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Fuzzy Esscher changes of measure and copula invariance in Lévy markets

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Abstract

In the context of a multidimensional exponential Lévy market, we focus on the Esscher change of measure and suggest a more flexible tool allowing for a fuzzy version of the standard Esscher transform. Motivated both by the empirical incompatibility of market data and the analytical form of the standard Esscher transform (see [8]) and by the desire to introduce a pricing technique under incompleteness conditions, we detect the impact of fuzziness in terms of measure change function and in contingent claims' pricing. In a multidimensional setting the fuzzy Esscher transform is a copula whose invariance, under margins' transformations induced by a change of measure, is investigated and connected to the notion of the absence of arbitrage opportunities. We highlight how Esscher transform, primarily used in pricing techniques, preserves the invariance of the aggregation operator and it can be generalized to the fuzzy version assuming that the measurable functions defining the Choquet marginal integrals are increasing. Furthermore, the empirical evidence seems to suggest that a weaker concept of invariance may be more suitable, i.e. the ε-measure invariance, coherent with the Esscher fuzzy copula tool. An empirical experiment for our model will make clear how this blurring technique fits the market data.

Keywords: Fuzzy copula; Esscher transform; Copula invariance; Lévy market; Pricing under uncertainty

1. Introduction

Extended literature is witness to the extraordinary efforts deployed to analytically define the Esscher transform in very general Lévy settings (see [25]), whose incompleteness makes the problem undetermined. Among several possible solutions, the exponential form was supported by Madan and Milne [23] due to approximation reasons and because this choice assures the closure of the Lévy class for changes of measure. Nevertheless, this shape of the Esscher transform seems to fail in consistency with the market data because it cannot capture the empirical U-shape feature of the change of measure function discussed by Carr et al. [8]. To overcome a super-imposed pricing technique and allow for the presence of unobservable variables justified by possible information incompleteness, here we propose

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a fuzzy version of the Esscher transform which translates the blurriness of the market into a corridor of Esscher transforms. It is worth mentioning the effort done in the last decades by many authors to combine randomness and fuzziness, especially in option pricing (see [27] for an in-depth literature review). Several contributions have been made to address direct problems in option pricing with fuzzy models, either in discrete or continuous time, which differ in the number of parameters taken as imprecise, in the type of fuzzy numbers used to model the parameters, and in the techniques used in the fuzzy computations. Among them, the seminal contribution of Muzzioli and Torricelli [29] proposed a fuzzy version of the Cox-Ross-Rubinstein model [13] where the price of the underlying asset is represented by both a triangular and an L-R type fuzzy number as well. In the financial literature of option pricing in continuous time, we can find many fuzzy versions of the well-known Black-Scholes model [5] where some parameters (for example the underlying asset, the risk-free rate, or the volatility) are represented by triangular fuzzy numbers as in Wu [43–45], to take into account the decision maker's subjective judgment. A fuzzy version of Merton's [24] jump-diffusion model has been proposed by several authors as Xu et al. [46], where fuzziness is used to characterize the uncertainty related to the number of jump times and the jump amplitudes and by using their crisp possibilistic mean value, eventually to recover the option's price. Always taking into account the possibility of jumps in the asset price, Nowak and Romaniuk [34] designed a fuzzy Lévy process to describe the dynamic of the underlying affected by different experts' opinions or imprecise estimates of parameters; Nowak and Romaniuk [35,36] proposed a pricing model for European options in a fuzzy-Lévy environment, based on the Esscher transformed, the mean correcting, and the minimal entropy martingale measure, giving a notable contribution to direct problems in fuzzy exponential-Lévy models. Other relevant contributions in the same field of research have been given by Nowak and Pawlowski [32,33], where a geometric Lévy model is analyzed and an analytical option valuation expression is obtained both in crisp and fuzzy case (here some model parameters are described in an imprecise way by fuzzy numbers), for European-style options by using as pricing measure the variance equivalent martingale measure. An extension to a multidimensional setting is provided by Wang and He [42] who priced n-fold compound options in a fuzzy geometric Lévy market with a martingale approach. To highlight the differences between our results and some previously discussed contributions showing many similarities in the setting of the market's structure, we need to mention another kind of literature contribution which addresses inverse problems instead of direct ones in a fuzzy setting: an inverse problem embraces a market approach which, given the market data, i.e. options' prices, aims to infer the underlying asset process.[1] As a matter of fact, our contributions are to inverse problems in fuzzy option pricing since our goal is to propose a model flexible enough to capture the market's feature and based on the decision-maker's uncertainty, to calibrate the pricing measure as a consequence. We propose a new class of pricing measures called *Choquet-fuzzy Esscher transforms*: they can be seen as a fuzzy version of the Esscher transformed martingale measure used to price options in [35,36]. The class of *Choquet-fuzzy Esscher transforms* is represented by a corridor of Esscher transforms, which depends on fuzzy variables adding flexibility to the model and allowing it to be more adherent to the empirical feature of the market; the proposed *Choquet-fuzzy Esscher transform* is a fuzzy version of the implied Esscher transform recovered by market data.[2] Therefore we can state that the goal of the paper is mainly to allow for a representation of market data as more adherent as possible, hence given the probability distributions under the historical and the risk-neutral measure implied by spot and options' prices respectively, we can detail the coherent transformed function very same explaining the data. Thus arguments of no-arbitrage lead us to define the fuzzy version of Esscher transform to assure at least for a weak absence of arbitrage assumption.[3]

In a multidimensional setting, the fuzzy Esscher transforms are connected by a copula function (i.e. the Esscher copula), whose invariance for marginals' change of measure, is studied. The invariance of the copula function for transformations of variables has been studied in the seminal paper by Schweizer and Wolff [41] where the invariance of bivariate copulas under a.s. strictly increasing transformations of variables has been proved. Invariance under increasing bijections on the unit interval and relationships with the maximum attractor are investigated in [20,21]. Here Archimax copulas are defined and their invariance properties are proved.

[1] For example in [26] and [28] the implied volatility smile function is recovered from options' prices through a fuzzy quadratic regression model, proving its superiority to the market practice approach based on standard cubic spline interpolation.

[2] A similar idea of Choquet integral has been proposed in [30] to calibrate an interval-valued implied tree by using information coming from call and put options.

[3] As it will be clarified in the following, the no-arbitrage condition is strictly related to invariance properties, involving a copula function in a multidimensional setting.

In [38] the invariance property of multidimensional dependencies under transformations of variables involved by some change of measure is investigated. Changes of probability measures are important in quantitative finance because they allow many derivatives prices (e.g. options) to be computed in closed form. This pricing technique consists in choosing a useful numeraire, for the problem at hand, to express market prices. This corresponds to a probability measure making the rescaled process of prices into a martingale. In this way, the recovered derivatives' prices are computed in an artificial world where an artificial probability measure assigns different probabilities to states of the world from the *true* probability measure. However, under some technical conditions assuring the respect of the absence of arbitrage opportunities (i.e. no free lunch or better no financial strategies assuring sure profit without risk is admitted into the market), the rescaling technique is just a way to simplify the calculus without modifying the final results that are real-world prices. Whenever the pricing problem involves a multidimensional random variable, the dependence structure among its components becomes necessarily the main ingredient that would be affected by the margins' transformations induced by the change of measure. In this paper, we generalize the argument to the Lévy markets and investigate the effects of Esscher transformations on the dependence structure itself. In the case of standard Esscher transforms we come to the invariance properties proved in [41] while in the case of not only mean-monotone transformations modeled in terms of fuzzy Esscher transforms, we generalize the invariance result introducing a weaker concept of invariance. Fuzzy Esscher transform allows us to formalize the implications of an incomplete setting in pricing, giving support to the lack of unicity through the definition of a corridor of Esscher transforms which translates the blurriness of the market itself. Esscher bounds are related to the concept of quasi-invariance, which we call ε-measure invariance, and which is based on the discrepancy to the strongest copula used to measure the distance between distributions in meaning very close to the *maximum mean discrepancy* (see [6,16,37]). A weaker request concerning the monotonicity of the change of measure transforms could justify the empirical evidence and motivate the pricing under the assumption of *quasi* absence of arbitrage opportunities based on the concept of ε-measure invariance. Rather than exogenously define the risk-neutral probabilities as fuzzy numbers (by fuzzifying the up/down jump factors as in [2,1]) and select weighted intervals of probabilities derived by no-arbitrage argument as proposed in [31], this paper accounts for a fuzzy Esscher transform able to represent several kinds of uncertainty/blurriness and at the same time to assure for a strong or weak no-arbitrage condition by invariance arguments. In support of our approach we provide, an empirical experiment based on market data to show how it is possible to properly calibrate a fuzzy system; it seems to suggest that an ε-measure invariance would be more suitable. Given that the literature focuses mainly on the theoretical aspects of the model's fuzzification, not paying much attention to the empirical investigation, the method proposed here seems relevant to us because the theoretical fuzzy structure is motivated by the need to make the model coherent with market data and risk-neutral pricing techniques, by eventually relaxing the definition of no-arbitrage if empirical data reveal critical features.

The plan of the paper is as follows: Section 2 defines the exponential Lévy market and discusses its incompleteness feature. The main analytical results are reviewed and the Esscher transform, along with its fuzzy version, is introduced. Section 3 discusses the concept of measure invariance under a fuzzy Esscher transform, while Section 4 focuses on an empirical example where both fuzziness calibration and pricing under blurred assumptions are investigated. Section 5 sums up our conclusions.

2. A multidimensional exponential Lévy market

We consider a frictionless continuous financial market whose stock price process is a n-dimensional exponential Lévy process, i.e.

$$\mathbf{S}_t = \mathbf{S}_0 \exp\{\mathbf{X}_t\}, t \in [0, T],$$

where $\mathbf{S}_0 > 0$ and $\mathbf{X} = (X_t)_{t\in[0,T]}$ is a $\mathbb{R}^n$ Lévy process on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with characteristic triple $(b, c, K)_{\mathbb{P}}$. We'll be working everywhere with a càdlàg modification of $\mathbf{X}$ so that the total number of jumps is at most countable and the number of jumps, whose size is *big*,[4] is finite.

[4] Given an arbitrary $\epsilon > 0$, the number of jumps whose size is in absolute value bigger than ϵ, is finite.

A very useful decomposition of a Lévy process is provided by the Lévy-Itô decomposition[5] of $\mathbf{X}$ into a deterministic drift, a continuous diffusion and a jump part, i.e.

$$\mathbf{X}_t = \mathbf{b}t + \sqrt{c}\mathbf{W}_t + h(x) * (\mu^X - \nu)_t + (x - h(x)) * \mu^X,$$

where $\mathbf{b} \in \mathbb{R}^n$, $c \in \mathbb{R}^n \times \mathbb{R}^n$ is a symmetric non-negative definite matrix, $\mathbf{W}$ is a n-dimensional Brownian motion and $h : \mathbb{R}^n \to \mathbb{R}^n$ is given by $h(x) = x\mathbf{1}_{0\le|x|\le 1}$. We observe that the *truncation function* $h(x)$ guarantees the existence of both the stochastic integral with respect to the jump measure μ^X and the compensated jump measure $(\mu^X - \nu)$, where ν stands for the intensity measure of the Poisson random measure μ^X.

It is worth recalling that a Lévy process is usefully represented in terms of its characteristic function. The well-known Lévy-Khinchin representation (see [40]) provides the characteristic function χ_t of $\mathbf{X}$, i.e. $\chi_t(z) = \exp\{t\psi(z)\}$ whose cumulant function is given by

$$\psi(\mathbf{z}) = i\mathbf{b}'\mathbf{z} - \frac{1}{2}\mathbf{z}'\mathbf{c}\mathbf{z} + \int_{\mathbb{R}^n} (e^{i\mathbf{z}'x} - 1 - i\mathbf{z}'h(x))K(dx),$$

where $K(A)$ is the Lévy measure of $\mathbf{X}$ and stands for the expected number of jumps per unit time having size in A. The Lévy measure allows us to define the intensity of the Poisson random measure μ^X on $[0, T] \times \mathbb{R}^n$ as $dtK(dx)$.

We point out that Lévy measure, which determines the jump intensity, though we may select three different classes of processes. The first one is a pure jump class with an infinite Lévy measure, i.e. it has an unlimited expected number of jumps in every finite time interval and infinite variation. The second is also a pure jump class having finite variation but the Lévy measure has an infinite mass as well. Finally, the third class represents the jump-diffusion process, mainly driven by the diffusive component whereas the jumps occur rarely, i.e. the Lévy measure has a finite mass.

2.1. Change of measure

The change of measure techniques is commonly used to transform the original process into a new one that satisfies the desired feature under a new probability measure. When we model securities' and derivatives' prices, we take as given some *true* probabilities $\mathbb{P}$, called *real-world measures* which assigns probabilities to different states of the world. Unfortunately, under $\mathbb{P}$ one cannot price derivatives as expected discounted dividend streams and no-arbitrage constructions or Feynman-Kac theorem can't give an explicit PDE. The change of measure is a very useful technique consisting in rescaling everything in order to have the same return of all assets, i.e. the remuneration for the asset without risk. The martingale measure of these rescaled prices is noted as $\mathbb{Q}$ and is called *risk-neutral measure*.

Unfortunately the incompleteness of the Lévy market prevents us from recovering a unique martingale measure ensuring the absence of arbitrage opportunity assumptions. It means that we can recover infinitely many martingale measures whose corresponding pricing rule preserves the market from the arbitrage strategies. The problem is in principle undetermined and we need to set up rules to make it manageable. We observe first of all that there is a simple parameterization of the change of measure if we impose to preserve the Lévy structure also under the new measure. In this case, the change of measure is realized through a deterministic variable changing the drift of the diffusive part and a deterministic positive function called *measure change function* which modifies the intensity of the jumps. Nevertheless, the possible solutions are infinitely many.

Jacod and Shiryaev [22] proved the necessary and sufficient conditions, respectively for all admissible parameterization preserving the closure of the Lévy market after martingale change of measure. We have in mind the exponential Lévy market and deal with necessary and sufficient conditions for the martingality of the stock prices discounted at the risk-free rate r. The change of measure is well defined by a deterministic vector $\beta \in \mathbb{R}^n$ and a deterministic function $y : \mathbb{R}^n \to \mathbb{R}^+$ such that the following conditions are satisfied:

$$\int_{|x|\ge 1} y(x)(e^{x^i} - 1)K(dx) \le \infty, \forall i = 1, 2, ..., n,$$

$$\int_{\mathbb{R}^n} (1 - \sqrt{y(x)})^2 K(dx) \le \infty.$$

[5] We invite the interested readers to see Cont and Tankov [12] for the proof of the decomposition.

Hence the new measure $\mathbb{Q}$ will be characterized by a new triplet $(\overline{b}, \overline{c}, \overline{K})$ defined as follows:

$$\overline{b} = b - r\mathbf{1} + c\beta + \frac{1}{2}\mathcal{C} + \int\limits_{\mathbb{R}^n} (J(x)y(x) - h(x))K(dx) = 0,$$

$$\overline{c} = c,$$

$$\frac{d\overline{K}}{dK}(x) = y(x),$$

where r is the risk-free rate, $J(x) = (e^{x^1} - 1, ..., e^{x^n} - 1) \in \mathbb{R}^n$ and $\mathcal{C} = (c^{11}, ..., c^{nn}) \in \mathbb{R}^n$.

A possible way to make the problem determined and ensure the uniqueness of the solution is given by imposing a simple parametrization for the measure change function as we do for the most popular Esscher function; this is an exponential function that depends on just one parameter, recovered to assure for the martingale property. This kind of function identifies the *Esscher martingale measures*.

The corresponding change of measure via Esscher transform is given by the following Radon-Nikodym derivative, i.e.

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\exp(\theta \mathbf{X}_T)}{\mathbb{E}^{\mathbb{P}}[\exp(\theta \mathbf{X}_T)]} = \exp\{\theta \mathbf{X}_T - T\psi(-i\theta)\},$$

where ψ is the cumulant function of $\mathbf{X}_1$ and $\theta \in \mathbb{R}^n$. The Esscher parameter θ could be thought as the inverse in sign of the relative risk aversion parameter of the representative agent having constant relative risk aversion utility function, in a general equilibrium model.

The corresponding measure change function is then given by

$$y(x) = e^{\theta x},$$

whose choice is approved by Madan and Milne [23] because it corresponds to the first-order Taylor approximation of the more general strictly positive function $y(x) = e^{f(x)}$ under the assumption $y(0) = 1$. Moreover, we observe that there is a large class of Lévy processes that are closed under an Esscher change of measure coinciding with the minimal entropy measure in the exponential Lévy market at hand. Despite the wide range of justifications for the previous choice, it does not seem flexible enough to capture effects empirically recovered as the U-shape feature of the change of measure function discussed by Carr et al. [8]. Hence finally this class of measures could fail in consistency with the market since it is a super-imposed pricing rule.

Example 1. Let us review very briefly an example of Esscher's change of measure for an exponential Lévy model. All that follows is very classical and can be found in a number of references on Lévy processes: the examples chosen here are detailed in [3].

It is well known that if $X = (X_t, \mathcal{F}_t)$ is a Lévy process with local characteristics (b, c, F) and with the characteristic function $\mathbb{E}e^{i\theta X_t} = e^{X_t \chi(\theta)}$, its local Fourier cumulant function is

$$\psi(-i\theta) = i\theta b - \frac{\theta^2}{2} + \int \left(e^{i\theta x} - 1 - i\theta h(x)\right) F(dx). \tag{1}$$

Here $\chi(\theta) = \psi(-i\theta)$, the usual switch between Fourier and Laplace transform. A typical model in applications arises when the financial price S_t is given as $S_t = S_0 e^{X_t}$, the exponential model. In Fig. 1 we can appreciate the behavior of the Laplace cumulant function in a normal setting.[6]

[6] The plotted behavior is obviously incompatible with a U-shaped change of measure function which seems supported by market data (see [8]).

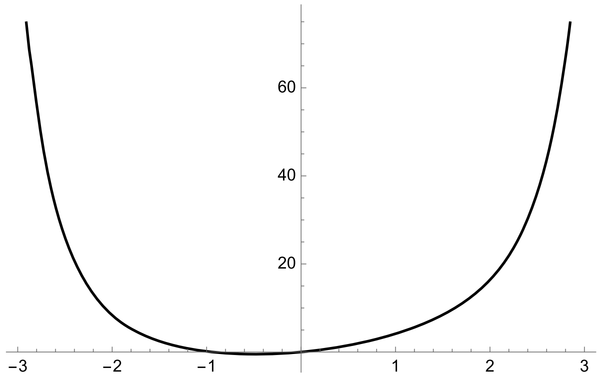


Fig. 1. Example of Laplace cumulant function with $F(dx)$ equal to a standard normal density, $b = 2, c = 3$.

The classical change of measure problem relative to this model can be formulated as the question of finding a new probability measure $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$, relative to which the process S_t above becomes a local martingale, or even a martingale.

We recall that the necessary and sufficient conditions for a Lévy process X to be a local martingale can be expressed as

$$b + \int_{\mathbb{R}} (x - h(x))F(dx) = 0.$$

When we apply this to the exponential model above we get that

$$b + \frac{c}{2} + \int_{\mathbb{R}} (e^x - 1 - h(x))F(dx) = 0.$$

If these conditions do not hold, it is still possible to formulate sufficient conditions allowing the construction of new probability measures for which the exponential model is again (local) martingale. This is done via the Esscher transform, a very quick recap, adapted to this model, follows.

With X_t given as before, it is known that for every $\theta \in \mathbb{R}$ the processes $Y^{(\theta)} = (Y_t^{(\theta)})_{0\le t\le T} = \frac{e^{\theta X_t}}{e^{t\chi(\theta)}} = e^{aX_t - t\chi(\theta)}$ are positive local martingales, assuming that $\int e^{x\theta} I(|x| > 1)F(dx) < \infty$. Here $\chi(\theta)$ denotes the Laplace's cumulant function defined in (1).

If we further assume that these $Y^{(\theta)}$ are martingales, then we can define a new Esscher measure $\mathbb{P}_T^{(\theta)}(A) = \int_A Y^{(\theta)}\mathbb{P}_T(d\omega)$, where $A \in \mathcal{F}_T$ and $\mathbb{P}_T = \mathbb{P}|\mathcal{F}_T$.

Let us consider this example. Take $X = (X_t, \mathcal{F}_t)$ as a Poisson difference process with drift:

$$X_t = \mu t + \alpha N_t^{(1)} - \beta N_t^{(2)} ,$$

where $\alpha > 0$, $\beta > 0$ and $N^{(1)} = (N_t^{(1)})_{t\ge 0}$ and $N^{(2)} = (N_t^{(2)})_{t\ge 0}$ are two independent Poisson processes with intensity parameters $\lambda_1 > 0$ and $\lambda_2 > 0$.

For this example, we can compute

$$\mathbb{E}X_t = (\mu + \alpha\lambda_1 - \beta\lambda_2)t,$$

and

$$\mathbb{V}X_t = (\alpha^2\lambda_1^2 - \beta^2\lambda_2^2)t.$$

The Laplace's cumulant function $\chi(\theta)$ can be computed to be

$$\chi(\theta) = \mu\theta + \lambda_1(e^{\alpha\theta} - 1) + \lambda_2(e^{-\beta\theta} - 1).$$

In the exponential model, the process is a martingale if the parameter $\tilde{a}$ is chosen as the solution of

$$\mu + \lambda_1 e^{\tilde{a}\alpha}(e^{\alpha} - 1) + \lambda_2 e^{-\tilde{a}\beta}(e^{-\beta} - 1) = 0.$$

If for instance $\mu = 0$ then

$$\tilde{a} = \frac{1}{\alpha+\beta} \log \frac{\lambda_2(1-e^{-\beta})}{\lambda_1(e^{\alpha}-1)}.$$

2.2. *Fuzzy change of measure*

In order to provide a more flexible alternative to the Esscher transform, we plan to construct a fuzzy version of the Esscher martingale measure.

The Esscher transformed vector of r.vs $(S^{\mathbb{Q}}_{1,t}, ..., S^{\mathbb{Q}}_{n,t})$ and the set of their marginal distributions is given under the new measure, i.e. $(u^{\mathbb{Q}}_{1,t}, ..., u^{\mathbb{Q}}_{n,t})$, where $u^{\mathbb{Q}}_{i,t} = \phi(u^{\mathbb{P}}_{i,t})$, $i = 1, ..., n$ and ϕ stands for the transform induced on the $\mathbb{P}$-cdf by the Esscher change of measure.

As proposed by Romagnoli [39] a degree of model ambiguity can be introduced through a fuzzy distortion of marginals distribution and the invariance of their dependence structure. Therefore in this kind of model, it is difficult to give a degree of ambiguity inducing an order rule. This way, in line with Gilboa and Schmeidler [18], it is possible to determine a range of variation of the fuzzy marginals through a particular tool, i.e. the Choquet integral (see [11]). The Choquet uncertainty approach (see [19,14]) is a special case of the fuzzification strategy, where the distortion is based on a Choquet-integral and originates from a rescaling of the rvs induced by a capacity variable, acting as a proxy for decision-makers attitudes towards ambiguity.

We recall the concept of Choquet integral.

Definition 2 *(Choquet integral).* Let $(\Omega, \mathcal{F}, \mu)$ be a nonadditive measure space and γ a measurable function on Ω. The Choquet integral of $\gamma : \Omega \to [0, +\infty)$ is defined as

$$(C)\int_{\Omega} \gamma d\mu = \int_{0}^{\infty} \mu(x|\gamma(x) \geq \alpha) d\alpha.$$

If $(C)\int_{\Omega} \gamma d\mu < +\infty$, γ said to be C-integrable in the space $(\Omega, \mathcal{F}, \mu)$. Moreover Choquet integral satisfies the following properties:

(i) if $\zeta \leq \gamma$, then $(C)\int_{\Omega} \zeta d\mu \leq (C)\int_{\Omega} \gamma d\mu$;
(ii) if $A \subset B$, $A, B \in \Omega$, then $(C)\int_{A} \gamma d\mu \leq (C)\int_{B} \gamma d\mu$;
(iii) if μ is lower semicontinuous and $\gamma_n \uparrow \gamma$ a.e. in Ω, then $(C)\int_{\Omega} \gamma_n d\mu \uparrow (C)\int_{\Omega} \gamma d\mu$;
(iv) if μ is upper semicontinuous, $\gamma_n \downarrow \gamma$ a.e. in Ω, and there exists a C-integrable function ζ such that $\gamma_1 \leq \zeta$, then $(C)\int_{\Omega} \gamma_n d\mu \downarrow (C)\int_{\Omega} \gamma d\mu$.

As we know, if the fuzzy measure μ is subadditive (we denote it $\underline{\mu}$ in the following), the previous definition refers to the so-called *lower Choquet integral*, that should correspond to the lowest bound of fuzzified marginals. Moreover, for a well-known property of nonadditive measures (see [17]), any subadditive measure is associated with a dual superadditive measure $\overline{\mu}$ by the relationship $\underline{\mu}(A) = 1 - \overline{\mu}(\overline{A})$, where $\overline{A}$ denotes the complement set of A.

Notice that the duality relationship provides a definition of a set of probability measures $\mathcal{C}$ said to be the core of measure μ, i.e. $\mathcal{C} = \{\xi : \underline{\mu}(A) \leq \xi(A) \leq \overline{\mu}(A), \underline{\mu}(A) = 1 - \overline{\mu}(\overline{A}), \forall A \in \mathcal{F}\}$. Based on the core of a fuzzy measure, we define the set of Choquet integral of a measurable function γ as $S_C = \{(C)\int_{\Omega} \gamma d\xi, \xi \in \mathcal{C}\}$; this is a bounded set delimited by the *lower and the upper Choquet integral*, respectively.

Definition 3 *(Choquet-fuzzy Esscher transforms).* Given the Esscher transformed vector $(S^{\mathbb{Q}}_{1,t}, \ldots, S^{\mathbb{Q}}_{n,t})$, and the corresponding set of Esscher cdfs, i.e.

$$\mathcal{ES}^{\mathbb{Q}}_t = \{(u^{\mathbb{Q}}_{1,t}, \ldots, u^{\mathbb{Q}}_{n,t}); u^{\mathbb{Q}}_{i,t} = \phi(u^{\mathbb{P}}_{i,t}), \forall i\},$$

where ϕ stands for the transform induced on the $\mathbb{P}$-cdf by the Esscher change of measure, the set of Choquet-fuzzy Esscher cdfs, corresponding to a set of core of a nonadditive [0, 1]-measure μ, one for every marginal projection of such measure, i.e. $\mathcal{C} = \{\mathcal{C}_i, i = 1, \dots, n\}$, where $\mathcal{C}_i = \{\xi_i : \underline{\mu}_i(A) \le \xi_i(A) \le \overline{\mu}_i(A), \underline{\mu}_i(A) = 1 - \overline{\mu}_i(\overline{A}), \forall A \in \mathcal{F}\}$ is

$$\mathcal{CFES}_t^{\mathbb{Q}} = \left\{ \left((C) \int_{\Omega} \tilde{u}_{1,t} d\xi_1, \dots, (C) \int_{\Omega} \tilde{u}_{n,t} d\xi_n \right), \xi_i \in \mathcal{C}_i, \forall i \right\}$$

where $\tilde{u}_{i,t} = \gamma_i(u_{i,t}^{\mathbb{Q}})$ is the fuzzy version of the ith marginal distribution for a measurable function $\gamma_i, \forall i$.

The fuzzy version of the Esscher transform could be seen as an ambiguous version of the standard one, where the transformed cdf of $S_{i,t}^{\mathbb{P}}, \forall i$ is assumed to be a function of a fuzzy variable. It can be motivated by several kinds of uncertainty affecting the model, i.e. lack of information (which implies measurability issues), distortion of the information signal due to the decision maker's beliefs, or also unreliable statistical estimates of specific model's parameters. A fuzzy variable lets us translate, for example, an expert's opinion about ambiguous parameters. The uncertainty is then introduced via a fuzzification of the change of measure transformation, i.e. either the parameter θ in equation (1) becomes fuzzy or we make fuzzy the shape of the exponent function $y(x)$. This way we make the change of measure more flexible than the standard Esscher transform and allow to be more adherent to the empirical feature of the market.

Example 4. Assume an arithmetic Lévy market where a risky asset is traded and whose price evolves as described by the following SDE[7]

$$dS_t = \sigma_t \hat{\theta}_t dt + \sigma_t d\hat{W}_t + \eta_t dI_t,$$

where $\hat{W}_t$ is a one-dimensional $\mathbb{P}$-Brownian motion, I_t is a jump process, independent from $\hat{W}_t$, while the other parameters are real-valued continuous functions. The jump process[8] is defined by its associated random jump measure $N(.,.)$ and compensator $\tilde{N}(.,.)$ and by the compensator measure noted as $l(.,.)$, i.e.

$$dI_t = \int_{\mathbb{R}} z \left(e^{\tilde{\theta}_t z} - 1 \right) l(dz, dt) + \int_{\mathbb{R}} z \tilde{N}(dz, dt).$$

Recall from arbitrage theory that to coherently solve any pricing problem, we need to identify all equivalent martingale measures which are risk-neutral in the market we are considering; this amounts to a rather wide class of potential pricing measures that we recognize to be the class of Esscher transforms. The Esscher transform introduces a set of parameters $\theta_t = (\hat{\theta}_t, \tilde{\theta}_t)$ and allows the description of the equivalent martingale measure $\mathbb{Q}^\theta$ by the density process Z_t^θ, i.e.

$$\frac{d\mathbb{Q}^\theta}{d\mathbb{P}}|_t = Z_t^\theta,$$

where the density can be decomposed into the Doleans-Dade martingales accounting for the continuous and the jump component, i.e. $Z_t^\theta = Z_t^{\hat{\theta}} \times Z_t^{\tilde{\theta}}$. Under technical condition ensuring that $Z_t^{\tilde{\theta}}$ is at least a local martingale with unitary expectation,[9] we resort to the $\mathbb{Q}^\theta$-dynamic of the risky asset, i.e.

$$dS_t = \sigma_t dW_t^\theta + \int_{\mathbb{R}} z \eta_t \tilde{N}^\theta(dz, dt),$$

7 It stands for Stochastic Differential Equation.

8 In order to assure that S_t has finite moments up to a certain order, it is necessary to impose a condition on the jump process I_t, i.e. it must be requested the existence of $c > 0$, such that $\int_{\mathbb{R}} \int_{|z| \ge 1} |z|^c l(du, dz) < \infty$. See [4].

9 This condition refers to the existence of $\nu > 0$ such that $\sup |\tilde{\theta}| + \nu \le c$, where c is the same positive constant assuring for the finitude of S_t moments.

where the change of measure acts for both the Brownian motion and the compensator, i.e.

$$W_t^\theta = W_t + \int_0^t \hat{\theta}_s ds,$$
$$\tilde{N}^\theta(dz, dt) = \tilde{N}(dz, dt) + (e^{\tilde{\theta}_t z} - 1)l(dz, dt).$$

The price of a call option, whose payoff is assumed to be integrable on the real line, can be recovered by the Fourier approach (see [4]). Therefore, the price of a call option written on S_t, with maturity T and strike price K, is given by

$$C(t, T, K) = e^{-r(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{G}_T(y)\Psi(t, T)e^{(a+iy)U_t^\theta} dy, \tag{2}$$

where $\hat{G}_T(y)$ is the Fourier transform of the dampened version of the payoff function[10] for a damping parameter $a > 0$, while

$$U_t^\theta = \int_0^t \sigma_s dW_s^\theta + \int_0^t z\eta_s \tilde{N}^\theta(dz, ds)$$
$$\ln \Psi(t, T) = \frac{1}{2}(a + iy)^2 \int_t^T \sigma_s^2 ds + \psi((y - ia)\eta_t),$$

where $\psi(.)$ is the cumulant function of I_t.

We observe that for every θ the call price in (2) can be seen as a solution to the crisp pricing problem; the fuzzy version of the same problem resorts if for example, we assume to deal with an uncertain set of parameters θ, which can be described as fuzzy variables. Therefore we define the Choquet-fuzzy Esscher transforms set, i.e.

$$\mathcal{CFES}_t^{\mathbb{Q}^\theta} = \left\{ (C)\int_\Omega \tilde{u}_t d\xi, \xi \in \mathcal{C} \right\},$$

where $\tilde{u}_t = \gamma(u_t^{\mathbb{Q}^\theta})$ is the fuzzy version of the risky asset's distribution $u_t = F_t^{\mathbb{Q}^\theta}(s)$ for a measurable function γ. It corresponds to the core of a nonadditive [0, 1]-measure μ, i.e. $\mathcal{C} = \{\xi : \underline{\mu}(A) \le \xi(A) \le \overline{\mu}(A), \underline{\mu}(A) = 1 - \overline{\mu}(\overline{A}), \forall A \in \mathcal{F}\}$. As a matter of fact, if we have an uncertain set of parameters, $\theta \in [\underline{\theta}, \overline{\theta}]$, it is supposed to induce a fuzzy set of Esscher transforms whose core allows to define the Esscher bounds, i.e.

$$S^U = (C)\int \gamma(u_t^{\mathbb{Q}^\theta})d\overline{\mu} = \int Z_t^{\overline{\theta}} dF^{\mathbb{P}}$$
$$S^L = (C)\int \gamma(u_t^{\mathbb{Q}^\theta})d\underline{\mu} = \int Z_t^{\underline{\theta}} dF^{\mathbb{P}},$$

and consequently to perform the call price at the bounds, i.e.

$$C^U(t, T, K) = e^{-r(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{G}_T(y)\Psi(t, T)e^{(a+iy)U_t^{\overline{\theta}}} dy$$
$$C^L(t, T, K) = e^{-r(T-t)} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{G}_T(y)\Psi(t, T)e^{(a+iy)U_t^{\underline{\theta}}} dy.$$

Finally the arbitrage-free corridor price is given by

$$C(t, T, K) \in [C^L(t, T, K), C^U(t, T, K)].$$

[10] The damping factor e^{-ax}, $a > 0$ allows to integrate the option's payoff function on the real line and to apply the Fourier approach. We need also to assure for an integrability condition, i.e. $c \ge a \sup_{[0,T]} |\eta_t|$, where c is the positive parameter whose existence proves that S_t has finite moments.

It is worth mentioning that the proposed model is able to represent the fogginess induced by market data and coherently define the core and the uncertain domain of parameters; however, as the empirical example will make clear later, to ensure the no-arbitrage condition it may be necessary to impose restrictions and resort to a *quasi*-absence of arbitrage opportunity.

Finally, we observe that the case of multiple priors can be thought of as a special case of Choquet uncertainty, where the Choquet capacity may be interpreted in terms of beliefs. As a matter of fact, if the behavior of a decision-maker in multiple-priors is represented by a set of probability measures overlapping the core of the Choquet capacity, the two models overlie; nonetheless, it works iff the decision-makers capacity is convex, while the behaviors described by a Choquet integral with respect to a non-convex capacity cannot be specified by a multiple-priors model (see [9]). Therefore we can consider the multiple-priors case as a sub-case of Choquet uncertainty, whose capacity's core is defined by the decision maker's beliefs.

3. Measure-invariance of the fuzzy Esscher transforms

In this section, we consider the multidimensional feature of our market and investigate some invariance properties of copula functions under change of measure transformations. This copula represents, at a given point in time, the dependence structure of n exponential Lévy variables.

Given the dependence structure of the stock prices at time t under the historical probability measure $\mathbb{P}$, i.e. given a copula function $C^{\mathbb{P}}_{\mathcal{I}_t}(u^{\mathbb{P}}_{1,t},\dots,u^{\mathbb{P}}_{n,t})$ where $\mathcal{I}_t=(S^{\mathbb{P}}_{1,t},\dots,S^{\mathbb{P}}_{n,t})$ is a vector of random variables defined in a complete filtered space $(\Omega,(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$[11] and $u^{\mathbb{P}}_{i,t}=F^{\mathbb{P}}_{S_{i,t}}(s_i),\forall i=1,\dots,n$ is the cdf of $S^{\mathbb{P}}_{i,t}$ under $\mathbb{P}$, we consider a transformation of margins, i.e. $\phi(u^{\mathbb{P}}_{i,t}),\forall i=1,\dots,n$, induced by a change of measure. What interests us is the resulting dependence structure under the new probability measure.

Let $\eta=\frac{d\mathbb{Q}}{d\mathbb{P}}|_t$ be the Esscher density satisfying the regularity conditions ensuring that it is at least an exponential local martingale. Moreover, if $\mathbb{E}^{\mathbb{P}}(\eta)=1$, the necessary conditions of the Radon-Nikodym theorem are satisfied and we are allowed to pass into measure $\mathbb{Q}$. The new margins will be $u^{\mathbb{Q}}_{i,t}=F^{\mathbb{Q}}_{S_{i,t}}(s_i)=F^{\mathbb{P}}_{S_{i,t}}(\alpha^{-1}_{i,t}(s_i))=\phi(u^{\mathbb{P}}_{i,t})$, $\forall i=1,...,n$, where $\alpha_{i,t}(.)$ and $\phi(.)$ are the transformations induced on the variable $S^{\mathbb{P}}_{i,t}$ and on the $\mathbb{P}$-cdf respectively[12] by the change of measure, i.e. $S^{\mathbb{Q}}_{i,t}=\alpha_{i,t}\left(S^{\mathbb{P}}_{i,t}\right)$. Given a transformation $\alpha_{i,t}(.)$ induced on the univariate r.vs $S^{\mathbb{P}}_{i,t}$ by a change of measure, we are now interested in the dependence structure of the transformed vector $(S^{\mathbb{Q}}_{1,t},\dots,S^{\mathbb{Q}}_{n,t})$, i.e. on the copula function whose marginals are the $\mathbb{Q}$-cdf of the transformed variables. Our main question concerns the relationship between the $\mathbb{Q}$-dependence structure and the original one linking the variables defined under the $\mathbb{P}$ measure.

Romagnoli [38] studied in depth this problem and defined the notion of copula m-invariance, i.e. of the dependence structure's invariance under the measure-change transformation of marginals. More precisely a necessary and sufficient condition for the m-invariance is proved to be the strictly increasing behavior of marginals' transformation, i.e. $C_{S^{\mathbb{P}}_{i,t},S^{\mathbb{Q}}_{i,t}}(u^{\mathbb{P}}_{i,t},u^{\mathbb{Q}}_{i,t})=u^{\mathbb{P}}_{i,t}\wedge u^{\mathbb{Q}}_{i,t},\forall i$.

In case of fuzzy Esscher change of measure, the m-invariance of copula function is assured iff all fuzzy marginals in S_{C_i} are strictly increasing transforms of $u^{\mathbb{P}}_i,\forall i$. Under this assumption we can say that the *conditional Choquet-fuzzy copula*, already introduced in [39], is m-invariant.

In order to simplify the notation, we assume $u_i=u^{\mathbb{P}}_i$, $\hat{u}_i=u^{\mathbb{Q}}_i$ while $\tilde{u}_i$ stands for the fuzzy version of the Esscher transform $\hat{u}_i$.

Definition 5 *(Fuzzy Esscher Copula).* A function $\tilde{S}:[0,1]^n\to[0,1]$ corresponding to a set of core of a nonadditive $[0,1]$-measure μ, one for every marginal projection of such measure, i.e. $\mathcal{C}=\{\mathcal{C}_i,i=1,\dots,n\}$, where $\mathcal{C}_i=\{\xi_i:$

[11] The sigma-algebra $\mathcal{F}_t$ is assumed to be generated by all r.vs, i.e. $\mathcal{F}_t=\bigvee_{i=1}^n\mathcal{F}_t^{S_i}$.

[12] The random variables $S^{\mathbb{P}}_{i,t}$ in $\mathcal{I}_t$ are not necessary equal as function nor do have the same probability law.

$\underline{\mu}_i(A) \leq \xi_i(A) \leq \overline{\mu}_i(A), \underline{\mu}_i(A) = 1 - \overline{\mu}_i(\overline{A}), \forall A \in \mathcal{F}\}$,[13] and to a measurable function γ is a *conditional Choquet-fuzzy copula*[14] called *Fuzzy Esscher Copula*, i.e. for $\xi_i \in \mathcal{C}_i, \forall i$

$$\tilde{S}(\tilde{u}_1, \dots, \tilde{u}_n) = \tilde{S}\left((C)\int_{\Omega} \gamma_1(\hat{u}_1)d\xi_1, \dots, (C)\int_{\Omega} \gamma_n(\hat{u}_n)d\xi_n\right),$$

where $\gamma_i(\hat{u}_i)$ is a measurable function of ith Esscher transformed marginal.

Theorem 6. *Let $(\mathbb{R}^n_+, \mathcal{B}(\mathbb{R}^n_+), \mu)$ be a nonadditive measure space, where $\mathcal{B}(\mathbb{R}^n_+)$ is the Borel set generated by the space $\mathbb{R}^n_+$ and $\gamma : [0,1]^n \to [0,1]^n$ be a strictly increasing measurable function. Then there exists a unique conditional Choquet-fuzzy copula $\tilde{S} : [0,1]^n \to [0,1]$ and*

$$\tilde{S}(\tilde{u}_1, \dots, \tilde{u}_n) = C^{\mathbb{P}}(\tilde{u}_1 \in S_{C_1}, \dots, \tilde{u}_n \in S_{C_n}),$$

where $\tilde{u}_i = \gamma_i(\hat{u}_i)$ and $S_{C_i} = \{(C)\int_{\omega} \gamma_i(\hat{u}_i)d\xi_i, \xi_i \in \mathcal{C}_i\}, \forall i$.

Proof. It is easily proved by observing that marginals are here increasingly transformed by taking the Choquet integrals of the marginals of the copula $\tilde{S}$. Hence the copula $C^{\mathbb{P}}$ is said to be invariant, or m-invariant if the transform γ is strictly increasing, knowing that its argument is the Esscher measure transform (which is increasing too). □

The previous theorem states that the m-invariance is guaranteed when we deal with *fuzzy Esscher copulas*, iff the fuzzy marginals depend on strictly increasing function γ. Different assumptions about the behavior of γ enable us to generalize the Esscher change of measure function and reproduce more general shapes.

Particular components of the set of *fuzzy Esscher copulas* are recovered, placing the marginals on the bounds; if the marginals are $\underline{u}_i = (C)\int_{\Omega} \gamma_i(\hat{u}_i)d\underline{\mu}_i, \forall i$, we talk about the *lower fuzzy Esscher copula*[15] (S^L for short) while if the marginals are $\overline{u}_i = (C)\int_{\Omega} \gamma_i(\hat{u}_i)d\overline{\mu}_i, \forall i$, we define the *upper fuzzy Esscher copula* (S^U in the following). We remark then that the level of ambiguity induced by a fuzzy measure μ is given by $a = S^U - S^L$. The following result is proved in [39].

Proposition 7. *Let $\mathcal{S}$ be the set of fuzzy Esscher copulas, i.e. $\mathcal{S} = \{\tilde{S}(\tilde{u}_1, \dots, \tilde{u}_n), u_i \in S_{C_i}, \forall i\}$. Then $S^L \leq \tilde{S}(\tilde{u}_1, \dots, \tilde{u}_n) \leq S^U, \forall \tilde{S} \in \mathcal{S}$.*

Proof. The statement is easily proved, thanks to the monotonicity of the Choquet integral, given the core's definition, and the monotonicity of copulas with respect to the margins. □

Taking into consideration the monotonicity of the fuzzy structure, we now focus on the *lower fuzzy Esscher copula* and on the request of strictly increasing monotonicity of its marginals with respect to the historical ones. More precisely if the fuzzy marginal $\underline{u}_i$, is a strictly increasing transform of the historical $u_i, \forall i$[16] then the dependence structure of the exponential Lévy market is invariant for change of measure transforms. We have the following condition for m-invariance of a market system.

[13] Here $\mathcal{F} = \mathcal{B}([0,1])$, i.e. the Borel set generated by $[0,1]$.

[14] The dependency structure is now a copula because Choquet integrals are defined by additive measures belonging to the core of non-additive μ. Obviously, this is not the case for Choquet integrals with respect to the core's extremes. Nevertheless, we talk here about conditional fuzzy copulas instead of semi-copulas.

[15] Here the transformation of margins is induced by a non-additive measure, implying that the corresponding dependency structure is a semi-copula.

[16] This condition holds true in case of marginal Esscher transform iff the Choquet integrals are functions of a strictly increasing function γ as proved before.

Proposition 8. *Let $(\mathbb{R}_+^n, \mathcal{B}(\mathbb{R}_+^n), \mu)$ be a nonadditive measure space, where $\mathcal{B}(\mathbb{R}_+^n)$ is the Borel set generated by the space $\mathbb{R}_+^n$ and $\gamma : [0,1]^n \to [0,1]^n$ be a strictly increasing measurable function. Then there exists a unique lower fuzzy Esscher copula S^L induced by a fuzzy Esscher measure transform of a system $\mathcal{M}$ and*

$$S^L(\tilde{u}_1, \dots, \tilde{u}_n) = C^{\mathbb{P}}(\tilde{u}_1 \in S_{C_1}, \dots, \tilde{u}_n \in S_{C_n}),$$

where $\tilde{u}_i = \gamma_i(\hat{u}_i)$ and $S_{C_i} = \{(C)\int_\omega \gamma_i(\hat{u}_i)d\xi_i, \xi_i \in \mathcal{C}_i\}, \forall i$. In this case the system $\mathcal{M}$ is m-invariant.

Proof. The argument follows easily by Theorem 6 and Proposition 7. □

Having in mind the thesis of Proposition 8, we can define the fuzzification process based on the market data, i.e. the bounds of the core, which are compatible with the m-invariance condition or with a weaker condition, as we will see in the following section.

3.1. m-Invariance and the absence of arbitrage opportunity assumption

The condition of m-invariance discussed above coincides with the closure of a class under change of measure transforms. This closure is frequently imposed in order to simplify the technical tractability of a model and has important implications on the assumption concerning the absence of arbitrage opportunity.

The fundamental theorems of asset pricing deal with the implications of the absence of arbitrage opportunity assumption on the risk-neutral pricing technique. More precisely: it can be proved that the existence of an equivalent martingale measure for the discounted prices, called *risk-neutral measure*, is essentially equivalent to the absence of arbitrage opportunities and that its uniqueness is assured in complete markets. In [38] a statistical test is proposed for the absence of arbitrages based on the concept of m-invariance. As a matter of fact, if the m-invariance is not verified by data, the historical and the risk-neutral distributions are not coherent in terms of the risk they are representing, implying that arbitrages are realizable by portfolios of derivatives and basic stocks bearing independent risks. Hence we can state that if the model verifies the m-invariance but is refused by the statistical test, then the no-arbitrage assumption should be refused too, or alternatively the model's assumptions, supporting the m-invariance, should be rejected. Therefore, finally, we can say that the m-invariance is necessary and sufficient for the stochastic dominance (at the first order) of the historical distribution towards the risk-neutral one and consequently for the absence of arbitrage opportunity assumption. This important connection between the m-invariance and the absence of arbitrages can be proved under the assumption of the market's perfection, i.e. ruling out any kind of market's imperfections caused for example by some friction as illiquidity, constraints on short sales, transaction costs or taxes.

Alternatively, we may imagine a connection between a weaker concept of m-invariance, seemingly more coherent with the empirical evidence, and an *almost everywhere* condition of absence of arbitrage opportunity. We observe that in the Esscher change of measure, the transformations of marginals implied are deterministic and always strictly increasing so that the m-invariance is warranted. Nevertheless, if we generalize the set of transforms allowing for a fuzzy Esscher change of measure they become a function of a capacity. In this case, the strictly increasing monotonicity is no longer directly implied by the analytical structure of the Esscher transform and we may be driven to search for a quasi-invariance property of copula function in order to relax the monotone condition on $\gamma_i, \forall i$. This more general setting can be useful to represent the information caught by real data.

For example if the transformation induced by an Esscher transform is not mean-monotone, we may determine some point-wise distance from the copula implied by the nearest monotone transformation. We define the mean discrepancy to the strongest copula as

$$D^M_{S_{i,t}} = \int_0^1 |C_{S^{\mathbb{P}}_{i,t}, S^{\mathbb{Q}}_{i,t}}(u^{\mathbb{P}}_{i,t}, u^{\mathbb{Q}}_{i,t}) - u^{\mathbb{P}}_{i,t} \wedge u^{\mathbb{Q}}_{i,t}| dF_i,$$

where $u^{\mathbb{P}}_{i,t} = F^{\mathbb{P}}_{S_{i,t}}(s_i)$ and $u^{\mathbb{Q}}_{i,t} = F^{\mathbb{Q}}_{S_{i,t}}(s_i), \forall i = 1, \dots, n$ and F_i stands for the empirical cdf of the ith discrepancy. Based on the mean discrepancy we would define a criterion to establish if dependencies, not explained by the strongest copula, are negligible or not. Given a small ε, if $D^M_{S_{i,t}} \leq \varepsilon, \forall i = 1, \dots, n$, then we confirm a quasi-invariance or ε-invariance of Esscher copula holds true.

Definition 9 *(ε-minvariance).* An Esscher copula is ε-measure invariant, ε-minvariant for short, if $D^M_{S_{i,t}} \leq \varepsilon, \forall i = 1, \dots, n$ and for a given small $\varepsilon \in \mathbb{R}$. We note the ε-minvariance operator as $\stackrel{\varepsilon}{=}$ and then we have $C^{\mathbb{Q}}_{\mathcal{I}_t}(u^{\mathbb{Q}}_{1,t}, \dots, u^{\mathbb{Q}}_{n,t}) \stackrel{\varepsilon}{=} C^{\mathbb{P}}_{\mathcal{I}_t}(u^{\mathbb{P}}_{1,t}, \dots, u^{\mathbb{P}}_{n,t})$.

A weaker condition of invariance could be compatible with a different shape of $\gamma_i, \forall i$ and able to model special empirical features of the market.

We assume for example to consider the ith marginal and to empirically recover the real-world and the risk-neutral cdfs; hence we evaluate their dependence structure through a parametric or an empirical method. Therefore we estimate the copula, i.e. $\tilde{C}(\tilde{F}^{\mathbb{P}}_S, \tilde{F}^{\mathbb{Q}}_S)$ and the empirical discrepancy, i.e.

$$\tilde{D}_{S,i} = |C_{S^{\mathbb{P}},S^{\mathbb{Q}}}(u^{\mathbb{P}}_i, u^{\mathbb{Q}}_i) - u^{\mathbb{P}}_i \wedge u^{\mathbb{Q}}_i|,$$

where $u^{\mathbb{P}}_i = \tilde{F}^{\mathbb{P}}_S(s_i)$ and $u^{\mathbb{Q}}_i = \tilde{F}^{\mathbb{Q}}_S(s_i), \forall i = 1, ..., n$. Moreover we recover the empirical mean discrepancy, i.e. $\tilde{D}^M_{S,i} = \int_0^1 \tilde{D}_{S,i} dF_i$.

The same idea applies with a multidimensional extension of the discrepancy which refers to the comparison between a copula representing the historical dependence and the Esscher copula.

The statistical test we propose provides a mechanism to determine whether there is enough evidence to reject a conjecture, called null hypothesis, stating the ε-minvariance of the copula, when we pass from the real-world measure to the risk-neutral one. If from a theoretical point of view, we know that they must be invariant, but empirically this hypothesis is refused, this then means that the data coming from the market are not coherent with the theoretical model based on the absence of arbitrage opportunity assumption and its implication in term of fundamental theorem of asset pricing. Hence the absence of arbitrage opportunity assumption cannot be accepted.

In order to set up a statistical hypothesis test, we need to specify the tolerance attributable to the concept of m-invariance. More precisely, since we talk about ε-minvariance, the test statistic, i.e. the mean discrepancy, will be specified by a chosen theoretical distribution and a set of parameters' values. Moreover, since the test-statistic D^M_S is always positive by definition, we put forward a one-sided test. Let us consider the null hypothesis

$$H_0 : \mathbb{P} \text{ and } \mathbb{Q} \text{ are } \varepsilon\text{-minvariant},$$

based on the test-statistic $D^M_S \sim \phi(\mathbf{d})$, where ϕ identifies a particular distribution function while $\mathbf{d}$ stands for a vector of parameters. Given a significance level α and let $\varepsilon_\alpha = \Phi^{-1}(\alpha)$, where $\Phi(.)$ stands for the theoretical cdf of the statistic, H_0 is not rejected iff $\varepsilon_\alpha \geq \tilde{D}^M_S$, where $\tilde{D}^M_S$ is the empirical mean discrepancy of the experiment. Therefore the acceptance region of H_0 is identified by all $D^M_S \in [0, \varepsilon_\alpha]$. In this case, we are justified in believing the absence of arbitrage opportunities assumption is true at the assumed confidence level.

If empirically we have support for a weaker concept of m-invariance, we are justified to preserve the incompleteness of the market with a fuzzy Esscher transform based on the non-strictly increasing transform of the standard Esscher change of measure.

4. Empirical experiment

Motivated by a pricing problem involving the market $\mathcal{M}$ characterized by a multidimensional set of n exponential Lévy random variables $\mathbf{S}$, we define a nonadditive measure space $(\Omega, \mathcal{F}, \mu)$ enabling us to perform a fuzzy Esscher risk neutral change of measure. The risk-neutral dependence structure is represented by a set of fuzzy copula functions $\tilde{S}$ because the market is ambiguous due to unobservable or unreliable estimates of the model's parameters. We can imagine dealing with a pricing problem under enlarged ambiguous filtration corresponding to the information related to unobservable variables.[17] To specify the fuzzification, we assume that μ is a fuzzy measure in the class of Sugeno g_λ-measure; this kind of measure, build starting from the objective measure of the market and a parameter $\lambda > 0$, is applied to financial markets in [10] and [39].

More formally if $F(.)$ is a probability distribution, a g_λ-measure is defined as $g_\lambda^F(]a,b]) = \frac{F(b)-F(a)}{1+\lambda F(a)}$ and then we have a one-to-one correspondence between distribution functions and the class of g_λ^F-measures; parameter λ will establish the degree of subadditivity of the measure itself. On the other hand, by the duality that holds between subadditive and superadditive measures, if we define $\lambda^* = -\frac{\lambda}{1+\lambda}$ then the $g_{\lambda^*}^F$-measure is the dual superadditive measure of g_λ^F.

We know that every pricing problem requires recovering the pricing kernel which will be fuzzy in our setting. If the market is incomplete, as in our setting, we'll have infinitely many prices as long as infinitely many martingale measures (whose complexity is given by the set of fuzzy Esscher transforms[18]). An ambiguous model is characterized by a greater level of risk concerning the subjective feelings and beliefs on the model assumptions. If, for example, we introduce some level of uncertainty on the parametrization of the univariate Esscher marginals involved in the contingent claim valuation, a suitable modelization would be in the same spirit as a fuzzified version of the bottom-up approach, whose representation is based on the conditional Choquet-fuzzy copula, we called *fuzzy Esscher copula*. Here the ambiguity is introduced into the model through a fuzzification of risk-neutral marginals whose corresponding pricing kernel will be called *conditional fuzzy Esscher kernel*. Obviously, given the incompleteness of such a market, the pricing kernel cannot be unique. Instead, we have a set of kernels $\mathcal{S}_\lambda$, for a set of measurable functions $\gamma_i, \forall i$ and the invariance property of copulas for conditional fuzzification, given by the following

$$\mathcal{S}_\lambda = \left\{ \tilde{S}\left((C)\int \gamma_1(\hat{u}_1)d\xi_1, \ldots, (C)\int \gamma_n(\hat{u}_n)d\xi_n \right), \xi_i \in \mathcal{C}_i, \forall i; S^L \le \tilde{S} \le S^U \right\},$$

where $\hat{u}_i = \mathbb{Q}(\Xi_i)$, for a general exercise set Ξ_i[19] and $\gamma_i(\hat{u}_i)$ is a measurable function of the Esscher risk neutral transforms $\hat{u}_i, \forall i$, $\mathcal{C}_i = \{\xi_i; g_\lambda^{\hat{u}_i} \le \xi_i \le g_{\lambda^*}^{\hat{u}_i}\}$ and where $S^L = \tilde{S}\left((C)\int \gamma_1(\hat{u}_1)dg_\lambda^{\hat{u}_1}, \ldots, (C)\int \gamma_n(\hat{u}_n)dg_\lambda^{\hat{u}_n}\right)$ and $S^U = \tilde{S}\left((C)\int \gamma_1(\hat{u}_1)dg_{\lambda^*}^{\hat{u}_1}, \ldots, (C)\int \gamma_n(\hat{u}_n)dg_{\lambda^*}^{\hat{u}_n}\right)$ are the lower and upper Esscher bounds, respectively.

The sufficient condition for the m-invariance is specified in Proposition 8 and corresponds to the strictly increasing monotonicity of $\gamma_i, \forall i$ in the lower Choquet bound. This way we can imagine placing ourselves in this setting and to calibrate the Sugeno parameter λ coherently with both the market data and the m-invariance condition. The steps to be followed are listed below:

1. given the market data and contingent market prices, we can uniquely calibrate the Esscher change of measure, i.e. we can recover $\hat{u}_i, \forall i$. We can have two different cases: i) $\hat{u}_i, \forall i$ lies above the corresponding objective cdfs, ii) $\hat{u}_i, \forall i$ goes below the corresponding objective cdfs, in particular domain's zones. The latter case identifies scenarios where the Esscher transform does not act as an increasing transformation;
2. we admit some ambiguity and make $\hat{u}_i, \forall i$ fuzzy, and in particular we recover the lower and upper bound of the fuzzy transforms, i.e. $\underline{u}_i, \overline{u}_i, \forall i$. More precisely we identify the marginal Esscher transforms as signals, whose envelopes permit us to calibrate the system's fuzziness and coherently recover the Sugeno parameters. In case ii) we can imagine performing pricing conditional to unobservable filtration, but under the assumption of invariance after Esscher transformation (which turns out to overlap a weaker version of the absence of arbitrage opportunities condition). The recovered price under relaxed conditions (about the absence of arbitrage) is evaluated under the upper Esscher bound, in the worst case;
3. we check for the condition of m-invariance, i.e. $C(u_i, \underline{u}_i) = u_i \wedge \underline{u}_i, \forall i$, where C stands for their dependence structure. The condition of m-invariance means that the Esscher marginals are perfectly positively correlated, i.e.

[17] The ambiguous parameters can be represented by any factor characterizing the underlying dynamic, i.e. the drift and the volatility as well, or the market where it is traded as the risk-free rate.

[18] We can imagine having an Esscher transform which depends on an uncertain parameter $a \in [\underline{a}, \overline{a}]$. This induces a fuzzy definition of the risk-neutral cdfs $\hat{u}_i \in [\underline{\hat{u}}_i, \overline{\hat{u}}_i], \forall i$, corresponding to a range $[\lambda, \lambda^*]$ of Sugeno parameters.

[19] If we consider an European put option, $\Xi_i = \{S_i \le K_i\}$, where K_i is the strike price.

Table 1
Descriptive statistics of CECE EURO and MSCI EAFE INDEX in two different periods.

	Mean	Min	Max	St. Dev.	Skewness	Kurtosis	JB test p-value
CECE EU							
period 1	1392.93	1095.50	1618.54	121.79	-0.75	2.55	1
period 2	1269.81	1128.43	1422.70	58.75	-0.05	3.53	0
MSCI EAFE							
period 1	2121.03	1780.11	2341.45	153.75	-0.65	2.12	1
period 2	1886.90	1784.91	2040.10	60.04	0.54	2.74	0

$\underline{u}_i = \phi_i(u_i), \forall i$ and strictly increasing ϕ_i. On the other hand, a weaker version of m-invariance can be considered if the mean discrepancies of the historical distributions and at least the upper Esscher transform $\overline{u}_i = \phi_i(u_i), \forall i$ allows us not to reject the ϵ-minvariance.

4.1. Fuzziness calibration

We propose an empirical experiment applied to two different periods to highlight how the proposed tool works in different scenarios. We use daily, publicly available data on the $MSCI$ index traded in the EAFE market and $CECE$ EURO index, traded in the EUREX market and on the corresponding options maturing in January 2021 and traded on December 10, 2020 (first period) and maturing in December 2023 and traded on September 6, 2022 (second period).[20] We use Thomson Financial Datastream data and we choose around 3 months of financial end-of-day quotes, from 10/09/2020 to 10/12/2020 (first period) and from 06/06/2022 to 06/09/2022 (second period) to recover the real-world cdf of both $MSCI$ and $CECE$ EURO index. The empirical real-world cdfs are computed through the Kaplan-Meier procedure.

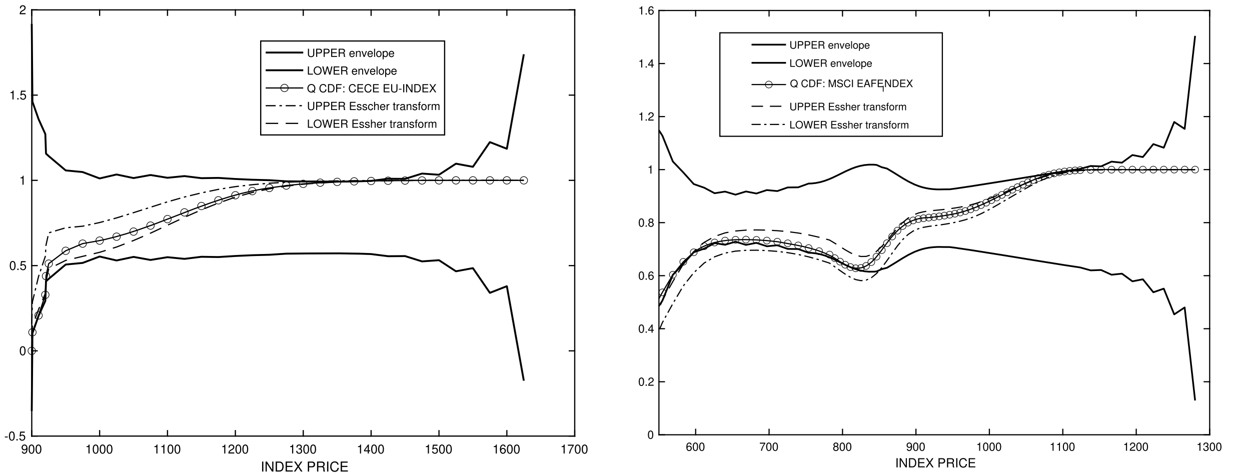
Table 1 contains the main descriptive statistics of the two indices considered for two analysis periods. Daily prices are decreasing and less volatile in the second period for both the time series considered; the MSCI EAFE index is always more volatile than the CECE EU index. As for the daily prices distribution, the statistics show skewness and kurtosis measures suggesting a left-skewed and leptokurtic distribution of the CECE EU index in the second period: the skewness value is always negative but decreasing from the first period and the kurtosis is greater than 3 only in the second period. On the other hand, the MSCI EAFE index changes from left to right-skewed in the second period and it is never leptokurtic. The p-values obtained from the Jarque Bera test indicate that the time series are normally distributed in the first periods but not in the second one.

On the other hand, end-of-day call prices traded on December 10, 2020, and September 6, 2022, are available for different strike prices[21] allowing us to recover the implied risk-neutral cdfs of their underlying via the Breeden and Litzenberger [7] approach. To implement this procedure we develop a preparation phase based on cubic interpolation to have a constant and equal to 25 bp (basis points) spacing between two strike prices.[22] Moreover, we identify the risk-neutral rate with Euribor 1 month recorded on December 10, 2020, and September 6, 2022, respectively. The risk-neutral cdfs are assumed to be signals yielding information about the multitude of Esscher transforms. Therefore we identify envelopes of real-valued signals and calibrate the fuzziness of the system as a function of the mode of the distribution of the distance between the upper and lower envelopes of signals. Eventually, we translate this information in a non-additive measure space with a fuzzy measure in the class of Sugeno g_λ-measures, whose parameter λ is exactly set at the level of the calibrated fuzziness. Given the sub-additivity level, by duality, we recover the dual

[20] The volume of options counts around 500 products in the first period while it is reduced of 20% in the second one.

[21] The end-of-day call prices traded on December 10, 2020, are available for a totally 112 different strike prices while on September 6, 2022, they are reduced to 70.

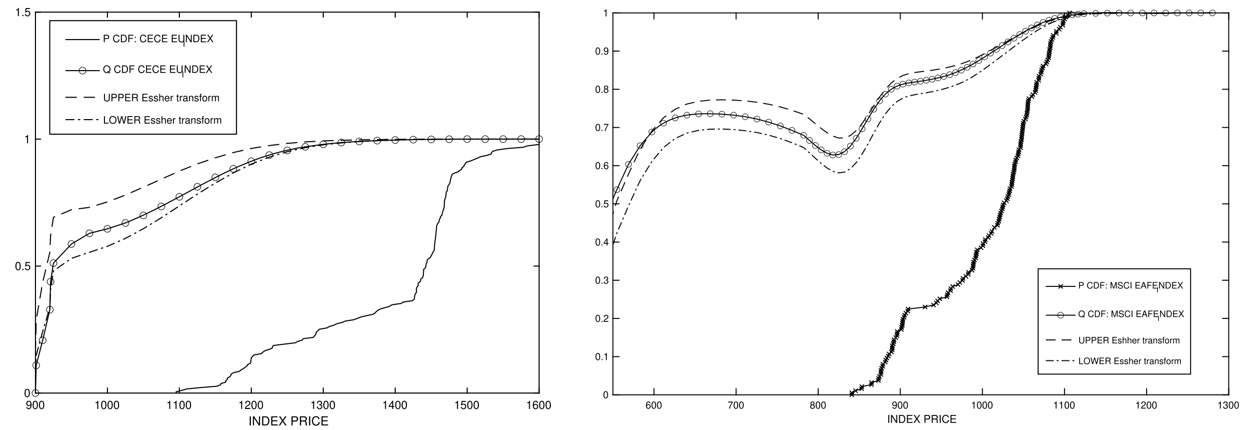
[22] This procedure to generate a call price surface (option price, strike, underlying price), returns a vector of interpolated option prices corresponding to the query points of equally spaced strike prices. The interpolated values are determined by cubic spline interpolation of strikes and option prices. In line with Fengler [15], we implement a kind of ex-post smoothing by imposing the price function to be convex and non-increasing in strikes, conditional on the knot points; as a matter of fact, the idea is not to modify market data in this completion phase, but to enlarge the data set without altering the feature of the market by including arbitrage opportunities.



(a) CECE EURO INDEX: Esscher bounds for $\lambda = 0.4517, \lambda^* = -0.3103$.

(b) MSCI EAFE INDEX: Esscher bounds for $\lambda = 0.2168, \lambda^* = -0.1781$.

Fig. 2. CECE EURO and MSCI EAFE INDEX: Risk-neutral cdf and signal envelope's bounds in the first period.



(a) CECE EURO INDEX: Esscher bounds for $\lambda = 0.4517, \lambda^* = -0.3103$.

(b) MSCI EAFE INDEX: Esscher bounds for $\lambda = 0.2168, \lambda^* = -0.1781$.

Fig. 3. CECE EURO and MSCI EAFE INDEX: Risk-neutral and real-world cdfs in the first period.

super-additive measure g_{λ^*}, where $\lambda^* = -\frac{\lambda}{1+\lambda}$. The computation of the lower and upper Esscher bounds is then implied by the fuzziness caught from the data.

4.1.1. First period: analysis at December 10, 2020

We start the fuzziness calibration, considering the first period. In Figs. 2(a) and 2(b), we report the evidence of the signals' bounds and the corresponding Esscher transforms' bounds in the first period considered.

In Figs. 3(a) and 3(b) both the real-world and the risk-neutral cdfs always concerning the first period are represented. For sake of comparison, the strictly increasing monotonicity of the lower Esscher transform should be assured.

We point out that a probable cause of unusual shapes of the empirical cdfs may be the lack of data, especially for some intervals of strike prices, forcing us to heavily resort to interpolation procedures to complete the data set. Still, increasing monotonicity in the MSCI EAFE index must be resorted to by approximating the Esscher transforms' bounds for the smallest concave upper bound and the upper convex lower bound of the Esscher's bounds set. The approximation procedure is articulated by identifying the endpoints, ϕ, ψ which solve the following optimal problem:

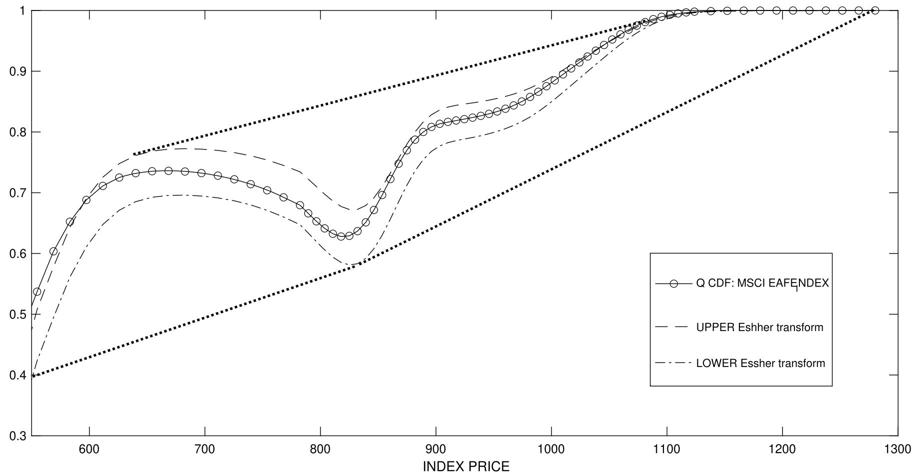
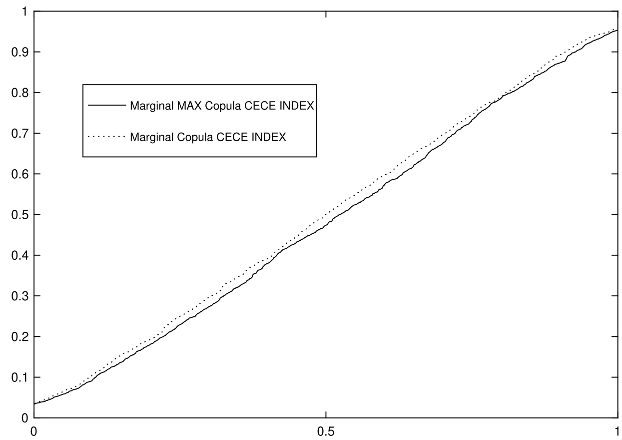
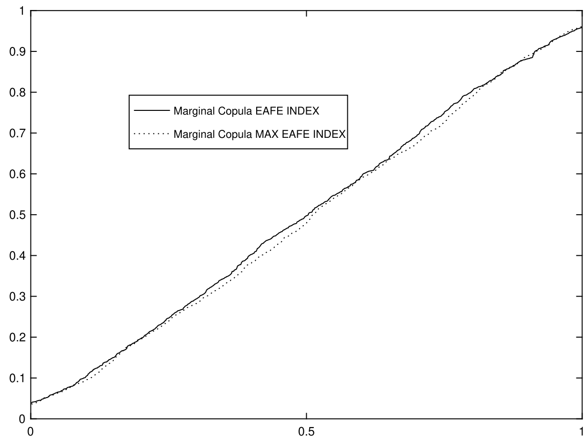


Fig. 4. MSCI EAFE INDEX: Risk-neutral cdfs, Esscher bounds and approximated Esscher corridor (dotted line).



(a) CECE EURO INDEX: empirical mean discrepancy.



(b) MSCI EAFE INDEX: empirical mean discrepancy.

Fig. 5. CECE EURO and MSCI EAFE INDEX: Kendal functions of the empirical mean discrepancy in the first period.

$$\sup_{\phi<\theta<\psi}\left\{f(\phi)\frac{\psi-\theta}{\psi-\phi}+f(\psi)\frac{\theta-\phi}{\psi-\phi}\right\},$$

and which allow to recover the *smallest concave majorant* of the upper Esscher transform $f(\theta)$. Similarly the *upper concave minorant* of the lower Esscher transform $f^*(\theta)$ is determined by the endpoints ϕ^*, ψ^* which solve the following optimal problem:

$$\inf_{\phi^*<\theta<\psi^*}\left\{f^*(\phi^*)\frac{\psi^*-\theta}{\psi^*-\phi^*}+f^*(\psi^*)\frac{\theta-\phi^*}{\psi^*-\phi^*}\right\}.$$

In Fig. 4 the approximated corridor of Esscher transforms is reported. It is worth mentioning that a first check about the marginal strict monotonous transformation acted by the change of measure is in order. As it is clear from Fig. 3, the Esscher transform of the CECE EURO index is strictly increasing hence its lower bound could be compared with the $\mathbb{P}$-marginal distribution. On the other hand, the MSCI EAFE index needs to rely on the approximated Esscher corridor to assure its increasing monotonicity; therefore we compare its approximated lower bound with the $\mathbb{P}$-marginal distribution. The evidences of the empirical mean discrepancies, concerning both the marginal distributions and the bivariate copula representing the dependence of the two indexes, are reported in Figs. 5 and 6, respectively. We remark how the order of the Kendal functions ensures the max copula (which stands for the case of perfect positive

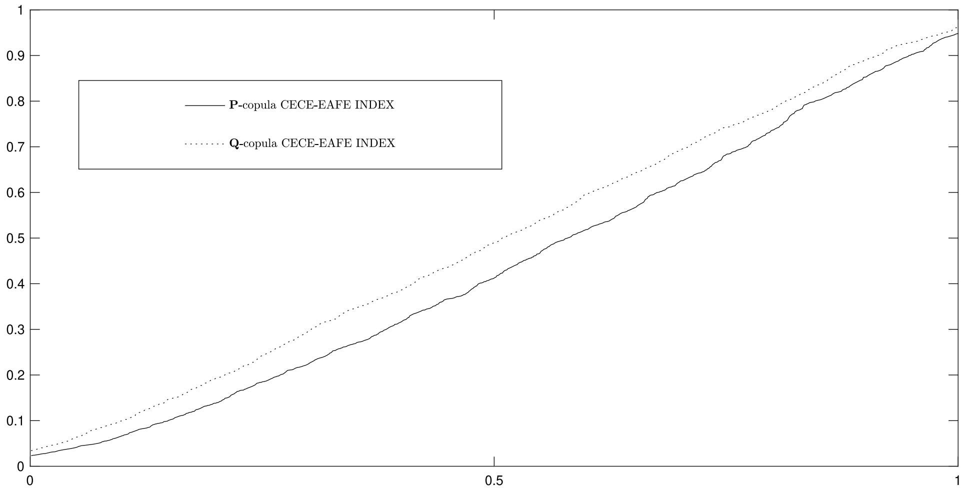


Fig. 6. Bivariate copula of CECE EURO and MSCI EAFE INDEX: Kendal functions of the empirical mean discrepancy in the first period.

Table 2
Marginal ϵ-measure invariance test results: CECE EURO and MSCI EAFE INDEX. Evidence on the invariance for the bivariate empirical copula in the first period.

	τ	ρ	D^M	α
CECE EURO INDEX	0.8961	0.9867	0.0100	0.5000
MSCI EAFE INDEX	0.9363	0.9950	0.0166	0.5100
Bivariate Copula	0.6600	0.8607	0.0540	0.5200

correlation between the distribution under the historical and the risk-neutral measure) be dominated by the empirical one, implying that the $\mathbb{P}$-distribution is dominated by the $\mathbb{Q}$-one (as clearly shown in Fig. 3).

The results of the test for marginal ϵ-minvariance for the first period are reported in Table 2.

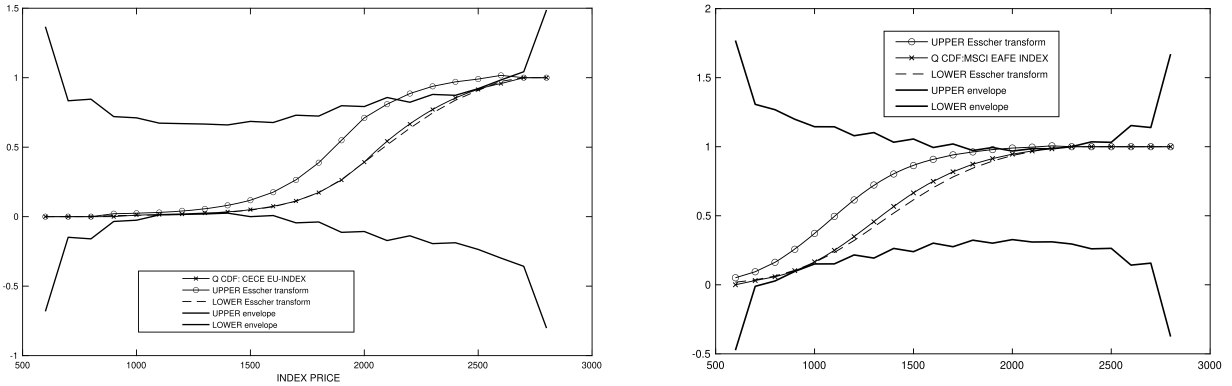
As we can see the ϵ-minvariance is not rejected for values of ϵ greater than or equal to the empirical mean discrepancy level. Therefore both the hypotheses are accepted with a confidence that is at least equal to 99.49%. The last row of Table 2 shows the measure invariance results for the bivariate empirical copula of the CECE EURO and the MSCI EAFE index; despite the greater empirical mean discrepancy, the significance level remains high and the invariance hypothesis results to be accepted with the confidence of 99.48%.

4.1.2. Second period: analysis at September 6, 2022

Now we pass to calibrate the fuzziness of the second period. In Fig. 7 we report the evidence of the signals' bounds and the corresponding Esscher transforms' bounds in the second period considered. We observe that the calibrated fuzziness is greater than in the first period, probably due to the turmoil characterizing the current financial market.

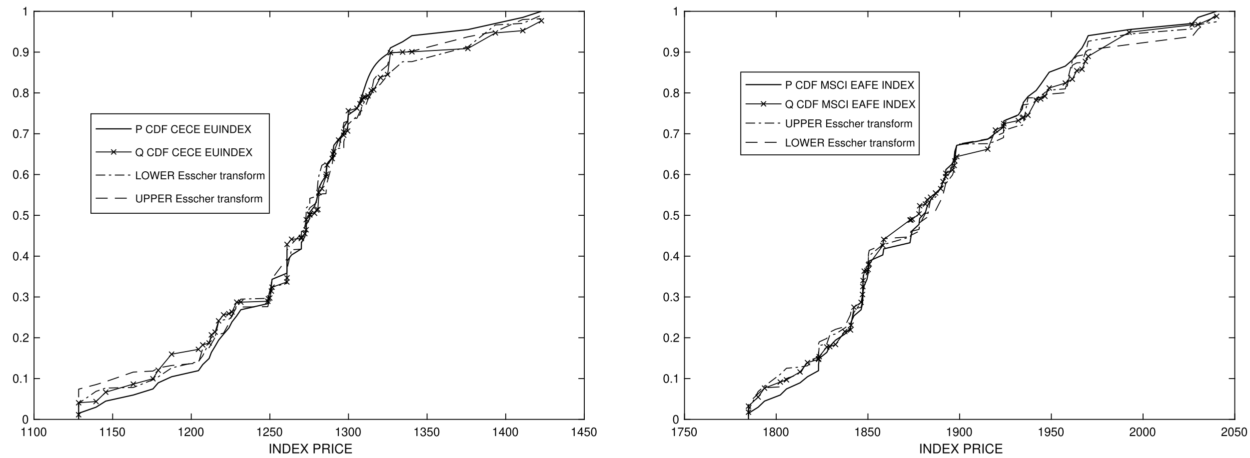
In Figs. 8(a) and 8(b) both the real-world and the risk-neutral cdfs always concerning the second period are represented. We notice that in this situation we are in presence of strictly increasing functions, but unfortunately, the risk-neutral cdfs and the Esscher transforms' bounds stand below the real-world function in some regions, suggesting the empirical rejection of the absence of arbitrage assumption in both the markets (see [38]).

It is worth mentioning that a first check about the marginal strict monotonous transformation acted by the change of measure is in order. As it is clear from Fig. 8, Esscher transforms of the CECE EURO and the MSCI EAFE index are strictly increasing but not dominating the $\mathbb{P}$-distribution; this implies that the upper bound must be compared with the $\mathbb{P}$-marginal distribution to check for a weaker condition on the absence of arbitrage opportunity assumption. The evidences of the empirical mean discrepancies (where the upper Esscher bound is taken into account), concerning both the marginal distributions and the bivariate copula representing the dependence of the two indexes, are reported in Figs. 9 and 10, respectively. We observe that the order of the Kendal functions does not guarantee the maximum copula (which stands for the case of perfect positive correlation between the distribution under the historical and the



(a) CECE EURO INDEX: Esscher bounds for $\lambda = 0.6346, \lambda^* = -0.3882$. (b) MSCI EAFE INDEX: Esscher bounds for $\lambda = 0.6517, \lambda^* = -0.3946$.

Fig. 7. CECE EURO and MSCI EAFE INDEX: Risk-neutral cdf and signal envelope's bounds in the second period.



(a) CECE EURO INDEX: Esscher bounds for $\lambda = 0.6346, \lambda^* = -0.3882$. (b) MSCI EAFE INDEX: Esscher bounds for $\lambda = 0.6517, \lambda^* = -0.3946$.

Fig. 8. CECE EURO and MSCI EAFE INDEX: Risk-neutral and real-world cdfs in the second period.

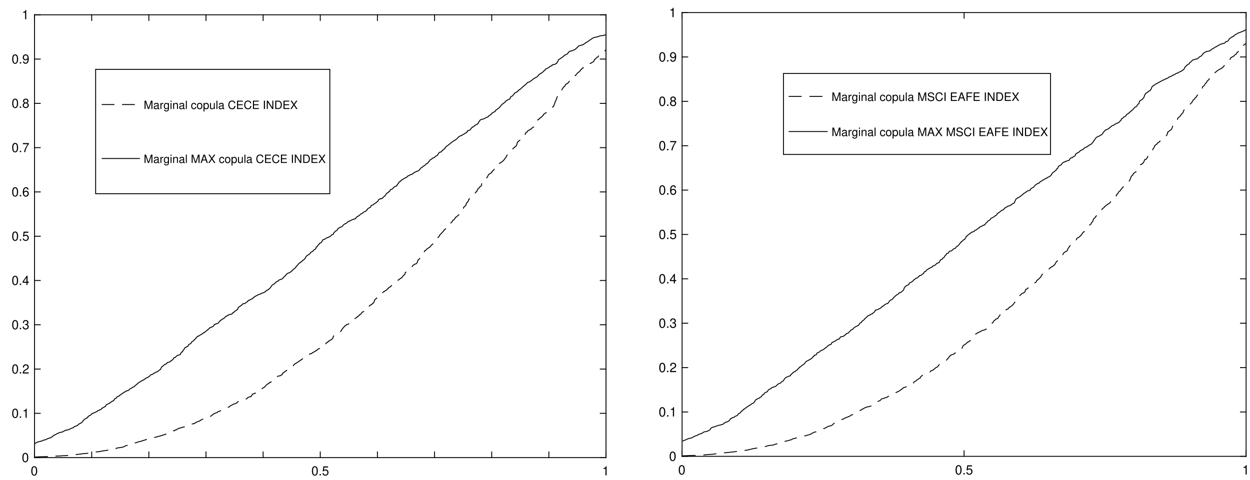


Fig. 9. CECE EURO and MSCI EAFE INDEX: Kendal functions of the empirical mean discrepancy in the second period.

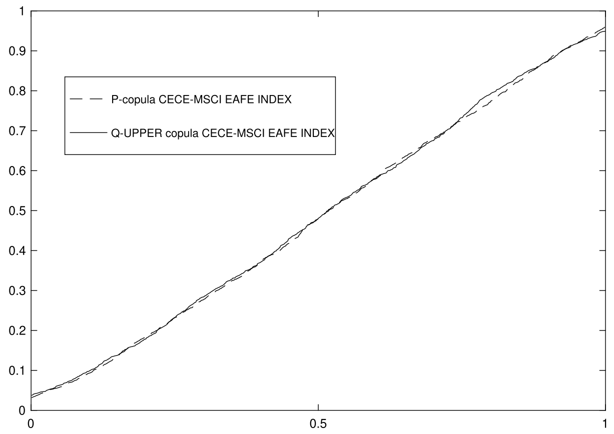


Fig. 10. Bivariate copula of CECE EURO and MSCI EAFE INDEX: Kendal functions of the empirical mean discrepancy in the second period.

Table 3
Marginal ϵ-measure invariance test results: CECE EURO and MSCI EAFE INDEX. Evidence on the invariance for the bivariate empirical copula in the second period.

	τ	ρ	D^M	α
CECE EURO INDEX	0.8721	0.9799	0.1474	0.5586
MSCI EAFE INDEX	0.9559	0.9976	0.1595	0.5634
Bivariate Copula	0.9131	0.9907	0.0054	0.5022

risk-neutral measure) is dominated by the empirical one, implying that actually the $\mathbb{P}$-distribution is not dominated by the $\mathbb{Q}$-one (as clearly shown in Fig. 8). On the other hand, the Kendal functions in Fig. 10 are mainly indistinguishable thanks to compensator effects in the marginals' combination.

The results of the test for marginal ϵ-minvariance (based on the upper Esscher bound and the $\mathbb{Q}$-copula of the upper bounds) for the second period are reported in Table 3.

As we can see the ϵ-minvariance is not rejected for values of ϵ greater than or equal to the empirical mean discrepancy level. Therefore both hypotheses are accepted with confidence which is at least equal to 99.44%, despite the greater empirical mean discrepancy. The last row of Table 3 shows the measure invariance results for the bivariate empirical copula of the upper Esscher transforms of the CECE EURO and the MSCI EAFE index; the empirical mean discrepancy is very low probably due to errors' compensation, hence the significance level is high and the invariance hypothesis results to be accepted with the confidence of 99.50%.

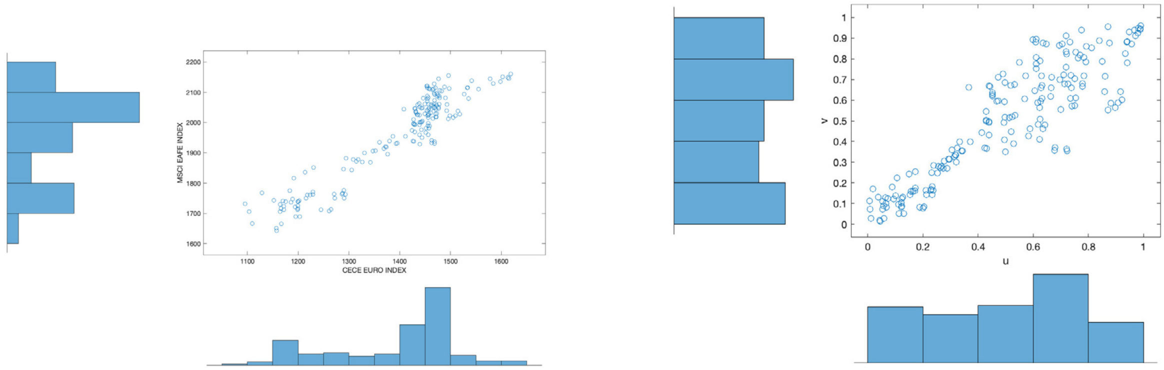
4.2. Bivariate pricing under enlarged ambiguous filtration

We assume to have a contingent claim whose drivers are MSCI EAFE and CECE EURO index. The pricing problem at hand is supposed to be dependent on unobservable variables[23] inducing a certain degree of fuzziness into the system. The Esscher transforms bounds, recovered after calibration of fuzziness from empirical data (both in the first and the second period), represent the first step to evaluate their Fuzzy Esscher Copula. Figs. 11 and 12 depict the scatter plot of the indices prices and the transformed data to copula scale using the kernel estimator of the cumulative distribution function in both the considered periods.

Assuming to work under the hypothesis of marginal ϵ-minvariance for $\epsilon = 0.0166$ and $\epsilon = 0.1595$[24] respectively in the two periods, we connect the indices by an Archimedean copula function and calibrate the parameter on the

[23] A non-Markovian system, where some model parameters are unobservable and unpredictable, is an example of such kind of ambiguous pricing problem.

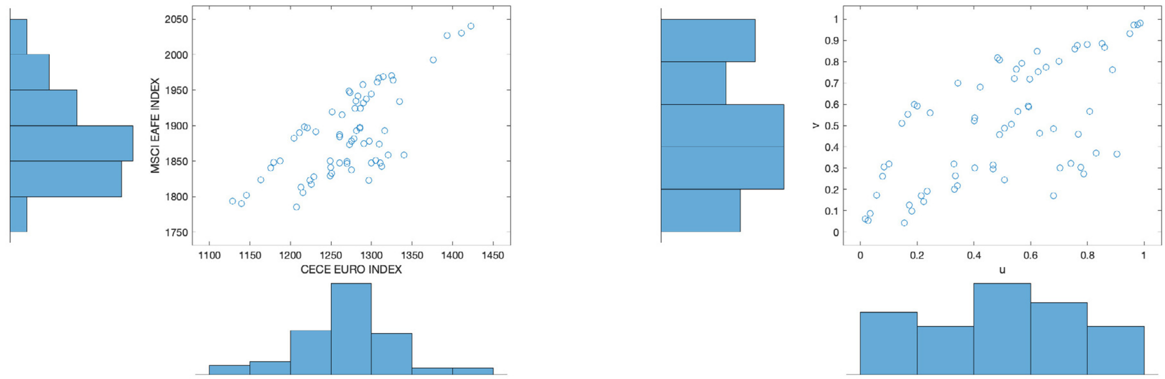
[24] The value of ϵ stands for a measure of the discrepancy from the standard absence of arbitrage assumption; greater ϵ, weaker is the m-invariance considered in defining the pricing measure. The value of ϵ is a big deal higher in the second period where the standard assumption of absence



(a) MSCI EAFE and CECE EURO INDEX: data scatter plot.

(b) MSCI EAFE and CECE EURO INDEX: transformed data scatter plot.

Fig. 11. CECE EURO and MSCI EAFE INDEX: data scatter plots concerning the first period.



(a) MSCI EAFE and CECE EURO INDEX: data scatter plot.

(b) MSCI EAFE and CECE EURO INDEX: transformed data scatter plot.

Fig. 12. CECE EURO and MSCI EAFE INDEX: data scatter plots concerning the second period.

corresponding dataset. In order to analyze the implication of the dependence structure, we assume to have every kind of tail dependence through the analysis of the comprehensive Archimedean family whose main components are Clayton copula, showing lower tail dependence, Gumbel copula, showing upper tail dependence, and Frank copula having no tail dependence. Figs. 13 and 14 show the copula pdf with different Archimedean dependences, i.e. for Clayton, Frank, and Gumbel copulas. The calibrated parameters are reported in the caption; they correspond to Kendall's tau of 0.66 and 0.54, respectively in the first and second periods.

Moreover in the marginal fuzzifying process, we define the cores through the previously recovered Sugeno measures. This way of blurring the model is related to assumptions concerning the marginals and is referred to as the conditional fuzzy approach (see [39]). In Figs. 15 and 16 we appreciate the range of the bivariate joint distribution, i.e. the Esscher transform corridors, for a different choice of Archimedean copulas and different periods. Table 4 reports goodness of fit evaluation results based on RMSE, a widely used measure of GOF focusing on the minimization of residuals, Akaike and Bayesian Information Criteria (AIC and BIC); the evidence shows a preference for Clayton copula (in the first period) and for Gumbel (in the second period), which explain at best the empirical dependence of

of arbitrage is rejected; as a matter of fact, here we resort to a weaker version of the absence of arbitrage (whose weaknesses' degree is given by $\epsilon = 0.1595$) which is not rejected for the upper Esscher bound pricing measure. On the other hand, the first period confirms the not rejection of the standard absence of arbitrage opportunity, assured by the dominance of the $\mathbb{Q}$-distribution and by the very low values of ϵ.

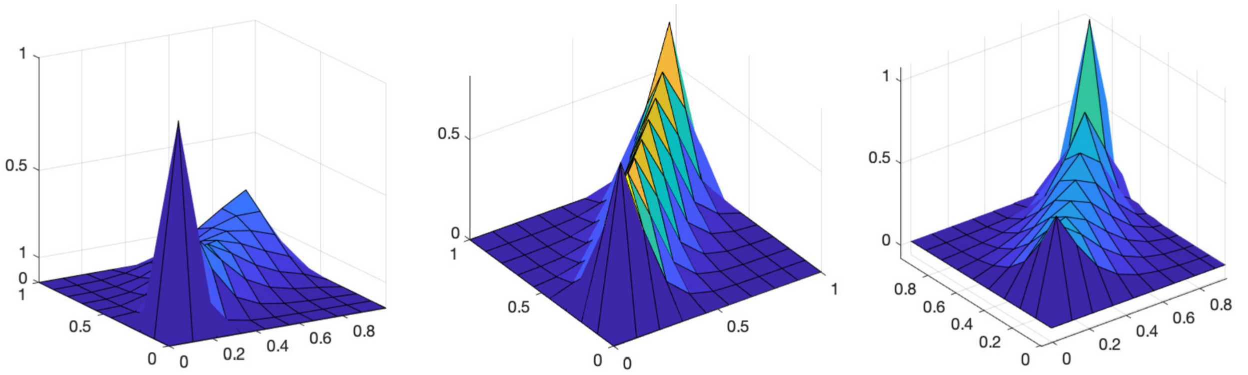


Fig. 13. MSCI EAFE and CECE EURO INDEX: Archimedean copula pdf for Clayton (3.62), Frank (12.58) and Gumbel (3.00). Evidences from the first period.

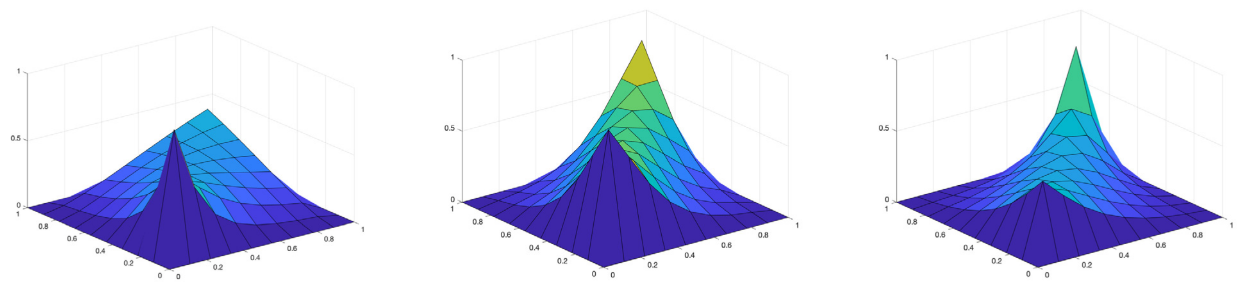


Fig. 14. MSCI EAFE and CECE EURO INDEX: Archimedean copula pdf or Clayton (1.33), Frank (5.62) and Gumbel (2.17) copula. Evidences from the second period.

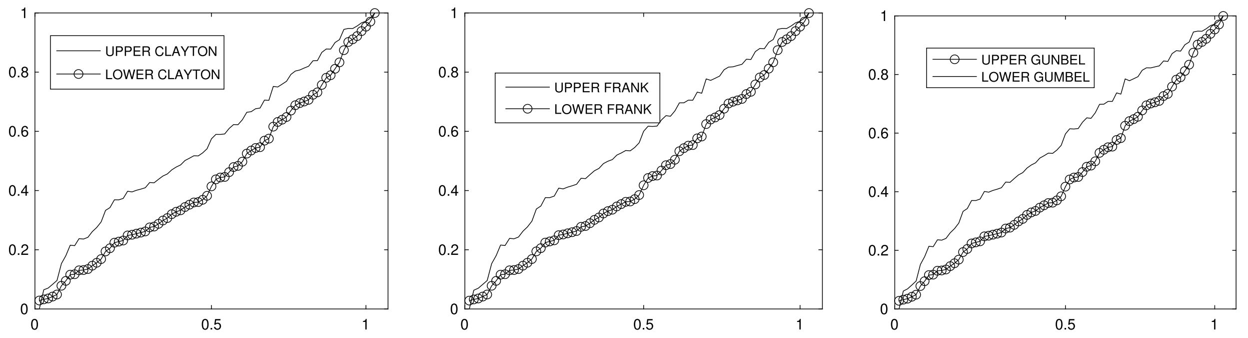


Fig. 15. MSCI EAFE and CECE EURO INDEX: Esscher transform corridors for Clayton (3.62), Frank (12.58) and Gumbel (3.00) copula. Evidence from the first period.

data. Lower and upper tail dependence parameters are also reported in Table 4; the impact of tail dependence (lower in the first period and upper in the second one) seems to prevail in the choice of the best copula family.

Yet Archimedean copulas seem to have in general a good performance (GOF are generally very similar in both periods); we observe that even if the properties of tail dependences of Archimedean copulas impact the differences on the ranges of Esscher corridors, their shapes seem very similar. The second period is characterized by irregular corridors due to a greater calibrated fuzziness in this period; nevertheless, we tested a weaker absence of arbitrage assumption which is not rejected just by the upper Esscher transform appointed as a good pricing measure (as a result of a *quasi* arbitrage pricing). Different choices inside the corridor will have a lower confidence not to be rejected and a greater mean discrepancy.

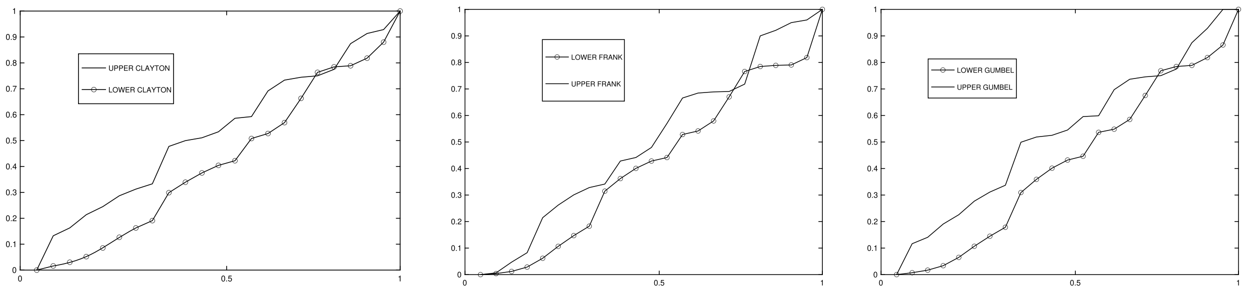


Fig. 16. MSCI EAFE and CECE EURO INDEX: Esscher transform corridors for Clayton (1.33), Frank (5.62) and Gumbel (2.17) copula. Evidences from the second period.

Table 4
Goodness of fit evaluation of the Archimedean dependence structure: CECE EURO and MSCI EAFE INDEX. Tail dependence evidence in the first and second periods.

	FRANK	CLAYTON	GUMBEL
RMSE			
first period	0.0169	0.0149	0.0168
second period	0.0532	0.0537	0.0518
AIC			
first period	5293.1	3641.5	3903.9
second period	1503.7	1944.8	1143.0
BIC			
first period	5295.7	3638.9	3906.5
second period	1505.9	1947.0	1145.2
Tail Dependence			
first period	0	$\lambda^L = 0.8257$	$\lambda^U = 0.7402$
second period	0	$\lambda^L = 0.5938$	$\lambda^U = 0.6237$

5. Conclusion

This paper's goal is to show how fuzzy Esscher transforms can help address invariance issues related to financial pricing and contingent-claims pricing in the context of uncertain and incomplete markets. Having in mind a pricing problem involving the market $\mathcal{M}$ characterized by a multidimensional set of n exponential Lévy random variables $\mathbf{S}$, we investigate the effects of a change of measure on the dependence structure. In the case of standard Esscher transforms we recover the invariance properties proved in [41]; however, they do not seem flexible enough to capture effects empirically recovered as the U-shape feature of the change of measure function discussed by Carr et al. [8]. This failure in consistency with the market can be overcome allowing for not only mean-monotone transformations modeled in terms of fuzzy Esscher transforms, which enables us to generalize the invariance result introducing a weaker concept of invariance. Therefore the consistency with market data justifies the interest in our research and the innovative proposal of the paper.

Through the (fuzzy) change of probability measure approach, we illustrate the advantage of considering Fuzzy Esscher transforms for the generalization of the invariance property to ambiguous and incomplete markets settings, thus introducing a new variant of the invariance property. A fuzzy Esscher transform permits us to formalize the implications of a pricing problem under enlarged ambiguous filtration corresponding to the information related to unobservable variables, giving support to the lack of completeness through the definition of a corridor of Esscher transforms which translates the blurriness of the market itself. Esscher bounds are related to the concept of quasi-invariance, which we call ε-measure invariance, and which is based on the discrepancy to the maximum copula used to measure the distance between distributions. A weaker request concerning the monotony of the change of measure transforms could justify the empirical evidence and motivate the pricing under the assumption of *quasi* absence of arbitrage opportunities based on the concept of ε-measure invariance. The proposed empirical experiments based on market data concerning two different periods allow us to show how to properly calibrate a fuzzy system taking advantage of the flexibility of the fuzzy Esscher transform, and seems to suggest an ε-measure invariance is indeed market-coherent.

Finally, we believe this paper contributes to the modelization of fuzzy tools in financial research whose future directions would be able to make a bridge between the analytical rigor of financial mathematics and the new frontiers, related to markets with no full disclosure of information about products and prices, where such a piece of blurry information causes uncertain and ambiguous feeling among market's agents, where some information is inaccessible, unreliable or misunderstood due to bias in estimation, big data or too small dataset issues, hence able to match a world which is everything but perfect.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Silvia Romagnoli reports article publishing charges were provided by University of Bologna. Silvia Romagnoli reports a relationship with University of Bologna that includes: employment.

Data availability

Data will be made available on request.

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