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Published Version:
Extensions and distortions of λ-fuzzy measures / Cherubini U.; Mulinacci S.. - In: FUZZY SETS AND SYSTEMS. - ISSN 0165-0114. - ELETTRONICO. - 412:(2021), pp. 27-40. [10.1016/j.fss.2020.02.014]

Availability:
This version is available at: https://hdl.handle.net/11585/757175 since: 2021-04-09

Published:
DOI: http://doi.org/10.1016/j.fss.2020.02.014

Extensions and distortions of λ-fuzzy measures

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Abstract

We propose extensions and distorsion techniques to improve the flexibility of λ-fuzzy measures. As for the extensions, we propose the use of the family of Archimedean t-conorms as generators of the fuzzy measure. As for distortions, we propose techniques based on the composition or patchwork of different generators. As an example of application, we show that in option pricing the techniques proposed substantially improve the flexibility of the model in order to reproduce features that are consistent with evidence expected on the market and that only one Archimedean generator would not be able to represent.

Keywords. Fuzzy measures, λ-measure, Archimedean t-conorms, Choquet pricing, Option pricing

1 Introduction

The class of λ-fuzzy measures was introduced by Sugeno in the 70s. Given a universe of events Ω, λ-measures are defined by the aggregation operator

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B) \quad (1)$$

for all disjoint sets A and B in Ω, and with λ a constant parameter defined in $[-1, \infty)$.

The advantage of the λ-measure is to provide a nice parametrization of sub and superaddivity. Applications have been made to clustering (Leszczinsky et al., 1985), option pricing (Cherubini, 1997), credit risk valuation (Cherubini and Della Lunga, 2001, Han and Zheng, 2005), and other industrial problems (see Liginlal and Ow, 2006, for a general review).

For some applications, the advantage of this parametric form is also that it is closed with respect to the duality between sub and superadditive measures. It is well known that for any sub-additive measure $\mu(A), A \in \Omega$ there exists a super-additive one such that $\mu^*(A^c) = 1 - \mu(A)$, where A^c denotes the complement set of A. We know that, if $\mu(.)$ is a fuzzy measure of the λ class with parameter λ in $(0, \infty)$, the dual measure is also of the same class, with parameter in $[-1, 0)$. More precisely

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B) - \frac{\lambda}{1+\lambda}\mu^*(A)\mu^*(B) \quad (2)$$

While the imposition of this specific functional shape, governed by a single parameter, is of great help in applications, it may turn out into a flaw for some of them which would require more flexibility. So, if for example we have $\mu(A) = \mu(B) = \mu(C)$ for the disjoint sets A, B and C, it follows that $\mu(A \cup B) = \mu(A \cup C) = \mu(B \cup C)$. This may be a flaw for some applications since, for example, it may happen that A and B are not clearly distinguishable one from the other while they are clearly separated from C. In this case, we would like to represent the union of A and B by a subadditive measure, and the union of each one of the two with event C by an additive one. Moreover, one could also have the need to represent models in which the uncertainty is mostly relevant in events that are rare, with respect to others that are more frequent, and for which therefore we have much more evidence.

In this paper we propose a generalization of the parametric form of the λ-measures. Resorting to the family of Archimedean t-conorms we will be able to extend the results of our analysis to a wide range of functional forms for $\mu(A \cup B)$ of which that in equation (1) is a specific case. In particular, we will be able to propose distortions of λ-measures to make the model more flexible for applications.

The plan of the paper is as follows. In section 2 we report the general technique to generalise the λ-measure to the class of Archimedean t-conorms generators. In section 3 we explore alternative parametric forms available to represent sub-(super-)additivity and the techniques to distort them. In section 4 we present an option pricing application showing an example of a case in which more flexibility is needed.

2 A generalization of the λ-measure

2.1 ψ-sum

Here we briefly recall the definition of a binary operator that generalizes the classical sum among real numbers in order to allow for super(sub)-additivity and that will be needed for the costruction of the generalization of the standard λ-measure.

The following is a result introduced in Aczél (1966), at p. 256.

Theorem 2.1. *Let us consider a binary operator in $\mathbb{R}$, $x * y$, such that x, y and $x * y$ lie in a given (possibly infinite) interval. We assume that the operator is reducible ($x * y = x * z$ or $y * w = z * w$ only if $z = y$). The general continuous solution of the functional equation $(x * y) * z = x * (y * z)$ (associative poperty) is*

$$x * y = \psi^{-1}(\psi(x) + \psi(y))$$

where ψ is a continuous and strictly monotone function.

Since ψ is determined up to a multiplicative non zero constant, we may restrict the analysis to the case in which ψ is strictly increasing. Since we are interested in extending the sum operator, we require that $x * 0 = x$, which implies ψ to be defined in 0 and $\psi(0) = 0$.

In the sequel we denote with I a generalized interval contained in $[0, +\infty]$ with $0 \in I$, meaning that or $I = [0, +\infty]$ or I is an interval contained in $[0, +\infty)$ with $0 \in I$.

Definition 2.2. *Let $\psi : I \to [0, +\infty]$ be strictly increasing, with $\psi(0) = 0$ and such that $\psi(x) + \psi(y) \in \psi(I)$ for all $x, y \in I$. We call* ψ-sum *on I with generator ψ the binary operator*

$$x \oplus_\psi y = \psi^{-1}(\psi(x) + \psi(y)),\ x, y \in I.$$

The following result is immediate:

Proposition 2.1. *The ψ-sum is sub-additive on I, that is*

$$x \oplus_\psi y \leq x + y,\ \text{for all } x, y \in I,$$

if and only if the function ψ is super-additive on I. The ψ-sum is super-additive that is

$$x \oplus_\psi y \geq x + y,\ \text{for all } x, y \in I,$$

if and only if the function ψ is sub-additive on I.

Moreover, sub and super-additivity of the operator can be obtained by selecting concave and convex $\psi(x)$ functions, respectively. The only case in which the operator is linear($x \oplus_\psi y = x + y$) corresponds to $\psi(x) = cx$ for every non zero constant c.

Definition 2.3. *Let us assume that a ψ-sum operator is defined on I. For $x, z \in I$ such that $x \geq z$, we define the ψ-difference $x \ominus_\psi z$ as*

$$x \ominus_\psi z = \psi^{-1}(\psi(x) - \psi(z)).$$

Notice that the ψ-difference is well defined and that it is the inverse operator of the ψ-sum. In fact

$$(x \ominus_\psi z) \oplus_\psi z = \psi^{-1}\left(\psi(x \ominus_\psi z) + \psi(z)\right) = x.$$

2.2 Fuzzy measures

Here we briefly introduce the main definitions, concepts and results that will be needed in the sequel. The reader interested in fuzzy measures is referred to Klir and Folger (1988), chapter 4, for an introductory treatment, and to Wang and Klir (1992) for a complete development of the topic.

Throughout the paper $(\Omega, \mathcal{F})$ will denote a measurable space.

Definition 2.4. *A* non-monotonic fuzzy measure *on $(\Omega, \mathcal{F})$ is a function $\mu : \mathcal{F} \to [0, +\infty]$ such that $\mu(\emptyset) = 0$.*

Definition 2.5. *A* fuzzy measure *on $(\Omega, \mathcal{F})$ is a function $\mu : \mathcal{F} \to [0, +\infty]$ such that*

1. *$\mu(\emptyset) = 0$,*
2. *if $A, B \in \mathcal{F}$ with $A \subset B$, then $\mu(A) \leq \mu(B)$.*
3. *for every monotonic sequence A_i*

$$\lim_{i \to \infty} \mu(A_i) = \mu\left(\lim_{i \to \infty} A_i\right)$$

If $\mu(\Omega) = 1$ the fuzzy measure is called regular.

Definition 2.6. *A regular fuzzy measure* μ *on* $(\Omega, \mathcal{F})$ *is said to be* decomposable *if there exists an operator* $\perp: [0,1] \times [0,1] \to [0,1]$ *such that for all* $A, B \in \mathcal{F}$ *with* $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) \perp \mu(B)$ *and* μ *is said to be generated by* $\perp$.

Definition 2.7. *A function* $\perp: [0,1] \times [0,1] \to [0,1]$ *is called a* t-conorm *if*

1. $a \perp b = b \perp a$ *for all* $a, b \in [0,1]$,
2. $a \perp 0 = a$ *for all* $a \in [0,1]$,
3. $(a \perp b) \perp c = a \perp (b \perp c)$ *for all* $a, b, c \in [0,1]$
4. *it is non-decreasing in each argument.*

Moreover, if $a \perp a > a$ *for all* $a \in (0,1)$ *then the* t*-conorm is called* Archimedean *and if* $\perp$ *it is strictly increasing in* $(0,1) \times (0,1)$ *the* t*-conorm is called* strict.

The following result is well known (see Schweizer and Sklar, 1961):

Theorem 2.8. *A function* $\perp: [0,1] \times [0,1] \to [0,1]$ *is a continuous Archimedean* t*-conorm if and only if there exists a strictly increasing function* $\psi : [0,1] \to [0, +\infty]$ *with* $\psi(0) = 0$ *such that*

$$a \perp b = \psi^{(-1)}(\psi(a) + \psi(b))$$

where $\psi^{(-1)}$ *is the pseudo-inverse of* ψ *defined as*

$$\psi^{(-1)}(x) = \begin{cases} \psi^{-1}(x), & \text{if } x \leq \psi(1) \\ 1, & \text{if } x > \psi(1) \end{cases}$$

Moreover, ψ *is called the generator of* $\perp$ *and* $\perp$ *is* strict *if and only if* $\psi(1) = +\infty$.

Remark 2.1. *The following facts are true.*

1. *A strict* t*-conorm is a particular specification of the* ψ*-sum operator.*
2. *Let* ψ *be defined on* I *with* $1 \in I$. *The* t*-conorm generated by* ψ *coincides with the* ψ*-sum on the pairs* $(x, y) \in [0,1] \times [0,1]$ *such that* $\psi(x) + \psi(y) \leq \psi(1)$.
3. *Let* ψ *be super-additive on* I *with* $1 \in I$. *If* $(x, y) \in [0,1] \times [0,1]$ *with* $x + y \leq 1$, *then* $x \perp y = x \oplus_\psi y$.

A way to construct fuzzy measures is as distortions of measures.

Definition 2.9. *A set function* μ *on* $(\Omega, \mathcal{F})$ *is called a* (non-monotonic) distorted fuzzy measure *if there exists a (non-monotonic) fuzzy measure* ν *and a non-decreasing function* $f : [0, \nu(\Omega)] \to [0, \nu(\Omega)]$ *with* $f(0) = 0$ *and* $f(\nu(\Omega)) = \nu(\Omega)$ *such that* $\mu(A) = f \circ \nu(A)$, *for all* $A \in \mathcal{F}$.
If ν *is a measure (probability) on* $(\Omega, \mathcal{F})$ *then* μ *is called* distorted measure (probability).

Remark 2.2. *Clearly if* ν *is a measure on* $(\Omega, \mathcal{F})$ *and* f *is strictly increasing, then* $\mu = f \circ \nu$ *is a strictly monotone fuzzy measure. In particular, if* ν *is a probability on* $(\Omega, \mathcal{F})$, *then* $\mu = f \circ \nu$ *is a strictly monotone, regular and decomposable fuzzy measure generated by the* t*-conorm generated by* f^{-1} *and, by 2. in Remark 2.1,* $\mu(A \cup B) = \mu(A) \oplus_{f^{-1}} \mu(B)$ *for* A *and* B *disjoint sets in* $\mathcal{F}$.

2.3 ψ-fuzzy measure

We now use the ψ-sum operator above to define a fuzzy measure that is meant to extend the well known λ-fuzzy measure.

Definition 2.10. *Let μ be a non-monotonic fuzzy measure on $(\Omega, \mathcal{F})$ and ψ a strictly increasing function with $\psi(0) = 0$ so that the corresponding ψ-sum operator is defined on the range of μ. We say that:*

- *μ satisfies the ψ-rule if*

$$\mu(A \cup B) = \mu(A) \oplus_\psi \mu(B)$$

provided that $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$

- *μ satisfies the $\sigma - \psi$-rule if*

$$\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) = \bigoplus_{\psi\, n=1}^{+\infty} \mu(A_n)$$

provided that $\{A_n\}_n \subset \mathcal{F}$ and $A_i \cap A_j = \emptyset$ if $j \neq j$.

Since $\psi(0) = 0$, the σ-ψ-rule implies the ψ-rule.

Definition 2.11. *μ is called a ψ-fuzzy measure on $(\Omega, \mathcal{F})$ if and only if it satisfies the σ-ψ-rule and it exists at least one set $\emptyset \neq A \in \mathcal{F}$ such that $\mu(A) < +\infty$*

A ψ-fuzzy measure is denoted with g_ψ. It is called *regular* if $g_\psi(\Omega) = 1$.

We show that the class of ψ-fuzzy measures g_ψ coincides with the class of distorted measures obtained through a strictly increasing distortion.

Proposition 2.2. *Let*

$$\theta_\psi(x) = \frac{\psi(x)}{k}, \tag{3}$$

where k is an arbitrary positive number.
g_ψ is a ψ-fuzzy measure on $(\Omega, \mathcal{F})$ if and only if $\theta_\psi \circ g_\psi$ is a measure on $(\Omega, \mathcal{F})$

Proof. Let $\{A_n\}_n$ be a disjoint sequence of elements in $\mathcal{F}$. We assume that g_ψ is a ψ-fuzzy measure. Then

$$\begin{aligned}\theta_\psi \circ g_\psi \left(\cup_{n=1}^{+\infty} A_n\right) = \theta_\psi \left(\psi^{-1}\left(\sum_{n=1}^{+\infty} \psi\left(g_\psi(A_n)\right)\right)\right) =\\ = \sum_{n=1}^{+\infty} \frac{1}{k}\psi\left(g_\psi(A_n)\right) =\\ = \sum_{n=1}^{+\infty} \theta_\psi\left(g_\psi(A_n)\right)\end{aligned}$$

that is $\theta_\psi \circ g_\psi$ is a measure. The converse can be proved similarly. □

Remark 2.3. *According to Remark 2.2, every ψ-fuzzy measure is strictly monotone. Nevertheless the class of ψ-fuzzy measures doesn't coincide with the class of strictly monotone fuzzy measures. In fact, if one considers the case of a fuzzy measure obtained by distorting a measure through a non strictly increasing distortion, the resulting distorted measure can be still strictly monotone even if it is not anymore a ψ-fuzzy measure.*
As an example in the case of a discrete Ω, one can consider Example 1 in Chateauneuf (1995). Here we will provide a more general example following the same idea. Let us consider a probability ν on $(\Omega, \mathcal{F})$ and a continuous distortion f such that $f(x) = 0$ for $0 \leq x \leq \bar{x} < 1$, with $\bar{x} \in (0,1)$, and f strictly increasing on $[\bar{x}, 1]$ with $f(1) = 1$. Let $\mu = f \circ \nu$ and $B = A \cup C$ with $\mu(C) = f(\nu(C)) > 0$ (that is $\nu(C) > \bar{x}$). Clearly $\nu(A \cup C) = \nu(A) + \nu(C) > \bar{x}$ and $\nu(A \cup C) = \nu(A) + \nu(C) > \nu(A)$. It follows that, if $\nu(A) > \bar{x}$, then, by the strict monotonicity of f on $[\bar{x}, 1]$. we have that

$$\mu(B) = f(\nu(A \cup C)) = f(\nu(A) + \nu(C)) > f(\nu(A)) = \mu(A)$$

while, if $\nu(A) \leq \bar{x}$, then

$$\mu(B) = f(\nu(A \cup C)) = f(\nu(A) + \nu(C)) > f(\bar{x}) = 0 = f(\nu(A)) = \mu(A)$$

and μ is strictly monotone.

2.4 Construction of a ψ-fuzzy measure on $(\mathbb{R}, \mathcal{B})$

Aim of this section is to construct a ψ-fuzzy measure g_ψ on the real line $\mathbb{R}$ equipped with the σ-field of Borel sets . Hence, the measurable space we are dealing with is $(\mathbb{R}, \mathcal{B})$

We start defining g_ψ on the set of intervals of type $[a, b) \subset \mathbb{R}$ and then we extend the definition to the whole $\mathcal{B}$.

Let $H(\cdot)$ be a cumulative distribution function on $\mathbb{R}$ and assume the ψ-sum generator ψ to be defined on an interval I such that $1 \in I$. We consider θ_ψ as defined in (3) with $k = \psi(1)$ and we define a new cumulative distribution function on $\mathbb{R}$ as

$$H'(y) = \theta_\psi \circ H(y) = \frac{\psi \circ H(y)}{\psi(1)}.$$

Let $[a, b) \subset \mathbb{R}$. Since

$$\theta_\psi^{-1}(z) = \psi^{-1}\left(\psi(1)z\right), \qquad z \in [0, 1],$$

we define

$$g_\psi\left([a, b)\right) = \theta_\psi^{-1}\left(H'(b) - H'(a)\right) = \psi^{-1}\left(\psi \circ H(b) - \psi \circ H(a)\right) = H(b) \ominus_\psi H(a) \tag{4}$$

Remark 2.4. *The function g_ψ defined on the set of intervals of type $[a, b)$, satisfies the ψ-rule on the set of contiguous intervals of type $[a, b)$. In fact, if $a \leq b \leq c$, then*

$$\begin{aligned} g_\psi([a, b)) \oplus_\psi g_\psi([b, c)) &= \psi^{-1}\left(\psi\left(g_\psi([a, b))\right) + \psi\left(g_\psi([b, c))\right)\right) = \\ &= \psi^{-1}\left(\psi(H(b)) - \psi(H(a)) + \psi(H(c)) - \psi(H(b))\right) = \\ &= g_\psi([a, c)). \end{aligned}$$

Let us now extend the definition of g_ψ to the whole $\mathcal{B}$. By definition (see (4)), $\theta_\psi \circ g_\psi$ coincides, on the intervals of type $[a, b)$, with the probability $\mathbb{P}'$ induced by H' on $(\mathbb{R}, \mathcal{B})$ (see Theorem Theorem 3.3 in Billingsley, 1995). As a consequence, g_ψ can be uniquely extended to $\mathcal{B}$ as a distorted probability by $g_\psi(A) = \theta_\psi^{-1}(\mathbb{P}'(A))$ for $A \in \mathcal{B}$. Clearly, by Proposition 2.2, g_ψ so defined is a ψ-fuzzy measure on $(\mathbb{R}, \mathcal{B})$.

Remark 2.5. *g_ψ is sub-additive if and only if ψ is super-additive.*

Remark 2.6. *Notice that when ψ is linear, we recover the probabilty distribution induced by the cumulative distribution function H.*

2.5 Duality

Lemma 2.1. *Let g_ψ be a regular ψ-fuzzy measure on $(\Omega, \mathcal{F})$. For all $A \in \mathcal{F}$,*

$$g_\psi(A^c) = 1 \ominus_\psi g_\psi(A). \tag{5}$$

Proof. It trivially follows from

$$1 = g_\psi(A) \oplus_\psi g_\psi(A^c).$$

□

Definition 2.12. *If g_ψ is a regular ψ-fuzzy measure, its* dual measure *μ on $(\Omega, \mathcal{F})$ is defined as*

$$\mu(A) = 1 - g_\psi(A^c)$$

for all $A \in \mathcal{F}$.

Proposition 2.3. *The dual measure of g_ψ satisfies the $\sigma - \hat{\psi}$-rule, where*

$$\hat{\psi}(z) = \psi(1) - \psi(1 - z) \tag{6}$$

for $z \in [0, 1]$.

Proof. Thanks to (5) and being $\hat{\psi}^{-1}(u) = 1 - \psi^{-1}(\psi(1) - u)$,

$$\begin{aligned}
\mu(\cup_n A_n) &= 1 - g_\psi((\cup_n A_n)^c) = \\
&= 1 - (1 \ominus_\psi g_\psi(\cup_n A_n)) = \\
&= 1 - \left(1 \ominus_\psi \left(\bigoplus_{n=1}^{+\infty}{}_\psi\, g_\psi(A_n)\right)\right) = \\
&= 1 - \psi^{-1}\left(\psi(1) - \sum_n \psi(g_\psi(A_n))\right) = \\
&= 1 - \psi^{-1}\left(\psi(1) - \sum_n \psi(1 \ominus g_\psi(A_n^c))\right) = \\
&= 1 - \psi^{-1}\left(\psi(1) - \sum_n [\psi(1) - \psi(1 - \mu(A_n))]\right) = \\
&= \hat{\psi}^{-1}\left(\sum_n \hat{\psi}(\mu(A_n))\right) = \\
&= \bigoplus_{n=1}^{+\infty}{}_{\hat{\psi}}\, \mu(A_n)
\end{aligned}$$

$\square$

Definition 2.13. *A family of distortions* $\{\psi_\theta\}_{\theta\in\Theta}$ *where* Θ *is the parameters set, is closed under duality if for every* $\theta \in \Theta$ *it exists a* $\theta^* \in \Theta$ *such that*

$$z = \psi_{\theta^*}^{-1}\left(\psi_\theta(1) - \psi_\theta(1-z)\right)$$

for all $z \in [0,1]$.

2.6 Symmetry

An interesting question is whether for some distortion $\psi(.)$ one could define some stronger property than closedness with respect to duality. One such property could be symmetry, that is the case in which $\hat{\psi}(x) = \psi(x)$ for all $x \in [0,1]$, that is the case in which both the distortions ψ and $\hat{\psi}$ belong to the same family and share the same parameter θ.

Proposition 2.4. *Let* ψ *be defined on* $I \supseteq [0,1]$ *and* $\hat{\psi}$ *given by (6).* $\hat{\psi}(x) = \psi(x)$ *for all* $x \in [0,1]$ *if and only if the graph of* ψ *is symmetric around the point* $\left(\frac{1}{2}, \frac{\psi(1)}{2}\right)$, *that is*

$$\psi(z) = \psi(1) - \psi(1-z), \text{ for all } z \in [0,1].$$

Proof. Since the dual measure can be recovered up to a constant, $\hat{\psi} \equiv \psi$ is equivalent to

$$c\psi(z) = \psi(1) - \psi(1-z)$$

Now, since the function $\psi(0) = 0$, we also have

$$c\psi(1) = \psi(1)$$

that implies $c = 1$ and we obtain

$$\psi(z) = \psi(1) - \psi(1-z)$$

which is the definition of symmetry around $\left(\frac{1}{2}, \frac{\psi(1)}{2}\right)$. $\square$

In particular, if ψ is convex on $\left[0, \frac{1}{2}\right]$, it has to be concave on $\left[\frac{1}{2}, 1\right]$ and viceversa.

Remark 2.7. *If* ψ *is globally convex or concave, then it satisfies the simmetry requirement if and only if it is linear.*

3 Specific functional forms and general distortions

We now show examples of the functional forms described above, starting from the classical λ-measure. This is obtained setting

$$\psi(x) = \frac{\ln(1+\theta x)}{\theta} \tag{7}$$

with $\theta \in (-1, \infty)$, where for $\theta = 0$ we have $\psi(x) = x$. Notice that the function is increasing and globally concave for $\theta > 0$ and convex for $\theta < 0$. It is easy to check that in this case

$$x \oplus_\psi y = x + y + \theta xy \tag{8}$$

and we obtain the λ-measure in equation (1). We know that this class is closed with respect to duality, meaning that the dual is the same function with parameter $\hat{\theta}$. Using (6) we have

$$\begin{aligned} \hat{\psi}(x) &= \frac{1}{\theta}\log(1+\theta) - \frac{1}{\theta}\log(1+\theta(1-x)) = \\ &= -\frac{1}{\theta}\log\left(1 - \frac{\theta}{1+\theta}x\right) \end{aligned}$$

and since the function $\hat{\psi}$ is defined up to a positive constant,

$$\hat{\psi}(x) = \frac{1}{\hat{\theta}}\log(1+\hat{\theta}x), \text{ with } \hat{\theta} = -\frac{\theta}{\theta+1}.$$

It is easy to find that other classes of distortions are not closed under duality. Consider for example the case

$$\psi(x) = x^{\frac{1}{\theta}} \tag{9}$$

for $\theta \in (0, \infty)$. In this case the fuzzy measure is generated by the operator

$$x \oplus_\psi y = \left(x^{\frac{1}{\theta}} + y^{\frac{1}{\theta}}\right)^\theta \tag{10}$$

It can be proved that the dual fuzzy measure is not a member of the same class. In fact

$$\hat{\psi}(x) = 1 - (1-x)^{\frac{1}{\theta}}$$

and, for $\theta \neq 1$, $\hat{\psi}'(0) = \frac{1}{\theta}$, while, for $\theta < 1$, $\psi'(0) = 0$ and, for $\theta > 1$, $\psi'(0) = +\infty$

We can also find an example that proves that the class of distortions generating the λ-measure is not the only class closed under duality. Consider in fact the case

$$\psi(x) = \frac{1}{\theta}\left(e^{\theta x} - 1\right) \tag{11}$$

In this case,

$$\begin{aligned} \hat{\psi}(x) &= \frac{1}{\theta}\left(e^\theta - 1\right) - \frac{1}{\theta}\left(e^{\theta(1-x)} - 1\right) = \\ &= \frac{1}{\theta}\left(e^\theta - e^{\theta - \theta x}\right) = \\ &= -\frac{e^\theta}{\theta}\left(e^{-\theta x} - 1\right) \end{aligned}$$

and since the function $\hat{\psi}$ is defined up to a positive constant,

$$\hat{\psi}(x) = \frac{1}{\hat{\theta}}\left(e^{\hat{\theta}x} - 1\right), \text{ with } \hat{\theta} = -\theta.$$

A comment is in order concerning why it may be preferable to work with ψ-fuzzy measures that are closed under duality. While this is not by any means a limitation in theoretical analysis, it may make life easier in applications. In fact, in many examples, including the one presented below in this paper, me may have to calibrate a set of ψ-fuzzy measures and their corresponding dual values. If both of them use the same function, this may simplify and speed up the computation, particularly if we have a large set of data.

3.1 Distorted parametric ψ-mesures

Starting from the parametric families introduced above we now show a way to construct more fexible distortions that depend on two parameters. This can be achieved by compounding or merging different ψ-measures.

For the sake of simplifying notation, we define $\psi_1(x) = \psi_{\theta_1}(x)$ and $\psi_2(x) = \psi_{\theta_2}(x)$, and specify θ_1 and θ_2 in every representation, when needed. So, for the composition of two distortions, we consider functions ψ of type

$$\psi(z) = \psi_2(\psi_1(z))$$

where ψ_1 and ψ_2 are different distortions, with parameters θ_1 and θ_2, respectively. Clearly, the new distortion is well defined if ψ_2 is defined on $[0, \psi_1(1)]$.

As an example, let ψ_1 be defined as in (7) with parameter $\theta_1 = \theta$ and ψ_2 as in (9) with parameter $\theta_2 = \eta$. Then

$$\psi(z) = \frac{\ln^{1/\eta}(1+\theta x)}{\theta^{1/\eta}}.$$

In the opposite example, if ψ_1 defined as in (9) with parameter $\theta_1 = \eta$ and ψ_2 is set as in (7) with parameter $\theta_2 = \theta$ we get

$$\psi(z) = \frac{\ln\left(1+\theta x^{1/\eta}\right)}{\theta}.$$

Another way to allow for more flexibility is by merging different distortions, that is by considering distortions ψ of type

$$\psi(z) = \begin{cases} \psi_1(z), & z \leq \alpha \\ \psi_2(z), & z > \alpha \end{cases}$$

where $\alpha \in (0,1)$ and with $\psi_1(\alpha) = \psi_2(\alpha)$. As a consequence, if $x \leq y$ (the complemetary case is similar)

$$x \oplus_\psi y = \begin{cases} x \oplus_{\psi_1} y, & x \leq y \leq \alpha \text{ and } \psi_1(x) + \psi_1(y) \leq \psi_1(\alpha) \\ \psi_2^{-1}(\psi_1(x) + \psi_1(y)), & x \leq y \leq \alpha \text{ and } \psi_1(x) + \psi_1(y) > \psi_1(\alpha) \\ \psi_2^{-1}(\psi_1(x) + \psi_2(y)), & x < \alpha \text{ and } y > \alpha \\ x \oplus_{\psi_2} y, & \alpha \leq x \leq y \end{cases}$$

As an example, if ψ_1 is defined as in (7) with parameter $\theta_1 = \theta > 0$ and ψ_2 as in (9) with parameter $\theta_2 = \eta = \frac{\ln \alpha}{\ln\left(\frac{\ln(1+\theta\alpha)}{\theta}\right)}$, we get

$$x \oplus_\psi y = \begin{cases} x + y + \theta xy, & x \leq y \leq \alpha \text{ and } x + y + \theta xy \leq \alpha \\ \frac{1}{\theta^\eta} \ln^\eta \left(1 + \theta(x+y) + \theta^2 xy\right), & x \leq y \leq \alpha \text{ and } x + y + \theta xy > \alpha \\ \left(\frac{\ln(1+\theta x)}{\theta} + y^{1/\eta}\right)^\eta & x < \alpha \text{ and } y > \alpha \\ \left(x^{1/\eta} + y^{1/\eta}\right)^\eta, & \alpha \leq x \leq y \end{cases}$$

4 Option pricing application

In this section we show that some of the fuzzy measures proposed in the sections above can be very useful to improve applications. One of the fields of application is option pricing, where λ-measures have been proposed as a specific parametric form for Choquet pricing. In other terms, option prices are obtained as Choquet integrals of super-additive measures. The reader interested in an in-depth treatment of the Choquet integral is referred to the Choquet (1953) paper and to Denneberg (1994) for the more general topic of integration with respect to non additive measures. Formally the Choquet integral is defined as

$$(c) \int f(x) d\mu = \int_{-\infty}^{0} (\mu(f(x) > u) - 1) du + \int_{0}^{\infty} \mu(f(x) > u) du \qquad (12)$$

where μ is a regular fuzzy measure. In our application, the set is restricted to the class of ψ-fuzzy measures studied in this paper.

These models were mainly introduced to allow for different prices for purchasing and selling the option (bid-ask prices). Theoretical models extending the standard no-arbitrage pricing theorem to non-additive pricing measures, that is fuzzy measures (also called capacities, in the literature), can be found in Chateauneuf Kast and Lapied (1996) and more recently in Cerreia-Vioglio, Maccheroni and Marinacci (2015). The λ-measure specification in option pricing applications was first proposed by Cherubini (1997).

Just for the sake of illustration, we show that the classical λ-measure may have features that are well suited for option pricing applications, but at the cost of other results that may be not consistent with the evidence we expect to find in the data. On the contrary, it may happen that switching to other distortion functions of the Archimedean class may solve the latter requirements, while not being compliant with the former. Hopefully, combining different Archimedean t-conorm fuzzy measures could blend the advantages of the individual fuzzy measures, or at last could ease their flaws.

In an option pricing application it would be natural to expect two kinds of evidence:

- **Smile/Skew effect**. A typical evidence that is commonly found in options markets is that the so called "*implied volatility*" is different for different strike prices. We remind that an option, say a call option, gives the owner the positive difference between the value of a risky asset S, called "the underlying asset", and a strike (or exercise) price K at time T, called the exercise date, and zero otherwise: $[S(T) - K]^+$. The value of the contract at any time $t < T$ may be expressed in term of units or currency, or in terms of "implied volatility": this is a parameter in the reference pricing formula, known as the Black & Scholes formula. While under the pricing model leading to the Black & Scholes formula the implied volatility should be the same across all the options with different strike prices K, after the Black Monday crisis of October 19 1987 a new evidence emerged of different implied volatility values for different strikes. For some market the typical relationship is parabolic, and it is known as "smile" and for some other, namely the equity markets, the relationship is decreasing, and it is known as "smirk", or more often as "skew", because it is referred to a skewed distribution of the percentage changes of $S(T)$. For the sake of example, in this analysis we focus on the "skew" shape.

- **High out-of-the-money bid-ask spreads**. Many practitioners observe that the market for options with low probability of being exercized, that techically are called "out-of-the-money", is typically less liquid than that for options for which the probability of exercize is in central part the distribution: these options are called "at-the-money" and are those with the stike price K around the current value of the underlying asset, $S(t)$. In our call options example, this means that the options "bid-ask spreads", that is the difference between prices at which the option is sold and bought, should be increasing with K. In other words, the higher K, the lower the probability of exercise of a call option, the higher the uncertainty of the price. This is also in line with the desired behaviour of the fuzzy measures that we declared among the motivations of this paper.

Our task is then to find a fuzzy measure model such that, even using the same reference volatility value,

- The call option implied volatilities computed at the mid-price, that is the average of bid and ask prices, are decreasing with higher values of the strike prices K
- The call option percentage bid-ask spreads, that is the bid-ask spread as a percentage of the mid-price, are increasing with higher values of the strike prices K

Our application consists in computing the price of a call option using a reference distribution function $H(y)$ for the underlying $S(T)$ and a distortion $\psi(.)$ to represent uncertainty. The bid and ask prices are obtained using the Choquet integral in equation (12).

Using our Archimedean fuzzy measures and remembering their duality relationships we find that the *bid* price of a call option is obtained computing

$$C_b(S,t) = \int_K^\infty \psi^{-1}\left(\psi(1) - \psi \circ H(y)\right) dy$$

and the *ask* price is

$$C_a(S,t) = \int_K^\infty \hat{\psi}^{-1}\left(\hat{\psi}(1) - \hat{\psi} \circ H(y)\right) dy.$$

Since in computation of the smile function it is quite often usual to resort to put options, we also report here the fuzzy version of it, both for bid and ask prices. They are simply obtained by substituting the survival function to the distribution function in the pricing formula.

$$P_b(S,t) = \int_0^K \psi^{-1}\left(\psi(1) - \psi \circ \bar{H}(y)\right) dy$$

and the *ask* price is

$$P_a(S,t) = \int_0^K \hat{\psi}^{-1}\left(\hat{\psi}(1) - \hat{\psi} \circ \bar{H}(y)\right) dy.$$

where $\bar{H}(x) = 1 - H(x)$.

Since our example is mainly for illustration purposes, we choose the simplest model for the probability distribution of $S(T)$, that is the log-normal that leads to the well known Black and Scholes formula. We then set

$$\Pr(S(T) \leq y) = H(y) = \Phi\left(-\frac{\log(S(t)/y) - 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)$$

where $\Phi(.)$ is the standard normal distribution. Without loss of generality, we assume $S(t) = 1$. The volatility parameter σ is equal to 16%.

As for the uncertainty model, we use two distortions $\psi(x)$:

- the standard λ-measure model with distortion:

$$\psi(x) = \frac{\ln(1+\theta x)}{\theta}$$

- the power model

$$\psi(x) = x^{\frac{1}{\theta}}$$

Finally, having considered the flaws of the two model above, we will consider their composition. More specifically, we will consider

- the composition model

$$\psi(x) = \frac{\ln^{1/\eta}(1+\theta x)}{\theta^{1/\eta}}$$

For the analysis below, we used options data for a representative day, October 9th 2018. Options are referred to the Italian stock index, FTSE-Mib. We collected bid-ask prices and we computed the implied volatility of the at-the-money quotes, for a value of 16%. Using this value we are going to show how the different fuzzy measures can be composed or mixed to generate different smile shapes and percentage bid ask spraad value. The parameters of the fuzzy measures were chosen to yield a shape of the bid-ask spread consistent with that observed for call options.

It must be stressed at this point that what we do here is for the mere purpose of illustration, without any serious attempt at calibration. Actually, nobody would ever beleive that the smile effect could be due to bid-ask spread only. In fact, for 30 years the literature has been proposing distributions to explain the smile (from local volatility, to stochastic volatility models, to Lévy processes). And indeed, even our data show that fuzzy measures parameters that are consistent with the bid-ask spreads are too small to generate the smile observed in the market. Nevertheless, they do affect the spread. Only for this reason we chose to illustrate this impact with respect to the standard Black and Scholes model, for which the relationship between strike prices and implied volatility is a flat curve. It must be however clear that in any meaningful practical application, the two terms of the problem, the distribution $H(x)$ and the fuzzy measure distortion, must be chosen and calibrated simultaneously.

In figure 1 we report the smile obtained from the standard λ-measure model. Notice that the standard λ-measure induces a "skew" effect on the smile, increasing the implied volatility for strike prices lower than the "at-the-money" level, and reducing it for higher strike prices. So, the λ-measure increases the skew shape that is typical of equity markets, like that used for our application.

A question is whether fuzzy measures only could be used to design a proper smile, with implied volatilities higher for both in-the-money and out-of-the money options, as it is typical in the foreign exchange and interest rate markets. We show in figure 2 that this smile can be generated by merging together the λ-fuzzy measure and the power one, as it is shown at the end of section 3.1. The point at which the two measures are patched is the at-the-money level. This model should be used if one beleive that liquidity issues accentuate the symmetric smile shape in markets in which this shape is typical.

In figure 3 we address the other requirement that we want to fulfil, that is an increasing relationship of the percentage bid-ask spread for higher levels of strike, representing higher uncertainty for extreme tail events. We see that the standard λ-measure model is not endowed with this property. The percentage bid-ask spread suddenly increases and remains flat for extreme events. On the contrary, the power model shows an increasing relationship with a slope that may also appear too steep.

We then report a third schedule, that is made using a composition of the two models. We see that the increasing percentage bid-ask spread property is maintained, even though the relationship is less steep than in the pure power-measure model. The final question is whether the composition model preserves the desired decreasing shape of the smile, that we would like to have in this equity market application. Figure 4 documents that this is actually the case. The decreasing shape of the smile is maintained even though the slope is less steep than that of the smile of the standard λ-measures models.

We may then conclude than in this option pricing applications, while neither of the fuzzy measure models applied is well suited to generate the desired features for the equity market application, a composition of the two model may provide the necessary flexibility to satisfy both the requirements. Merging the two models may produce symmetric smiles that are observed in other markets.

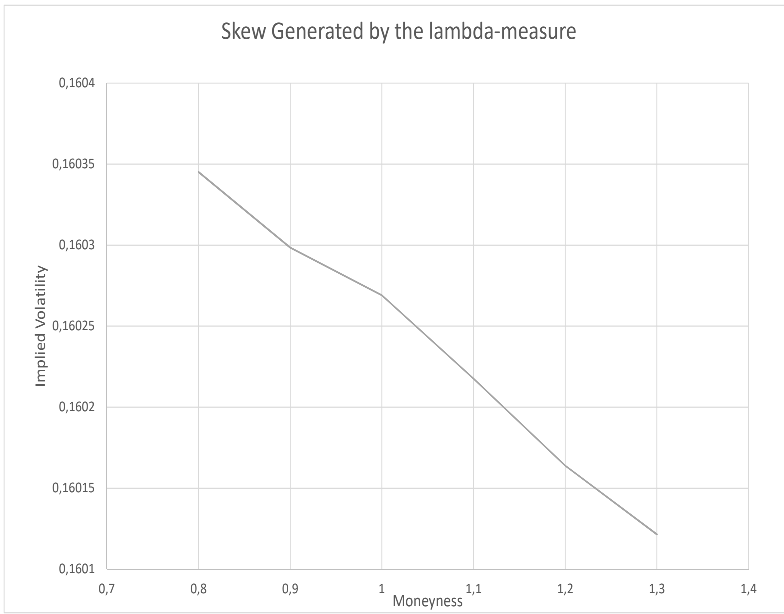

Figure 1: Skew-shaped smile of the λ measure model with parameter $\theta = 0.1$.

5 Conclusions

In this paper we extended the classical λ-measure approach to allow for the flexibility needed in some applications. The extension is carried out along the same lines of the standard λ-measure, that is by means of distortion of a reference probability measure by a function. Our approach is to extend the choice of the distortion function to the class of Archimedean t-conorms. Such operators are defined by the relationship

$$x * y = \psi^{-1}(\psi(x) + \psi(y))$$

and for this reason we call this the class of ψ-measures or Archimedean measures. The classical λ-measure is a specific instance of the Archimedean measure class.

The extension of the fuzzy measure class provides more flexibility in two ways. The first is the possibility to use a variety of different distortion functions resorting to the class of Archimedean t-conorms. The second

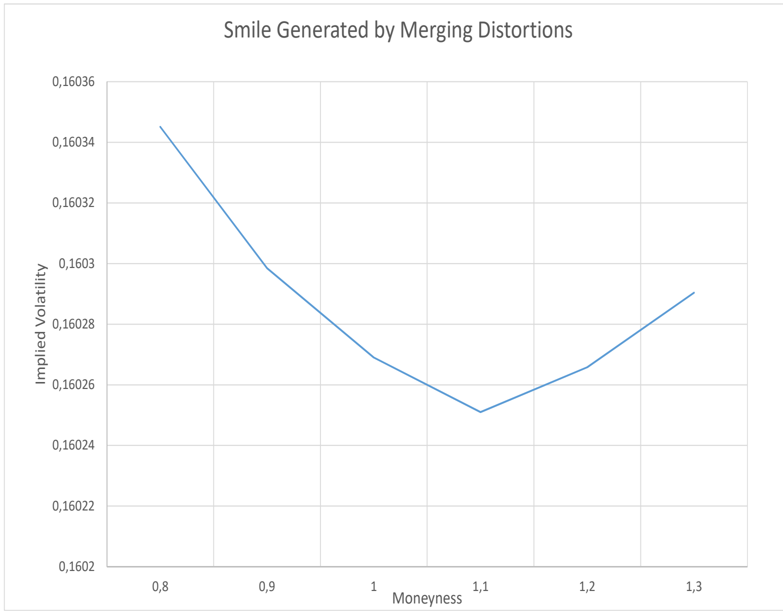

Figure 2: Symmetric smile obtained merging the λ-measure with parameter $\theta = 0.1$ for strikes lower than the at-the-money strike, and power fuzzy measure with parameter $\theta = 1.0285$ for the lower ones.

is the chance to combine several Archimedean fuzzy measures. The combination can be achieved either by patchwork or composition of different fuzzy measures.

As an example of the relevance of flexibility for some applications we show that in option pricing the classical λ-measure is not completely well suited to represent features that we expect to find in the data. In fact, on one hand the λ-measure spontaneously generates the phenomenon of higher "*implied volatility*" for options with lower probability of being exercised, a feature called "*smile*" or "*skew*" in the option pricing jargon. On the other hand the uncertainty that is measured by the percentage of bid-ask spreads, that is the percentage difference of prices for sale and purchase of the option contracts, are not significantly higher for options with lower exercise probability, as expected by industry practitioners. We show that another member of the Archimedean fuzzy measure class, the power fuzzy measure, is able to generate this feature, while generating *implied volatility* schedules that are opposite to what is expected. By combining the two measures by composition we find that we are able to build a generator that at the same time produces lower volatility and higher

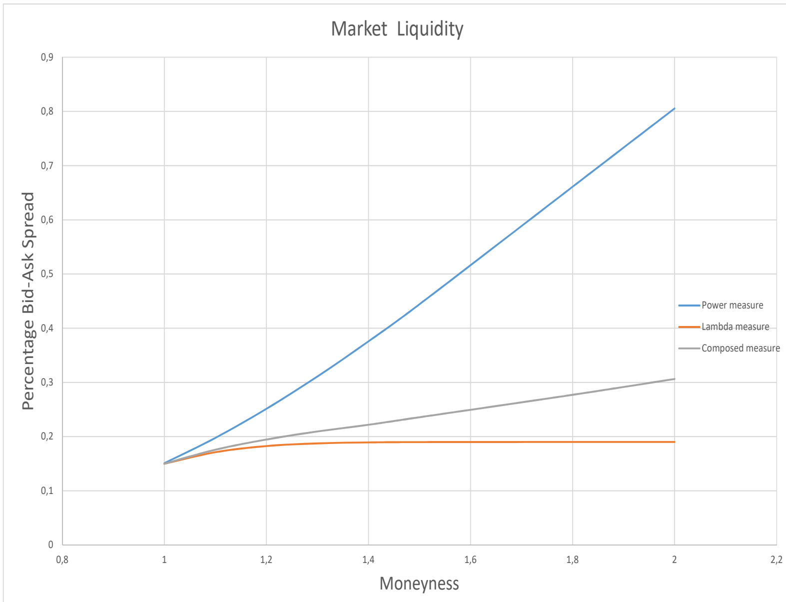

Figure 3: Market liquidity measured as the percentage bid-ask spread for three model: λ-measure, power measure and a composition of the two, with two parameters equal to 0.0814 and 0.995.

uncertainty for call options with lower probability of exercise. Moreover, we also show that by a simple patchwork of the two fuzzy measures we are also allowed to generate symmetric smiles that are typical of the foreign exchange and interest rates markets.

While our option application is only an example, in our opinion the fact that the degree of uncertainty could be generally higher for very rare events could be considered a general concept. For this reason, we expect that this requirement could arise of many other applications, and could lead to further practical and theoretical developments on this line of research.

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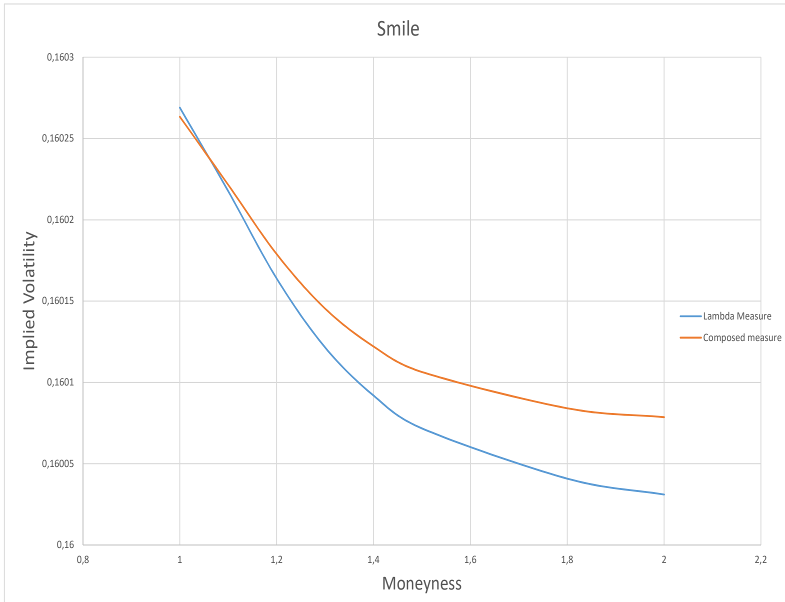

Figure 4: Smile of the λ-measure in the model given by the composition of it with the power model, with the two parameters equal to 0.0814 and 0.995.

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