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Stability and Stabilizability of Discrete-time Structured Linear Systems

Elena Zattoni * **Anna Maria Perdon** ** **Giuseppe Conte** **

* *Department of Electrical, Electronic, and Information Engineering "G. Marconi", Alma Mater Studiorum Università di Bologna, Bologna, Italy (e-mail: elena.zattoni@unibo.it)*
** *Accademia Marchigiana di Scienze, Lettere ed Arti, Ancona, Italy (e-mail: a.m.perdon@ieee.org; gconte@univpm.it)*

Abstract: This work presents a graph theoretic approach to the investigation of stability and stabilizability of discrete-time structured linear systems – i.e., discrete-time dynamical systems defined by linear maps whose entries are only known to be either zero or nonzero (unknown) values. The main result consists in a necessary and sufficient condition for each element of the family of systems represented by a given discrete-time structured linear system to be asymptotically stable. In particular, under the stated condition, convergence to zero of the free state evolution of each system of the family is shown to be achieved in a finite number of steps, through what will be referred to as a dead-beat behavior. The notions of essential state feedback and essential output injection are then introduced and a sufficient condition for stabilizability by essential state feedback and by essential output injection, respectively, is given. An obstruction to stabilizability by essential state feedback or by essential output injection, respectively, is also pointed out.

Keywords: Structured linear systems, discrete-time systems, stability, graph-theoretic approach.

1. INTRODUCTION

Introduced in the mid seventies by Lin (1974) and extensively investigated in the following thirthy years (see, e.g., Dion et al., 2003), structured linear systems are now attracting renewed attention and a considerable deal of research effort, as is documented, e.g., in (Ramos et al., 2022). A main reason for this renovated interest lies in the effectiveness of structured linear systems in modeling and analyzing multiagent and networked systems and in the ubiquitous role these latter have surged to (see, e.g., Commault, 2020; Commault et al., 2020; Zhang et al., 2021; Jia et al., 2022).

Structured linear systems are dynamical systems whose state-space description is given by linear maps whose entries are only known to be either zero or nonzero (unknown) values, where zero means that there is no connection between the related variables, while nonzero means that there is a connection, but the corresponding value is unknown. Thus, any structured linear system is representative of a whole family of linear systems in the usual sense. In this framework, it is of interest to investigate properties which are true for almost any value of the parameters and, therefore, are said to be *generic* or *structural* (Dion et al., 2003), or even for all values of the parameters.

Structured linear systems can be easily described by directed graphs. Indeed, several classic analysis and synthesis problems have been investigated in the context of structured linear systems, often by graph theoretic methods: notably, structural controllability and structural observability (Lin, 1974); the generic structure at infinity and the generic number of invariant zeros (or generic finite structure) (van der Woude, 1991); noninteracting control (Dion and Commault, 1993) and disturbance decoupling (Commault et al., 1997).

The literature of the eighties and nineties was mainly focused on structured systems as a means to ascertain properties and solve control problems by exploiting the poor information provided only by the location of the fixed zero entries in the system matrices. Instead, as mentioned above, the literature of the last two decades tends to emphasize the efficacy of structured systems and of the associated graphs for modeling and studying the interactions between a plurality of agents in complex systems and, more generally, in systems of systems (see, e.g., Rahmani et al., 2009; Chapman and Mesbahi, 2013; Gao et al., 2014; Monshizadeh et al., 2015; Harrison, 2016; Chen and Ren, 2021).

Indeed, a consolidated approach to the study of networks of dynamical agents leverages on the network topology to derive the network properties, while the behavior of each agent is modelled by relatively simple dynamics. In this topological perspective, multiagent networks are formalized by graphs where agents are represented by nodes and their mutual relations by edges (see, e.g., Mesbahi and Egerstedt, 2010). In the simplest case (see, e.g., Commault and Kibangou, 2020), each agent is described by a single integrator, while the direct connections between agents are only known to be either existent or nonexistent.

For the structured linear systems considered in this work, it is assumed that the input distribution matrix has one and only one nonzero entry in each of its columns, while

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the output distribution matrix has one and only one nonzero entry in each of its rows. These assumptions restrict the class of structured linear systems considered herein with respect to that dealt with in (Dion et al., 2003). However, this class, also considered, e.g., in (van Waarde et al., 2020; Trefois and Delvenne, 2015; Monshizadeh et al., 2014), has attracted noticeable interest since it can be effectually used to model networks of agents (Mesbahi and Egerstedt, 2010), quantum systems (Burgarth et al., 2013) and biological systems (Califano et al., 2018).

This work deals with stability and stabilizability in the discrete-time case, along the same lines of (Kirkoryan and Belabbas, 2014), of the class of structured linear systems defined above. The formal mathematical description of this class of discrete-time structured linear systems and the corresponding graphs are illustrated in Section 2. In Section 3, the main contribution, which is a necessary and sufficient condition for each element of the family of linear systems represented by a given discrete-time structured linear system to be asymptotically stable, is presented (Theorem 1). In particular, it is shown that, under the stated condition, convergence to zero of the free state evolution of each system of the considered family is achieved in a finite number of steps (dead-beat behavior). Then, the notions of essential state feedback and essential output injection are introduced. A sufficient condition for stabilizability by essential state feedback and by essential output injection, respectively, is given (Theorems 2 and 3). An obstruction to stabilizability by essential state feedback or by essential output injection, respectively, is also pointed out (Proposition 1). Some illustrative examples are worked out in Section 4. The conclusions are discussed in Section 5.

2. STRUCTURED LINEAR SYSTEMS AND GRAPHS

Structured linear systems are dynamical objects defined (in the discrete time) by equations of the form

$$\Sigma \equiv \begin{cases} x(t+1) = A\,x(t) + B\,u(t), \\ y(t) = C\,x(t), \end{cases} \qquad (1)$$

where $t \in \mathbb{N}$ is the discrete-time variable, $x \in \mathbb{R}^n$ the state, $u \in \mathbb{R}^m$ the input and $y \in \mathbb{R}^p$ the output. The matrices

$$A = [a_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,n}}, \; B = [b_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}, \; C = [c_{ij}]_{\substack{i=1,\dots,p\\ j=1,\dots,n}},$$

have entries which are either 0 or real, mutually independent, nonzero parameters. Moreover, it is assumed herein that B and C have one single element different from 0 in each column and, respectively, in each row. This means that each input variable appears in only one state equation and, likewise, each output variable coincides, except for a possible scale factor, with one state variable.

Structured linear systems can be used to model families of linear systems whose elements are characterized by specific values of the parameters appearing in (1) or uncertain systems where the parameters describe unknown relations between variables. Consequently, the variables x, u, y represent the state, the input and the output, respectively, of each single system in the family or of the uncertain system. The structured system captures the dynamical features that are common to all the members of the family of systems or that are independent of the variation of the uncertain parameters.

From another point of view, structured systems can be used to model specific aspects of the overall dynamics of a network of dynamical agents or of a complex system of systems. Hence, the component x_i of x gives a description of the status of the i-th agent in the network or of the i-th system. Agents or systems mutually influence each other and exchange information with the external environment through communication lines whose weights are expressed, respectively, by the parameters in the matrices A, B, C. The structured system captures the dynamical features of the mutual interaction between the agents or the systems and between them and the external environment, provided that these mutual interactions depend on the topology of the network and not on the values of the weights.

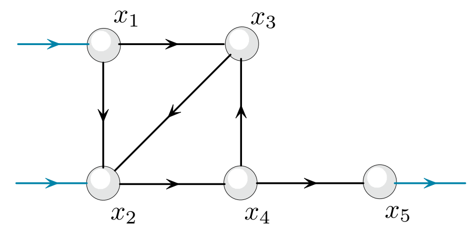

Fig. 1. A directed graph with $G^{in} = \{x_1, x_2\}$, $G^{out} = \{x_5\}$

Structured systems can be conveniently represented by graphs whose vertices are associated to the components of x and whose edges are associated to the nonzero elements of A, B, C. More precisely, given a structured linear system Σ of the form (1), let us consider a directed graph $(G, \mathcal{E})$, without multiple edges, where $G = \{x_1, \dots, x_n\}$ is the set of the vertices, while the set of the edges $\mathcal{E} \subseteq G \times G$ is defined by

$$\mathcal{E} = \{(x_j, x_i) \in G \times G \text{ such that } a_{ij} \neq 0\}.$$

Namely, $(x_j, x_i) \in \mathcal{E}$ means that the edge with the tail in x_j and the head in x_i belongs to $\mathcal{E}$. Moreover, let us define the two subsets of vertices $G^{in} \subseteq G$ and $G^{out} \subseteq G$ by

$$\begin{aligned} G^{in} &= \{x_{i_1}, \dots, x_{i_m} \in G, \text{ such that } x_{i_h} = x_i \text{ and } b_{ih} \neq 0\} \\ G^{out} &= \{x_{j_1}, \dots, x_{j_p} \in G, \text{ such that } x_{j_h} = x_j \text{ and } c_{hj} \neq 0\} \end{aligned}$$

Hence, the triple $((G, \mathcal{E}), G^{in}, G^{out}))$, which contains the same information of the parametric state-space representation (1), can be graphically represented by the directed graph $(G, \mathcal{E})$ in which the elements of G^{in} are marked by ingoing arrows and the element of G^{out} are marked by outgoing arrows (see, e.g., Fig. 1). In light of the correspondence between the parametric state-space representation (1) and the graph representation $((G, \mathcal{E}), G^{in}, G^{out})$, the notation $\Sigma((G, \mathcal{E}), G^{in}, G^{out})$ will be used henceforth to indicate the given structured linear system in either form. Also, the specification of the triplet may be dropped if it is clear from the context.

It is worth stressing that, through the graph representation, it is possible to relate dynamical properties of interest to graph theoretical ones, thus gaining in simplicity and understanding. The graph essentially captures the dynamical properties that do not depend on the specific values of the parameters appearing in the state space representation. These properties are generally referred to as *structural properties*.

3. MAIN RESULTS

The one-dimensional structured system with state-space parametric representation $\Sigma \equiv \{x(t+1) = ax(t)$ shows that, in general, the stability of the elements of the family of systems described by a structured system depends on the actual values of the parameters. However, it is interesting to investigate conditions under which all the elements, or, possibly, almost all, are asymptotically stable. To state the following result, which is a contribution in this direction, it is worth recalling that a discrete-time system Σ has a dead-beat behavior if, starting from any initial condition, the free state evolution reaches 0 in a finite number of time units and this is no greater than the system's dimension.

Theorem 1. Given a structured linear system Σ with state-space parametric representation (1) or, equivalently, with graph representation $((G,\mathcal{E}),G^{in},G^{out})$, the following properties are equivalent:

i) there are no loops in $((G,\mathcal{E}),G^{in},G^{out})$;
ii) all the elements of the family of systems described by (1) have a dead-beat behavior;
iii) all the elements of the family of systems described by (1) are asymptotically stable.

Proof. $i) \Rightarrow ii)$ Nilpotency of A in (1) for all the values of the parameters means that all the elements of the family of systems described by the structured linear system Σ have a dead-beat behavior. So, let us prove the implication by contradiction, assuming that A is not nilpotent and deriving that there is a loop in $((G,\mathcal{E}),G^{in},G^{out})$. If A is not nilpotent, then there is an element $a_{ij} \neq 0$ in A^n. Since $a_{ij} = \sum_h a_{ih}\, a_{hj}$, where, for $h = 1,\ldots,n$, a_{ih} is an element of A^{n-1} and a_{hj} is an element of A, $a_{ij} \neq 0$ implies that there exists $h_1 \in \{1,\ldots,n\}$ such that $a_{ih_1} \neq 0$ and $a_{h_1 j} \neq 0$. The last inequality means that there is an edge $(x_j, x_{h_1}) \in \mathcal{E}$. In turn, since $a_{ih_1} = \sum_h a_{ih}\, a_{hh_1}$, where, for $h = 1,\ldots,n$, a_{ih} is an element of A^{n-2} and a_{hh_1} is an element of A, by the same argument previously applied, it ensues that, for some $h_2 \in \{1,\ldots,n\}$, there is an edge $(x_{h_1}, x_{h_2}) \in \mathcal{E}$. By iterating the same reasoning, we find that there is a path $\{(x_j, x_{h_1}),\ldots,(x_{h_{n-1}}, x_{h_n})\}$ in $(G,\mathcal{E})$ that crosses $n+1$ vertices. Since there are n vertices in G, the path must be a loop. $ii) \Rightarrow iii)$ It is obvious. $iii) \Rightarrow i)$ We show by contradiction that the presence of loops is an obstruction to asymptotic stability. Assume that there is a loop $\{(x_{i_1}, x_{i_2}), (x_{i_2}, x_{i_3}),\ldots,(x_{i_k}, x_{i_1})\}$ in $(G,\mathcal{E})$ and take all the nonzero entries a_{ij} of A in such a way that $a_{ij} = \bar{a}_{ij} \geq \alpha > 1$. By initializing the corresponding linear system, say $\bar{\Sigma}$, at a point $x(0)$ such that $x_{i_1}(0) > 0$ and all the other coordinates are greater than or equal to 0, we have that the i_1-th coordinate of $x(k) = A^k x(0)$ satisfies $x_{i_1}(k) \geq \alpha^k x_{i_1}(0)$ and all the other coordinates are greater than or equal to 0. Then, $x_{i_1}(t)$ diverges as $t \to +\infty$ and $\bar{\Sigma}$ is not asymptotically stable. □

Remark 1. It is well known that no results similar to Theorem 1 hold for continuous-time structured systems (Dion et al., 2003, Section 2.3). Moreover, the presence of divergent behaviors in continuous-time structured systems is not necessarily related to the presence of cycles in their graph. For instance, the structured linear system

$$\Sigma \equiv \begin{cases} \dot{x}_1(t) = a_{12} x_2(t), \\ \dot{x}_2(t) = 0, \end{cases}$$

has no cycles in its graph. However, the motion originating at $x(0) = (a\ b)^\top$ with $b \neq 0$, i.e. $x(t) = (a + a_{12} b\, t\ \ b)^\top$, diverges for all the values of $a_{12} \neq 0$.

Remark 2. From Theorem 1, it ensues that the presence of a single unstable element in the family of systems described by a discrete-time structured linear system Σ implies the presence of a loop in $((G,\mathcal{E}),G^{in},G^{out})$. Hence, the family contains an unstable element like $\bar{\Sigma}$ considered in the *Only-if* part of the proof. In the space of the nonzero parameters of A, the point corresponding to $\bar{\Sigma}$ has a neighborhood of points with coordinates greater than 1, which define unstable systems. Then, asymptotic stability is not a generic property: unstable elements either are not present in the family or are in infinite number.

Remark 3. To illustrate the relevance of Theorem 1, the situation in which Σ describes the evolution of the displacement from an equilibrium point of a given dynamics is considered. In practical situations, the equilibrium point can represent a desired behavior or a situation of consensus among interconnected agents and the displacement from it can be due, e.g., to noise, unknown inputs or failures. The absence of loops in $((G,\mathcal{E}),G^{in},G^{out})$ guarantees that, after the disturbance has vanished, the dynamics returns to the equilibrium in n time instants at most for any value of the parameters. Although this requirement is quite strong, it is often desirable (e.g., for safety or security reasons). If it is not satisfied, the displacement from the equilibrium point originated by a perturbation with finite duration may diverge.

Another important property related to asymptotic stability is feedback stabilizability. In the case of a family of systems described by a structured linear system Σ, it is interesting to investigate if all the elements, or almost all of them, are stabilizable by a state feedback, which, obviously, depends on the specific value of the parameters. Since feedback stabilizability is implied by reachability, an obvious remark is that almost all the systems of the family are feedback stabilizable if Σ satisfies one of the equivalent conditions, expressed in terms of the graph representation, in (Dion et al., 2003, Theorem 1): i.e., if, in the terminology of that paper, Σ is generically controllable. However, the question is more complicated if structural reachability does not hold. A partial result can be given herein on the basis of Theorem 1 and the notion of *essential state feedback*, which the authors introduced for structured linear systems in (Conte et al., 2019).

Definition 1. (Conte et al., 2019, Definition 3) Given a structured linear system Σ, with parametric state-space representation (1) and graph representation $((G,\mathcal{E}),G^{in},G^{out})$, an *essential state feedback* is a set of edges

$$\mathcal{F} \subseteq G \times G^{in}.$$

The application of an essential state feedback $\mathcal{F}$ to the structured linear system $\Sigma((G,\mathcal{E}),G^{in},G^{out})$ gives rise to the compensated structured linear system $\Sigma^{\mathcal{F}}((G,\mathcal{E} \setminus\!\setminus \mathcal{F}),G^{in},G^{out})$, where $\mathcal{E} \setminus\!\setminus \mathcal{F} = (\mathcal{E} \cup \mathcal{F}) \setminus (\mathcal{E} \cap \mathcal{F}) = \{(x_j, x_i) \in \mathcal{E} \cup \mathcal{F}$ such that $(x_j, x_i) \notin \mathcal{E} \cap \mathcal{F}\}$.

An essential state feedback $\mathcal{F}$ modifies the graph representation $((G,\mathcal{E}),G^{in},G^{out})$ of Σ either by adding to $\mathcal{E}$ new edges of the form (x_j, x_i) with $x_j \in G$ and $x_i \in G^{in}$ or by removing from $\mathcal{E}$ existing edges of the same form.

In order to associate to $\mathcal{F}$ the relation $u = F\,x$, where

$$F = [f_{kj}]_{\substack{k=1,\dots,m\\ j=1,\dots,n}}$$

is an $m \times n$ matrix whose entries are real, mutually independent parameters, let us take the parameters f_{kj} so as to satisfy the following conditions:

- $f_{kj} \neq 0$ if and only if $(x_j, x_i) \in \mathcal{F}$ with $x_j \in G$ and $x_i = x_{i_k} \in G^{in}$ (i.e., x_i is the k-th element of G^{in});
- $f_{kj} = -a_{ij}/b_{ik}$ if $(x_j, x_i) \in \mathcal{F} \cap \mathcal{E}$ with $x_j \in G$ and $x_i = x_{i_k} \in G^{in}$ (note that x_i being the k-th element of G^{in} implies $b_{ik} \neq 0$ and $(x_j, x_i) \in \mathcal{E}$ implies $a_{ij} \neq 0$).

Note that, as a consequence of the second condition above, some of the nonzero parameters of F are constrained in that they are function of the parameters in A and B.

The compensated system $\Sigma^{\mathcal{F}}((G, \mathcal{E} \setminus\setminus \mathcal{F}), G^{in}, G^{out})$ is given in parametric form by the equations

$$\Sigma^{\mathcal{F}} \equiv \begin{cases} x(t+1) = (A + B\,F)\,x(t) + B\,u(t), \\ y(t) = C\,x(t). \end{cases} \quad (2)$$

Then, the following result about state feedback stabilization, which, in particular, applies if Σ is not structurally reachable, can be shown.

Theorem 2. The elements of the family of systems described by the structured linear system Σ with parametric state-space representation (1) and graph representation $((G, \mathcal{E}), G^{in}, G^{out})$ are all stabilizable by state feedback if each cycle in the graph representation $((G, \mathcal{E}), G^{in}, G^{out})$ crosses a vertex in G^{in}. Moreover, under the same hypothesis, they can be compensated by a state feedback in such a way to have a dead-beat behavior.

Proof. Let $((x_{i_1}, x_{i_2}), \dots, (x_{i_h}, x_{i_1}))$ be a simple loop in the graph representation $((G, \mathcal{E}), G^{in}, G^{out})$ and assume, without loss of generality, that x_{i_1} is the k-th element x_{i_k} of G^{in}. Then, both $a_{i_1 i_h} \in A$ and $b_{i_1 k} \in B$ are different from 0. By applying the essential state feedback $\mathcal{F}$ with $\mathcal{F} = \{(x_{i_h}, x_{i_1})\}$, whose associated $1 \times n$ matrix F is such that $f_{k i_h} = -a_{i_1 i_h}/b_{i_1 k}$ and all other elements are equal to 0, the edge (x_{i_h}, x_{i_1}) originally belonging to $\mathcal{E}$ does no longer appear in $\mathcal{E} \setminus\setminus \mathcal{F}$. Therefore, all the simple loops containing the edge (x_{i_h}, x_{i_1}) are not present anymore in the graph representation $((G, \mathcal{E} \setminus\setminus \mathcal{F}), G^{in}, G^{out})$ of the compensated structured linear system $\Sigma^{\mathcal{F}}$. By iterating the same procedure, all the loops in $((G, \mathcal{E}), G^{in}, G^{out})$ can be eliminated from the graph representation of the resulting compensated system. Hence, by virtue of Theorem 1, all the systems of the resulting family are asymptotically stable and, in particular, have a dead-beat behavior for all values of the parameters. □

The notion of essential state feedback finds its dual one in that of *essential output injection*, defined as follows.

Definition 2. Given a structured linear system Σ, with parametric state-space representation (1) and graph representation $((G, \mathcal{E}), G^{in}, G^{out})$, an *essential output injection* is a set of edges

$$\mathcal{K} \subseteq G^{out} \times G.$$

The application of an essential output injection $\mathcal{K}$ to the structured linear system $\Sigma((G, \mathcal{E}), G^{in}, G^{out})$ gives rise to the new system $\Sigma^{\mathcal{K}}((G, \mathcal{E} \setminus\setminus \mathcal{K}), G^{in}, G^{out})$, where $\mathcal{E} \setminus\setminus \mathcal{K} = (\mathcal{E} \cup \mathcal{K}) \setminus (\mathcal{E} \cap \mathcal{K}) = \{(x_j, x_i) \in \mathcal{E} \cup \mathcal{K}$ such that $(x_j, x_i) \notin \mathcal{E} \cap \mathcal{K}\}$.

Note that output injection is essentially a design tool and cannot be physically applied to any dynamical system. By an abuse of terms, the expression used in Definition 2 means that $\Sigma((G, \mathcal{E}), G^{in}, G^{out})$ and $\mathcal{K}$ are combined in the design of the new abstract system $\Sigma^{\mathcal{K}}((G, \mathcal{E} \setminus\setminus \mathcal{K}), G^{in}, G^{out})$. In particular, the new system $\Sigma^{\mathcal{K}}((G, \mathcal{E} \setminus\setminus \mathcal{K}), G^{in}, G^{out})$ is obtained either by adding to $\mathcal{E}$ new edges of the form (x_j, x_i) with $x_j \in G^{out}$ and $x_i \in G$ or by removing from $\mathcal{E}$ existing edges of the same form.

In order to associate to $\mathcal{K}$ the relation $x = K\,y$, where

$$K = [k_{i\ell}]_{\substack{i=1,\dots,n\\ \ell=1,\dots,p}}$$

is an $n \times p$ matrix whose entries are real, mutually independent parameters, let us take the parameters $k_{i\ell}$ so as to satisfy the following conditions:

- $k_{i\ell} \neq 0$ if and only if $(x_j, x_i) \in \mathcal{K}$ with $x_i \in G$ and $x_j = x_{i_\ell} \in G^{out}$ (i.e., x_j is the ℓ-th element of G^{out});
- $k_{i\ell} = -a_{ij}/c_{\ell j}$ if $(x_j, x_i) \in \mathcal{K} \cap \mathcal{E}$ with $x_i \in G$ and $x_j = x_{i_\ell} \in G^{out}$ (note that x_j being the ℓ-th element of G^{out} implies $c_{\ell j} \neq 0$ and $(x_j, x_i) \in \mathcal{E}$ implies $a_{ij} \neq 0$).

Like in the case of the essential state feedback, as a consequence of the second condition above, some of the nonzero parameters of K are function of those appearing in (1).

According to the conditions above, the structured linear system $\Sigma^{\mathcal{K}}((G, \mathcal{E} \setminus\setminus \mathcal{K}), G^{in}, G^{out})$ has the state-space parametric form

$$\Sigma^{\mathcal{K}} \equiv \begin{cases} z(t+1) = (A + K\,C)\,z(t) + B\,u(t), \\ y(t) = C\,z(t). \end{cases} \quad (3)$$

In (3), the state variable is indicated by z (instead of x as in (1)) to remark that $\Sigma^{\mathcal{K}}((G, \mathcal{E} \setminus\setminus \mathcal{K}), G^{in}, G^{out})$ is obtained by an abstract construction which does not correspond to any physical operation on $\Sigma((G, \mathcal{E}), G^{in}, G^{out})$.

Indeed, the construction of new structured linear systems by means of essential output injections is related to the synthesis of observers. In this regard, the following theorem is given.

Theorem 3. The elements of the family of systems described by the structured linear system Σ with parametric state-space representation (1) and graph representation $((G, \mathcal{E}), G^{in}, G^{out})$ are all stabilizable by output injection if each cycle in the graph representation $((G, \mathcal{E}), G^{in}, G^{out})$ crosses a vertex in G^{out}. Moreover, under the same hypothesis, for each element of Σ there exists an output injection such that the corresponding element of the resulting new structured linear system has a dead-beat behavior.

Proof. It follows from Theorem 2 by duality arguments. □

Remark 4. It is important to note that, under the hypothesis of Theorem 3, for each element of the family of systems described by Σ it is possible to synthesize a dead-beat observer, namely one whose observation error goes to 0 in finite time.

We recall (Conte et al., 2019, Definition 1) that, given a structured linear system Σ with graph representation $((G, \mathcal{E}), G^{in}, G^{out})$, a subset of vertices $V \subseteq G$ is said to be invariant for Σ if $(x_j, x_i) \in \mathcal{E}$ with $x_j \in V$ implies $x_i \in V$. An invariant subset V for Σ defines a subsystem of Σ, denoted by $\Sigma((V, \mathcal{E}|_V), V^{in}, V^{out})$, where

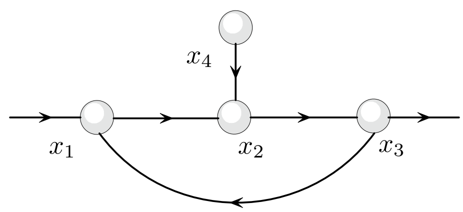

Fig. 2. Graph representation of the unobservable and unreachable structured linear system Σ described by (4)

$\mathcal{E}|_V = \{(x_j, x_i) \in \mathcal{E}$ such that $x_j, x_i \in V\}$, $V^{in} = V \cap G^{in}$, $V^{out} = V \cap G^{out}$. Also, one can consider the dynamics induced by that of Σ on $G \setminus V$. In particular, the smallest invariant subset $V_R \subseteq G$ containing G^{in} defines the smallest structured linear subsystem Σ_R containing the reachable subsystem of each element of the family defined by Σ. Similarly, the largest invariant subset $V_O \subseteq G$ contained in $G \setminus G^{out}$ defines the largest structured subsystem Σ_O contained in the unobservable subsystem of each element of the family defined by Σ. With these notions, the following results, which points out an obstruction to stabilizability can stated.

Proposition 1. Given a structured linear system Σ with parametric state-space representation (1) and graph representation $((G, \mathcal{E}), G^{in}, G^{out})$, there are infinitely many elements of the family of systems described by Σ that are not stabilizable by output injection if $\mathcal{E}|_{V_O}$ contains at least one cycle. Similarly, there are infinitely many elements of the family of systems described by Σ that are not stabilizable by state feedback if $\mathcal{E}|_{G \setminus V_R}$ contains at least one cycle.

Proof. The conclusions follow from Theorem 3 and Theorem 2, respectively. □

4. EXAMPLES

The examples of this section illustrate the effect of essential state feedbacks and of essential output injections in relation to the stabilization of structured linear systems in the light of Theorem 2 and Theorem 3.

4.1 Example 1

The structured linear system

$$\Sigma \equiv \begin{cases} x_1(t+1) = a_{13}\, x_3(t) + b_{11}\, u_1(t), \\ x_2(t+1) = a_{21}\, x_1(t) + a_{24}\, x_4(t), \\ x_3(t+1) = a_{32}\, x_2(t), \\ x_4(t+1) = 0, \\ \quad y_1(t) = c_{13}\, x_3(t), \end{cases} \tag{4}$$

whose graph representation is shown in Fig. 2, is unreachable and unobservable. In particular, the state $x = (0\ \ 0\ \ 0\ \ 1)^\top$ is not reachable and the state $x = (a_{24}\ \ 0\ \ 0\ \ -a_{21})$ is not observable for any value of the parameters. Moreover, due to the presence of the cycle $\{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}$ that crosses the vertices x_1, x_2, x_3, not all the systems of the family described by Σ are asymptotically stable (Theorem 1). However, since $x_1 \in G^{in}$, all the systems of the family described by Σ are stabilizable by state feedback (Theorem 2). Moreover, since $x_3 \in G^{out}$, all the systems of the family described by Σ are stabilizable by output injection (Theorem 3). The essential state feedback $\mathcal{F} = \{(x_3, x_1)\} \subseteq G \times G^{in}$, whose associated matrix is $F = (0\ 0\ {-a_{13}/b_{11}}\ 0)$, actually removes the edge (x_3, x_1), thus eliminating the cycle that can cause instability. The compensated system $\Sigma^{\mathcal{F}}$ is described by

$$\Sigma^{\mathcal{F}} \equiv \begin{cases} x_1(t+1) = 0, \\ x_2(t+1) = a_{21}\, x_1(t) + a_{24}\, x_4(t), \\ x_3(t+1) = a_{32}\, x_2(t), \\ x_4(t+1) = 0, \\ \quad y(t) = c_{13}\, x_3(t), \end{cases}$$

and its graph is shown in Fig. 3. Similarly, the essential output injection $\mathcal{K} = \{(x_3, x_1)\} \subseteq G^{out} \times G$, whose associated matrix is $K = (-a_{13}/c_{13}\ \ 0\ \ 0\ \ 0)$, also removes the edge (x_3, x_1), thus eliminating the cycle which can cause instability. The new system $\Sigma^{\mathcal{K}}$ obtained by the output injection is described by the same equations of $\Sigma^{\mathcal{F}}$, where, consistently with (3), the state variables are denoted by z_i, with $i = 1, \ldots, 4$.

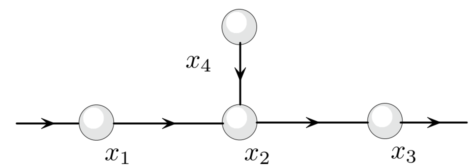

Fig. 3. Graph representation of the system obtained by state feedback or by output injection from the structured system Σ described by (4)

4.2 Example 2

The structured linear system

$$\Sigma \equiv \begin{cases} x_1(t+1) = b_{11}\, u_1(t), \\ x_2(t+1) = a_{21}\, x_1(t) + a_{23}\, x_3(t), \\ x_3(t+1) = a_{32}\, x_2(t), \\ \quad y_1(t) = c_{12}\, x_2(t), \end{cases} \tag{5}$$

whose graph representation is shown in Fig. 4, is unobservable for any value of the parameters. Moreover, neither the condition of Theorem 2 nor the condition of Theorem 3 are satisfied.

In particular, the subset V_O (i.e., the largest invariant subset for Σ cointained in $G \setminus G^{out}$) is given by $\{x_2, x_3\}$. Moreover, $\Sigma((V_O, \mathcal{E}|_{V_O}), V_O^{in}, V_O^{out})$ contains the cycle $\{(x_2, x_3), (x_3, x_2)\}$. Hence, by Proposition 1, there are infinitely many elements in the family of systems described by Σ that are not stabilizable by output injection. Nevertheless, each element in the family of systems described by Σ is stabilizable by state feedback and it can be compensated in such a way to get a dead-beat behavior. In fact, the system $\Sigma^{\mathcal{F}}$ obtained with the application of the essential state feedback $\mathcal{F} = \{(x_2, x_1)\}$, is described by

$$\Sigma \equiv \begin{cases} x_1(t+1) = b_{11} f_{12}\, x_2(t), \\ x_2(t+1) = a_{21}\, x_1(t) + a_{23}\, x_3(t), \\ x_3(t+1) = a_{32}\, x_2(t), \\ \quad y_1(t) = c_{11} x_1(t) \end{cases}$$

and, if f_{12} is taken such that $f_{12} = -a_{23}\, a_{32} (b_{11} a_{21})^{-1}$, the dynamic matrix of $\Sigma^{\mathcal{F}}$ becomes nilpotent for all values of the parameters, since $(A + B\, F)^3 = 0$. This example suggests that the condition of Theorem 2 can be weakened. In particular, one can conjecture that the divergent behavior associated to a cycle that cannot be eliminated by an

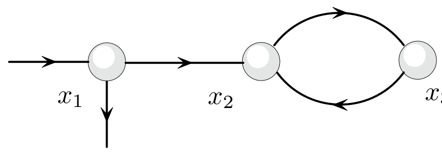

Fig. 4. Graph representation of the unobservable and unreachable structured system Σ described by (5)

essential state feedback can be counteracted by choosing, for each value of the parameters, a suitable state feedback, if there exists a single path from a vertex of G^{in} to a vertex of the cycle and other conditions are satisfied. These are that there is no path from any vertex of the cycle to a vertex of any other cycle, that the path between the vertex of G^{in} and the vertex of the cycle has a length equal to that of the cycle minus 1 and that there is no path between any of its vertices and a vertex of any other cycle.

5. CONCLUSIONS

It has been shown that stability and stabilization of discrete-time structured linear systems can be investigated and characterized in terms of graph representations. This facilitates the analysis of such properties and it allows the statement of results that hold for the elements of the family of systems described by a given structured linear systems. The condition for stabilizability by state feedback or by output injection of all the elements of a family of systems described by a linear structured system given in Theorem 2 and in Theorem 3 can be weakened, as suggested by Example 4, by asking that each cycle in the graph representation satisfies certain constraints. Investigations in this direction will be the object of future work.

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