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Classical central limit theorem via conditional expectations

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ABSTRACT

Let (X_n) be a sequence of real, square integrable random variables. Define $S_n = \sum_{j=1}^n X_j$ and suppose $E(X_n) = 0$ and σ_n^2 =Var$(S_n) > 0$. Conditions for $\frac{S_n}{\sigma_n} \to N(0,1)$ stably are provided. The main of such conditions is that $E(X_{n+1} \mid X_1,\dots,X_n) \to 0$ (in some sense) at a suitable rate. Various examples are given as well.

1. Introduction

Throughout, $(X_n : n \geq 1)$ is a sequence of real, square integrable random variables. We let $S_n = \sum_{j=1}^n X_j$ and we assume

$$E(X_n) = 0 \quad \text{and} \quad \sigma_n^2 = \text{Var}(S_n) > 0. \tag{1}$$

Under such condition, we say that the *classical central limit theorem* (classical CLT) holds if the probability distribution of $\frac{S_n}{\sigma_n}$ converges weakly to $N(0,1)$ as $n \to \infty$.

If (X_n) is independent, a sufficient condition for the classical CLT is

$$\frac{\sum_{j=1}^n X_j^2}{\sigma_n^2} \xrightarrow{P} 1 \quad \text{and} \quad \frac{\max_{1\leq j\leq n} E(X_j^2)}{\sigma_n^2} \longrightarrow 0. \tag{2}$$

Moreover, if (X_n) is i.i.d., condition (2) is necessary as well; see e.g. Gut (2006). Hence, in addition to (1), we also assume condition (2).

This note stems from the following (naive) question. Define $\mathcal{F}_n = \sigma(X_1,\dots,X_n)$ and suppose that

$$E(X_{n+1} \mid \mathcal{F}_n) \longrightarrow 0, \qquad \text{in some sense, as } n \to \infty. \tag{3}$$

Does the classical CLT hold under conditions (1), (2) and (3) ? Sometimes, the answer is yes.

Example 1. Suppose $0 < E(X_1^2) < \infty$ and (X_n) is *conditionally identically distributed* (c.i.d.) in the sense of Berti et al. (2004), i.e., $(X_1,\dots,X_n,X_k) \sim (X_1,\dots,X_n,X_{n+1})$ for all $k > n \geq 0$. Then, both $E(X_{n+1} \mid \mathcal{F}_n)$ and $E(X_{n+1}^2 \mid \mathcal{F}_n)$ converge a.s. and one can define $V \overset{a.s.}{=} \lim_n E(X_{n+1} \mid \mathcal{F}_n)$ and $U \overset{a.s.}{=} \lim_n E(X_{n+1}^2 \mid \mathcal{F}_n)$. By Theorem 3.1 of Berti et al. (2004), $n^{-1/2}(S_n - nV)$ converges stably to the kernel $N(0, U - V^2)$ provided the sequence $\big(X_{k+n} - V : n \geq 1\big)$ is still c.i.d. for some $k \geq 0$. Unless (X_n) is exchangeable, the latter condition is quite strong and rarely satisfied. However, such condition is trivially true if $E(X_{n+1} \mid \mathcal{F}_n) \xrightarrow{P} 0$, for in this case $V = 0$

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a.s. Moreover, under (2), one obtains $U = u$ a.s. where u is the constant $u = \lim_n \sigma_n^2/n$. To sum up, if (X_n) is c.i.d., $0 < E(X_1^2) < \infty$ and conditions (2)–(3) hold, then

$$\frac{S_n}{\sigma_n} = \frac{\sqrt{n}}{\sigma_n}\,\frac{S_n}{\sqrt{n}} \xrightarrow{stably} \frac{1}{\sqrt{u}}\,N(0,u) = N(0,1).$$

In particular, this result applies if (X_n) is exchangeable. In this case, condition (3) is equivalent to $E(X_1X_2) = 0$. Hence, exploiting (3), we obtain an alternative formulation of a well known CLT; see e.g. Klass and Teicher (1987).

Generally, however, conditions (1), (2) and (3) do not imply the classical CLT. Interestingly, this may happen even if (X_n) is pairwise independent.

Example 2. Following Janson (1988), define $Y_n = \mathrm{RE}(UV^{n-1})$ where U and V are independent complex random variables, both uniformly distributed on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Among other things, it is easily seen that (Y_n) is pairwise independent and identically distributed, with $|Y_1| \le 1$, $E(Y_1) = 0$ and $E(Y_1^2) > 0$. In addition, $\left|\sum_{j=1}^n Y_j\right| \le L$ a.s. where L is a suitable random variable. Next, define $X_n = n^{-1/2}Y_n$. Then, (X_n) is still pairwise independent and condition (3) is trivially true since $|X_n| \le n^{-1/2}$. On noting that $E(X_iX_j) = 0$ for $i \ne j$, one obtains $\sigma_n^2 = \sum_{j=1}^n E(X_j^2) = E(Y_1^2)\sum_{j=1}^n (1/j)$. Letting $A_n = \sigma_n^{-2}\sum_{j=1}^n X_j^2$, it follows that $E(A_n) = 1$ and

$$\mathrm{Var}(A_n) \le \frac{\sum_{j=1}^n E(X_j^4)}{\sigma_n^4} = \frac{E(Y_1^4)}{E(Y_1^2)^2}\,\frac{\sum_{j=1}^n (1/j^2)}{\left\{\sum_{j=1}^n (1/j)\right\}^2} \longrightarrow 0.$$

Hence, $A_n \xrightarrow{L_2} 1$, so that conditions (1) and (2) are satisfied. However, the classical CLT fails. To see this, let $L_n = \sum_{j=1}^n Y_j$. Then,

$$S_n = \sum_{j=1}^n j^{-1/2}Y_j = n^{-1/2}L_n + \sum_{j=1}^{n-1}\left\{j^{-1/2} - (j+1)^{-1/2}\right\}L_j$$

so that

$$|S_n| \le n^{-1/2}L + \sum_{j=1}^{n-1}\left\{j^{-1/2} - (j+1)^{-1/2}\right\}L = n^{-1/2}L + (1 - n^{-1/2})\,L = L \quad \text{a.s.}$$

Therefore,

$$\left|\frac{S_n}{\sigma_n}\right| \le \frac{L}{\sqrt{E(Y_1^2)\,\sum_{j=1}^n (1/j)}} \xrightarrow{a.s.} 0.$$

It is worth noting that the sequence (X_n) of Example 2 is pairwise independent but not identically distributed. There are actually a number of examples where the classical CLT fails even if

$$(X_n) \text{ is pairwise independent, identically distributed, } E(X_1) = 0 \text{ and } 0 < E(X_1^2) < \infty. \tag{4}$$

In such examples, however, condition (3) fails; see e.g. Avanzi et al. (2021), Janson (1988), Pruss (1998), Raic (2025) and references therein. Therefore, an (intriguing) open problem is whether the classical CLT holds whenever conditions (3) and (4) are both true. Another open problem is mentioned in Remark 10.

Coming back to the general case, despite Example 2, condition (3) has to do with the classical CLT. In fact, we will prove that the classical CLT holds if $E(X_{n+1} \mid \mathcal{F}_n)$ converges at a suitable rate. For instance, under some conditions, it holds if $\sqrt{n}\,E(X_{n+1} \mid \mathcal{F}_n) \xrightarrow{L_1} 0$.

This note investigates the connections between condition (3) and the classical CLT. Our main results are Theorem 3 and its corollaries, which give conditions for $\frac{S_n}{\sigma_n} \xrightarrow{stably} N(0,1)$. One of such conditions is just that $E(X_{n+1} \mid \mathcal{F}_n)$ converges to 0 quickly enough. Another possible condition is based on the sign of the differences $E(X_iX_j) - E\Big(E(X_i \mid \mathcal{F}_{i-1})\,E(X_j \mid \mathcal{F}_{j-1})\Big)$ for $i < j$; see Corollary 7. Various examples are given as well.

Finally, due to space constraints, all the proofs are gathered in the "Supplementary Material".

2. Results

We first briefly recall some well known facts. Let $(\Omega, \mathcal{A}, P)$ denote the probability space where (X_n) is defined.

A *kernel* on $\mathbb{R}$ (or a *random probability measure* on $\mathbb{R}$) is a map K on $\Omega \times \mathcal{B}(\mathbb{R})$ such that:

– $K(\omega, \cdot)$ is a probability measure on $\mathcal{B}(\mathbb{R})$ for fixed $\omega \in \Omega$;

– the map $\omega \mapsto K(\omega, B)$ is $\mathcal{A}$-measurable for fixed $B \in \mathcal{B}(\mathbb{R})$.

A popular example is $K = \delta_X$, where X is a real random variable and δ_x denotes the unit mass at the point x. Another example is the Gaussian kernel $K = N(0, L)$, where L is a real non-negative random variable and $N(0,0) = \delta_0$.

Say that X_n *converges stably to* K, where K is a kernel on $\mathbb{R}$, if the probability measures $P(X_n \in \cdot \mid H)$ converge weakly to $E\{K(\cdot) \mid H\}$ for all $H \in \mathcal{A}$ with $P(H) > 0$. Obviously, stable convergence implies convergence in distribution (just let $H = \Omega$). In addition, stable convergence is connected with convergence in probability. In fact, $X_n \xrightarrow{P} X$ if and only if X_n converges stably to the kernel $K = \delta_X$.

A last general remark is that, as apparent from the proofs, a fundamental tool for this note is the martingale CLT; see e.g. Berti et al. (2004, p. 2042) and the "Supplementary Material".

We next turn to our main result. In the sequel, $\mathcal{F}_0$ *is the trivial* σ*-field.*

Theorem 3. $\frac{S_n}{\sigma_n} \xrightarrow{stably} N(0,1)$ *provided:*

(i) Conditions (1)–(2) *hold;*
(ii) $\frac{1}{\sigma_n} \max_{1\le j\le n} |X_j| \xrightarrow{L_1} 0$;
(iii) $\liminf_n \frac{\sigma_n^2}{n} > 0$;
(iv) $E(X_{n+1} \mid \mathcal{F}_n) \xrightarrow{L_2} 0$;
(v) $\frac{\sum_{j=1}^n E(X_j|\mathcal{F}_{j-1})}{\sigma_n} \xrightarrow{P} 0.$

Moreover, under (iii), a sufficient condition for (v) is $\sqrt{n}\, E(X_{n+1} \mid \mathcal{F}_n) \xrightarrow{L_1} 0.$

In a nutshell, Theorem 3 states that, under some standard conditions (such as (i) and (ii)), the classical CLT holds whenever $E(X_{n+1} \mid \mathcal{F}_n)$ vanishes at a suitable rate and the ratio σ_n^2/n is lower bounded by a strictly positive constant. Note that condition (ii) trivially holds if $\sigma_n \to \infty$ and the X_n are uniformly bounded. Similarly, condition (iii) certainly holds if $u := \inf_n E(X_n^2) > 0$ and $E(X_i X_j) \ge 0$ for all i, j. In this case in fact $\sigma_n^2 \ge n\,u$. Moreover, as shown in Corollaries 6 and 7, conditions (ii), (iii) and (v) may be replaced by some other assumptions.

Our feeling is that, apart from minor technicalities, the conditions of Theorem 3 cannot be substantially weakened. The next two examples support this claim.

Example 4. In Example 2, condition (iii) and the classical CLT both fail, but all the other conditions of Theorem 3 hold true. We briefly discuss how to check (v). Recall that $X_n = n^{-1/2}\,\mathrm{RE}(U V^{n-1})$ with U and V i.i.d. complex random variables uniformly distributed on the unit circle. Mimicking Remark 2 of Janson (1988), write $U = \exp\big(i\,W\big)$ and $V = \exp\big(i\,Z\big)$ where W and Z are i.i.d. random variables uniformly distributed on $(-\pi, \pi)$. On noting that $\cos(W) = \cos(|W|)$,

$$\mathcal{F}_1 = \sigma(X_1) = \sigma\big(\cos(W)\big) = \sigma\big(\cos(|W|)\big) = \sigma(|W|).$$

Define $Y = 1_{\{W\neq 0\}}\,\frac{Z}{W}$. As shown below, Y is $\mathcal{F}_3$-measurable. Moreover, on the set $\{W \neq 0\}$, one obtains

$$n^{1/2}\,X_n = \cos\big\{W + (n-1)Z\big\} = \cos\big\{|W|\,\big(1 + (n-1)\,Y\big)\big\}.$$

Therefore, $E(X_{n+1} \mid \mathcal{F}_n) = X_{n+1}$ a.s. for each $n \ge 3$. Arguing as in Example 2, it follows that

$$\frac{\sum_{j=1}^n E(X_j \mid \mathcal{F}_{j-1})}{\sigma_n} = \frac{\sum_{j=1}^3 E(X_j \mid \mathcal{F}_{j-1})}{\sigma_n} + \frac{\sum_{j=4}^n X_j}{\sigma_n} \xrightarrow{a.s.} 0.$$

It remains to show that Y is $\mathcal{F}_3$-measurable. First note that

$$\mathcal{F}_3 = \sigma(X_1, X_2, X_3) = \sigma\Big(\cos(W),\ \cos(W+Z),\ \cos(W+2Z)\Big)$$
$$= \sigma\Big(\cos(|W|),\ \cos(|W+Z|),\ \cos(|W+2Z|)\Big) = \sigma\Big(|W|,\ |W+Z|,\ |W+2Z|\Big).$$

Hence,

$$\{Z = 0\} = \big\{|W| = |W+Z| = |W+2Z|\big\} \in \mathcal{F}_3,$$

which in turn implies $\{Y = 0\} = \{W = 0\} \cup \{W \neq 0,\ Z = 0\} \in \mathcal{F}_3$. Finally, on the set $\{W \neq 0,\ Z \neq 0\}$, one obtains

$$\frac{1}{Y} = \frac{W}{Z} = \frac{2WZ + Z^2}{2Z^2} - \frac{1}{2} = \frac{(W+Z)^2 - W^2}{W^2 - 2(W+Z)^2 + (W+2Z)^2} - \frac{1}{2}.$$

Therefore, Y is $\mathcal{F}_3$-measurable.

Example 5. Say that condition (i*) holds if condition (1) holds, $\sigma_n^{-2} \max_{1\le j\le n} E(X_j^2) \longrightarrow 0$ and $\sigma_n^{-2} \sum_{j=1}^n X_j^2 \xrightarrow{P} c$, where the constant c is strictly positive but not necessarily 1. We now give an example where the classical CLT fails even if conditions (i*), (ii), (iii), (iv) are satisfied. Suppose

$$X_n \in \{-1, 1\}, \quad E(X_1) = 0 \quad \text{and} \quad E(X_{n+1} \mid \mathcal{F}_n) = n^{-1/2} X_1 \quad \text{for } n > 0. \tag{5}$$

Then,

$$q_n := 2 \sum_{1\le i<j\le n} E(X_i X_j) = 2 \sum_{j=2}^n \frac{1}{\sqrt{j-1}} + 2 \sum_{i=2}^{n-1} \frac{1}{\sqrt{i-1}} \sum_{j=i+1}^n \frac{1}{\sqrt{j-1}}.$$

Since $\sigma_n^2 = n + q_n$ and $\frac{q_n}{n} \longrightarrow 4$, conditions (i*), (ii), (iii) and (iv) are easily seen to be true. In particular, $\frac{1}{\sigma_n^2} \sum_{j=1}^n X_j^2 = \frac{n}{n+q_n} \longrightarrow \frac{1}{5}$. Arguing as in the proof of Theorem 3 and using (i*), (ii), (iii) and (iv), one obtains

$$\frac{\sum_{j=1}^n \{X_j - E(X_j \mid \mathcal{F}_{j-1})\}}{\sigma_n} \xrightarrow{stably} N\left(0, \frac{1}{5}\right).$$

In addition,

$$\frac{\sum_{j=1}^n E(X_j \mid \mathcal{F}_{j-1})}{\sigma_n} = X_1 \frac{\sum_{j=1}^{n-1} \frac{1}{\sqrt{j}}}{\sqrt{n+q_n}} \xrightarrow{a.s.} \frac{2\, X_1}{\sqrt{5}}.$$

Hence, the classical CLT fails. Define in fact the probability measure $\nu = \frac{1}{2}\left\{ N\left(\frac{2}{\sqrt{5}}, \frac{1}{5}\right) + N\left(-\frac{2}{\sqrt{5}}, \frac{1}{5}\right)\right\}$. Since $P(X_1 = 1) = P(X_1 = -1) = 1/2$, stable convergence yields

$$E\left\{f\left(\frac{S_n}{\sigma_n}\right)\right\} = \frac{1}{2}\, E\left\{f\left(\frac{S_n}{\sigma_n}\right) \mid X_1 = 1\right\} + \frac{1}{2}\, E\left\{f\left(\frac{S_n}{\sigma_n}\right) \mid X_1 = -1\right\} \longrightarrow \int f\, d\nu$$

for each bounded continuous function $f : \mathbb{R} \to \mathbb{R}$. Hence, the probability distribution of $\frac{S_n}{\sigma_n}$ converges weakly to ν (and not to $N(0,1)$, as required by the classical CLT). A last remark is that, to obtain a sequence (X_n) satisfying condition (5), it suffices to let

$$X_{n+1} = 2\, 1_{H_n}(U) - 1 \qquad \text{for all } n \geq 0,$$

where the random variable U is uniformly distributed on $(0,1)$ and

$$H_0 = \left(0, \frac{1}{2}\right) \quad \text{and} \quad H_n = \left(0, \frac{\sqrt{n}+1}{4\sqrt{n}}\right) \bigcup \left(\frac{1}{2}, \frac{1}{2} + \frac{\sqrt{n}-1}{4\sqrt{n}}\right).$$

We next give two corollaries of Theorem 3.

Corollary 6. *Assume conditions (i), (iv), (v). Then,* $\frac{S_n}{\sigma_n} \xrightarrow{stably} N(0,1)$ *provided*

$$\inf_n E(X_n^2) > 0, \quad \sigma_n^{-2} \sum_{j=1}^n E(X_j^2) \longrightarrow 1, \quad \textit{the sequence } \left(\frac{X_n^2}{E(X_n^2)} : n \geq 1\right) \textit{ is uniformly integrable.} \tag{6}$$

Note that, in particular, condition (6) holds if the X_n are identically distributed and $\frac{\sigma_n^2}{n} \longrightarrow E(X_1^2) > 0$.

Corollary 7. *Assume conditions (i)–(iv) and* $\sigma_n^{-2} \sum_{j=1}^n E(X_j^2) \longrightarrow 1$. *Then,* $\frac{S_n}{\sigma_n} \xrightarrow{stably} N(0,1)$ *if*

$$\liminf_n \frac{1}{\sigma_n^2} \sum_{1 \leq i < j \leq n} \left\{ E(X_i X_j) - E\Big(E(X_i \mid \mathcal{F}_{i-1})\, E(X_j \mid \mathcal{F}_{j-1})\Big)\right\} \geq 0,$$

or if

$$\lim_n \frac{1}{\sigma_n^2} \sum_{j=1}^{n-1} \left\{ E\Big[E\Big(S_n - S_{n-j} \mid \mathcal{F}_{n-j}\Big)^2\Big] - E\Big[E\Big(S_n - S_{n-j} \mid \mathcal{F}_{n-j-1}\Big)^2\Big]\right\} = 0.$$

We close this note with two examples and a remark. Even if trivial, the examples are possibly to be stressed. The remark gives some hints for possible future work.

Example 8. A meaningful special case is when $E(X_{n+1} \mid \mathcal{F}_n)$ is a linear combination of $(X_1, \ldots, X_n)$, say

$$E(X_{n+1} \mid \mathcal{F}_n) = \sum_{r=1}^n a_{n,r} X_r \quad \text{a.s.}$$

where the $a_{n,r}$ are real constants. In this case, $\frac{S_n}{\sigma_n} \xrightarrow{stably} N(0,1)$ provided conditions (i), (ii), (iv) hold, $E(X_n^2) = 1$ for all n, and $E(X_i X_j) = 0$ whenever $|i-j| > k$ for some integer $k \geq 1$. To prove this fact, for the sake of simplicity, we assume $k = 1$, but the argument used below works for any k. By condition (iv),

$$\frac{1}{n} \Big| \sum_{i=1}^{n-1} E(X_i X_{i+1}) \Big| = \frac{1}{n} \Big| \sum_{i=1}^{n-1} E\{X_i\, E(X_{i+1} \mid \mathcal{F}_i)\} \Big| \leq \frac{1}{n} \sum_{i=1}^{n-1} \sqrt{E\{E(X_{i+1} \mid \mathcal{F}_i)^2\}} \longrightarrow 0.$$

Since $k = 1$ and $E(X_n^2) = 1$ for all n, then $\sigma_n^2 = n + 2 \sum_{i=1}^{n-1} E(X_i X_{i+1})$. Hence, $\frac{\sigma_n^2}{n} \to 1$, which in turn implies condition (iii) and $\sigma_n^{-2} \sum_{j=1}^n E(X_j^2) \to 1$. Moreover, for all $i < j$,

$$E\Big(E(X_i \mid \mathcal{F}_{i-1})\, E(X_j \mid \mathcal{F}_{j-1})\Big) = E\{X_j\, E(X_i \mid \mathcal{F}_{i-1})\} = \sum_{r=1}^{i-1} a_{i-1,r}\, E(X_j X_r) = 0.$$

Therefore, it suffices to apply Corollary 7 after noting that

$$\frac{1}{\sigma_n^2}\sum_{1\le i<j\le n}\Big\{E(X_iX_j)-E\Big(E(X_i\mid\mathcal{F}_{i-1})\,E(X_j\mid\mathcal{F}_{j-1})\Big)\Big\}=\frac{n}{\sigma_n^2}\,\frac{1}{n}\sum_{i=1}^{n-1}E(X_iX_{i+1})\longrightarrow 0.$$

Example 9. By Corollary 6, the classical CLT holds whenever $E(X_{n+1}\mid\mathcal{F}_n)=0$ a.s. for all $n\ge 0$ and

$$E(X_n^2)=1 \text{ for all } n,\quad E(X_i^2X_j^2)\le 1 \text{ for all } i\ne j,\quad \sup_n E(X_n^4)<\infty. \tag{7}$$

In fact, since $E(X_{n+1}\mid\mathcal{F}_n)=0$ a.s., conditions (iv)–(v) are trivially true and $E(X_iX_j)=0$ for $i\ne j$. Hence, $\sigma_n^2=n$ and condition (6) is straightforward. Moreover, $\frac{\sum_{j=1}^n X_j^2}{n}\xrightarrow{a.s.}1$ due to $\mathrm{cov}(X_i^2,X_j^2)\le 0$ and $\sup_n E(X_n^4)<\infty$. However, $E(X_{n+1}\mid\mathcal{F}_n)=0$ a.s. is actually a strong assumption. Hence, we assume condition (7) only and we replace (X_n) with a related sequence (Y_n). Let $Y_n=X_nZ_n$, where (Z_n) is i.i.d., it is independent of (X_n) and $P(Z_1=1)=P(Z_1=-1)=1/2$. Then, the classical CLT holds for (Y_n), i.e., $s_n^{-1}\sum_{j=1}^n Y_j\xrightarrow{stably}N(0,1)$ where $s_n^2=\mathrm{Var}\Big(\sum_{j=1}^n Y_j\Big)$. In fact, (Y_n) satisfies (7) and

$$E(Y_{n+1}\mid Y_1,\dots,Y_n)=E\Big\{E(Z_{n+1})\,E(X_{n+1}\mid X_1,Z_1,\dots,X_n,Z_n)\mid Y_1,\dots,Y_n\Big\}=0\qquad\text{a.s.}$$

Remark 10. The *predictive distributions* of (X_n) are $\alpha_n(\cdot)=P(X_{n+1}\in\cdot\mid\mathcal{F}_n)$, where each α_n is meant as a random probability measure on $\mathcal{B}(\mathbb{R})$. The α_n play a role in various frameworks, including Bayesian inference, machine learning and species sampling models. A basic question is whether α_n converges (in some sense) as $n\to\infty$; see e.g. Berti et al. (2013, 2025), Efron (2020), Fong et al. (2023) and Pitman (1996). Now, on noting that $E(X_{n+1}\mid\mathcal{F}_n)=\int x\,\alpha_n(dx)$ a.s., the arguments of this note could be developed to obtain CLTs (not necessarily of the classical type) under some assumptions on the asymptotic behavior of α_n. For instance, in Leisen et al. (2025), conditions for α_n to converge weakly in probability are provided. This means that there is a random probability measure α on $\mathcal{B}(\mathbb{R})$ such that

$$E\big(f(X_{n+1})\mid\mathcal{F}_n\big)=\int f\,d\alpha_n\xrightarrow{P}\int f\,d\alpha\quad\text{for each bounded continuous function } f:\mathbb{R}\to\mathbb{R}. \tag{8}$$

Suppose now that

$$E(X_n)=0,\quad E(X_n^2)=1,\quad \lim_n\frac{\sigma_n^2}{n}=1 \text{ and } (X_n^2) \text{ uniformly integrable.} \tag{9}$$

Despite (9), condition (8) does not imply the classical CLT. As an example, suppose (X_n) is exchangeable with $E(X_1)=E(X_1X_2)=0$, $E(X_1^2)=1$ and $E(X_1^2X_2^2)>1$. Then, conditions (8)–(9) hold but $\frac{S_n}{\sigma_n}$ converges stably to a mixture of centered normal distributions (and not to the $N(0,1)$). An open problem is whether the classical CLT holds under conditions (8)–(9) with α non-random, i.e., $\alpha=\alpha_0$ a.s. for some (fixed) probability measure α_0 on $\mathcal{B}(\mathbb{R})$.

Appendix A. Proofs

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.spl.2026.110689.

Data availability

No data was used for the research described in the article.

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