

RECEIVED: March 7, 2023

REVISED: April 17, 2023

ACCEPTED: April 28, 2023

PUBLISHED: May 15, 2023

Quantum rotating black holes

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ABSTRACT: We consider black holes generically sourced by quantum matter described by regular wavefunctions. This allows for integrable effective energy densities and the removal of Cauchy horizons in spherically symmetric configurations. Moreover, we identify the ultrarigid rotation of the Kerr spacetime as causing the existence of an inner horizon in rotating systems, and describe general properties for quantum matter cores at the centre of rotating black holes with integrable singularities and no Cauchy horizon.

KEYWORDS: Black Holes, Classical Theories of Gravity, Models of Quantum Gravity, Spacetime Singularities

ARXIV EPRINT: [2303.02713](https://arxiv.org/abs/2303.02713)

Contents

1	Introduction and motivation	1
2	Spherical black holes	3
3	Rotating black holes	4
3.1	The ultrarigid Kerr metric	4
3.2	From spherical black holes to rotating black holes	4
4	Concluding remarks	6
A	Geodesics in spherical spacetime	7
B	Effective energy-momentum tensor	8
B.1	Kerr-like case	9
B.2	Central quantities	10

1 Introduction and motivation

In the search for regular black holes one usually imposes regularity conditions inspired by classical physics, like finite (effective) energy density and scalar invariants (for recent reviews, see refs. [1, 2]). These conditions allow one to remove the singularities that plague known black hole solutions [3] but usually bring back (or do not allow to remove) a seemingly undesirable inner Cauchy horizon. This fact is easily seen in the static spherically symmetric case, for which one can always introduce a Killing time t and the areal radial coordinate r in which the metric reads¹

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{1.1}$$

where $g_{tt} = -e^\varphi g^{rr}$, with $\varphi = \varphi(r)$ a regular function and

$$-g^{rr} = 1 - \frac{2m(r)}{r}. \tag{1.2}$$

In the above, the Misner-Sharp-Hernandez mass function [4, 5] is given by

$$m(r) = 4\pi \int_0^r \epsilon(x) x^2 dx, \tag{1.3}$$

where $\epsilon = \epsilon(r)$ is the proper energy density of the source. We recall that the mass function approaches the ADM mass [6] M in asymptotically flat space, that is $m(r \rightarrow \infty) = M$. For

¹We use units with $c = G_N = 1$ and metric signature $(+ - - -)$.

the Schwarzschild vacuum solution, $m = M$ and constant, and both the component g_{tt} of the metric and its Kretschmann scalar diverge for $r \rightarrow 0$. The latter result signals that tidal gravitational forces also diverge towards the centre.

If we require that $\epsilon = \epsilon(r)$ is regular for $r \rightarrow 0$, we find $m \sim r^3$ and both the components of the metric in the chosen frame and its Ricci and Kretschmann scalars remain finite in $r = 0$ [1, 2]. The central Schwarzschild singularity is removed but one necessarily finds a Cauchy horizon if there is an event horizon. In fact, horizons are located at values of $r = r_H$ such that $g_{tt}(r_H) = g^{rr}(r_H) = 0$. Since $g_{tt} > 0$ for $r > r_+ = r_H$, it must be negative just inside the event horizon. However, for $m \sim r^3$ one necessarily finds $g_{tt}(0) = e^{\varphi(0)} > 0$, which implies that there must exist a second zero $r = r_-$, with $0 < r_- \leq r_+$.²

In the quantum theory, one can allow for milder conditions to apply for the energy density of the source. In particular, if one considers that

$$\epsilon \propto |\Psi|^2, \tag{1.4}$$

where $\Psi = \Psi(r)$ is the wavefunction of the static matter source, the fundamental requirement is that Ψ be integrable, since a wavefunction must yield finite probability densities. For any finite r , one then must have³

$$4\pi \int_0^r |\Psi(x)|^2 x^2 dx < \infty. \tag{1.5}$$

This accommodates for the milder condition $\epsilon \sim r^{-2}$ and $m \sim r$, which still ensures that $m(0) = 0$. We shall recall in section 2 that this behaviour is tame enough to both replace the central singularity with an integrable one and not allow for the emergence of a Cauchy inner horizon for electrically neutral black holes [7]. Moreover, the Cauchy horizon of the Reissner-Nordström black hole can also be eliminated by this prescription [8]. Note that by *integrable singularity* we mean regions where the curvature invariants and the effective energy-momentum tensor diverge, while their “volume” integrals remain finite [9].

When one considers rotating systems, however, the above condition on Ψ , hence on m , is not sufficient to remove the inner horizon that appears in the Kerr black hole. As we shall show in section 3, this is due to the ultrarigid nature of the vacuum general relativistic solution, which is characterised by a constant specific angular momentum $a = J/M$ (like its generalisations [10–12]). On the contrary, a radial dependent specific angular momentum is indeed more natural for extended bodies. For example, an homogeneous sphere with mass function $m \sim r^3$ rotating with angular velocity ω has angular momentum $J \sim m r^2 \omega \sim \omega r^5$ and $a \sim r^2$. We will show that the simultaneous tempering of the ring singularity and removal of the inner horizon are accomplished by assuming that the quantum state Ψ is such that $a \sim m \sim r$ for $r \rightarrow 0$.

²The region of negative g_{tt} shrinks to zero volume in the extremal case $r_- = r_+$.

³One might argue that the correct volume measure must contain the determinant of the spatial metric, which implies an extra factor of $\sqrt{-g_{tt}}$ for $r < r_+$. Such a factor would then depend on Ψ itself according to eq. (1.3). However, we notice that the condition (1.5) remains valid if $|g_{tt}|$ is finite everywhere inside r_+ .

2 Spherical black holes

Consider a spherically symmetric and static spacetime described by the line element (1.1) with $\varphi = 0$, that is

$$g_{tt} = -g^{rr} = 1 - \frac{2m(r)}{r}, \quad (2.1)$$

where the mass aspect $m = m(r)$ is not fixed a priori. The corresponding energy-momentum tensor is obtained from

$$8\pi T_{\mu\nu} = G_{\mu\nu}, \quad (2.2)$$

with $G_{\mu\nu}$ denoting the Einstein tensor computed from the line element (2.1). By introducing the tetrad

$$\begin{aligned} e_t^\mu &= \left(\sqrt{g^{tt}}, 0, 0, 0 \right), & e_r^\mu &= \left(0, \sqrt{g_{rr}}, 0, 0 \right), \\ e_\theta^\mu &= \left(0, 0, r^{-1}, 0 \right), & e_\phi^\mu &= -\left(0, 0, 0, (r \sin \theta)^{-1} \right), \end{aligned} \quad (2.3)$$

the effective energy-momentum tensor can be written as

$$T^{\mu\nu} = \epsilon e_t^\mu e_t^\nu + p_r e_r^\mu e_r^\nu + p_\theta e_\theta^\mu e_\theta^\nu + p_\phi e_\phi^\mu e_\phi^\nu, \quad (2.4)$$

with ϵ the energy density and p_r , p_θ , and p_ϕ are pressure functions. One therefore finds

$$\epsilon = -p_r = \frac{m'(r)}{4\pi r^2}, \quad (2.5)$$

and

$$p_\theta = p_\phi = -\frac{m''(r)}{8\pi r}. \quad (2.6)$$

Furthermore, from the metric functions in eq. (2.1) one can easily compute the Ricci scalar

$$R = -2 \frac{r m'' + 2 m'}{r^2}, \quad (2.7)$$

and the Kretschmann scalar

$$R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = 4 \frac{r^4 (m'')^2 + 4[2r^2 (m')^2 - 4r m m' + 3m^2] + 4r^2(m - r m')m''}{r^6}. \quad (2.8)$$

For $m \sim r^3$, it is easy to see that the Ricci scalar (2.7) and the Kretschmann scalar (2.8), as well as the effective energy density (2.5) and pressures (2.6) remain finite for $r \rightarrow 0$. The singularity is thus removed and tidal forces do not diverge by approaching what would be the singular centre of the Schwarzschild black hole along radial geodesics.

Under the weaker condition $m \sim r$, one can likewise check that all of the above quantities are still (at least) integrable, and should therefore not be discarded in the quantum context. Furthermore, it is easy to prove (see appendix A for a detailed discussion) that radial geodesics can be extended past $r = 0$, namely the position of the would-be classical singularity. These properties are clearly displayed by the models introduced in refs. [7, 8] and, e.g., refs. [13–16], but are not enjoyed by all regular black holes in the

literature [17]. In particular, the coherent quantum states in refs. [7, 8] are meant to represent (simplified) gravity states in any quantum theory of gravity, from which the effective classical geometry emerges as a mean field for any matter source. In this respect, it would be very interesting to investigate the transition from the more classical behaviour $m \sim r^3$ to $m \sim r$ which must happen in the interior of a collapsing body when it becomes a black hole.

3 Rotating black holes

There exist several procedures to map a spherically symmetric metric into a rotating one, like the Janis-Newman algorithm [10]. In the end, all of these procedures lead to a metric of the Kerr form with the constant ADM mass M replaced by the mass function $m = m(r)$ of the spherically symmetric seed metric [11].

3.1 The ultrarigid Kerr metric

Let us start by recalling that the Kerr metric reads

$$ds^2 = \left[1 - \frac{2 r M}{\rho^2}\right] dt^2 + \frac{4 a r M \sin^2 \theta}{\rho^2} dt d\phi - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{\Sigma \sin^2 \theta}{\rho^2} d\phi^2, \quad (3.1)$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta \quad (3.2)$$

$$\Delta = r^2 - 2 r M + a^2 \quad (3.3)$$

$$\Sigma = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \quad (3.4)$$

and $a = J/M$, with J is the angular momentum of the system.

The above metric contains a ring singularity located at $\rho^2 = 0$ (i.e., $r = 0$ and $\theta = \pi/2$), where the Kretschmann scalar $R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$ diverges (see appendix B.1). For $a^2 < M^2$, the Kerr metric also has the horizons

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}, \quad (3.5)$$

corresponding to the two zeros of $\Delta = \Delta(r)$. Note in particular that the existence of the inner (Cauchy) horizon follows from the ‘‘ultrarigid’’ rotation described by a constant a . In fact, one has

$$\Delta(0) = a^2 > 0, \quad (3.6)$$

which implies that $\Delta = \Delta(r)$ must change sign twice going inward, from positive to negative across $r_H = r_+$ and then back to positive across r_- .

3.2 From spherical black holes to rotating black holes

We next consider the rotating metric obtained by replacing the constant M in eq. (3.1) with a mass function $m = m(r)$ [11]. In order to attenuate the ring singularity, one could impose $m \sim r^3$. Like in the spherically symmetric case, the weaker condition $m \sim r$ is again

sufficient to make the relevant scalar quantities integrable around $r = 0$. However, it is not sufficient to remove the inner horizon, since eq. (3.6) still holds for a constant.

Let the mass function be analytic and related to an integrable energy density in the neighbourhood of $r = 0$, that is

$$m = m_1 r + M \sum_{k=2}^{\infty} m_k \left(\frac{r}{M}\right)^k, \quad (3.7)$$

where all the coefficients $m_1 \geq 0$ and m_k are dimensionless. Further, we require a less than ultrarigid rotation by assuming that $a/(r/M)^\alpha$ is also analytic for fixed values of $\alpha \geq 0$ in the neighbourhood of $r = 0$. In other words, we assume a specific angular momentum given by

$$a = M \left(\frac{r}{M}\right)^\alpha \sum_{k=0}^{\infty} a_{k+1} \left(\frac{r}{M}\right)^k, \quad (3.8)$$

in the neighbourhood of $r = 0$, where the coefficients a_k are also dimensionless. Our metric will represent a black hole with one horizon at $r = r_H > 0$ if $\Delta = \Delta(r)$ has *only one (strictly) positive real root* and $\Delta(0) \leq 0$. With the above expansions, on assuming $m_1 \neq 0$ and $a_1 \neq 0$, we have

$$\Delta \simeq \left(\frac{r}{M}\right)^{2\alpha} (a_1 M + a_2 r)^2 - \frac{2r^2}{M} (m_1 M + m_2 r) + r^2. \quad (3.9)$$

For $0 \leq \alpha < 1$, we would have $\Delta \simeq (r/M)^{2\alpha} a_1^2 M^2$, which is always positive near $r = 0$ since $a_1 \neq 0$. If we assume $\alpha = 1$, then $\Delta \simeq (1 + a_1^2 - 2m_1) r^2$, which is negative inside the horizon provided

$$2m_1 > 1 + a_1^2. \quad (3.10)$$

Lastly, $\Delta \simeq (1 - 2m_1) r^2$, for $\alpha > 1$, which is consistent with the condition $\Delta < 0$ inside the horizon if $2m_1 > 1$. Note that for $\alpha \geq 1$, we always have $\Delta(0) = 0$, so that the location of the would-be inner Cauchy horizon is in fact pushed to the centre of the system.

To clarify the above result, let us consider a simple example given by

$$m = \frac{2M}{\pi} \arctan\left(\frac{r}{M}\right) \quad \text{and} \quad a = \frac{2A}{\pi} \arctan\left(\frac{r}{M}\right), \quad (3.11)$$

which smoothly interpolate between the required behaviour near $r = 0$ and the asymptotic ADM mass M and specific angular momentum A . In particular,

$$m = \frac{2}{\pi} r + \mathcal{O}(r^2) \quad \text{for } r \rightarrow 0^+, \quad \lim_{r \rightarrow \infty} m(r) = M, \quad (3.12)$$

which implies $m_1 = 2/\pi$, and

$$a = \frac{2A}{\pi M} r + \mathcal{O}(r^2) \quad \text{for } r \rightarrow 0^+, \quad \lim_{r \rightarrow \infty} a(r) = A, \quad (3.13)$$

which yields $a_1 = 2A/\pi M$. Furthermore, one finds that

$$\Delta = r^2 + \frac{4A^2}{\pi^2} \arctan^2\left(\frac{r}{M}\right) - \frac{4Mr}{\pi} \arctan\left(\frac{r}{M}\right), \quad (3.14)$$

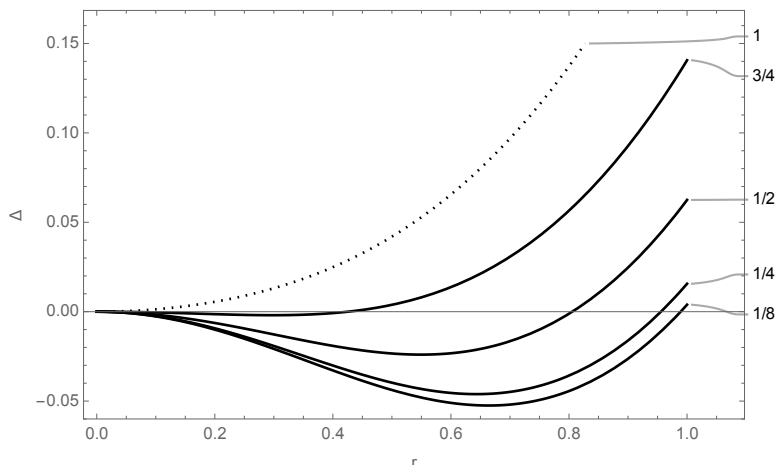


Figure 1. $\Delta = \Delta(r)$ in eq. (3.14) for $M = 1$ for different values of A satisfying the condition (3.10) (solid lines) and a case violating eq. (3.10) (dotted line). All solid lines display one horizon where $\Delta(r_H > 0) = 0$.

which admits at most one horizon at $r_H > 0$ for values of A satisfying the condition (3.10), that is $0 < A < \sqrt{(1 - \pi/4)\pi} M$, as is shown in figure 1.

The effective energy-momentum tensor for a rotating metric with general $m = m(r)$ and $a = a(r)$ is given in appendix B. In particular, the behaviour near the centre for mass and specific angular momentum of the forms in eqs. (3.7) and (3.8) with $\alpha = 1$ is analysed in appendix B.2.

4 Concluding remarks

In refs. [7] and [8] coherent states were constructed to reproduce the classical Schwarzschild and Reissner-Nordström metrics. The condition of normalisability of such quantum states implies the necessary departure from the respective exact classical geometries, an effect that one can understand with the existence of an extended (quantum) matter core [13, 15, 16, 19, 20]. The similar quantum condition (1.5) has been analysed here in more generality to show that the classical singularity can be generically replaced by an integrable singular structure in the interior of a spherically symmetric black hole if $m \sim r$ near $r = 0$.

For a spherically symmetric system, the above condition on the mass function can be implemented in order to also avoid the presence of inner horizons (see ref. [8] for the details). However, the inner horizon in rotating black holes can be circumvented only if one further assumes that the specific angular momentum a is not constant throughout space but vanishes sufficiently fast towards the centre. This additional condition is in fact natural if one considers that the geometry is again sourced by a quantum core [21], for even classical extended bodies do not rotate as rigidly as the vacuum Kerr spacetime. We showed that the conditions $m \sim a \sim r$ are sufficient to ensure an integrable interior without Cauchy horizon. One could therefore use the quantum metrics from refs. [7, 8] as seeds for generating the rotating integrable versions of the Kerr and Kerr-Newman geometries by

means of the algorithms described in refs. [11, 18] with the additional generalisation $a \sim m$ (or faster decay towards the centre).

The absence of Cauchy horizons cures the issue of mass inflation that plagues most regular black hole geometries present in the literature [2]. This suggests that the quantum picture of black holes developed in this work could indeed represent (meta)stable final configurations for the gravitational collapse of compact astrophysical objects. Since our analysis is limited to stationary configurations, the effects due to matter accretion, or the merging of two black holes, and the Hawking evaporation must be left for future developments. In particular, we notice that the global size of the core is not relevant for the issues discussed here, as long as $m \sim a \sim r$ in a sufficiently large neighbourhood of the centre. However, we expect that departures from the outer classical geometry (usually termed “quantum hair”) are highly sensitive to the relative size of the core radius with respect to the horizon radius, as shown explicitly in refs. [7, 8], and that all of black hole phenomenology is therefore related to this aspect of the interior.

Acknowledgments

R.C. is partially supported by the INFN grant FLAG. A.G. is supported by the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Actions (grant agreement No. 895648-CosmoDEC). The work of R.C. and A.G. has also been carried out in the framework of activities of the National Group of Mathematical Physics (GNFM, INdAM). J.O. is partially supported by ANID FONDECYT grant No. 1210041.

A Geodesics in spherical spacetime

Let us consider the Lagrangian for the motion of a point-like particle in the metric (1.1) with $\varphi = 0$,

$$\mathcal{L} = \left(1 - \frac{2m}{r}\right) \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\theta}^2 - r^2 \sin^2 \theta \dot{\phi}^2, \tag{A.1}$$

where dots denote derivatives with respect to the an affine parameter λ . This Lagrangian is constant along geodesics and takes the value $\mathcal{L} = 1$ for timelike trajectories and $\mathcal{L} = 0$ for the null case. Since \mathcal{L} does not depend explicitly on t and ϕ , the corresponding conjugate momenta are conserved and read

$$E = g_{tt} \dot{t} = \left[1 - \frac{2m(r)}{r}\right] \dot{t} \quad \text{and} \quad L = r^2 \sin^2 \theta \dot{\phi}. \tag{A.2}$$

Furthermore, taking advantage of the spherical symmetry one can fix $\theta = \pi/2$ for all λ , without loss of generality. By plugging in the conserved quantities E and L , one finds the radial equation

$$\frac{\dot{r}^2}{2} + V_{\text{eff}} = \frac{E^2}{2}, \tag{A.3}$$

with

$$V_{\text{eff}} = \frac{1}{2} \left[1 - \frac{2m(r)}{r}\right] \left(\mathcal{L} + \frac{L^2}{r^2}\right). \tag{A.4}$$

For radial geodesics with $L = 0$, and assuming $m = m_1 r + \mathcal{O}(r^2)$ as $r \rightarrow 0^+$, we find

$$V_{\text{eff}} = -\left(m_1 - \frac{1}{2}\right)\mathcal{L} + \mathcal{O}(r) \quad \text{for } r \rightarrow 0^+, \quad (\text{A.5})$$

which guarantees the extendibility of radial geodesics past $r = 0$. Furthermore, g_{tt} must change sign only once (from positive to negative inward), say at $r = r_H$, in order to guarantee the existence of one horizon. This implies that the component g_{tt} cannot be positive at $r = 0$ and the condition $m_1 \geq 1/2$.

We should remark that a process of gravitational collapse leading to a spherically symmetric configuration must satisfy the same degree of symmetry throughout its evolution. In other words, in order to potentially form the geometry (1.1), a self-gravitating system must undergo a spherical collapse. As a result, the relevant causal structure for the system is determined by radial null geodesics. Geodesics with non-vanishing angular momentum are of limited interest since the presence of matter moving along such trajectories would require to abandon the spherical symmetry in order to provide a fully consistent description, particularly inside the core sourcing the geometry.

B Effective energy-momentum tensor

For a metric of the form (3.1) with mass function $M = m(r)$ and $a = a(r)$, it is convenient to introduce the tetrads [22]

$$\begin{aligned} e_t^\mu &= \frac{(r^2 + a^2, 0, 0, a)}{\sqrt{\rho^2 \Delta}}, & e_r^\mu &= \frac{\sqrt{\Delta}(0, 1, 0, 0)}{\sqrt{\rho^2}} \\ e_\theta^\mu &= \frac{(0, 0, 1, 0)}{\sqrt{\rho^2}}, & e_\phi^\mu &= -\frac{(a \sin^2 \theta, 0, 0, 1)}{\sqrt{\rho^2} \sin \theta}, \end{aligned} \quad (\text{B.1})$$

so that the effective source can be written in the form (2.4), where the energy density is given by

$$\begin{aligned} \epsilon &= \frac{r^2 m'}{4 \pi \rho^4} - \frac{r + m - r m' - (3r + r m' - 9m) \cos^2 \theta}{8 \pi \rho^6} r^2 a a' \\ &+ \frac{3r + m + r m' - (r - m - r m') \cos^2 \theta}{8 \pi \rho^6} a^3 a' \cos^2 \theta \\ &- \frac{9r^2 + 14r m + 8(2r^2 + a^2) \cos^2 \theta - (r^2 - 2mr) \cos 4\theta}{64 \pi \rho^6} a^2 (a')^2 \\ &- \frac{r^2 - 3m^2 + (r^2 + 3m^2 - 4rm) \cos^2 \theta}{8 \pi \rho^6} r^2 (a')^2 \\ &- \frac{\Delta (1 + \cos^2 \theta)}{8 \pi \rho^4} a a'', \end{aligned} \quad (\text{B.2})$$

and the radial pressure reads

$$\begin{aligned} p_r &= -\frac{r^2 m'}{4 \pi \rho^4} + \frac{r^3 + r^2 m - r^3 m' + (r - m - r m') a^2 \cos^2 \theta}{8 \pi \rho^6} (1 + \cos^2 \theta) a a' \\ &+ \frac{r^2 m^2 + [(r^2 + 2rm) a^2 - r^2 m^2] \cos^2 \theta + a^4 \cos^4 \theta}{8 \pi \rho^6} (a')^2. \end{aligned} \quad (\text{B.3})$$

The two tensions are given by

$$\begin{aligned}
 p_\theta = & -\frac{a^2 m' \cos^2 \theta}{4 \pi \rho^4} - \frac{r^3 + r^2 m - r^3 m' + (r - m - r m') a^2 \cos^2 \theta}{8 \pi \rho^6} (1 + \cos^2 \theta) a a' \\
 & + \frac{r^4 - r^2 m^2 + [(r^2 - 2 r m) a^2 - r^2 m^2] \cos^2 \theta}{8 \pi \rho^6} (a')^2 \\
 & - \frac{r m''}{8 \pi \rho^2} + \frac{a a''}{8 \pi \rho^2}
 \end{aligned} \tag{B.4}$$

and

$$\begin{aligned}
 p_\phi = & -\frac{a^2 m' \cos^2 \theta}{4 \pi \rho^4} + \frac{r - 3 m + 3 r m' - (3 r - m + r m') \cos^2 \theta}{8 \pi \rho^6} r^2 a a' \\
 & - \frac{(r - m - r m')(3 - \cos^2 \theta)}{8 \pi \rho^6} a^3 a' \cos^2 \theta \\
 & + \frac{4 r^4 \cos^2 \theta + 4 r^2 m^2 \sin^2 \theta - a^4 \sin^2 2\theta}{32 \pi \rho^6} (a')^2 \\
 & + \frac{(4 \cos 2\theta + \sin^2 2\theta) r - (9 - \cos 4\theta) m}{32 \pi \rho^6} r a^2 (a')^2 \\
 & - \frac{r m''}{8 \pi \rho^2} + \frac{2 r m - a^2 + (r^2 - 2 r m - 2 a^2) \cos^2 \theta}{8 \pi \rho^4} a a'',
 \end{aligned} \tag{B.5}$$

and we further notice that $p_\theta = p_\phi$ on the axis (that is, for $\theta = 0$ or π).

Other quantities of interest are the Ricci scalar

$$\begin{aligned}
 R = & 2 \frac{2 a a'' + (a')^2 - r m'' - 2 m'}{\rho^6} a^4 \cos^4 \theta - 2 \frac{r m'' + 2 m'}{\rho^6} r^4 \\
 & + 2 \frac{[3 r + (r - 2 m) \sin^2 \theta] (a')^2 \cos^2 \theta - (r m'' + 2 m') \sin^2 \theta}{\rho^6} r a^2 \\
 & + \frac{r a'' [r(3 + \cos 2\theta) - 4 m \cos^2 \theta] - 2 a' [2 r - r(3 + m') \sin^2 \theta - (4 - 5 \sin^2 \theta) m]}{\rho^6} r^2 a \\
 & + \frac{r a'' [r(7 + \cos 2\theta) - 4 m \cos^2 \theta] - 2 a' [2 r + r(1 - m') \sin^2 \theta - m \sin^2 \theta]}{\rho^6} a^3 \cos^2 \theta \\
 & + 2 \frac{2 r - (r - m) \sin^2 \theta}{\rho^6} r^2 (a')^2,
 \end{aligned} \tag{B.6}$$

and the Kretschmann scalar $R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$, whose expression is really too cumbersome to display.

For consistency, we will show below that the expressions for constant a and m are correctly recovered.

B.1 Kerr-like case

The above expressions reduce to those given in ref. [18] for $a(r) = a$ constant, that is

$$\epsilon = -p_r = \frac{r^2 m'}{4 \pi \rho^4} \tag{B.7}$$

and

$$p_\theta = p_\phi = -\frac{r m''}{8 \pi \rho^2} - \frac{a^2 \cos^2 \theta m'}{4 \pi \rho^4}. \quad (\text{B.8})$$

Moreover, the Ricci scalar reads

$$R = -2 \frac{r m'' + 2 m'}{\rho^2} \quad (\text{B.9})$$

and

$$\begin{aligned} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = & 4 \frac{r^2 (m'')^2}{\rho^4} + 16 \frac{m' (2r^4 - 5a^2 r^2 \cos^2 \theta + a^4 \cos^4 \theta) - r m'' (r^4 - a^4 \cos^4 \theta)}{\rho^8} m' \\ & + 16 \frac{r^2 - 3a^2 \cos^2 \theta}{\rho^8} r^2 m m'' - 64 \frac{r^4 - 8a^2 r^2 \cos^2 \theta + 3a^4 \cos^4 \theta}{\rho^{10}} r m m' \\ & + 48 m^2 \frac{r^6 - 15a^2 r^4 \cos^2 \theta + 15a^4 r^2 \cos^4 \theta - a^6 \cos^6 \theta}{\rho^{12}}. \end{aligned} \quad (\text{B.10})$$

For the standard Kerr solution, one of course has the Ricci scalar $R = 0$ and

$$R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = 48 M^2 \frac{r^6 - 15a^2 r^4 \cos^2 \theta + 15a^4 r^2 \cos^4 \theta - a^6 \cos^6 \theta}{\rho^{12}}, \quad (\text{B.11})$$

which diverges for $r = 0$ at the equator $\theta = \pi/2$.

B.2 Central quantities

We can finally study the leading order behaviour of the energy-momentum tensor near $r = 0$ on assuming the expansions (3.7) and (3.8) with $\alpha = 1$. Since the general expressions are very cumbersome, we shall just consider the effective energy density and pressure terms on the equator ($\theta = \pi/2$) and on the axis of symmetry ($\theta = 0$).

For $\theta = \pi/2$, the leading terms in r/M for the energy density are given by

$$\begin{aligned} \epsilon \simeq & \frac{2m_1 - (1 + 2m_1)a_1^4 - (2 - 3m_1^2)a_1^2}{8 \pi r^2} \\ & + \frac{m_2(4 + a_1^2 - 6m_1 a_1^2 - 2a_1^4) - (9 - 4m_1 - 12m_1^2 + 12m_1 a_1^2 + 8a_1^2)a_1 a_2}{8 \pi M r}. \end{aligned} \quad (\text{B.12})$$

The radial pressure reads

$$p_r \simeq -\frac{2m_1 - (1 + m_1^2)a_1^2}{8 \pi r^2} - \frac{m_2(4 - 2m_1 a_1^2 + a_1^2) - (3 + 4m_1^2)a_1 a_2}{8 \pi M r}, \quad (\text{B.13})$$

the tensions

$$p_\theta \simeq -\frac{a_1^2 m_1^2}{8 \pi r^2} - \frac{2m_2 - (1 - 2m_1)m_1 a_1^2 - (3 - 4m_1^2)a_1 a_2}{8 \pi M r} \quad (\text{B.14})$$

and

$$\begin{aligned} p_\phi \simeq & \frac{a_1^2(1 + m_1^2 - 2m_1 a_1^2 - a_1^2)}{8 \pi r^2} \\ & - \frac{m_2(2 - 3a_1^2 - 2m_1 a_1^2 + 2a_1^4) - (3 + 4m_1 + 4m_1^2 - 12m_1 a_1^2 - 8a_1^2)a_1 a_2}{8 \pi M r}. \end{aligned} \quad (\text{B.15})$$

For $\theta = 0$, we find

$$\epsilon \simeq \frac{2 m_1 (1 - a_1^2 + a_1^4) - a_1^4 - a_1^6}{8 \pi (1 + a_1^2)^3 r^2} \tag{B.16}$$

$$+ \frac{m_2 (1 + a_1^2) (2 + a_1^2 + 2 a_1^4) - [3 + 2 m_1 (2 - 8 a_1^2 + a_1^4) + 11 a_1^2 (1 + a_1^2) + 3 a_1^6] a_1 a_2}{4 \pi (1 + a_1^2)^4 M r},$$

$$p_r \simeq - \frac{2 m_1 - a_1^2 (1 - a_1^2) (2 - 2 m_1 + a_1^2)}{8 \pi (1 + a_1^2)^3 r^2}$$

$$- \frac{m_2 (2 + 5 a_1^2 + 5 a_1^4 + 2 a_1^6) - [3 + 2 m_1 (2 + a_1^4) + 5 a_1^2 + 3 a_1^4 + a_1^6] a_1 a_2}{4 \pi (1 + a_1^2)^4 M r} \tag{B.17}$$

and

$$p_\theta = p_\phi \simeq - \frac{a_1^2 (1 + 2 m_1 + a_1^2)}{8 \pi (1 + a_1^2)^3 r^2}$$

$$- \frac{m_2 [1 + a_1^2 (2 + a_1^2)^2] + [2 m_1 (1 - 2 a_1^2) - 3 a_1^2 - 4 a_1^4 - a_1^6] a_1 a_2}{4 \pi (1 + a_1^2)^4 M r}. \tag{B.18}$$

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References

- [1] H. Maeda, *Quest for realistic non-singular black-hole geometries: regular-center type*, *JHEP* **11** (2022) 108 [[arXiv:2107.04791](https://arxiv.org/abs/2107.04791)] [[INSPIRE](#)].
- [2] R. Carballo-Rubio, F. Di Filippo, S. Liberati and M. Visser, *Singularity-free gravitational collapse: From regular black holes to horizonless objects*, [arXiv:2302.00028](https://arxiv.org/abs/2302.00028) [[INSPIRE](#)].
- [3] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press (1973) [[DOI:10.1017/cbo9780511524646](https://doi.org/10.1017/cbo9780511524646)].
- [4] C.W. Misner and D.H. Sharp, *Relativistic equations for adiabatic, spherically symmetric gravitational collapse*, *Phys. Rev.* **136** (1964) B571 [[INSPIRE](#)].
- [5] W.C. Hernandez and C.W. Misner, *Observer Time as a Coordinate in Relativistic Spherical Hydrodynamics*, *Astrophys. J.* **143** (1966) 452 [[INSPIRE](#)].
- [6] R.L. Arnowitt, S. Deser and C.W. Misner, *Dynamical Structure and Definition of Energy in General Relativity*, *Phys. Rev.* **116** (1959) 1322 [[INSPIRE](#)].
- [7] R. Casadio, *Geometry and thermodynamics of coherent quantum black holes*, *Int. J. Mod. Phys. D* **31** (2022) 2250128 [[arXiv:2103.00183](https://arxiv.org/abs/2103.00183)] [[INSPIRE](#)].
- [8] R. Casadio, A. Giusti and J. Ovalle, *Quantum Reissner-Nordström geometry: Singularity and Cauchy horizon*, *Phys. Rev. D* **105** (2022) 124026 [[arXiv:2203.03252](https://arxiv.org/abs/2203.03252)] [[INSPIRE](#)].
- [9] V.N. Lukash and V.N. Strokov, *Space-Times with Integrable Singularity*, *Int. J. Mod. Phys. A* **28** (2013) 1350007 [[arXiv:1301.5544](https://arxiv.org/abs/1301.5544)] [[INSPIRE](#)].

- [10] E.T. Newman and A.I. Janis, *Note on the Kerr spinning particle metric*, *J. Math. Phys.* **6** (1965) 915 [INSPIRE].
- [11] M. Gurses and G. Feza, *Lorentz Covariant Treatment of the Kerr-Schild Metric*, *J. Math. Phys.* **16** (1975) 2385 [INSPIRE].
- [12] C. Bambi and L. Modesto, *Rotating regular black holes*, *Phys. Lett. B* **721** (2013) 329 [arXiv:1302.6075] [INSPIRE].
- [13] R. Casadio, M. Lenzi and A. Ciarfella, *Quantum black holes in bootstrapped Newtonian gravity*, *Phys. Rev. D* **101** (2020) 124032 [arXiv:2002.00221] [INSPIRE].
- [14] R. Brustein, A.J.M. Medved and T. Simhon, *Black holes as frozen stars*, *Phys. Rev. D* **105** (2022) 024019 [arXiv:2109.10017] [INSPIRE].
- [15] Y. Yokokura, *Entropy-Area Law from Interior Semi-classical Degrees of Freedom*, arXiv:2207.14274 [INSPIRE].
- [16] R. Casadio, *Quantum dust cores of black holes*, arXiv:2304.06816 [INSPIRE].
- [17] T. Zhou and L. Modesto, *Geodesic incompleteness of some popular regular black holes*, *Phys. Rev. D* **107** (2023) 044016 [arXiv:2208.02557] [INSPIRE].
- [18] E. Contreras, J. Ovalle and R. Casadio, *Gravitational decoupling for axially symmetric systems and rotating black holes*, *Phys. Rev. D* **103** (2021) 044020 [arXiv:2101.08569] [INSPIRE].
- [19] R. Casadio, *A quantum bound on the compactness*, *Eur. Phys. J. C* **82** (2022) 10 [arXiv:2103.14582] [INSPIRE].
- [20] R. Casadio et al., *Configurational entropy of black hole quantum cores*, *Class. Quant. Grav.* **40** (2023) 075014 [arXiv:2206.10398] [INSPIRE].
- [21] R. Casadio and L. Tabarroni, *Slowly rotating quantum dust cores and black holes*, *Eur. Phys. J. Plus* **138** (2023) 104 [arXiv:2212.05514] [INSPIRE].
- [22] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, W.H. Freeman, San Francisco (1973) [INSPIRE].