurence of α_p in the ith row.

By the *Straightening Law* of Grosshans, Rota and Stein ([20], Proposition 10, see also

[2], Thm. 8.1), each summand in the two sums equals

$$(-1)^{n-2}\frac{1}{2}\left(\begin{array}{lllll} 1 & 2 & \cdots & \alpha_p & \alpha_p \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & \cdots & n \\ 1\,2 & \cdots & & & \\ \cdots & \cdots & \cdots & & \end{array}\right|$$

and, hence,

$$e_{\alpha_p n-1}\cdot\left(\begin{array}{lllll} 1 & 2 & \cdots & n-1 & \alpha_p \\ 1 & 2 & \cdots & \cdots & n \\ \cdots & \cdots & \cdots & & \\ \cdots\cdots\cdots & & & & \\ 1\,2 & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & & \end{array}\right| = (-1)^{n-2}\frac{(\mu_{n-1}+1)}{2}\left(\begin{array}{lllll} 1 & 2 & \cdots & \alpha_p & \alpha_p \\ 1 & 2 & \cdots & n-1 & n \\ \cdots & \cdots & \cdots & & \\ 1 & 2 & \cdots & \cdots & n \\ 1\,2 & \cdots & & & \\ \cdots & \cdots & \cdots & & \end{array}\right|.$$

By iterating this argument, we obtain:

$$e_{\alpha_p j}\cdot\Big(\frac{1}{(n-j)!}\left(\begin{array}{llllll} 1 & 2 & \cdots & j & & \alpha_p^{n-j} \\ 1 & 2 & \cdots & j & \cdots & n \\ \cdots & \cdots & \cdots & & & \\ 1 & 2 & \cdots & j & \cdots & n \\ 1 & 2 & \cdots & & & \\ \cdots & \cdots & & & & \end{array}\right|\Big) =$$

$$= (-1)^{j-1}\ \frac{\mu_j+n-j}{(n-j+1)!}\ \left(\begin{array}{lllllll} 1 & 2 & \cdots & j-1 & & & \alpha_p^{n-j+1} \\ 1 & 2 & \cdots & j-1 & j & \cdots & n \\ \cdots & \cdots & \cdots & & & & \\ 1 & 2 & \cdots & j-1 & j & \cdots & n \\ 1 & 2 & \cdots & & & & \\ \cdots & \cdots & & & & & \end{array}\right|.$$

By iterating this procedure,

$$e_{\alpha_p 1}\cdots e_{\alpha_p n}\cdot(Der_{\tilde{\mu}}|Der_{\tilde{\mu}}^P) =$$

$$= \frac{(-1)^{\binom{n}{2}}}{n!}\ (\mu_1+n-1)(\mu_2+n-2)\cdots\mu_n\left(\begin{array}{llll} \alpha_p & \alpha_p & \cdots & \alpha_p \\ 1 & 2 & \cdots & n \\ \cdots & \cdots & \cdots & \\ 1 & 2 & \cdots & n \\ 1 & 2\cdots & & \\ \cdots & \cdots & & \end{array}\right|$$

and

$$e_{Coder_{n^p},Der_{n^p}} \cdot (Der_{\tilde{\mu}}|Der^P_{\tilde{\mu}}) =$$

$$= \left(\prod_{i=0}^{p-1} (\mu_1 - i + n - 1)\cdots(\mu_n - i)\right) \ \frac{(-1)^{\binom{n}{2}p}}{(n!)^p} \ \left(\begin{array}{llll} \alpha_p & \alpha_p & \cdots & \alpha_p \\ \alpha_{p-1} & \alpha_{p-1} & \cdots & \alpha_{p-1} \\ \cdots & \cdots & \cdots & \\ \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ 1 & 2 & \cdots & \\ \cdots & \cdots & & \end{array}\right| =$$

$$= \left(\prod_{i=0}^{p-1} (\mu_1 - i + n - 1)\cdots(\mu_n - i)\right) \ \frac{(-1)^{\binom{n}{2}p+\binom{p}{2}n}}{(n!)^p} \ \left(\begin{array}{llll} \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ \cdots & \cdots & \cdots & \\ \alpha_{p-1} & \alpha_{p-1} & \cdots & \alpha_{p-1} \\ \alpha_p & \alpha_p & \cdots & \alpha_p \\ 1 & 2 & \cdots & \\ \cdots & & & \end{array}\right| .$$

Since

$$e_{Der^*_{n^p},Coder_{n^p}} \cdot \left(\begin{array}{llll} \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ \cdots & \cdots & \cdots & \\ \alpha_{p-1} & \alpha_{p-1} & \cdots & \alpha_{p-1} \\ \alpha_p & \alpha_p & \cdots & \alpha_p \\ 1 & 2 & \cdots & \\ \cdots & & & \end{array}\right| \ = \ (-1)^{\binom{n}{2}p}(n!)^p \ (Der_{\tilde{\mu}}|Der^P_{\tilde{\mu}}) =$$

$$= \mathbf{K^p_n}(v_\mu) = \mathbf{K^p_n} \cdot (Der_{\tilde{\mu}}|Der^P_{\tilde{\mu}}) = e_{Der^*_{n^p},Coder_{n^p}} \ e_{Coder_{n^p},Der_{n^p}} \cdot (Der_{\tilde{\mu}}|Der^P_{\tilde{\mu}}) =$$

$$= \left(\prod_{i=0}^{p-1} (\mu_1 - i + n - 1)\cdots(\mu_n - i)\right) \ \frac{(-1)^{\binom{n}{2}p}}{(n!)^p} \ (-1)^{\binom{p}{2}n} \times$$

$$\times \, e_{Der^*_{n^p},Coder_{n^p}} \cdot (\left(\begin{array}{llll} \alpha_1 & \alpha_1 & \cdots & \alpha_1 \\ \cdots & \cdots & \cdots & \\ \alpha_{p-1} & \alpha_{p-1} & \cdots & \alpha_{p-1} \\ \alpha_p & \alpha_p & \cdots & \alpha_p \\ 1 & 2 & \cdots & \\ \cdots & & & \end{array}\right|) =$$

$$= \left(\prod_{i=0}^{p-1} (\mu_1 - i + n - 1)\cdots(\mu_n - i)\right) (-1)^{\binom{p}{2}n}(Der_{\tilde{\mu}}|Der^P_{\tilde{\mu}}).$$

Notice that, if $\mu_n < p$, then $\mathbf{K^p_n}(v_\mu) = 0$.

□

9 Appendix. A glimpse on the superalgebraic method of virtual variables

In this section, we summarize the main features of the superalgebraic method of virtual variables. We follow [8] and [9].

9.1 The general linear Lie super algebra $gl(m|n)$

Given a vector space V of dimension n, we will regard it as a subspace of a $\mathbb{Z}_2-$graded vector space $V_0 \oplus V_1$, where $V_1 = V$. The vector spaces V_0 (we assume that $dim(V_0) = m$ is "sufficiently large") is called the *positive virtual (auxiliary) vector space* and V is called the *(negative) proper vector space*.

The inclusion $V \subset V_0 \oplus V_1$ induces a natural embedding of the ordinary general linear Lie algebra $gl(n)$ of V_n into the *auxiliary* general linear Lie *superalgebra* $gl(m|n)$ of $V_0 \oplus V_1$ (see, e.g. [23], [39]).

Let $\mathcal{A}_0 = \{\alpha_1, \ldots, \alpha_{m_0}\}$, $\mathcal{L} = \{x_1, x_2, \ldots, x_n\}$ denote *fixed bases* of V_0 and $V = V_1$, respectively; therefore $|\alpha_s| = 0 \in \mathbb{Z}_2$, and $|i| = 1 \in \mathbb{Z}_2$.

Let

$$\{e_{a,b}; a, b \in \mathcal{A}_0 \cup \mathcal{L}\}, \qquad |e_{a,b}| = |a| + |b| \in \mathbb{Z}_2$$

be the standard $\mathbb{Z}_2-$homogeneous basis of the Lie superalgebra $gl(m|n)$ provided by the elementary matrices. The elements $e_{a,b} \in gl(m|n)$ are $\mathbb{Z}_2-$homogeneous of $\mathbb{Z}_2-$degree $|e_{a,b}| = |a| + |b|$.

The superbracket of the Lie superalgebra $gl(m_0|m_1+n)$ has the following explicit form:

$$[e_{a,b}, e_{c,d}] = \delta_{bc}\ e_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}\ e_{c,b},$$

$a, b, c, d \in \mathcal{A}_0 \cup \mathcal{L}$.

For the sake of readability, we will frequently write $\mathcal{L} = \{1, 2, \ldots, n\}$ in place of $\mathcal{L} = \{x_1, x_2, \ldots, x_n\}$.

The elements of the sets $\mathcal{A}_0, \mathcal{L}$ are called *positive virtual symbols* and *negative proper symbols*, respectively.

9.2 The supersymmetric algebra $\mathbb{C}[M_{m|n,d}]$

For the sake of readability, given $n, d \in \mathbb{Z}^+$, $n \leq d$, we write

$$M_{n,d} = [(i|j)]_{i=1,\ldots,n, j=1,\ldots,d} = \begin{pmatrix} (1|1) & \ldots & (1|d) \\ \vdots & & \vdots \\ (n|1) & \ldots & (n|d) \end{pmatrix}$$

in place of

$$M_{n,d} = [x_{ij}]_{i=1,\ldots,n;j=1,\ldots,d} = \begin{bmatrix} x_{11} & \ldots & x_{1d} \\ x_{21} & \ldots & x_{2d} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nd} \end{bmatrix}.$$

(compare with eq. (3)) and, consistently,

$$\mathbb{C}[M_{n,d}] = \mathbb{C}[(i|j)]_{i=1,\ldots,n,j=1,\ldots,d}$$

in place of

$$\mathbb{C}[M_{n,d}] = \mathbb{C}[x_{ij}]_{i=1,\ldots,n,j=1,\ldots,d}$$

for the polynomial algebra in the (commutative) entries $(i|j)$ of the matrix $M_{n,d}$.

We regard the commutative algebra $\mathbb{C}[M_{n,d}]$ as a subalgebra of the *"auxiliary" supersymmetric algebra*

$$\mathbb{C}[M_{m|n,d}]$$

generated by the ($\mathbb{Z}_2$-graded) variables

$$(a|j), \quad a \in \mathcal{A}_0 \cup \mathcal{L}, \quad j \in \mathcal{P} = \{j = 1, \ldots, d; |j| = 1 \in \mathbb{Z}_2\},$$

with $|(a|j)| = |a| + |j| \in \mathbb{Z}_2$, subject to the commutation relations:

$$(a|h)(b|k) = (-1)^{|(a|h)||(b|k)|} \, (b|k)(a|h).$$

In plain words, $\mathbb{C}[M_{m|n,d}]$ is the free supersymmetric algebra

$$\mathbb{C}\big[(\alpha_s|j), (i|j)\big]$$

generated by the ($\mathbb{Z}_2$-graded) variables $(\alpha_s|j), (i|j)$, $j = 1, 2, \ldots, d$, where all the variables commute each other, with the exception of pairs of variables $(\alpha_s|j), (\alpha_t|j)$ that skew-commute:

$$(\alpha_s|j)(\alpha_t|j) = -(\alpha_t|j)(\alpha_s|j).$$

In the standard notation of multilinear algebra, we have:

$$\mathbb{C}[M_{m|n,d}] \cong \Lambda\big[V_0 \otimes P_d\big] \otimes \mathrm{Sym}\big[V_1 \otimes P_d\big]$$

where $P_d = (P_d)_1$ denotes the trivially $\mathbb{Z}_2-$graded vector space with distinguished basis $\mathcal{P} = \{j = 1, \ldots, d; |j| = 1 \in \mathbb{Z}_2\}$.

9.3 Left superderivations and left superpolarizations

A *left superderivation* D^l ($\mathbb{Z}_2-$homogeneous of degree $|D^l|$) (see, e.g. [39], [23]) on $\mathbb{C}[M_{m|n,d}]$ is an element of the superalgebra $End_{\mathbb{C}}[\mathbb{C}[M_{m|n,d}]]$ that satisfies "Leibniz rule"

$$D^l(\mathbf{p} \cdot \mathbf{q}) = D^l(\mathbf{p}) \cdot \mathbf{q} + (-1)^{|D^l||\mathbf{p}|}\mathbf{p} \cdot D^l(\mathbf{q}),$$

for every $\mathbb{Z}_2-$homogeneous of degree $|\mathbf{p}|$ element $\mathbf{p} \in \mathbb{C}[M_{m|n,d}]$.

Given two symbols $a, b \in \mathcal{A}_0 \cup \mathcal{L}$, the *left superpolarization* $D^l_{a,b}$ of b to a is the unique *left* superderivation of $\mathbb{C}[M_{m|n,d}]$ of $\mathbb{Z}_2-$degree $|D^l_{a,b}| = |a| + |b| \in \mathbb{Z}_2$ such that

$$D^l_{a,b}\left((c|j)\right) = \delta_{bc}\ (a|j),\ c \in \mathcal{A}_0 \cup \mathcal{L},\ j = 1, \ldots, n.$$

Informally, we say that the operator $D^l_{a,b}$ *annihilates* the symbol b and *creates* the symbol a.

9.4 The superalgebra $\mathbb{C}[M_{m|n,d}]$ as a $\mathbf{U}(gl(m|n))$-module

Since

$$D^l_{a,b}D^l_{c,d} - (-1)^{(|a|+|b|)(|c|+|d|)}D^l_{c,d}D^l_{a,b} = \delta_{b,c}D^l_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{a,d}D^l_{c,b},$$

the map

$$e_{a,b} \mapsto D^l_{a,b}, \qquad a, b \in \mathcal{A}_0 \cup \mathcal{L}$$

is a Lie superalgebra morphism from $gl(m|n)$ to $End_{\mathbb{C}}\left[\mathbb{C}[M_{m|n,d}]\right]$ and, hence, it uniquely defines a representation:

$$\varrho : \mathbf{U}(gl(m|n)) \to End_{\mathbb{C}}[\mathbb{C}[M_{m|n,d}]],$$

where $\mathbf{U}(gl(m|n))$ is the enveloping superalgebra of $gl(m|n)$.

In the following, we always regard the superalgebra $\mathbb{C}[M_{m|n,d}]$ as a $\mathbf{U}(gl(m|n))-$supermodule, with respect to the action induced by the representation ϱ:

$$e_{a,b} \cdot \mathbf{p} = D^l_{a,b}(\mathbf{p}),$$

for every $\mathbf{p} \in \mathbb{C}[M_{m|n,d}]$.

We recall that $\mathbf{U}(gl(m|n))-$module $\mathbb{C}[M_{m|n,d}]$ is a semisimple module, whose simple submodules are - up to isomorphism - *Schur supermodules* (see, e.g. [4], [5], [2]. For a more traditional presentation, see also [15]).

Clearly, $\mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n))$ is a subalgebra of $\mathbf{U}(gl(m|n))$ and the subalgebra $\mathbb{C}[M_{n,d}]$ is a $\mathbf{U}(gl(n))-$submodule of $\mathbb{C}[M_{m|n,d}]$.

9.5 The virtual algebra $Virt(m, n)$ and the virtual presentations of elements in $\mathbf{U}(gl(n))$

We say that a product

$$e_{a_m,b_m} \cdots e_{a_1,b_1} \in \mathbf{U}(gl(m|n)), \quad a_i, b_i \in \mathcal{A}_0 \cup \mathcal{L}, \ i = 1, \ldots, m$$

is an *irregular expression* whenever there exists a right subword

$$e_{a_i,b_i} \cdots e_{a_2,b_2} e_{a_1,b_1},$$

$i \leq m$ and a virtual symbol $\gamma \in \mathcal{A}_0$ such that

$$\#\{j; b_j = \gamma, j \leq i\} > \#\{j; a_j = \gamma, j < i\}.$$

The meaning of an irregular expression in terms of the action of $\mathbf{U}(gl(m|n))$ by left superpolarization on the algebra $\mathbb{C}[M_{m|n,d}]$ is that there exists a virtual symbol γ and a right subsequence in which the symbol γ is *annihilated* more times than it was already *created* and, therefore, the action of an irregular expression on the algebra $\mathbb{C}[M_{n,d}]$ is *zero*.

Example 7. *Let $\gamma \in \mathcal{A}_0$ and $x_i, x_j \in \mathcal{L}$. The product*

$$e_{\gamma,x_j} e_{x_i,\gamma} e_{x_j,\gamma} e_{\gamma,x_i}$$

is an irregular expression.

□

Let **Irr** be the *left ideal* of $\mathbf{U}(gl(m|n))$ generated by the set of irregular expressions.

Proposition 19. *The superpolarization action of any element of* **Irr** *on the subalgebra* $\mathbb{C}[M_{n,d}] \subset \mathbb{C}[M_{m|n,d}]$ *- via the representation ϱ - is identically zero.*

Proposition 20. *The sum* $\mathbf{U}(gl(0|n)) + \mathbf{Irr}$ *is a direct sum of vector subspaces of* $\mathbf{U}(gl(m|n))$.

Proposition 21. *The direct sum vector subspace* $\mathbf{U}(gl(0|n)) \oplus \mathbf{Irr}$ *is a* subalgebra *of* $\mathbf{U}(gl(m|n))$.

The subalgebra

$$Virt(m, n) = \mathbf{U}(gl(0|n)) \oplus \mathbf{Irr} \subset \mathbf{U}(gl(m|n)).$$

is called the *virtual algebra*.

Proposition 22. *The left ideal* $\mathbf{Irr}$ *of* $\mathbf{U}(gl(m|n))$ *is a two sided ideal of* $Virt(m,n)$.

The *Capelli devirtualization epimorphism* is the surjection

$$\mathfrak{p} : Virt(m,n) = \mathbf{U}(gl(0|n)) \oplus \mathbf{Irr} \twoheadrightarrow \mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n))$$

with $Ker(\mathfrak{p}) = \mathbf{Irr}$.

Any element in $\mathbf{M} \in Virt(m,n)$ defines an element in $\mathbf{m} \in \mathbf{U}(gl(n))$ - via the map $\mathfrak{p}$ - and $\mathbf{M}$ is called a *virtual presentation* of $\mathbf{m}$.

Furthermore,

Proposition 23. *The subalgebra* $\mathbb{C}[M_{n,d}] \subset \mathbb{C}[M_{m|n,d}]$ *is invariant with respect to the action of the subalgebra* $Virt(m,n)$.

Proposition 24. *For every element* $\mathbf{m} \in \mathbf{U}(gl(n))$*, the action of* $\mathbf{m}$ *on the subalgebra* $\mathbb{C}[M_{n,d}]$ *is the same of the action of any of its virtual presentation* $\mathbf{M} \in Virt(m,n)$. *In symbols,*

$$if \quad \mathfrak{p}(\mathbf{M}) = \mathbf{m} \quad then \quad \mathbf{m} \cdot \mathbf{P} = \mathbf{M} \cdot \mathbf{P}, \quad for\ every\ \mathbf{P} \in \mathbb{C}[M_{n,d}].$$

Since the map $\mathfrak{p}$ a surjection, any element $\mathbf{m} \in \mathbf{U}(gl(n))$ admits several virtual presentations. In the sequel, we even take virtual presentations as the *definition* of special elements in $\mathbf{U}(gl(n))$, and this method will turn out to be quite effective.

The superalgebra $\mathbf{U}(gl(m|n))$ is a Lie module with respect to the adjoint representation $Ad_{gl(m|n)}$. Since $gl(n) = gl(0|n)$ is a Lie subalgebra of $gl(m|n)$, then $\mathbf{U}(gl(m|n))$ is a $gl(n)-$module with respect to the adjoint action $Ad_{gl(n)}$ of $gl(n)$.

Proposition 25. *The virtual algebra* $Virt(m,n)$ *is a submodule of* $\mathbf{U}(gl(m|n))$ *with respect to the adjoint action* $Ad_{gl(n)}$ *of* $gl(n)$.

Proposition 26. *The Capelli epimorphism*

$$\mathfrak{p} : Virt(m,n) \twoheadrightarrow \mathbf{U}(gl(n))$$

is an $Ad_{gl(n)}-$equivariant *map.*

Corollary 18. *The isomorphism* $\mathfrak{p}$ *maps any* $Ad_{gl(n)}-$*invariant element* $\mathbf{m} \in Virt(m,n)$ *to a* central *element of* $\mathbf{U}(gl(n))$.

Balanced monomials are elements of the algebra $\mathbf{U}(gl(m|n))$ of the form:

– $e_{i_1,\gamma_{p_1}} \cdots e_{i_k,\gamma_{p_k}} \cdot e_{\gamma_{p_1},j_1} \cdots e_{\gamma_{p_k},j_k}$,

– $e_{i_1,\theta_{q_1}} \cdots e_{i_k,\theta_{q_k}} \cdot e_{\theta_{q_1},\gamma_{p_1}} \cdots e_{\theta_{q_k},\gamma_{p_k}} \cdot e_{\gamma_{p_1},j_1} \cdots e_{\gamma_{p_k},j_k}$,

– and so on,

where $i_1, \ldots, i_k, j_1, \ldots, j_k \in L$, i.e., the $i_1, \ldots, i_k, j_1, \ldots, j_k$ are k proper (negative) symbols, and the $\gamma_{p_1}, \ldots, \gamma_{p_k}, \ldots, \theta_{q_1}, \ldots, \theta_{q_k}, \ldots$ are virtual symbols. In plain words, a balanced monomial is product of two or more factors where the rightmost one *annihilates* (by superpolarization) the k proper symbols $j_1, \ldots, j_k$ and *creates* (by superpolarization) some virtual symbols; the leftmost one *annihilates* all the virtual symbols and *creates* the k proper symbols $i_1, \ldots, i_k$; between these two factors, there might be further factors that annihilate and create virtual symbols only.

Proposition 27. *Every balanced monomial belongs to $Virt(m, n)$. Hence, the Capelli epimorphism $\mathfrak{p}$ maps balanced monomials to elements of $\mathbf{U}(gl(n))$.*

9.6 Bitableaux monomials and Capelli bitableaux in $\mathbf{U}(gl(n))$

We will introduce two classes of remarkable elements of the enveloping algebra $\mathbf{U}(gl(n))$, that we call *bitableaux monomials*, *Capelli bitableaux*, respectively.

Let $\lambda \vdash h$ be a partition, and label the boxes of its Ferrers diagram with the numbers $1, 2, \ldots, h$ in the following way:

$$\begin{array}{lllll} 1 & 2 & \cdots & \cdots & \lambda_1 \\ \lambda_1+1 & \lambda_1+2 & \cdots & \lambda_1+\lambda_2 & \\ \cdots & \cdots & \cdots & & \\ \cdots & \cdots & h & & \end{array} .$$

A *Young tableau* T of shape λ over the alphabet $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{L}$ is a map $T : \underline{h} = \{1, 2, \ldots, h\} \to \mathcal{A}$; the element $T(i)$ is the symbol in the cell i of the tableau T.

The sequences

$$\begin{array}{l} T(1)T(2)\cdots T(\lambda_1), \\ T(\lambda_1+1)T(\lambda_1+2)\cdots T(\lambda_1+\lambda_2), \\ \cdots\cdots \end{array}$$

are called the *row words* of the Young tableau T.

We will also denote a Young tableau by its sequence of rows words, that is $T = (\omega_1, \omega_2, \ldots, \omega_p)$. Furthermore, the *word of the tableau* T is the concatenation

$$w(T) = \omega_1\omega_2\cdots\omega_p.$$

The *content* of a tableau T is the function $c_T : \mathcal{A} \to \mathbb{N}$,

$$c_T(a) = \sharp\{i \in \underline{h};\ T(i) = a\}.$$

Given a shape/partition λ, we assume that $|\mathcal{A}_0| = m \geq \widetilde{\lambda}_1$, where $\widetilde{\lambda}$ denotes the conjugate shape/partition of λ. Let us denote by $\alpha_1, \ldots, \alpha_p \in \mathcal{A}_0$ an *arbitrary* family of *distinct positive symbols*. Set

$$C_\lambda^* = \left(\begin{array}{l} \alpha_1 \ldots\ldots\ldots \alpha_1 \\ \alpha_2 \ldots\ldots \alpha_2 \\ \ldots\ldots \\ \alpha_p \ldots \alpha_p \end{array} \right). \qquad (16)$$

The tableaux of kind (16) are called *virtual Coderuyts tableaux* of shape λ,.

Let S and T be two Young tableaux of same shape $\lambda \vdash h$ on the alphabet $\mathcal{A}_0 \cup \mathcal{L}$:

$$S = \left(\begin{array}{l} z_{i_1} \ldots\ldots\ldots z_{i_{\lambda_1}} \\ z_{j_1} \ldots\ldots z_{j_{\lambda_2}} \\ \ldots\ldots \\ z_{s_1} \ldots z_{s_{\lambda_p}} \end{array} \right), \qquad T = \left(\begin{array}{l} z_{h_1} \ldots\ldots\ldots z_{h_{\lambda_1}} \\ z_{k_1} \ldots\ldots z_{k_{\lambda_2}} \\ \ldots\ldots \\ z_{t_1} \ldots z_{t_{\lambda_p}} \end{array} \right).$$

To the pair (S, T), we associate the *bitableau monomial*:

$$e_{S,T} = e_{z_{i_1}, z_{h_1}} \cdots e_{z_{i_{\lambda_1}}, z_{h_{\lambda_1}}} e_{z_{j_1}, z_{k_1}} \cdots e_{z_{j_{\lambda_2}}, z_{k_{\lambda_2}}} \cdots\cdots e_{z_{s_1}, z_{t_1}} \cdots e_{z_{s_{\lambda_p}}, z_{t_{\lambda_p}}}$$

in $\mathbf{U}(gl(m|n))$.

Given a pair of Young tableaux S, T of the same shape λ on the proper alphabet L, consider the elements

$$e_{S, C_\lambda^*} \; e_{C_\lambda^*, T} \in \mathbf{U}(gl(m|n)).$$

Since these elements are *balanced monomials* in $\mathbf{U}(gl(m|n))$, then they belong to the *virtual subalgebra* $Virt(m, n)$.

Hence, we can consider their images in $\mathbf{U}(gl(n))$ with respect to the Capelli epimorphism $\mathfrak{p}$.

We set

$$\mathfrak{p}\Big(e_{S, C_\lambda^*} \; e_{C_\lambda^*, T} \Big) = [S|T] \in \mathbf{U}(gl(n)), \qquad (17)$$

and call the element $[S|T]$ a *Capelli bitableau*.

The elements defined in (17) do not depend on the choice of the virtual Coderuyts tableau C_λ^*.

9.7 Biproducts and bitableaux in $\mathbb{C}[M_{m|n,d}]$

Embed the algebra

$$\mathbb{C}[M_{m|n,d}] = \mathbb{C}[(\alpha_s|j), (i|j)]$$

into the (supersymmetric) algebra $\mathbb{C}[(\alpha_s|j), (i|j), (\gamma|j)]$ generated by the ($\mathbb{Z}_2$-graded) variables $(\alpha_s|j), (i|j), (\gamma|j)$, $j = 1, 2, \ldots, d$, where

$$|(\gamma|j)| = 1 \in \mathbb{Z}_2 \quad for\ every\ j = 1, 2, \ldots, d,$$

and denote by $D^l_{z_i,\gamma}$ the superpolarization of γ to z_i.

Let $\omega = z_1 z_2 \cdots z_p$ be a word on $\mathcal{A}_0 \cup \mathcal{L}$, and $\varpi = j_{t_1} j_{t_2} \cdots j_{t_q}$ a word on the alphabet $P = \{1, 2, \ldots, d\}$. The *biproduct*

$$(\omega|\varpi) = (z_1 z_2 \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_q})$$

is the element

$$D^l_{z_1,\gamma} D_{z_2,\gamma} \cdots D^l_{z_p,\gamma}\Big((\gamma|j_{t_1})(\gamma|j_{t_2}) \cdots (\gamma|j_{t_q})\Big) \in \mathbb{C}[M_{m|n,d}]$$

if $p = q$ and is set to be zero otherwise.

Claim 1. *The biproduct* $(\omega|\varpi) = (z_1 z_2 \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_q})$ *is supersymmetric in the z's and skew-symmetric in the j's. In symbols*

1. $(z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_q}) = (-1)^{|z_i||z_{i+1}|}(z_1 z_2 \cdots z_{i+1} z_i \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_q})$
2. $(z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_i} j_{t_{i+1}} \cdots j_{t_q}) = -(z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_{t_1} \cdots j_{t_{i+1}} j_{t_i} \cdots j_{t_q})$.

Proposition 28. ***(Laplace expansions)*** *We have*

1. $(\omega_1\omega_2|\varpi) = \Sigma_{(\varpi)}\ (-1)^{|\varpi_{(1)}||\omega_2|}\ (\omega_1|\varpi_{(1)})(\omega_2|\varpi_{(2)})$.
2. $(\omega|\varpi_1\varpi_2) = \Sigma_{(\omega)}\ (-1)^{|\varpi_1||\omega_{(2)}|}\ (\omega_{(1)}|\varpi_1)(\omega_{(2)}|\varpi_2.)$

where

$$\triangle(\varpi) = \Sigma_{(\varpi)}\ \varpi_{(1)} \otimes \varpi_{(2)}, \quad \triangle(\omega) = \Sigma_{(\omega)}\ \omega_{(1)} \otimes \omega_{(2)}$$

denote the coproducts in the Sweedler notation *of the elements* ϖ *and* ω *in the supersymmetric Hopf algebra of W* (see, e.g. [2]) *and in the free exterior Hopf algebra generated by* $j = 1, 2, \ldots, d$, *respectively.*

Let $\omega = i_1 i_2 \cdots i_p$, $\varpi = j_1 j_1 \cdots j_p$ be words on the negative alphabet $\mathcal{L} = \{1, 2, \ldots, n\}$ and on the negative alphabet $\mathcal{P} = \{1, 2, \ldots, d\}$.

From Proposition 28, we infer

Corollary 19. *The biproduct of the two words ω and ϖ*

$$(\omega|\varpi) = (i_1 i_2 \cdots i_p | j_1 j_2 \cdots j_p)$$

is the signed minor*:*

$$(\omega|\varpi) = (-1)^{\binom{p}{2}} \ det\Big(\ (i_r|j_s) \ \Big)_{r,s=1,2,\ldots,p} \in \mathbb{C}[M_{n,d}].$$

Following the notation introduced in the previous sections, let

$$Super[V_0 \oplus V_1] = Sym[V_0] \otimes \Lambda[V_1]$$

denote the *(super)symmetric* algebra of the space

$$V_0 \oplus V_1$$

(see, e.g. [39]).

By multilinearity, the algebra $Super[V_0 \oplus V_1]$ is the same as the superalgebra $Super[\mathcal{A}_0 \cup \mathcal{L}]$ generated by the "variables"

$$\alpha_1, \ldots, \alpha_{m_0} \in \mathcal{A}_0, \quad 1, \ldots, n \in L,$$

modulo the congruences

$$zz' = (-1)^{|z||z'|} z'z, \quad z, z' \in \mathcal{A}_0 \cup \mathcal{L}.$$

Let $d^l_{z,z'}$ denote the (left)polarization operator of z' to z on

$$Super[W] = Super[\mathcal{A}_0 \cup \mathcal{L}],$$

that is the unique superderivation of $\mathbb{Z}_2$-degree

$$|z| + |z'| \in \mathbb{Z}_2$$

such that

$$d^l_{z,z'}(z'') = \delta_{z',z''} \cdot z,$$

for every $z, z', z'' \in \mathcal{A}_0 \cup \mathcal{L}$.

Clearly, the map

$$e_{z,z'} \to d^l_{z,z'}$$

is a Lie superalgebra map and, therefore, induces a structure of

$$gl(m|n) - module$$

on $Super[\mathcal{A}_0 \cup \mathcal{L}] = Super[V_0 \oplus V_1]$.

Proposition 29. *Let $\varpi = j_{t_1} j_{t_2} \cdots j_{t_q}$ be a word on $P = \{1, 2, \ldots, d\}$. The map*

$$\Phi_\varpi : \omega \mapsto (\omega|\varpi),$$

ω any word on $\mathcal{A}_0 \cup \mathcal{L}$, uniquely defines $gl(m|n)-$equivariant linear operator

$$\Phi_\varpi : Super[\mathcal{A}_0 \cup \mathcal{L}] \to \mathbb{C}[M_{m|n,d}],$$

that is

$$\Phi_\varpi\big(e_{z,z'} \cdot \omega\big) = \Phi_\varpi\big(d^l_{z,z'}(\omega)\big) = D^l_{z,z'}\big((\omega|\varpi)\big) = e_{z,z'} \cdot (\omega|\varpi),$$

for every $z, z' \in \mathcal{A}_0 \cup \mathcal{L}$.

With a slight abuse of notation, we will write (29) in the form

$$D^l_{z,z'}\big((\omega|\varpi)\big) = (d^l_{z,z'}(\omega)|\varpi). \tag{18}$$

Let $S = (\omega_1, \omega_2, \ldots, \omega_p$ and $T = (\varpi_1, \varpi_2, \ldots, \varpi_p)$ be Young tableaux on $\mathcal{A}_0 \cup \mathcal{L}$ and $P = \{1, 2, \ldots, d\}$ of shapes λ and μ, respectively.

If $\lambda = \mu$, the *Young bitableau* $(S|T)$ is the element of $\mathbb{C}[M_{m|n,d}]$ defined as follows:

$$(S|T) = \left(\begin{array}{c|c} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{array}\right) = \pm\ (\omega_1)|\varpi_1)(\omega_2)|\varpi_2) \cdots (\omega_p)|\varpi_p),$$

where

$$\pm = (-1)^{|\omega_2||\varpi_1| + |\omega_3|(|\varpi_1|+|\varpi_2|) + \cdots + |\omega_p|(|\varpi_1|+|\varpi_2|+\cdots+|\varpi_{p-1}|)}.$$

If $\lambda \neq \mu$, the *Young bitableau* $(S|T)$ is set to be zero.

By naturally extending the slight abuse of notation (18), the action of any polarization on bitableaux can be explicitly described:

Proposition 30. *Let $z, z' \in \mathcal{A}_0 \cup \mathcal{L}$, and let $S = (\omega_1, \ldots, \omega_p)$, $T = (\varpi_1, \ldots, \varpi_p)$. We have the following identity:*

$$\begin{aligned} e_{z,z'} \cdot (S\,|\,T) \ &= \ D^l_{z,z'}\ \Big(\left(\begin{array}{c|c} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{array}\right) \Big) \\ &= \ \sum_{s=1}^{p}\ (-1)^{(|z|+|z'|)\epsilon_s} \ \left(\begin{array}{c|c} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ d^l_{z,z'}(\omega_s) & \vdots \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{array}\right), \end{aligned}$$

where

$$\epsilon_1 = 1, \quad \epsilon_s = |\omega_1| + \cdots + |\omega_{s-1}|, \quad s = 2, \ldots, p.$$

Example 8. *Let* $\alpha_i \in \mathcal{A}_0$, $1, 2, 3, 4 \in L$, $|D_{\alpha_i,2}| = 1$. *Then*

$$e_{\alpha_i,2} \cdot \left(\begin{array}{l|l} 1\ 3\ 2 & 1\ 2\ 3 \\ 2\ 3 & 2\ 3 \\ 4\ 2 & 3\ 1 \end{array} \right) = D^l_{\alpha_i,2} \left(\left(\begin{array}{l|l} 1\ 3\ 2 & 1\ 2\ 3 \\ 2\ 3 & 2\ 3 \\ 4\ 2 & 3\ 1 \end{array} \right) \right) =$$

$$= \left(\begin{array}{l|l} 1\ 3\ \alpha_i & 1\ 2\ 3 \\ 2\ 3 & 2\ 3 \\ 4\ 2 & 3\ 1 \end{array} \right) - \left(\begin{array}{l|l} 1\ 3\ 2 & 1\ 2\ 3 \\ \alpha_i\ 3 & 2\ 3 \\ 4\ 2 & 3\ 1 \end{array} \right) + \left(\begin{array}{l|l} 1\ 3\ 2 & 1\ 2\ 3 \\ 2\ 3 & 2\ 3 \\ 4\ \alpha_i & 3\ 1 \end{array} \right).$$

References

[1] M. Atiyah, R. Bott and V. Patodi, On the heat equation and the Index Theorem, *Invent. Math.* **19**(1973), 279-330

[2] A. Brini, Combinatorics, superalgebras, invariant theory and representation theory, *Séminaire Lotharingien de Combinatoire* **55** (2007), Article B55g, 117 pp.

[3] A. Brini, Superalgebraic Methods in the Classical Theory of Representations. Capelli's Identity, the Koszul map and the Center of the Enveloping Algebra $\mathbf{U}(gl(n))$, in *Topics in Mathematics, Bologna*, Quaderni dell' Unione Matematica Italiana n. 15, UMI, 2015, pp. 1 – 27

[4] A. Brini, A. Palareti, A. Teolis, Gordan–Capelli series in superalgebras, *Proc. Natl. Acad. Sci. USA* **85** (1988), 1330–1333

[5] A. Brini, A. Teolis, Young–Capelli symmetrizers in superalgebras, *Proc. Natl. Acad. Sci. USA* **86** (1989), 775–778.

[6] A. Brini, A. Teolis, Capelli bitableaux and $\mathbb{Z}$-forms of general linear Lie superalgebras, *Proc. Natl. Acad. Sci. USA* **87** (1990), 56–60

[7] A. Brini, A. Teolis, Capelli's theory, Koszul maps, and superalgebras, *Proc. Natl. Acad. Sci. USA* **90** (1993), 10245–10249

[8] A. Brini, A. Teolis, Young-Capelli bitableaux, Capelli immanants in $\mathbf{U}(gl(n))$ and the Okounkov quantum immanants, *Journal of Algebra and Its Applications*, Vol. 20, No. 7 (2021) 2150123 (44 pages), Preprint: arXiv: 1807.10045v3, 45 pp.

[9] A. Brini, A. Teolis, On the action of the Koszul map on the enveloping algebra of the general linear Lie algebra, *Communications in Algebra* **48**(12) (2020), 5254-5281 Preprint, arXiv: 1906.02516v3, 33 pp.

[10] A. Brini, A. Teolis, Quantum immanants, double Young-Capelli bitableaux and Schur shifted symmetric functions, *Journal of Algebra and its Applications*, to appear. Preprint, arXiv:2107.10205v3, 52 pp.

[11] A. Capelli, Ueber die Zurückführung der Cayley'schen Operation Ω auf gewöhnliche Polar-Operationen, *Math. Ann.* **29** (1887), 331-338

[12] A. Capelli, Sul sistema completo delle operazioni di polare permutabili con ogni altra operazione di polare fra le stesse serie di variabili, *Rend. Regia Acc. Scienze Napoli* vol. VII (1893), 29 - 38

[13] A. Capelli, Dell'impossibilità di sizigie fra le operazioni fondamentali permutabili con ogni altra operazione di polare fra le stesse serie di variabili, *Rend. Regia Acc. Scienze Napoli*, vol. VII (1893), 155 - 162

[14] A. Capelli, *Lezioni sulla teoria delle forme algebriche,* Pellerano, Napoli, 1902.

[15] S.-J. Cheng, W. Wang, Howe duality for Lie superalgebras, *Compositio Math.* **128** (2001), 55–94

[16] C. De Concini, D. Eisenbud, C. Procesi, Young diagrams and determinantal varieties, *Invent. Math.* **56** (1980), 129–165.

[17] J. Désarménien, J. P. S. Kung, G.-C. Rota, Invariant theory, Young bitableaux and combinatorics, *Adv. Math.* **27** (1978), 63–92

[18] J. Dixmier, *Enveloping algebras,* Graduate Studies in Mathematics 11, American Mathematical Society, Providence, RI, 1996.

[19] P. Doubilet, G.-C. Rota, J. A. Stein, On the foundations of combinatorial theory IX. Combinatorial methods in invariant theory, *Studies in Appl. Math.* **53** (1974), 185–216

[20] F. D. Grosshans, G.-C. Rota and J. A. Stein, *Invariant Theory and Superalgebras*, AMS, 1987

[21] R. Howe, Remarks on classical invariant theory, *Trans. Amer. Math. Soc.* **313** (1989), 539–570

[22] R. Howe, T. Umeda, The Capelli identity, the double commutant theorem, and multiplicity-free actions, *Math. Ann.* 290 (1991), 565-619

[23] V. Kac, Lie Superalgebras, *Adv. Math.* **26** (1977), 8–96

[24] B Kostant and S. Sahi, The Capelli identity, Tube Domains and the Generalized Laplace Transform, *Adv. Math.* **87** (1991), 71–92

[25] B Kostant and S. Sahi, Jordan algebras and Capelli identities, *Invent. Math.* **112** (1993), 657–664

[26] J.-L. Koszul, Les algébres de Lie graduées de type sl(n,1) et l'opérateur de A. Capelli, *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), no. 2, 139-141

[27] A.I. Molev, Yangians and their applications, in *Handbook of Algebra, vol.* 3 (M.Hazewinkel, Ed.), pp. 907 – 960, Elsevier, 2003

[28] A.I. Molev, *Yangians and Classical Lie Algebras*, Mathematical Surveys and Monographs, **143**, Amer. Math. Soc., Providence RI, 2007

[29] A.I. Molev and M. Nazarov, Capelli identities for classical Lie algebras, *Math. Ann.* 313 (1999), 315-357

[30] M. Nazarov, Quantum Berezinian and the classical Capelli identity, *Lett. Math. Phys.* 21 (1991), 123-131

[31] A. Okounkov, Quantum immanants and higher Capelli identities, *Transformation Groups* 1 (1996), 99-126

[32] A. Okounkov, Young basis, Wick formula, and higher Capelli identities, *Intern. Math. Res. Notices* (1996), no. 17, 817–839

[33] A. Okounkov, G. I. Olshanski, Shifted Schur functions, *Algebra i Analiz* **9**(1997), no. 2, 73–146 (Russian); English translation: *St. Petersburg Math. J.* **9** (1998), 239–300

[34] G. I. Olshanski, Extension of the algebra U(g) for infinite-dimensional classical Lie algebras g, and the Yangians Y (gl(m)), *Soviet Math. Dokl.* **36** (1988), 569–573.

[35] G. I. Olshanski, Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians, in *Topics in Representation Theory* (A. A. Kirillov, Ed.), Advances in Soviet Math. **2**, AMS, Providence RI, 1991, pp. 1–66

[36] C. Procesi, *Lie Groups. An approach through invariants and representations*, Universitext, Springer, 2007

[37] F. Regonati, On the combinatorics of Young-Capelli symmetrizers, *Séminaire Lotharingien de Combinatoire* **62** Article B62d, (2009), 36 pp.

[38] S. Sahi, The Spectrum of Certain Invariant Differential Operators Associated to a Hermitian Symmetric Space, in *Lie theory and Geometry: in honor of Bertram Kostant*, (J.-L. Brylinski, R.. Brylinski, V. Guillemin, V. Kac, Eds.), Progress in Mathematics, Vol. 123, pp. 569–576, Birkhauser, 1994

[39] M. Scheunert, *The theory of Lie superalgebras: an introduction,* Lecture Notes in Math., vol. 716, Springer Verlag, New York, 1979

[40] T. Umeda, The Capelli identity one century after, in: *Selected Papers on Harmonic Analysis, Groups and Invariants*, pp. 51-78, Amer. Math. Soc. Transl. Ser. 2, **183**, AMS, Providence, RI, 1998

[41] T. Umeda, On the proof of the Capelli identities, *Funkcialaj Ekvacioj* **51** (2008), 1-15

[42] H. Weyl, *The Classical Groups*, 2nd ed., Princeton University Press, 1946