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# MANY CUSPED HYPERBOLIC 3-MANIFOLDS DO NOT BOUND GEOMETRICALLY

ALEXANDER KOLPAKOV, ALAN W. REID, AND STEFANO RIOLO

**ABSTRACT.** In this note, we show that there exist cusped hyperbolic 3-manifolds that embed geodesically, but cannot bound geometrically. Thus, being a geometric boundary is a non-trivial property for such manifolds. Our result complements the work by Long and Reid on geometric boundaries of compact hyperbolic 4-manifolds, and by Kolpakov, Reid and Slavich on embedding arithmetic hyperbolic manifolds.

*Key words:* 3-manifold, 4-manifold, hyperbolic geometry, cobordism, geometric boundary.

*2010 AMS Classification:* 57R90, 57M50, 20F55, 37F20.

## 1. INTRODUCTION

In the sequel, all hyperbolic manifolds are assumed to be connected, orientable, complete, and of finite volume. We are particularly interested in *cusped*, i.e. non-compact, such manifolds.

A hyperbolic  $n$ -manifold  $M$  *bounds geometrically* if it is isometric to  $\partial W$ , for a hyperbolic  $(n+1)$ -manifold  $W$  with totally geodesic boundary, c.f. [18], and also [14, 16, 19, 20, 22, 28, 29] for further progress in this topic. A hyperbolic  $n$ -manifold  $M$  is said to *embed geodesically* if there exists a hyperbolic  $(n+1)$ -manifold  $N$  that contains a totally geodesic hypersurface isometric to  $M$ . We remark that many arithmetic hyperbolic 3-manifolds of simplest type embed geodesically by [15].

A geometrically bounding manifold embeds geodesically, but the converse is not necessarily true. Indeed, the Euler characteristic  $\chi(M)$  of a geometrically bounding manifold  $M$  must be even. This can be seen by taking a hyperbolic  $(n+1)$ -manifold  $N$  with totally geodesic boundary  $\partial N = M$ , and doubling it along  $M$  in order to obtain a hyperbolic manifold  $DN$ . By the excision property, we have for the Euler characteristic that  $\chi(DN) = 2\chi(N) - \chi(M)$ . If  $n$  is odd, we have that  $\chi(M) = 0$ , while if  $n$  is even, then  $\chi(DN) = 0$ , and  $\chi(M)$  is thus even. The fact that an odd-dimensional cusped hyperbolic manifold has  $\chi = 0$  follows from Margulis' Lemma and the 1<sup>st</sup> Bieberbach Theorem.

Thus, the thrice-punctured sphere cannot bound geometrically. On the other hand, this manifold is arithmetic and of even dimension, so by [15] it embeds geodesically. This same discussion also applies when  $n = 4, 6$ , since the respective minimal-volume arithmetic manifolds constructed in [7, 27] have Euler characteristic  $\chi = \pm 1$ . Note that such an argument becomes vacuous if  $n$  is odd.

The aim of this note is to provide examples of hyperbolic 3-manifolds that embed geodesically, but fail to bound geometrically, thereby explicitly showing that bounding is much more non-trivial to arrange for  $n = 3$  too.

In particular, we show that several well-known cusped hyperbolic 3-manifolds cannot bound geometrically. Namely, we prove the following theorems, the first of which should

be contrasted with [29], which shows that the figure-eight knot complement bounds geometrically.

**Theorem 1.1.** *The figure-eight knot sibling 3-manifold embeds geodesically but does not bound geometrically.*

The figure-eight knot complement and its “sibling” manifold are precisely the cusped hyperbolic 3-manifolds of smallest volume [3]. Both of them are known to be arithmetic [21] with invariant trace-field  $\mathbb{Q}(\sqrt{-3})$ . Our methods also show:

**Theorem 1.2.** *A single-cusped hyperbolic 3-manifold with invariant trace-field of odd degree does not bound geometrically.*

There are many such examples of single-cusped hyperbolic 3-manifolds, indeed even arising as knot complements in  $S^3$ . We record the following corollary of Theorem 1.2. This follows automatically from [11] which establishes that if  $K_m$  is the  $m$ -twist knot (see Figure 1) the degree of the invariant trace-field is given by  $\text{cr}(K_m) - 2$  where  $\text{cr}(K_m)$  is the crossing number of  $K_m$  (here  $m \neq -2, -1, 0, 1$ ). Note that the figure-eight knot is the 2-twist knot according to Figure 1. If we assume that  $m \geq 2$ , then  $\text{cr}(K_m) - 2 = m$ .

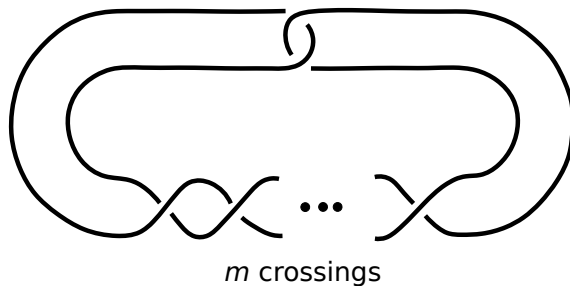


FIGURE 1. The twist knot  $K_m$  ( $m > 0$ ) in its alternating (and thus minimal) projection.

**Corollary 1.3.** *Let  $K_m$  be the  $m$ -twist knot with  $m > 1$  odd. Then  $\mathbb{S}^3 \setminus K_m$  does not bound geometrically.*

Note that the figure-eight knot’s sibling manifold does not satisfy Theorem 1.2, since as noted above its invariant trace-field has degree two. In addition, at present we do not know *a single example* of a finite volume hyperbolic 3-manifold with odd degree invariant trace-field that even embeds geodesically.

The proofs of Theorems 1.1 and 1.2 essentially follow from a simple observation about the cusps of manifolds that bound geometrically, which applies in all dimensions. In the case of  $n = 3$ , this implies that the cusp of a single-cusped hyperbolic 3-manifold that bounds geometrically must be “rectangular” (c.f. Proposition 2.8). We refer to Section 2 for the terminology. In the case of  $n = 4$  this gives another proof that the minimal volume hyperbolic 4-manifolds of [27] do not bound geometrically (see Section 2.4).

*Remark 1.4.* Nimershiem [26] proved that the cusp shapes of single-cusped hyperbolic 3-manifolds form a dense subset in the moduli space of the 2-torus. Since the set of rectangular tori is nowhere dense, this lends credence to the claim that “most single-cusped hyperbolic 3-manifolds do not bound geometrically”. Similar considerations were known to hold in the

compact setting. Indeed, in [18] it is shown that if a closed hyperbolic 3-manifold bounds geometrically, then it has integral  $\eta$ -invariant. On the other hand, Meyerhoff [24] showed that a reduction modulo  $\frac{1}{3}$  of the  $\eta$ -invariant of closed hyperbolic 3-manifolds takes values in a dense subset of the circle.

As a possible measure of geometric “complexity” of embedding geodesically a hyperbolic  $n$ -manifold  $M$  into a hyperbolic  $(n+1)$ -manifold  $N$ , or making  $M$  bound a hyperbolic  $(n+1)$ -manifold  $W$  geometrically, we introduce the following quantities:

$$\varepsilon\text{-vol}(M) = \min_N \text{vol}(N), \quad \partial\text{-vol}(M) = \min_W \text{vol}(W),$$

where “vol”, here and below, means hyperbolic volume. It is easy to see that if  $M$  bounds, then  $\varepsilon\text{-vol}(M) \leq 2 \cdot \partial\text{-vol}(M)$ .

In [29], Slavich proved that the figure-eight knot complement has  $\partial\text{-vol} = \frac{4\pi^2}{3}$ , i.e. the minimum possible by the Gauß-Bonnet theorem. Concerning  $\varepsilon\text{-vol}$ , we adopt his technique to improve Theorem 1.1 as follows:

**Theorem 1.5.** *The figure-eight knot complement and its sibling manifold have  $\varepsilon\text{-vol} = \frac{4\pi^2}{3}$ .*

Note that if  $M$  bounds geometrically, resp. embeds geodesically, then Miyamoto’s work [25] implies that  $\partial\text{-vol}(M) \geq d_{n+1} \text{vol}(M)$ , resp.  $\varepsilon\text{-vol}(M) \geq 2 d_{n+1} \text{vol}(M)$ , where  $d_n$  is the optimal horoball packing density in  $\mathbb{H}^n$ , c.f. [13, Table 3]. The former inequality holds also if  $M$  is disconnected, and the latter follows by cutting any  $N$  in which  $M$  embeds along the respective hyper-surface isometric to  $M$ , and considering the resulting manifold  $N_{//}M$  with boundary  $M \sqcup (-M)$ . The equality, in the case of bounding manifolds, can be attained only in dimensions  $n = 2$  and  $3$ .

The paper is organised as follows: in Section 2 we prove Theorems 1.1 and 1.2, in Section 3 we prove Theorem 1.5.

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## 2. CUSP SECTIONS OF GEOMETRIC BOUNDARIES

In this section, we provide a simple obstruction for a non-compact manifold to bound geometrically, formulated in Propositions 2.5 and 2.8, which we then use in order to prove Theorems 1.1 and 1.2.

**2.1. Cusps with boundary.** In this section, we analyse the ends of a hyperbolic manifold with totally geodesic boundary. We basically follow [10, 2.10.C].

**Definition 2.1.** An  $(n+1)$ -dimensional *cusp with boundary* is a Riemannian warped product  $C = F \times_f (0, +\infty)$ , where  $F$  is a compact connected flat  $n$ -manifold with totally geodesic boundary and  $f(r) = e^{-r}$ .

This means that  $F \times (0, +\infty)$  is endowed with the Riemannian metric  $e^{-2r}g + dr^2$ , where  $g$  is the flat metric on  $F$ .

**Definition 2.2.** A *section* of a cusp  $C$  as above is a level set  $F \times \{r_0\} \subset C$ .

Note that all sections of  $C$  are homothetic.

**Definition 2.3.** The *shape* of a cusp  $C$  is the similarity class of a section.

The following fact is well known:

**Proposition 2.4.** [10, 2.10.D] *Let  $W$  be a complete, finite-volume hyperbolic  $(n+1)$ -manifold with (possibly empty) totally geodesic boundary. There is a compact subset of  $W$  whose complement is isometric to a disjoint union of cusps with (possibly empty) boundary.*

Given  $W$  as above, we call *boundary cusps* the cusps of  $W$  with non-empty boundary. Each cusp of the hyperbolic  $n$ -manifold  $\partial W$  is a boundary component of a boundary cusp of  $W$ .

**2.2. Cusps of bounding manifolds.** We now furnish an obstruction for a hyperbolic manifold to bound geometrically.

**Proposition 2.5.** *If a cusped hyperbolic  $n$ -manifold  $M$  bounds geometrically, then the cusps of  $M$  that do not admit a fixed-point-free orientation-reversing isometric involution are isometric in pairs.*

In particular, the number of such cusps has to be even, possibly zero. The proof will follow easily from a simple lemma about flat manifolds with totally geodesic boundary:

**Lemma 2.6.** *Let  $F$  be a compact connected orientable flat  $n$ -manifold with non-empty totally geodesic boundary, such that  $F$  is not isometric to a product with an interval. Then,  $\partial F$  is connected and has a fixed-point-free orientation-reversing isometric involution.*

*Proof.* The Riemannian universal cover of any constant-curvature manifold with totally geodesic boundary embeds isometrically into the model space as the intersection of some half-spaces with pairwise disjoint boundaries. In the flat case, the number of such half-spaces can only be 0, 1, or 2. The first case is excluded because  $\partial F \neq \emptyset$ , while the second one is excluded because  $F$  is compact.

Thus, the universal cover  $\tilde{F}$  of  $F$  is isometric to a strip  $\mathbb{R}^{n-1} \times I \subset \mathbb{R}^n$ , where  $I = [-a, a]$ , and  $F$  is isometric to a quotient of this strip by a discrete group of Euclidean isometries acting on it. Every isometry of the strip  $\tilde{F}$  must preserve the  $I$ -fibration and the 0-section  $\mathbb{R}^{n-1} \times \{0\}$ . This implies that there is an  $I$ -bundle  $\pi: F \rightarrow B$  whose 0-section  $B_0 \subset F$  is a totally geodesic hypersurface. Note that  $\pi|_{\partial F}$  is a Riemannian double covering.

By hypothesis the bundle  $\pi$  is non-trivial, so  $B_0$  is a one-sided hypersurface inside  $F$ , and  $B$  is non-orientable. Thus,  $\pi|_{\partial F}$  is the orientation double cover of  $B$ , and  $\partial F$  has a fixed-point-free orientation-reversing involution.  $\square$

*Remark 2.7.* It follows from the proof of Lemma 2.6 that if  $n = 3$  and  $\partial F$  is connected, then  $F$  is diffeomorphic to  $K \tilde{\times} I$ , which is the orientable manifold arising as a twisted  $I$ -bundle over the Klein bottle.

We are ready to prove Proposition 2.5.

*Proof of Proposition 2.5.* Let  $M = \partial W$  for a hyperbolic  $(n+1)$ -manifold  $W$  with totally geodesic boundary, and let  $C' \subset M$  be a cusp of  $M$  with section  $F'$ . Then,  $F' \subset \partial F$  is a boundary component of a section  $F$  of a boundary cusp  $C$  of  $W$ . If  $F'$  has no fixed-point-free orientation-reversing involution, by Lemma 2.6 we have  $F \cong F' \times I$ , and  $F'$  is isometric to a section of another cusp of  $M$ .  $\square$

**2.3. Rectangular tori.** In this section, we give a more precise characterisation of cusp shapes of geometrically bounding manifolds in the case  $n = 3$ , and then prove Theorems 1.1 and 1.2.

A cusp of a hyperbolic 3-manifold has section a flat 2-torus. Recall that a flat torus  $T = \mathbb{R}^2/\Gamma$  has a fixed-point-free orientation-reversing isometric involution if and only if a conjugate of the lattice  $\Gamma$  is generated by two vectors that span a rectangle or a rhombus. We call such flat tori respectively *rectangular* or *rhombic*.

In the usual fundamental domain

$$\mathcal{D} = \{|z| \geq 1, |\operatorname{Re}(z)| \leq 1/2, \operatorname{Im}(z) > 0\} \subset \mathbb{C}$$

for the moduli space of tori (c.f. for instance [8, §12.2] and [23, §4.2]), the rectangular and rhombic ones correspond to the curves  $\mathcal{D} \cap \{\operatorname{Re}(z) = 0\}$  and  $\mathcal{D} \cap \{|z| = 1 \text{ or } |\operatorname{Re}(z)| = 1/2\}$ , respectively. Thus, we call a cusp of a hyperbolic 3-manifold *rectangular* or *rhombic* depending on the shape of its section.

With these definitions in hand, we can now improve Proposition 2.5:

**Proposition 2.8.** *If a cusped hyperbolic 3-manifold  $M$  bounds geometrically, then the non-rectangular cusps of  $M$  are isometric in pairs.*

*Proof.* Suppose again  $M = \partial W$ , and let  $C \cong F \times_f (0, +\infty)$  be a boundary cusp of  $W$  with connected boundary. By Remark 2.7,  $F$  is diffeomorphic to  $K \tilde{\times} I$  and  $\partial F$  is a flat torus. We now show that  $\partial F$  is rectangular. To that end, we have

$$F \cong (\mathbb{R}^2 \times [-a, a]) / \Gamma,$$

where we can assume that the group  $\Gamma < \operatorname{Isom}(\mathbb{R}^2 \times [-a, a]) < \operatorname{Isom}(\mathbb{R}^3)$  is generated by a parallel translation  $T_x$  along  $(2b, 0, 0)$ , a translation  $T_y$  along  $(0, c, 0)$ , and a roto-translation  $R_x: (x, y, z) \mapsto (x + b, -y, -z)$  (c.f. for instance [31, Theorem 3.5.5, item 2]). In particular, we have

$$\partial F \cong (\mathbb{R}^2 \times \{a\}) / \langle T_x, T_y \rangle,$$

and the lattice  $\langle T_x, T_y \rangle$  is generated by two vectors spanning a rectangle. □

We can now prove Theorems 1.1 and 1.2.

*Proof of Theorem 1.1:* Let  $\mathbb{H}^3/\Gamma$  be the figure-eight's sibling manifold. Up to conjugation,  $\Gamma$  is an index 12 subgroup in  $PSL_2(\mathcal{O}_3)$ , c.f. [21, §13.7.1(vi)]. Thus  $\Gamma$  can be embedded in  $SO(q, \mathbb{Z})$  for a quadratic form of signature  $(3, 1)$ , c.f. [4, §3] and [6, 12]. The argument given in [15, §9.1] now applies to show that the sibling manifold embeds geodesically.

However, from [30], one sees that the modulus of the cusp is  $\omega = \frac{-1+\sqrt{-3}}{2}$ , and hence is not rectangular. Thus, by Proposition 2.8, the figure-eight knot's sibling manifold does not bound geometrically. □

*Remark 2.9.* The cusp shape of the figure-eight knot sibling can also be computed using SnapPy [5]. By setting `M = Manifold('m003')`, where 'm003' is the entry for the figure-eight knot's sibling manifold in the Callahan-Hildebrand-Weeks census [2], and issuing the command `M.cusp_info(0).modulus`, one sees that the cusp section of the sibling manifold is not rectangular (a numerical estimate suffices).

*Proof of Theorem 1.2:* Let  $T$  be the cusp section of a single-cusped hyperbolic 3-manifold  $M = \mathbb{H}^3/\Gamma$  whose invariant trace field has odd degree. We shall show that the odd degree assumption precludes  $T$  from being rectangular. Thus, assume to the contrary that  $T$  is

rectangular. Let  $K$  be the trace-field of  $\Gamma$ , and  $k$  its invariant trace field, i.e. the trace field of the group  $\Gamma^{(2)} = \langle g^2 \mid g \in \Gamma \rangle$ .

By [21, Theorem 4.2.3] we may assume that, up to conjugation,  $\Gamma \subset PSL_2(K)$ , and moreover  $\pi_1(T) = \langle a, b \rangle$ , with

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

while there exists an element  $x \in \Gamma$  such that

$$x = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

The complex number  $s$  above is the modulus parameter for the torus  $T$ .

Since  $a^2, b^2, x^2 \in \Gamma^{(2)}$ , it follows that  $\text{tr}(a^2 x^2) \in k$ , hence  $t \in k$ . Furthermore,  $\text{tr}(b^2 x^2) = 2 + 4st \in k$ . Thus the cusp parameter  $s$  of  $M$  belongs to  $k$ , which implies that  $s$  has odd degree over  $\mathbb{Q}$ . Hence  $F = \mathbb{Q}(s) \subset k$  is a sub-field of  $k$ , having odd degree. By Proposition 2.8,  $T$  is a rectangular torus, and thus its modulus belongs to the imaginary axis  $i\mathbb{R}$ . However, if  $s = ir$ , for some  $r \in \mathbb{R}$ , then  $F$  is preserved by complex conjugation, implying that the real sub-field  $F \cap \mathbb{R}$  has degree 2 in  $F$ , contradicting the fact that the degree of  $F$  is odd. This completes the proof.  $\square$

**2.4. Minimal volume hyperbolic 4-manifolds.** The Ratcliffe-Tschanz census [27] contains most of the known cusped hyperbolic 4-manifolds of minimal volume. All of these manifolds are arithmetic. In particular, by [15] they all embed geodesically.

By [27, Table 2], each of the 22 Ratcliffe-Tschanz orientable 4-manifolds has an odd number of cusps with section diffeomorphic to the so-called *Hantzsche-Wendt manifold* (denoted by  $F$  in [27] and by  $\mathcal{G}_6$  in [31]). This flat 3-manifold has no fixed-point-free orientation-reversing self-homeomorphism (c.f. [31, Theorem 3.5.9] and also [32]). Thus, by Proposition 2.5, none of the manifolds from [27, Table 2] bounds geometrically.<sup>1</sup>

### 3. EMBEDDING THE FIGURE-EIGHT'S SIBLING

Although Theorem 1.1 shows that the figure eight knot sibling embeds in a hyperbolic 4-manifold, it gives little control on the topology of the latter. The purpose of this section is to prove Theorem 1.5 using an approach due to Slavich [29], which will afford additional control. We start with some necessary definitions.

**Definition 3.1.** A 4-dimensional *triangulation*  $\mathcal{T}$  is a pair  $(\{\Delta_i\}_{i=1}^{2k}, \{g_j\}_{j=1}^{5k})$ , where  $k$  is a positive integer, the  $\Delta_i$ 's are copies of the standard 4-dimensional simplex, and the  $g_j$ 's are simplicial pairings between all the  $10k$  facets of the  $\Delta_i$ 's.

**Definition 3.2.** A triangulation  $\mathcal{T}$  is *orientable* if it is possible to choose an orientation for each  $\Delta_i$  so that all the  $g_j$ 's are orientation-reversing (c.f. also [17, Definition 4.2]).

**Definition 3.3.** A 4-dimensional triangulation  $\mathcal{T}$  is *6-valent* if all cycles of 2-faces in  $\mathcal{T}$  have length exactly 6.

With each cycle  $c$  of 2-faces in  $\mathcal{T}$  there is a naturally associated return map  $r_c$  from a 2-simplex to itself. In order to obtain it, one has to follow the simplicial pairings from one 4-simplex to the next one, until the cycle closes up.

Our proof will make essential use of the fact stated below.

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<sup>1</sup>One can also arrive at this conclusion by the Euler characteristic argument mentioned in Section 1.



**Proposition 3.4** (Proposition 3.9 in [29]). *Let  $M$  be a hyperbolic 3-manifold obtained by glueing the sides of some copies of the regular ideal hyperbolic tetrahedron via isometries. If this glueing can be realised as the link of a vertex in a 6-valent orientable 4-dimensional triangulation  $\mathcal{T}$  with trivial return maps, then  $M$  embeds geodesically. Moreover,*

$$\varepsilon\text{-vol}(M) \leq \frac{4\pi^2}{3} \cdot \frac{2k}{3},$$

where  $2k$  is the number of 4-simplices in  $\mathcal{T}$ .

*Sketch of proof.* By replacing each 4-simplex of  $\mathcal{T}$  with an ideal hyperbolic rectified 5-cell, one gets a hyperbolic 4-manifold  $W$  with totally geodesic boundary  $\partial W$  tessellated by regular ideal tetrahedra. The link of each vertex of  $\mathcal{T}$  gives the tessellation into tetrahedra of a boundary component of  $W$ . The manifold  $M$  embeds geodesically in the double of  $W$ . Finally, the volume of the ideal rectified 4-simplex is  $\frac{2\pi^2}{9}$  [17].  $\square$

We are ready to prove Theorem 1.5.

*Proof of Theorem 1.5:* We will adopt the usual ideal triangulations of the figure-eight knot complement and its sibling manifold by regular ideal hyperbolic tetrahedra. Each of them consists of two such tetrahedra,  $A$  and  $B$ , with the following glueing maps between their 2-faces.

For the figure-eight knot complement, depicted in Figure 2–(i), we set:

$$(1) \quad \begin{array}{ccc} A & & B \\ (1, 2, 3) & \leftrightarrow & (3, 2, 1) \\ (1, 2, 4) & \leftrightarrow & (1, 4, 2) \\ (1, 3, 4) & \leftrightarrow & (3, 4, 2) \\ (2, 3, 4) & \leftrightarrow & (4, 1, 3), \end{array}$$

while for the figure-eight sibling manifold, depicted in Figure 2–(ii), we set:

$$(2) \quad \begin{array}{ccc} A & & B \\ (1, 2, 3) & \leftrightarrow & (4, 1, 2) \\ (1, 2, 4) & \leftrightarrow & (3, 4, 1) \\ (1, 3, 4) & \leftrightarrow & (1, 3, 2) \\ (2, 3, 4) & \leftrightarrow & (2, 4, 3). \end{array}$$

Let  $Y$  be the cone over the 3-dimensional triangulation in Figure 2–(i), and  $X, Z$  be two copies of the cone over the triangulation in Figure 2–(ii).

This means that each of  $X, Y$  and  $Z$  consists of two 4-simplices  $A'$  and  $B'$  whose facets are identified as follows:

$$(3) \quad Y : \left\{ \begin{array}{ccc} A' & & B' \\ (1, 2, 3, 5) & \leftrightarrow & (3, 2, 1, 5) \\ (1, 2, 4, 5) & \leftrightarrow & (1, 4, 2, 5) \\ (1, 3, 4, 5) & \leftrightarrow & (3, 4, 2, 5) \\ (2, 3, 4, 5) & \leftrightarrow & (4, 1, 3, 5), \end{array} \right.$$

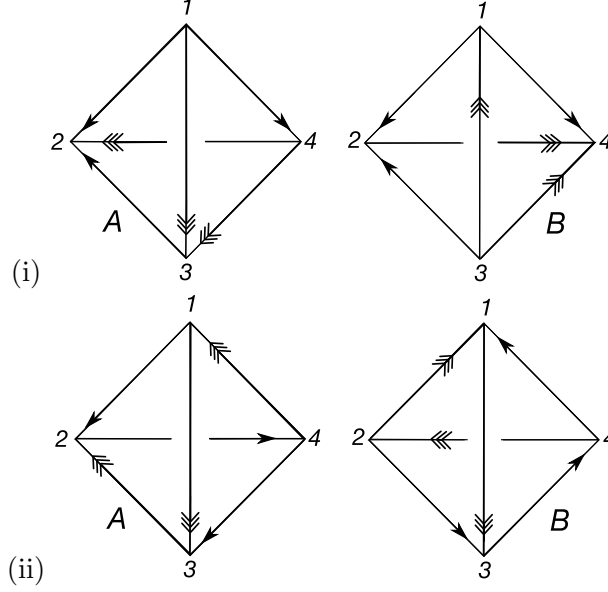


FIGURE 2. Ideal triangulations: (i) figure-eight knot complement; (ii) figure-eight sibling manifold.

and

$$(4) \quad X, Z : \begin{cases} A' & B' \\ (1, 2, 3, 5) & \leftrightarrow (4, 1, 2, 5) \\ (1, 2, 4, 5) & \leftrightarrow (3, 4, 1, 5) \\ (1, 3, 4, 5) & \leftrightarrow (1, 3, 2, 5) \\ (2, 3, 4, 5) & \leftrightarrow (2, 4, 3, 5). \end{cases}$$

Observe that each of  $X$ ,  $Y$  and  $Z$  has two remaining facets  $A$  and  $B$  with vertices  $\{1, 2, 3, 4\}$  unidentified. We shall build a 4-dimensional triangulation  $\mathcal{T}$  by pairing these free facets of  $X$ ,  $Y$  and  $Z$  as depicted in Figure 3. The map  $\sigma_{XY}$  will be used in order to identify facet

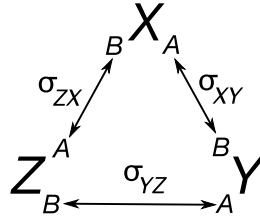


FIGURE 3. Identifying the free facets of  $X$ ,  $Y$ , and  $Z$ .

$A$  of  $X$  to  $B$  of  $Y$ , and an analogous notation  $\sigma_{YZ}$  and  $\sigma_{ZX}$  is adopted for the remaining maps. We set:

$$(5) \quad \begin{aligned} \sigma_{XY} &: (1, 2, 3, 4) \rightarrow (3, 1, 4, 2) \\ \sigma_{YZ} &: (1, 2, 3, 4) \rightarrow (3, 4, 2, 1) \\ \sigma_{ZX} &: (1, 2, 3, 4) \rightarrow (2, 4, 1, 3). \end{aligned}$$

Now we check that the resulting 4-dimensional triangulation  $\mathcal{T}$  satisfies the conditions of Proposition 3.4. First of all,  $\mathcal{T}$  is orientable because (3), (4) are cones over orientable

triangulations, and the pairing maps in (5) are identified with odd permutations in the symmetric group  $\mathfrak{S}_4$ . Second, the condition on the cycles of 2-faces should be satisfied.

By using the glueing equations (3), (4) and (5), together with the diagram in Figure 3, we can compute the cycles of 2-faces with no vertex labelled 5:

$$\begin{aligned} X_A : (1, 2, 3) \rightarrow Y_B : (3, 1, 4) \rightarrow Y_A : (4, 3, 2) \rightarrow Z_B : (1, 2, 4) \rightarrow \\ \rightarrow Z_A : (2, 3, 1) \rightarrow X_B : (4, 1, 2) \rightarrow X_A : (1, 2, 3), \end{aligned}$$

$$\begin{aligned} X_A : (1, 2, 4) \rightarrow Y_B : (3, 1, 2) \rightarrow Y_A : (1, 3, 2) \rightarrow Z_B : (3, 2, 4) \rightarrow \\ \rightarrow Z_A : (4, 2, 3) \rightarrow X_B : (3, 4, 1) \rightarrow X_A : (1, 2, 4), \end{aligned}$$

$$\begin{aligned} X_A : (1, 3, 4) \rightarrow Y_B : (3, 4, 2) \rightarrow Y_A : (1, 3, 4) \rightarrow Z_B : (3, 2, 1) \rightarrow \\ \rightarrow Z_A : (3, 4, 1) \rightarrow X_B : (1, 3, 2) \rightarrow X_A : (1, 3, 4), \end{aligned}$$

$$\begin{aligned} X_A : (2, 3, 4) \rightarrow Y_B : (1, 4, 2) \rightarrow Y_A : (1, 2, 4) \rightarrow Z_B : (3, 4, 1) \rightarrow \\ \rightarrow Z_A : (1, 2, 4) \rightarrow X_B : (2, 4, 3) \rightarrow X_A : (2, 3, 4). \end{aligned}$$

All such cycles have length 6 and trivial return maps. The same conclusion holds for the cycles of 2-faces containing vertex 5, since they correspond to the glueing of edges of simplices  $A$  and  $B$  in the manifold triangulations from Figure 2.

By using Regina [1] we can conclude that  $\mathcal{T}$  has 4 vertices.<sup>2</sup> Two of their links are isomorphic to the sibling manifold triangulation, one to the figure-eight triangulation, and the remaining fourth link is isomorphic to the triangulation of the manifold  $O = \text{otet24\_00260}$  from the census of tetrahedral manifolds [9]. Thus, as described in the proof of Proposition 3.4, we have

$$\partial W \cong K \sqcup L \sqcup L \sqcup O,$$

where  $K$  is the figure-eight knot complement, and  $L$  is its sibling manifold.

The figure-eight knot complement  $K$  is the orientation double-cover of the non-orientable Gieseking manifold, while  $O$  is the orientation double-cover of the non-orientable manifold  $\text{ntet12\_00019}$  from [9] (as one can verify by SnapPy). Thus, we can quotient the  $O$  and  $K$  boundary components of  $W$ , in order to obtain a hyperbolic manifold  $W'$  with two boundary components, each isometric to  $L$ . By identifying the  $L$  boundary components of  $W'$ , we obtain a hyperbolic 4-manifold  $N$  of volume  $\frac{4\pi^2}{3}$ , in which the sibling manifold  $L$  embeds geodesically.

Similarly, Slavich produced a hyperbolic 4-manifold  $W''$  with totally geodesic boundary

$$\partial W'' \cong K \sqcup K \sqcup K \sqcup O'$$

with  $\text{vol}(W'') = \frac{4\pi^2}{3}$ , where  $O'$  is another tetrahedral 3-manifold with an orientation-reversing fixed-point-free involution [29, Remark 4.4]. To conclude the proof for the figure-eight knot complement, we glue together two  $K$ -components of  $\partial W''$  via an isometry, and quotient the remaining boundary components as before.

□

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<sup>2</sup>A word of caution to the reader: Regina does not recognise  $\mathcal{T}$  as a valid triangulation, since it does not allow reverse identifications of edges. However,  $\mathcal{T}$  does not have to satisfy this condition. The links of vertices are valid triangulations for Regina, as it should be.

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