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(SEMI-)GLOBAL ANALYTIC HYPOELLIPTICITY FOR A CLASS OF "SUMS OF SQUARES" WHICH FAIL TO BE LOCALLY ANALYTIC HYPOELLIPTIC

GREGORIO CHINNI

Abstract. The global and semi-global analytic hypoellipticity on the torus is proved for two classes of sums of squares operators, introduced in [1] and [2], satisfying the Hörmander condition and which fail to be neither locally nor microlocally analytic hypoelliptic.

1. Introduction

Our aim, in this work, is to prove global and semi-global, i.e. local in some variables and global in others, analytic hypoellipticity on the torus for some models of sums of squares of vector fields with real valued and real analytic coefficients which satisfy Hörmander condition, [5].

In two recent papers, [1] and [2], Albano, Bove and Mughetti and Bove and Mughetti produced and studied the first models of sums of squares operators not consistent with the (micro-)local Treves conjecture, [9]. They showed that the sufficient part of the Treves' conjecture, for details on the subjet see [9], does not hold neither locally nor microlocally. More precisely, in [1] the authors studied the model

$$P_{ABM}(x,D) = D_1^2+D_2^2+x_1^{2(r-1)}\left(D_3^2+D_4^2\right) + x_2^{2(p-1)}D_3^2 + x_2^{2(q-1)}D_4^2\,, \tag{1.1}$$

on Ω, open neighborhood of the origin in $\mathbb{R}^4$, where r, p and q are positive integers such that $1<r<p<q$. They showed that even if P_{ABM} has a single symplectic stratum, in meaning of the Poisson-Treves stratification, it is Gevrey hypoelliptic of order $s = r(q-1)[q-1+(r-1)(p-1)]^{-1}$ and not better.

In [2] the authors investigated the following operator

$$P_{BM}(x,D) = D_1^2+x_1^{2(r+\ell-1)}\left(D_3^2+D_4^2\right)+x_1^{2\ell}\left(D_2^2+x_2^{2(p-1)}D_3^2+x_2^{2(q-1)}D_4^2\right), \tag{1.2}$$

on Ω, open neighborhood of the origin in $\mathbb{R}^4$, with $1<r<p<q$. They proof that even if the codimention of the characteristic manifold of P_{BM} is 2 and the related stratification, in the sense of Treves, is made up by two symplectic strata the operator is not analytic hypoelliptic. It is Gevrey hypoelliptic of order $s = (\ell+r)(q-1)[(q-1)(\ell+1)+(r-1)(p-1)]^{-1}$ and not better.

Our purpose will be analyze the global and the semi-global analytic regularity on the four dimensional torus for two classes of operators which include as particular

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cases the global version of the operators P_{ABM} and P_{BM}.
Statement of the results.

Theorem 1.1. *Let $P_1(x,D) = \sum_{j=1}^{6} X_j^2(x,D)$ be the operator given by*

$$D_1^2 + D_2^2 + a^2(x_1)\left(D_3^2 + D_4^2\right) + b_1^2(x_2)D_3^2 + b_2^2(x_2)D_4^2 \tag{1.3}$$

on $\mathbb{T}^4$ where a, b_1 and b_2 are real value real analytic functions not identically zero. Then given any sub-interval $\mathcal{I} \subset \mathbb{T}^2_{x'}$, $x' = (x_1, x_2)$, and given any u in $\mathcal{D}'(\mathcal{I}\times\mathbb{T}^2_{x''})$, $x'' = (x_3, x_4)$, the condition $P_1 u \in C^\omega(\mathcal{I} \times \mathbb{T}^2_{x''})$ implies $u \in C^\omega(\mathcal{I} \times \mathbb{T}^2_{x''})$.

Theorem 1.2. *Let $P_1(x,D)$ be as in (1.3). Assume that a, b_1 and b_2 are 0 at zero and the zero order at $x_2 = 0$ of b_2 is strictly greater than that of b_1. Let $\mathcal{I}$ an open neighborhoods of $(x_1, x_2) = (0, 0)$ and $\mathcal{U}$ a sub-interval of $\mathbb{T}^1_{x_4}$. Then if $P_1 u = f$, with f real analytic on $\mathcal{I} \times \mathbb{T}^1_{x_3} \times \mathcal{U}$ then u is analytic on $\mathcal{I} \times \mathbb{T}^1_{x_3} \times \mathcal{U}$.*

A few remarks are in order.

(a) If we take $a(x_1) = (\sin x_1)^{r-1}$, $b_1(x_2) = (\sin x_2)^{p-1}$ and $b_2(x_2) = (\sin x_2)^{q-1}$, with r, p and q positive integers such that $1 < r < p < q$, the operator $P_1(x,D)$ is the global version on the torus of the operator P_{ABM}, (1.1), which is not local analytic hypoelliptic.

(b) We point out that if the zero order at 0 of b_2 is equal than that of b_1 then the operator $P_1(x,D)$ is microlocally anlytic hypoelliptic as showed in [1], hence also global analytic hypoelliptic. Otherwise if the zero order at 0 of b_2 is smaller than that of b_1 then the role of the directions x_3 and x_4 is exchanged, i.e. the operator $P_1(x,D)$ is locally analytic hypoelliptic with respect to the variables x_1, x_2 and x_3 but globally analytic hypoelliptic with respect to the variable x_4.

(c) The operator P_1, (1.3), belongs to the class studied by Cordaro and Himonas, [3], therefore it is globally analytic hypoelliptic.

For completeness we recall the result proved in [3].

Theorem ([3])**.** *Let P be a sum of squares operator, $P = \sum_1^\nu X_j$, on the torus $\mathbb{T}^N = \mathbb{T}^m \times \mathbb{T}^n$ with variables, (x', x''), $x' = (x_1, \dots, x_m)$, $x'' = (x_{m+1}, \dots, x_N)$ and*

$$X_j = \sum_{k=1}^{n} a_{jk}(x'')\frac{\partial}{\partial x_{m+k}} + \sum_{k=1}^{m} b_{jk}(x'')\frac{\partial}{\partial x_k}$$

are real vector fields with coefficients in $C^\omega(\mathbb{T}^n)$. If the following two conditions hold:

(i) *$X_1, \dots, X_\nu$ and their brackets of length at most r span the tangent space at every point on $\mathbb{T}^N$, i.e. they satisfy the Hörmander condition,*

(ii) *the vectors $\sum_{k=1}^n a_{jk}(x'')\partial_{x_{m+k}}$ span $T_{x''}(\mathbb{T}^n)$ for every $x'' \in \mathbb{T}^n$,*

then the operator P is globally analytic hypoelliptic on $\mathbb{T}^N$.

Next we look at the global and semi-global analytic regularity for operators which are a global version on the four dimensional torus of the operator studied in [2].

Theorem 1.3. *Let $P_2(x,D) = \sum_{j=1}^{6} X_j^2(x,D)$ be the operator given by*

$$D_1^2 + a_1^2(x_1)\left(D_3^2 + D_4^2\right) + a_2^2(x_1)\left(D_2^2 + b_1^2(x_2)D_3^2 + b_2^2(x_2)D_4^2\right) \tag{1.4}$$

on $\mathbb{T}^4$, where a_j and b_j, $j = 1, 2$, are real valued real analytic functions not identically zero. We have:

- **(i)** *Let x_1^0 be a common zero of a_1 and a_2 and assume that the zero order at x_1^0 of a_2 is strictly greater than that of a_1. Let $\mathcal{I}_1$ an open neighborhood of x_1^0 and $\mathcal{I}_2$ a sub-interval of $\mathbb{T}^1_{x_2}$. The condition $P_2 u \in C^\omega(\mathcal{I}_1 \times \mathcal{I}_2 \times \mathbb{T}^2_{x''})$, $x'' = (x_3, x_4)$, implies $u \in C^\omega(\mathcal{I}_1 \times \mathcal{I}_2 \times \mathbb{T}^2_{x''})$.*
- **(ii)** *Let (x_1^0, x_2^0) be a zero of a_i and b_i, $i = 1, 2$, and assume that the zero order at x_1^0 of a_1 is strictly greater than that of a_2 and that the zero order at x_2^0 of b_2 is strictly greater than that of b_1. Let $\mathcal{I}$ an open neighborhood of (x_1^0, x_2^0) and $\mathcal{U}$ a sub-interval of $\mathbb{T}^1_{x_4}$. The condition $P_2 u \in C^\omega(\mathcal{I} \times \mathbb{T}^1_{x_3} \times \mathcal{U})$, implies $u \in C^\omega(\mathcal{I} \times \mathbb{T}^1_{x_3} \times \mathcal{U})$.*

Moreover, with the aid of the partition of unity we have:

Corollary 1.1. *Let $P_2(x, D)$ be as in (1.4). Then the operator $P_2(x, D)$ is globally analytic hypoelliptic on $\mathbb{T}^4$.*

Some remarks are in order.

- (a) If we take $a_1(x_1) = (\sin x_1)^{r+\ell-1}$, $a_2(x_1) = (\sin x_1)^\ell$ $b_1(x_2) = (\sin x_2)^{p-1}$ and $b_2(x_2) = (\sin x_2)^{q-1}$, with r, p, q and ℓ positive integers such that $1 < r < p < q$, the operator $P_2(x, D)$ is the global version on the torus of the operator $P_{_{BM}}$, (1.2), which is not local analytic hypoelliptic.
- (b) The operator P_2, (1.4), does not belong to the class studied by Cordaro and Himonas, [3].
- (c) Theorem 1.3-**(ii)**: if the zero order at x_2^0 of b_2 is equal than that of b_1 then the operator $P_2(x, D)$ is microlocally anlytic hypoelliptic as showed in [2], hence also global analytic hypoelliptic. Otherwise if the zero order at x_2^0 of b_2 is smaller than that of b_1 then the role of the directions x_3 and x_4 is exchanged, i.e. the operator $P_2(x, D)$ is locally analytic with respect to the variables x_1, x_2 and x_3 and globally analytic with respect to the variable x_4.

Remark. The results obtained are "consistent" with the global version of the Treves conjecture, [9]. In both case the (semi-)global analytic hypoellipticity is due to the fact that the bicharacteristic leaf of the missing stratum, see Remark 2.1[2], $\tilde{\Sigma} = \{(0, 0, x_3, x_4; 0, 0, 0, \xi_4) | \xi_4 \neq 0\}$ is compact.

The interest in this work was inspired by the seminal works of Cordaro and Himonas, [3] and [4], and Tartakoff, [8]. To obtain the results we will follows the ideas in [3], proof of the Theorem 1.1, and the ideas in [8], proof of the Theorems 1.2 and 1.3.

2. Proof of the theorem 1.1

Without loss of generality we assume that $x' = (0, 0)$ is a zero for the functions a, b_1 and b_2, $\mathcal{I} \doteq \mathcal{I}_1 \times \mathcal{I}_2 =] - \delta_1, \delta_1[\times] - \delta_2, \delta_2[$, $\delta_i > 0$, $a(x_1) \neq 0$ for $x_1 \in \mathcal{I}_1 \setminus \{0\}$ and $b_j(x_2) \neq 0$ for $x_2 \in \mathcal{I}_2 \setminus \{0\}$, $j = 1, 2$. By Hörmander theorem, [5], P_1 is hypoelliptic, therefore we can assume $u \in C^\infty(\mathcal{I} \times \mathbb{T}^2_{x''})$. Taking the Fourier transform with respect to x'' we have

$$\widehat{P_2u}(x',\xi'')=\widehat{D_1^2u}(x',\xi'')+\widehat{D_2^2u}(x',\xi'')+\left[a^2(x_1)|\xi''|^2+b_1^2(x_2)\xi_3^2+b_2^2(x_2)\xi_4^2\right]\widehat{u}(x',\xi'').$$

We multiply by $\bar{\widehat{u}}$ and integrate in $\mathcal{I}$:

$$\int_{\mathcal{I}} \widehat{P_1u}(x',\xi'')\bar{\widehat{u}}(x',\xi'')dx' = \int_{\mathcal{I}} \left((\widehat{\partial_1^2u})(x',\xi'')+(\widehat{\partial_2^2u})(x',\xi'')\right)\bar{\widehat{u}}(x',\xi'')dx'$$
$$+\int_{\mathcal{I}} \left[a^2(x_1)|\xi''|^2+b_1^2(x_2)\xi_3^2+b_2^2(x_2)\xi_4^2\right]|\widehat{u}(x',\xi'')|^2dx'.$$

We have

$$\begin{aligned}(2.1)\quad &\int_{\mathcal{I}}\left[a^2(x_1)|\xi''|^2+b_1^2(x_2)\xi_3^2+b_2^2(x_2)\xi_4^2\right]|\widehat{u}(x',\xi'')|^2dx'+\int_{\mathcal{I}}|\widehat{u}_{x_1}(x',\xi'')|^2dx'\\ &+\int_{\mathcal{I}}|\widehat{u}_{x_2}(x',\xi'')|^2dx' = \int_{\mathcal{I}_1}\widehat{u}_{x_2}(x',\xi'')\bar{\widehat{u}}(x',\xi'')\Big|_{x_2=-\delta_2}^{x_2=\delta_2}dx_2\\ &+\int_{\mathcal{I}_2}\widehat{u}_{x_1}(x',\xi'')\bar{\widehat{u}}(x',\xi'')\Big|_{x_1=-\delta_1}^{x_1=\delta_1}dx_2\int_{\mathcal{I}}\widehat{P_2u}(x',\xi'')\bar{\widehat{u}}(x',\xi'')dx',\end{aligned}$$

where $\widehat{u}_{x_i} \doteq \widehat{\partial_i u}$, $i=1,2$. Since $Pu \in C^\omega(\mathcal{I}\times\mathbb{T}^2_{x''})$ and P is elliptic away from $(0,0)$ we can estimate the left hand side of the above equality by $Ce^{-\varepsilon|\xi''|}$, with C and ε suitable positive constants.
In order to complete the proof we need of an analogous, in two variable, of the Lemma 4.1 in [3].

Lemma 2.1. *For $f \in C^\infty(\overline{\mathcal{I}})$ let*

$$\|f\|_g^2 = \int_{\mathcal{I}} g^2(x')|f(x')|^2dx' + \int_{\mathcal{I}} |(\partial_1 f)(x')|^2+|(\partial_2 f)(x')|^2\,dx',$$

where g is a real analytic function on $\mathcal{I}$ not identically zero such that $g(0)=0$ and $g(x')\neq 0$ for every $x'\in\overline{\mathcal{I}}\setminus\{0\}$. Then there is a positive constant depending on g such that

$$\|f\|_0^2 \le C\|f\|_g^2. \qquad (2.2)$$

Proof. We have

$$f(x_1,x_2)=f(y_1,y_2)+\int_{y_1}^{x_2}(\partial_2 f)(y_1,t_2)\,dt_2+\int_{y_1}^{x_1}(\partial_1 f)(t_1,x_2)\,dt_1.$$

Since $g(y')\neq 0$ for every $y'\in\overline{\mathcal{I}}\setminus\{0\}$, there exists $\alpha>0$ on $]\frac{\delta_1}{2},\delta_1\times]\frac{\delta_1}{2},\delta_2[$ such that $g^2(y')>\alpha^2$, we have

$$|f(x_1,x_2)|^2\le C\left(\int_{\mathcal{I}}g^2(y')|f(y')|^2\,dy'+\int_{\mathcal{I}}(\partial_2 f)(y_1,t_2)\,dt_2\,dy_1+\int_{-\delta_1}^{\delta_1}(\partial_1 f)(t_1,x_2)\,dt_1\right),$$

where C depends on α, δ_1 and δ_2. By integrating the above inequality on $\mathcal{I}$ with respect to x' we obtain (2.2). □

Applying the above Lemma with $f(x')=\widehat{u}(x',\xi'')$ and $g^2(x')=a^2(x_1)|\xi''|^2+b_1^2(x_2)\xi_3^2+b_2^2(x_2)\xi_4^2$, $\xi''\neq 0$, we can estimate from below the right hand side of (2.1), equal to $\|\widehat{u}(\cdot,\xi'')\|_g^2$, with $\|\widehat{u}(\cdot,\xi'')\|^2$. We have

$$\|\widehat{u}(\cdot,\xi'')\|_0 \le Ce^{-\varepsilon|\xi''|},\quad \xi''\in\mathbb{Z}^2, \qquad (2.3)$$

where C and ε are suitable positive constants.
Let $\phi \in C_0^\infty(\mathcal{I})$ with $\phi \equiv 1$ in $\mathcal{I}_1$, $\mathcal{I}_1$ neighborhood of the origin compactly contained in $\mathcal{I}$. Let $u_1(x', x'') = \phi(x')u(x', x'')$, we have

$$|\widehat{u}_1(\xi', \xi'')| = \Big| \int_{\mathbb{T}^2_{x'}} e^{-i\langle x', \xi' \rangle} \widehat{u}_1(x', \xi'') \, dt \Big| C_1 \leq \|\widehat{u}(\cdot, \xi'')\|_0 \leq C_2 e^{-\varepsilon_1 |\xi|},$$

for every $(\xi', \xi'') \in \mathbb{Z}^4$ with $|\xi''| \neq 0$ and $|\xi'| < c|\xi''|$, $c > 0$. This shows that the points of the form $(x', x'', \xi', \xi'') \in T^*(\mathcal{I} \times \mathbb{T}^2_{x''}) \setminus \{0\}$ with $\xi'' \neq 0$ and $|\xi'| < c|\xi''|$ do not belong to $WF_a(u)$, the analytic wave front set of u. Therefore there is no points in $Char(P_1)$, the characteristic variety of P_1, which belong to $WF_a(u)$. By the Theorem 8.6.1 in [6] we conclude that the analytic wave front set of u is empty.

3. Proof of the theorem 1.2

Since the vector fields $X_1, \ldots, X_6$ satisfy the Hörmander condition, [5], P_1 is hypoelliptic. Furthermore with the aid of the partition of unity the operator P_1 satisfies the following subelliptic a priori estimate:

$$\|u\|^2_{\frac{1}{r}} + \sum_{j=1}^{6} \|X_j u\|^2 \leq C|\langle P_1 u, u\rangle| + C^{N+1}\|u\|^2_{-N}, \tag{3.1}$$

for every $N \in \mathbb{Z}_+$. Here u is a smooth function on $\mathcal{I} \times \mathbb{T}_{x_3} \times \mathcal{U}$ with compact support with respect to x_1, x_2 and x_4. $\|\cdot\|_s$ denotes the Sobolev norm of order s and r the length of the iterated commutator such that the vector fields, their commutators, their triple commutators etcetera up to the commutators of length r generate a Lie algebra of dimension equal to that of the ambient space. More precisely $r-1$ is the minimum between the zero order at 0 of a and that at 0 of b_1. The above estimate was proved first by Hörmander in [5] for a Sobolev norm of order $r^{-1} + \varepsilon$ and up to order r^{-1} subsequently by Rothschild and Stein [7].
To achieve the result, we want show the analytic growth of high order derivatives of the solutions in L^2-norm. As a matter of fact we estimate a suitable localization of a high derivative of the solutions using (3.1).
Let $\phi_N(x_1, x_2, x_4)$ be a cutoff function of Ehrenpreis-Hörmander type: ϕ_N in $C_0^\infty(\mathcal{I} \times \mathcal{U})$ non negative such that $\phi_N \equiv 1$ on $\mathcal{U}_0$, $\mathcal{U}_0$ neighborhood of the origin compactly contained in $\mathcal{I} \times \mathcal{U}$, and exist a constant C such that for every $|\alpha| \leq 2N$ we have $|D^\alpha \phi_N(x)| \leq C^{\alpha+1} N^\alpha$, $\alpha \in \mathbb{Z}^3$.
We may assume that ϕ_N is independent of the x_1 and x_2-variable: every x_1, x_2-derivative landing on ϕ_N would leave a cut off function supported where x_1 or x_2 is bounded away from zero, where the operator is elliptic. As in [8], to gain the result we have to show the analytic growth of $\phi_N D_j^N u$, $j = 1, 2, 3, 4$, via (3.1). It will be sufficient analyze the direction D_4. As matter of fact D_3 commutes with P_1 and, moreover, following the same strategy employed to analyze the case D_4, we can transform powers of D_1 and D_2 in powers of D_3 and D_4.
We replace u in (3.1) by $\phi_N D_4^N u$. We have

$$\|\phi_N D_4^N u\|^2_{1/r} + \sum_{j=1}^{6} \|X_j \phi_N D_4^N u\|^2 \leq C\,|\langle P_1 \phi_N D_4^N u, \phi_N D_4^N u\rangle| + C^{N+1}\|\phi_N D_4^N u\|_{-N}. \tag{3.2}$$

The last term on the right hand side gives analytic growth. The scalar product:

$$\langle \phi_N D_4^N P_1 u, \phi_N D_4^N u\rangle + \sum_{j=1}^{6}\langle [X_j^2, \phi_N D_4^N]u, \phi_N D_4^N u\rangle$$
$$= 2\sum_{j=1}^{6}\langle [X_j, \phi_N D_4^N]u, X_j\phi_N D_4^N u\rangle + \sum_{j=1}^{6}\langle [X_j, [X_j, \phi_N D_4^N]]u, \phi_N D_4^N u\rangle$$
$$+ \langle \phi_N D_4^N P_2 u, \phi_N D_4^N u\rangle.$$

With regard to the last scalar product on the right hand side we have

$$|\langle \phi_N D_4^N P_1 u, \phi_N D_4^N u\rangle| \leq \left(\frac{1}{2C}\right)\|\phi_N D_4^N u\|^2_{\frac{1}{r}} + (2C)^{rN}\|\varphi_i \phi_N D_4^N u\|^2_{-N}$$
$$+ \|\phi_N D_4^N P_2 u\|^2.$$

The last two terms give analytic growth, $P_1 u \in C^\omega$; the first one can be absorbed on the left hand side of (3.2).

Since ϕ_N depend only by x_4 we have to analyze the commutators with, X_3, and X_6. Since the same strategy can be used to handle the case involving X_3 and X_6, we will give the details only of the case X_3. We have

$$2|\langle [X_3, \phi_N D_4^N]u, X_3\phi_N D_4^N u\rangle| + |\langle [X_3, [X_3, \phi_N D_4^N]]u, \phi_N D_4^N u\rangle|$$
$$= 2|\langle a_1\phi_N^{(1)} D_4^N u, X_3\phi_N D_4^N u\rangle| + |\langle a_1^2\phi_N^{(2)} D_4^N u, \phi_N D_4^N u\rangle|. \tag{3.3}$$

The first term can be estimate by

$$|\langle a_1\phi_N^{(1)} D_4^N u, X_3\varphi_N D_4^N u\rangle| \leq \sum_{j=1}^{N} C_j\|X_3\phi_N^{(j)} D_4^{N-j}u\|^2 + \sum_{j=1}^{N+1}\frac{1}{C_j}\|X_3\phi_N D_4^N u\|^2$$
$$+ C_{N+1}\|\phi_N^{(N+1)}u\|^2,$$

The constants C_j are arbitrary, we make the choice $C_j = \varepsilon^{-1}2^j$, ε suitable small positive constant. The terms of the form $C_j^{-1}\|X_3\phi_N D_4^N u\|^2$ can be absorbed on the right hand side of (3.2). The last term gives analytic growth. Finally we observe that the terms in the first sum have the same form as $\|X_3\phi_N D_4^N u\|^2$ where one or more x_4-derivatives have been shifted from u to ϕ_N; on these terms we can take maximal advantage from the sub-elliptic estimate restarting the process.

With regard to the second term on the right hand side of (3.3) we have

$$|\langle a_1^2\phi_N^{(2)} D_4^N u, \phi_N D_4^N u\rangle| \leq \frac{1}{2N^2}\|X_3\phi_N^{(2)} D_4^{N-1}u\|^2 + \frac{N^2}{2}\|X_3\phi_N D_4^{N-1}u\|^2$$
$$+ |\langle a_1\phi_N^{(2)} D_3^{N-1}u, X_3\phi_N^{(1)} D_4^{N-1}u\rangle|$$
$$+ |\langle N^{-1}a_1\phi_N^{(3)} D_4^{N-1}u, NX_3\phi_N D_4^{N-1}u\rangle|$$
$$+ |\langle a_1^2\phi_N^{(3)} D_4^{N-1}u, \phi_N^{(1)} D_4^{N-1}u\rangle|.$$

On the first two terms we can take maximal advantage from the sub-elliptic estimate restarting the process. The “ weight” N introduced above helps to balance the number of x_4-derivatives on u with the number of derivatives on ϕ_N, we take the factor N as a derivative on ϕ_N and $N^{-1}\phi_N^{(2)}$ as $\phi_N^{(1)}$. The second and the third terms have the same form of the first term on the right hand side of (3.3), the third one with the help of the weight N; we can handled both in the same way. The last term is the same of the left hand side in which one x_4-derivative has been shifted from u to ϕ_N on both side. Restarting the process we can estimate the left hand side of the above inequality by

$$\frac{1}{2N^2}\sum_{j=1}^{N}\|X_3\phi_N^{(j+1)}D_4^{N-j}u\|^2+\frac{N^2}{2}\sum_{j=1}^{N}\|X_3\phi_N^{(j-1)}D_4^{N-j}u\|^2$$
$$+\sum_{j=1}^{N}\sum_{\ell=j}^{N}\|X_3\phi_N^{(N-\ell+j+1)}D_4^{\ell-j}u\|^2+\sum_{j=1}^{N}2^{j+1}\|X_3\phi_N^{(N-j+1)}D_4^{j-1}u\|^2$$
$$+\frac{1}{N^2}\sum_{j=1}^{N}\sum_{\ell=j}^{N}\|X_3\phi_N^{(N-\ell+j+2)}D_4^{\ell-j}u\|^2+N^2\sum_{j=1}^{N}C^{j+1}\|X_3\phi_N^{(N-j)}D_4^{j-1}u\|^2$$
$$+2^{N+1}\left(\|a_1\phi_N^{(N+1)}u\|^2+\|a_1\phi_N^{(N+2)}u\|^2\right)+\|a_1\phi_N^{(N)}u\|^2.$$

The last terms give analytic growth, the others, in the sums, have the same form as $\|X_3\phi_N D_4^N u\|^2$, we can restart the process without the help of the sub-ellipticity. Therefore at any step of the process we obtain or terms which give analytic growth or terms from which we can take maximum advantage from the sub-elliptic estimate. We can conclude

$$\|\phi_N D_4^N u\|_{1/r}^2+\sum_{j=1}^{6}\|X_j\phi_N D_4^N u\|^2\le C^{N+1}(N)^{2N},$$

where C is independent by N but depends on u and a_1. This conclude the proof.

4. Proof of the theorem 1.3

Part (i), Theorem 1.3. Without loss of generality we assume that $x_1^0=0$ and $\mathcal{I}_1\times\mathcal{I}_2$ is a neighborhood of the point $x'=(0,0)$. Since the vector fields $X_1,\dots,X_6$ satisfy the Hörmander condition P_2 is hypoelliptic, it has the following sub-elliptic estimate:

$$\|u\|_{1/r}^2+\sum_{j=1}^{6}\|X_ju\|^2\le C\left(|\langle P_2u,u\rangle|+\|u\|_0^2\right), \tag{4.1}$$

where u is a smooth function on $\mathcal{I}_1\times\mathcal{I}_2\times\mathbb{T}^2_{x''}$ with compact support with respect to x'. Here $r-1$ is the zero order at 0 of a_2.

As in the proof of the Theorem 1.2 the result will be achieved via the L^2 estimate of suitable localizations of high derivatives of the solutions. Even if not strictly necessary in this situation we will follow a little bit different strategy which will involve the partition of unity, as done in [8]. This more general approach would allow us, without particular technical efforts, to extend the results to a more general setting in which the two-dimensional torus is replaced by a compact real analytic manifold, M, without boundary and the vector fields D_3 and D_4 are replaced by a couple of real analytic vector fields X_3 and X_4 on M such that they span TM at each point.

Let $\phi_N(x_2)$ be a cutoff function of Ehrenpreis-Hörmander type. ϕ_N is taken independent of the x_1-variable since every x_1-derivative landing on ϕ_N would leave a cut off function supported where x_1 is bounded away from zero, where the operator is elliptic.

Let $\{\mathcal{V}_j\}$ be a finite covering of $\mathbb{T}^2_{x''}$, $j=1,\dots,k$, and $\{\varphi_j\}$ a partition of unity

subordinate to to such a cover, $\varphi_j \in C_0^\infty(\mathcal{V}_j)$, $\varphi_j \geq 0$ and $\sum \varphi_j = 1$.
We replace u in (4.1) by $\varphi_j(x_3, x_4)\phi_N(x_2)D_2^N u$. We have

$$\|\varphi_j \phi_N D_2^N u\|^2_{\frac{1}{r}} + \sum_{i=1}^{6} \|X_i \varphi_j \phi_N D_2^N u\|_0^2 \leq C|\langle P_2 \varphi_j \phi_N D_2^N u, \varphi_j \phi_N D_2^N u\rangle| + C^{N+1}\|\varphi_j \phi_N D_2^N u\|^2_{-N}. \tag{4.2}$$

The last term on the right hand side gives analytic growth. As done in the proof of the Theorem 1.2 we have to handle the scalar product on the right hand side, more precisely we have to study terms of type

$$\langle [X_i, \varphi_j \phi_N D_2^N] u, X_i \varphi_j \phi_N D_2^N u\rangle, \quad \langle [X_i, [X_i, \varphi_j \phi_N D_2^N]] u, \varphi_j \phi_N D_2^N u\rangle,$$

$i = 2, \dots, 6$. The case $X_4 = a_2(x_1)D_2$ can handled following the same strategy used in the proof of Theorem 1.2, see (3.3), in this case we can take maximal advantage from the sub-elliptic estimate, therefore it gives analytic growth. Concerning the other cases it is sufficient study the case $X_2 = a_1(x_1)D_3$, the remaining cases can be handled following the same strategy [1]. We have to estimate

$$2|\langle a_1 \varphi_j^{(1)} \phi_N D_2^N u, X_2 \varphi_j \phi_N D_2^N u\rangle| + |\langle a_1^2 \varphi_j^{(2)} \phi_N D_2^N u, \varphi_j \phi_N D_2^N u\rangle| \doteq I_1 + I_2, \tag{4.3}$$

where $\varphi_j^{(\ell)} = \partial_3^\ell \varphi_j$. Here we can not take maximum advantage from the sub-elliptic estimate. In the local case would be this term which would give Gevrey growth. The argument that we will use to handle these two terms is the reason because the results is global and not local with respect to the x_3-variable. We have

$$\begin{aligned} I_1 &\leq 4C\|a_1 \varphi_j^{(1)} \phi_N D_2^N u\|^2 + \frac{1}{4C}\|X_2 \varphi_j \phi_N D_2^N u\|^2 \\ &\leq 4C\|a_1\|_\infty^2 \sup_j \|\varphi_j^{(1)}\|_\infty^2 \|\phi_N D_2^N u\|^2 + \frac{1}{4C}\|X_2 \varphi_j \phi_N D_2^N u\|^2 \\ &\leq 4C\|a_1\|_\infty^2 \sup_j \|\varphi_j^{(1)}\|_\infty^2 C_1 \sum_{j=1}^{k} \|\varphi_j \phi_N D_2^N u\|^2_{1/r} + \frac{1}{4C}\|X_2 \varphi_j \phi_N D_2^N u\|^2 \\ &\quad + 4C\|a_1\|_\infty^2 \sup_j \|\varphi_j^{(1)}\|_\infty^2 C_1^{-rN} \sum_{j=1}^{k} \|\varphi_j \phi_N D_2^N u\|^2_{-N}, \end{aligned} \tag{4.4}$$

where the constant C_1 is arbitrary. The second term on the right hand side can be absorbed on the left hand side of (4.2), the last one gives analytic growth. The term I_2 in (4.3) can be estimate by

$$\begin{aligned} &\|a_1^2\|_\infty^2 \sup_i \|\varphi_i^{(2)}\|_\infty^2 \left(C_2 \sum_{j=1}^{k} \|\varphi_j \phi_N D_2^N u\|^2_{1/r} + C_2^{-rN} \sum_{i=1}^{k} \|\varphi_j \phi_N D_2^N u\|^2_{-N} \right) \\ &\quad + \frac{1}{2C}\|\varphi_j \phi_N D_2^N u\|^2_{1/r} + (2C)^{rN}\|\varphi_j \phi_N D_2^N u\|^2_{-N}, \end{aligned} \tag{4.5}$$

where the constant C_2 is arbitrary. The last term gives analytic growth and the second to last can be absorbed on the left hand side of (4.2). Summing (4.2) over

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[1] We remark that the terms involving the fields X_5 and X_6 could be handled taking maximum advantage from the sub-elliptic estimate, this could be done choosing a partition of unity subordinate to the cover, whose elements are cutoff functions of Ehrenpreis-Hörmander type.

j and choosing C_1 and C_2 small enough so that the first term in (4.4) and the first one in (4.5) can be absorbed on the left, we can conclude

$$\|\phi_N D_2^N u\|_{1/r}^2 + \sum_{j=1}^{6} \|X_j \phi_N D_2^N u\|^2 \leq C^{N+1}(N)^{2N},$$

where C is independent by N but depends on u. This conclude the proof.

Part (ii), Theorem 1.3. We can assume that $(x_1^0, x_2^0) = (0,0)$ and $\mathcal{U}$ is an open neighborhood of the zero. Since the vector fields satisfy the Hörmander condition at the step r, for some $r \in \mathbb{Z}_+$, the following a priori estimate holds:

$$\|u\|_{\frac{1}{r}}^2 + \sum_{j=1}^{6} \|X_j u\|_0^2 \leq C\left(|\langle P_2 u, u\rangle| + C^N \|u\|_{-N}^2\right), \quad \forall N \in \mathbb{Z}_+. \tag{4.6}$$

Here u is a smooth function on $\mathcal{I} \times \mathbb{T}_{x_3} \times \mathcal{U}$ with compact support with respect to x_1, x_2 and x_4. The result is obtained via estimate of suitable localization of high derivatives, that is estimating $\phi_N(x_4) D_4^N u$ through (4.6). We will not give the details since the proof can be easily archived following the same strategies used in the proofs of the Theorem 1.2 and Theorem 1.3-(i). We only remark that the cutoff function of Ehrenpreis-Hörmander type, ϕ_N, can be assumed independent of the x_1 and x_2-variable: every x_1-derivative landing on ϕ_N would leave a cut off function supported where x_1 is bounded away from zero, where the operator is elliptic; every x_2-derivative landing on ϕ_N would leave a cut off function supported where x_2 is bounded away from zero, in this region the operator P_2 behaves like the operator $D_1^2 + a_2^2(x_1)\left(D_2^2 + D_3^2 + D_4^2\right) + a_1^2(x_1)\left(D_3^2 + D_4^2\right)$, which is (micro-)locally analytic hypoelliptic, therefore (semi-)globally analytic hypoelliptic.

References

1. P. Albano and A. Bove and M. Mughetti, *Analytic Hypoellipticity for Sums of Squares and the Treves Conjecture*, Preprint, http://arxiv.org/abs/ 1605.03801, 2016.
2. A. Bove and M. Mughetti, *Analytic Hypoellipticity for Sums of Squares and the Treves Conjecture. II*, Anal. PDE **10** (2017), no. 7, 1613–1635.
3. P.D. Cordaro and A.A. Himonas, *Global analytic hypo-ellipticity of a class of degenerate elliptic operator on the torus*, Math. Res. Lett., **1** no. 4 (1994), 501–510.
4. P.D. Cordaro and A.A. Himonas, *Global analytic regularity for sum of squares of vector fields*, Trans. Amer. Math. Soc., **350** (1998), 4993–5001.
5. L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147-171.
6. L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **256**. Springer-Verlag, Berlin, 1983. ix+391 pp.
7. L. Preiss Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247-320.
8. D.S. Tartakoff, *Global (and local) analyticity for second order operators constructed from rigid vector felds on products of tori*, Trans. Amer. Math. Soc., **348** (1996), 2577–2583.
9. F. Treves, *On the analyticity of solutions of sums of squares of vector fields*, Phase space analysis of partial differential equations, 315-329, Progr. Nonlinear Differential Equations Appl., **69**, Birkhäuser Boston, Boston, MA, 2006.

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