

Research Article

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Existence Results for the Conformal Dirac–Einstein System

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Abstract: We consider the coupled system given by the first variation of the conformal Dirac–Einstein functional. We will show existence of solutions by means of perturbation methods.

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1 Introduction

Let $(M, g, \Sigma M)$ be a closed (compact, without boundary) three-dimensional Riemannian Spin manifold where ΣM is its spin bundle (we refer the reader to [10] for a good introduction to spin geometry and to [14] for an introduction and basic tools used in conformal geometry that will be used later in this paper). We denote by L_g the conformal Laplacian of g and by D_g the Dirac operator. We consider the energy functional

$$E_M(v, \psi) = \frac{1}{2} \int_M (vL_g v + \langle D_g \psi, \psi \rangle - |v|^2 |\psi|^2) \, d\text{vol}_g$$

and we take its first variation on the related Sobolev space $H^1(M) \times H^{\frac{1}{2}}(\Sigma M)$; therefore its critical points satisfy the coupled system

$$\begin{cases} L_g v = |\psi|^2 v, \\ D_g \psi = |v|^2 \psi, \end{cases} \quad \text{on } M.$$

This functional arises as the conformal version in the description of a super-symmetric model consisting of coupling gravity with fermionic interaction and it generalizes the classical Hilbert–Einstein energy functional, see for instance [4, 9, 13].

Indeed, the total energy functional consists of the Hilbert–Einstein energy which is the total curvature, coupled with a fermionic action. Now, since the energy of the system is invariant under the group of diffeomorphisms of M , when one restricts it to a fixed conformal class of a given Riemannian metric g , the functional E_M appears.

In particular, due to the conformal invariance, the Palais–Smale compactness condition is violated by this functional and in addition, due to the presence of the Dirac operator, it is strongly indefinite.

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Regarding the first issue, in [20] the authors studied the lack of compactness and gave a precise description of the bubbling phenomena, characterizing the behavior of the Palais–Smale sequences, in the spirit of classical works [3, 5, 15, 16, 22–24]. For the strong indefiniteness difficulty, in [17–19] general functionals with these features are studied by using methods based on a homological approach. Notice that in our situation one cannot apply these homological approaches because of the violation of compactness stated above.

In this paper, we are concerned with the existence of solutions to the coupled system, by using a perturbation approach, starting from the sphere \mathbb{S}^3 equipped with its standard metric $g_{\mathbb{S}^3}$. Therefore, let K be a function of the form $K = 1 + \varepsilon k$, where k is a function that satisfies suitable assumptions to be determined later; we consider the functional

$$\mathcal{E}(v, \psi) = \frac{1}{2} \int_{\mathbb{S}^3} (vL_{g_{\mathbb{S}^3}}v + \langle D_{g_{\mathbb{S}^3}}\psi, \psi \rangle - K|v|^2|\psi|^2) d \operatorname{vol}_{g_{\mathbb{S}^3}}$$

and we will focus on the existence of solutions to the following coupled system:

$$\begin{cases} L_{g_{\mathbb{S}^3}}v = K|\psi|^2v, \\ D_{g_{\mathbb{S}^3}}\psi = Kv^2\psi, \end{cases} \quad \text{on } \mathbb{S}^3. \quad (1.1)$$

Notice that these solutions converge to the standard bubbles when the parameter ε tends to zero; here we called standard bubbles the solutions of our equation in \mathbb{S}^3 (or \mathbb{R}^3 via stereographic projection), see [20]. This is expected from the description of the Palais–Smale sequences of the functional E_M , but it remains open whether all the solutions on the sphere with positive scalar component are in fact standard ones.

Let us denote by $\pi : \mathbb{S}^3 \setminus \{\operatorname{sp}\} \rightarrow \mathbb{R}^3$ the stereographic projection, where sp is the south pole. Our main result is the following:

Theorem 1.1. *Let $k \in C^2(\mathbb{S}^3)$ be a Morse function on \mathbb{S}^3 such that the south pole is not a critical point. Let us set $h = k \circ \pi^{-1}$ and suppose that*

- (i) $\Delta h(\xi) \neq 0$ for all $\xi \in \operatorname{crit}[h]$,
- (ii) $\sum_{\xi \in \operatorname{crit}[h], \Delta h(\xi) < 0} (-1)^{m(h, \xi)} \neq -1$,

where Δ is the standard Laplacian operator on \mathbb{R}^3 , $\operatorname{crit}[h]$ denotes the set of critical points of h and $m(h, \xi)$ is the Morse index of h at a critical point ξ . Then there exists $\varepsilon_0 > 0$ such that for $K = 1 + \varepsilon k$ and $|\varepsilon| < \varepsilon_0$, system (1.1) has a solution.

The condition on the critical point at the south pole of the sphere is needed since we are going to use the standard stereographic projection π , however this condition can be always satisfied by making a unitary transformation which does not affect the generality of the result. In fact, assumptions (i) and (ii) can be formulated using directly the function k on the sphere and its Laplacian $\Delta_{\mathbb{S}^3}k$, but since we are shifting the analysis on \mathbb{R}^3 we stated the assumptions using h .

The previous result is the analogous of several ones obtained with this kind of hypothesis of Bahri–Coron type on the function k : for instance, for the standard Riemannian case of prescribing the scalar curvature and its generalization to the Q_γ curvature see [2, 6, 7]; in the case of prescribing the Webster curvature in the CR setting and its fractional generalization see [21] and [8]; for the spinorial Yamabe type equations involving the Dirac operator on the sphere see [12].

The idea of the proof follows the abstract perturbation method introduced in [1]. The difficulties in dealing with such system of equations come from the strongly indefiniteness of one of the operator involved and the characterization of the critical manifold of the unperturbed problem. Fortunately, the first difficulty does not affect our analysis in this situation (even though it was a crucial difficulty to circumvent in the general problem as in [20]). The second difficulty is actually central in our situation. In fact, even after characterizing the critical manifold and showing its non-degeneracy, when going through the finite-dimensional reduction of the functional, another kind of degeneracy appears which is due to the invariance with respect to one of the parameters of the problem (see Remark 3.6). This constitute a major bifurcation from the problems found in the literature.

2 Notations and Definitions

Let (M, g) be a closed (compact, without boundary) three-dimensional Riemannian manifold.

We start to describe briefly the first operator appearing in the system. We denote by L_g the conformal Laplacian acting on functions

$$L_g = -\Delta_g + \frac{1}{8}R_g.$$

Here Δ_g is the standard Laplace–Beltrami operator and R_g is the scalar curvature. L_g is a conformally covariant operator. More precisely, given a metric $\tilde{g} = f^2g$ in the conformal class of g , we have

$$L_{\tilde{g}}u = f^{-\frac{5}{2}}L_g(f^{\frac{1}{2}}u).$$

We recall that the usual Sobolev space on M , denoted by $H^1(M)$, continuously embeds in $L^p(M)$ for $1 \leq p \leq 6$. Moreover, for $1 \leq p < 6$, the embedding is compact. In particular, if we assume M to be the sphere

$$\mathbb{S}^3 = \{(x', x_4) \in \mathbb{R}^3 \times \mathbb{R} : |x'|^2 + x_4^2 = 1\}$$

equipped with its standard metric $g_{\mathbb{S}^3}$, then it is possible to identify $\mathbb{S}^3 \setminus \{\text{sp}\}$, being $\text{sp} = (0, -1)$ the south pole, with \mathbb{R}^3 , by means of the stereographic projection

$$\begin{aligned} \pi : \mathbb{S}^3 \setminus \{\text{sp}\} &\rightarrow \mathbb{R}^3, \\ (x, x_4) &\mapsto y = \frac{x}{1 + x_4}. \end{aligned}$$

The standard metric $g_{\mathbb{R}^3}$ on \mathbb{R}^3 and the metric $\tilde{g} = (\pi^{-1})^*g_{\mathbb{S}^3}$ are conformal, more precisely $\tilde{g} = f^2g_{\mathbb{R}^3}$, with $f = \frac{2}{1+|y|^2}$. Thus the standard conformal Laplacian on the sphere $L_{g_{\mathbb{S}^3}}$ and the one on \mathbb{R}^3 , which we denote as usual $L_{g_{\mathbb{R}^3}} = -\Delta$, are related by the following identity:

$$L_{g_{\mathbb{S}^3}}v = [f^{-\frac{5}{2}}(-\Delta)(f^{\frac{1}{2}}v \circ \pi^{-1})] \circ \pi, \quad v \in H^1(\mathbb{S}^3).$$

Now, let us describe the second operator involved. Let ΣM be the canonical spinor bundle associated to M , whose sections are simply called spinors on M . This bundle is endowed with a natural Clifford multiplication

$$\text{Cliff} : C^\infty(TM \otimes \Sigma M) \rightarrow C^\infty(\Sigma M),$$

a hermitian metric and a natural metric connection

$$\nabla^\Sigma : C^\infty(\Sigma M) \rightarrow C^\infty(T^*M \otimes \Sigma M).$$

We denote by D_g the Dirac operator acting on spinors

$$D_g : C^\infty(\Sigma M) \rightarrow C^\infty(\Sigma M), \quad D_g = \text{Cliff} \circ \nabla^\Sigma,$$

where the composition $\text{Cliff} \circ \nabla^\Sigma$ is meaningful provided that we identify $T^*M \simeq TM$ by means of the metric g . We also have a conformal invariance that in our situation, $\tilde{g} = f^2g$, reads as follows: there exists an isomorphism of vector bundles $F : \Sigma(M, g) \rightarrow \Sigma(M, \tilde{g})$ such that

$$D_{\tilde{g}}\psi = F[f^{-2}D_g(fF^{-1}\psi)]. \quad (2.1)$$

The functional space that we are going to define is the Sobolev space $H^{\frac{1}{2}}(\Sigma M)$. First we recall that the Dirac operator D_g on a compact manifold is essentially self-adjoint in $L^2(\Sigma M)$, has compact resolvent and there exists a complete L^2 -orthonormal basis of eigenspinors $\{\psi_i\}_{i \in \mathbb{Z}}$ of the operator

$$D_g\psi_i = \lambda_i\psi_i,$$

and the eigenvalues $\{\lambda_i\}_{i \in \mathbb{Z}}$ are unbounded, that is, $|\lambda_i| \rightarrow \infty$ as $|i| \rightarrow \infty$. In this way every function in $L^2(\Sigma M)$ has a representation in this basis, namely:

$$\psi = \sum_{i \in \mathbb{Z}} a_i\psi_i, \quad \psi \in L^2(\Sigma M).$$

We define the unbounded operator $|D_g|^s : L^2(\Sigma M) \rightarrow L^2(\Sigma M)$ by

$$|D_g|^s(\psi) = \sum_{i \in \mathbb{Z}} a_i |\lambda_i|^s \psi_i$$

and we denote by $H^s(\Sigma M)$ the domain of $|D_g|^s$, namely $\psi \in H^s(\Sigma M)$ if and only if

$$\sum_{i \in \mathbb{Z}} a_i^2 |\lambda_i|^{2s} < +\infty.$$

Note that $H^s(\Sigma M)$ coincides with the usual Sobolev space $W^{s,2}(\Sigma M)$ and for $s < 0$, $H^s(\Sigma M)$ is defined as the dual of $H^{-s}(\Sigma M)$.

For $s > 0$, we define the following inner product: for $\psi, \phi \in H^s(\Sigma M)$,

$$\langle \psi, \phi \rangle_s = \langle |D_g|^s \psi, |D_g|^s \phi \rangle_{L^2},$$

which induces an equivalent norm in $H^s(\Sigma M)$; we will take

$$\langle \psi, \psi \rangle := \langle \psi, \psi \rangle_{\frac{1}{2}} = \|\psi\|^2$$

as our standard norm for the space $H^{\frac{1}{2}}(\Sigma M)$. In this case as well, the embedding $H^s(\Sigma M) \hookrightarrow L^p(\Sigma M)$ is continuous for $1 \leq p \leq 3$ and it is compact if $1 \leq p < 3$.

Then we decompose $H^{\frac{1}{2}}(\Sigma M)$ in a natural way. Let us consider the L^2 -orthonormal basis of eigenspinors $\{\psi_i\}_{i \in \mathbb{Z}}$: we denote by ψ_i^- the eigenspinors with negative eigenvalue, ψ_i^+ the eigenspinors with positive eigenvalue and ψ_i^0 the eigenspinors with zero eigenvalue; we also recall that the kernel of D_g is finite dimensional. Now we set

$$H^{\frac{1}{2},-} := \overline{\text{span}\{\psi_i^-\}_{i \in \mathbb{Z}}}, \quad H^{\frac{1}{2},0} := \text{span}\{\psi_i^0\}_{i \in \mathbb{Z}}, \quad H^{\frac{1}{2},+} := \overline{\text{span}\{\psi_i^+\}_{i \in \mathbb{Z}}},$$

where the closure is taken with respect to the $H^{\frac{1}{2}}$ -topology. Therefore we have the orthogonal decomposition of $H^{\frac{1}{2}}(\Sigma M)$, which reads as

$$H^{\frac{1}{2}}(\Sigma M) = H^{\frac{1}{2},-} \oplus H^{\frac{1}{2},0} \oplus H^{\frac{1}{2},+}.$$

Also, we let P^+ and P^- be the projectors on $H^{\frac{1}{2},+}$ and $H^{\frac{1}{2},-}$, respectively.

Again, if we assume M to be the sphere \mathbb{S}^3 and we identify \mathbb{S}^3 minus the south pole with \mathbb{R}^3 via stereographic projection, the conformal invariance of the Dirac operator reads as

$$D_{g_{\mathbb{S}^3}} \psi = F\{[f^{-2}D(fF^{-1}(\psi \circ \pi^{-1}))] \circ \pi\}, \quad \psi \in H^{\frac{1}{2}}(\Sigma \mathbb{S}^3),$$

where $D_{g_{\mathbb{S}^3}}$ and $D_{g_{\mathbb{R}^3}} = D$ denote the Dirac operators on the standard sphere and \mathbb{R}^3 , respectively; moreover $f = \frac{2}{1+|y|^2}$ and $F : \Sigma(\mathbb{R}^3, g_{\mathbb{R}^3}) \rightarrow \Sigma(\mathbb{S}^3, g_{\mathbb{S}^3})$ the isomorphism of vector bundles in (2.1).

In the sequel we will need the following function spaces on \mathbb{R}^3 :

$$\begin{aligned} D^{\frac{1}{2}}(\Sigma \mathbb{R}^3) &= \{\psi \in L^3(\Sigma \mathbb{R}^3) : |\xi|^{\frac{1}{2}} |\widehat{\psi}| \in L^2(\mathbb{R}^3)\}, \\ D^1(\mathbb{R}^3) &= \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}. \end{aligned}$$

Here $\widehat{\psi}$ is the Fourier transform of ψ . Moreover, for a more detailed exposition on conformal and spin geometry as well as on the functional spaces involved we address the reader to [10, 12, 13] and the references therein.

3 Proof of the Main Result

Our existence result will be obtained by means of the abstract perturbation method illustrated in [1]. We recall it in the following theorem and then we will show how it can be applied in our setting.

Theorem 3.1. (see [1]) *Let A be a Hilbert space and assume $J_0 \in C^2(A, \mathbb{R})$ satisfies the following conditions:*

- (1) J_0 has a finite-dimensional manifold Z of critical points,
- (2) $J_0''(z)$ is a Fredholm operator of index zero for every $z \in Z$,
- (3) $T_z Z = \ker J_0''(z)$ for every $z \in Z$.

For $G \in C^2(A, \mathbb{R})$, let $\varepsilon > 0$ small enough, we denote by $J_\varepsilon = J_0 - \varepsilon G$ the perturbed functional, by V the orthogonal complement of $T_z Z$ in A and by $P : A \rightarrow V$ the orthogonal projection. Then, for any $z \in Z$, there exists $v(z) \in V$ such that $P(J'_\varepsilon(z + v(z))) = 0$.

Moreover, if there exists a compact set $\Omega \subset Z$ such that $J_\varepsilon|_Z$ has a critical point $z \in \Omega$, then $z + v(z)$ is a critical point of the perturbed functional J_ε in A .

In order to apply the previous result to our situation, we introduce the map

$$H^1(\mathbb{S}^3) \times H^{\frac{1}{2}}(\Sigma\mathbb{S}^3) \ni (v, \psi) \mapsto (u, \phi) = (f^{\frac{1}{2}}v \circ \pi^{-1}, fF^{-1}(\psi \circ \pi^{-1})),$$

which gives a one to one correspondence between solutions to (1.1) on \mathbb{S}^3 and solutions to the equivalent system on \mathbb{R}^3

$$\begin{cases} -\Delta u = H|\phi|^2 u, \\ D\psi = Hu^2 \phi, \end{cases} \quad \text{on } \mathbb{R}^3,$$

where we set $H = K \circ \pi^{-1}$. Hence let us consider this last problem and let us denote

$$A = D^1(\mathbb{R}^3) \times D^{\frac{1}{2}}(\Sigma\mathbb{R}^3).$$

We take $w = (u, \psi) \in A$ and we set

$$\begin{aligned} J_0(w) &= \frac{1}{2} \int_{\mathbb{R}^3} \left(-u\Delta u + \langle D\phi, \phi \rangle - |u|^2|\phi|^2 \right), \\ G(w) &= \frac{1}{2} \int_{\mathbb{R}^3} h|u|^2|\phi|^2, \\ J_\varepsilon(w) &= J_0(w) - \varepsilon G(w) \end{aligned}$$

with $h = k \circ \pi^{-1}$. We are going to define the manifold of critical points of J_0 . Let $\lambda \in \mathbb{R}^+$, $y, \xi \in \mathbb{R}^3$, $a \in \Sigma\mathbb{R}^3$ with $|a| = 1$, it is well known that the functions

$$\bar{U}_{\lambda, \xi}(y) = \sqrt[4]{3} \frac{\lambda^{\frac{1}{2}}}{(\lambda^2 + |y - \xi|^2)^{\frac{1}{2}}}$$

are a family of positive solutions to $-\Delta u = u^5$ in \mathbb{R}^3 and the spinors

$$\bar{\Phi}_{\lambda, \xi, a}(x) = \frac{2\lambda}{(\lambda^2 + |y - \xi|^2)^{\frac{3}{2}}} (\lambda - (y - \xi)) \cdot a$$

solve $D\phi = \frac{3}{2}|\phi|\phi$ in $\Sigma\mathbb{R}^3$. Using this fact, and the equality $|\bar{\Phi}_{\lambda, \xi, a}| = \frac{2}{1+|y|^2}$, one can check that the pairs

$$(U_{\lambda, \xi}, \Phi_{\lambda, \xi, a}) = \left(\sqrt[4]{3} \bar{U}_{\lambda, \xi}, \frac{\sqrt{3}}{2} \bar{\Phi}_{\lambda, \xi, a} \right) \in A$$

are critical points of J_0 . Hence

$$Z = \{W_{\lambda, \xi, a} = (U_{\lambda, \xi}, \Phi_{\lambda, \xi, a}) : \lambda \in \mathbb{R}^+, \xi \in \mathbb{R}^3 \text{ and } a \in \Sigma\mathbb{R}^3, |a| = 1\} \subset A$$

is a seven-dimensional manifold of critical points of J_0 . Let us fix any $a_0 \in \Sigma\mathbb{R}^3$ with $|a_0| = 1$, in the sequel we will use the notation $U_0 = U_{1,0}$, $\Phi_0 = \Phi_{1,0,a_0}$ and $W_0 = (U_0, \Phi_0)$.

Now we will check assumption (2) in Theorem 3.1. We have

$$\begin{aligned} \langle J''_0(W_{\lambda, \xi, a})[w_1], w_2 \rangle &= \int_{\mathbb{R}^3} \left(-u_2 \Delta u_1 - u_2 u_1 |\Phi_{\lambda, \xi, a}|^2 - 2u_2 U_{\lambda, \xi} \langle \Phi_{\lambda, \xi, a}, \phi_1 \rangle \right) \\ &\quad + \int_{\mathbb{R}^3} \left(\langle D\phi_1 - |U_{\lambda, \xi}|^2 \phi_1, \phi_2 \rangle - 2U_{\lambda, \xi} u_1 \langle \Phi_{\lambda, \xi, a}, \phi_2 \rangle \right). \end{aligned}$$

Therefore J''_0 is a compact perturbation of the identity, hence it is a Fredholm operator of index zero for all $W_{\lambda, \xi, a} \in Z$.

Now it remains to check that $T_{W_{\lambda, \xi, a}} Z = \ker J''_0(W_{\lambda, \xi, a})$ for every $\lambda \in \mathbb{R}^+$, $\xi \in \mathbb{R}^3$ and $a \in \Sigma\mathbb{R}^3$ with $|a| = 1$. Since J''_0 is invariant with respect to translations and dilations, it will be enough to prove $T_{W_0} Z = \ker J''_0(W_0)$. We will need the following remark.

Remark 3.2. Let $\lambda_1 = \frac{3}{4}$ and $\mu_1 = \frac{3}{2}$. The map $(v, \psi) \mapsto (v, \eta) = (\mu_1^{-\frac{1}{2}}v, \lambda_1^{-\frac{1}{2}}\psi)$ is a one-to-one correspondence between solution to (1.1) on \mathbb{S}^3 and the equivalent rescaled system

$$\begin{cases} L_{g_{\mathbb{S}^3}} v = \lambda_1 |\eta|^2 v, \\ D_{g_{\mathbb{S}^3}} \eta = \mu_1 v^2 \eta, \end{cases} \quad \text{on } \mathbb{S}^3,$$

which in turn it is equivalent to

$$\begin{cases} -\Delta u = \lambda_1 |\phi|^2 u, \\ D\phi = \mu_1 u^2 \phi, \end{cases} \quad \text{on } \mathbb{R}^3 \quad (3.1)$$

by means of the stereographic projection. Notice that (3.1) arises as the first variation of the functional

$$\tilde{J}_0(w) = \frac{1}{2} \int_{\mathbb{R}^3} (-\lambda_1^{-1} u \Delta u + \mu_1^{-1} \langle D\phi, \phi \rangle - |\phi|^2 |u|^2)$$

and since $(U_{\lambda, \xi}, \Psi_{\lambda, \xi, a})$ are critical points of J_0 , it follows that

$$\tilde{W}_{\lambda, \xi, a} = (\mu_1^{-\frac{1}{2}} U_{\lambda, \xi}, \lambda_1^{-\frac{1}{2}} \Psi_{\lambda, \xi, a})$$

are critical points of \tilde{J}_0 .

Lemma 3.3. We have $T_{W_0} Z = \ker J_0''(W_0)$.

Proof. It is standard to check that $T_{W_0} Z \subseteq \ker J_0''(W_0)$, so it suffices to prove the inclusion $\ker J_0''(W_0) \subseteq T_{W_0} Z$. Moreover, since $\dim(T_{W_0} Z) = 7$, it is enough to show that

$$\dim(\ker J_0''(W_0)) \leq 7$$

and, by means of Remark 3.2, this is equivalent to

$$\dim(\ker \tilde{J}_0''(\tilde{W}_0)) \leq 7.$$

On the sphere \mathbb{S}^3 , the linearization of (3.1) at \tilde{W}_0 reads as

$$\begin{cases} L_{g_{\mathbb{S}^3}} v = \lambda_1 v |\Psi_1|^2 + 2\lambda_1 V_1 \langle \Psi_1, \eta \rangle, \\ D_{g_{\mathbb{S}^3}} \eta = \mu_1 |V_1|^2 \eta + 2\mu_1 v V_1 \Psi_1, \end{cases} \quad (3.2)$$

where

$$(V_1, \Psi_1) = (\mu_1^{-\frac{1}{2}} (f^{-\frac{1}{2}} U_{\lambda, \xi}) \circ \pi, \lambda_1^{-\frac{1}{2}} (f \circ \pi)^{-1} F(\Phi_{\lambda, \xi, a} \circ \pi)) = (1, \Psi_1).$$

Notice that Ψ_1 satisfies

$$D_{g_{\mathbb{S}^3}} \Psi_1 = \frac{3}{2} |\Psi_1| \Psi_1 \quad \text{and} \quad |\Psi_1| = 1, \quad (3.3)$$

so it is an eigenspinor of $D_{g_{\mathbb{S}^3}}$ with eigenvalue $\frac{3}{2}$. We set $\eta = \sum_{k \in \mathbb{Z}} f_k \Psi_k$, where Ψ_k is a trivialization with Killing spinors of the spinor bundle of \mathbb{S}^3 and we write $f_1 = g_1 + ih_1$, where g_1 and h_1 are real-valued functions; here by trivialization we mean a global basis of the spinor bundle of \mathbb{S}^3 formed by Killing spinors. For more details regarding such a basis, we refer the reader to [10] or [12, formula (5.14)]. We will first find f_1 . Since $f_1 = \langle \eta, \Psi_1 \rangle$, we have (see [12, Lemma 5.2 and formula (5.16)])

$$\Delta_{g_{\mathbb{S}^3}} f_1 = \langle \Delta_{g_{\mathbb{S}^3}} \eta, \Psi_1 \rangle + \langle \eta, \Delta_{g_{\mathbb{S}^3}} \Psi_1 \rangle + \langle D_{g_{\mathbb{S}^3}} \eta, \Psi_1 \rangle.$$

Notice now that, by (3.3) and the Lichnerowicz's formula on the sphere

$$D_{g_{\mathbb{S}^3}}^2 = -\Delta_{g_{\mathbb{S}^3}} + \frac{3}{2},$$

we have $-\Delta_{g_{\mathbb{S}^3}} \Psi_1 = \frac{3}{4} \Psi_1$ and

$$\begin{aligned} -\Delta_{g_{\mathbb{S}^3}} \eta &= D_{g_{\mathbb{S}^3}}^2 \eta - \frac{3}{2} \eta \\ &= D_{g_{\mathbb{S}^3}} \left(\frac{3}{2} \eta + 3v \Psi_1 \right) - \frac{3}{2} \eta = \frac{3}{2} \left(\frac{3}{2} \eta + 3v \Psi_1 \right) + 3\nabla v \cdot \Psi_1 + \frac{9}{2} v \Psi_1 - \frac{3}{2} \eta \\ &= \frac{3}{4} \eta + 9v \Psi_1 + 3\nabla v \cdot \Psi_1. \end{aligned}$$

Therefore

$$-\Delta_{g_{\mathbb{S}^3}} f_1 = \frac{3}{4} f_1 + 9\nu + 3\langle \nabla \nu \cdot \Psi_1, \Psi_1 \rangle + \frac{3}{4} f_1 - \frac{3}{2} f_1 - 3\nu = 6\nu + 3\langle \nabla \nu \cdot \Psi_1, \Psi_1 \rangle.$$

Since the last addend in the previous equality is purely imaginary, we take the real and imaginary part to have

$$-\Delta_{g_{\mathbb{S}^3}} g_1 = 6\nu$$

and

$$-\Delta_{g_{\mathbb{S}^3}} h_1 = -3i\langle \nabla \nu \cdot \Psi_1, \Psi_1 \rangle.$$

In particular, recalling that $L_{g_{\mathbb{S}^3}} = -\Delta_{g_{\mathbb{S}^3}} + \frac{3}{4}$ and the first equation in (3.2), we have the system

$$\begin{cases} -\Delta_{g_{\mathbb{S}^3}} \nu = \frac{3}{2} g_1, \\ -\Delta_{g_{\mathbb{S}^3}} g_1 = 6\nu. \end{cases}$$

Hence,

$$\Delta_{g_{\mathbb{S}^3}}^2 g_1 = 9g_1$$

from which we deduce that g_1 is the first eigenfunction of the Laplacian on the sphere and $\nu = \frac{g_1}{2}$. So, the first equation in (3.2) becomes

$$L_{g_{\mathbb{S}^3}} \frac{g_1}{2} = \frac{3}{4} \frac{g_1}{2} + 3f_1$$

and recalling the definition of $L_{g_{\mathbb{S}^3}}$, from the quality above we get

$$f_1 = g_1.$$

Using this fact, system (3.2) becomes

$$\begin{cases} \nu = \frac{g_1}{2}, \\ D_{g_{\mathbb{S}^3}} \eta = \frac{3}{2} \eta + \frac{3}{2} \langle \eta, \Psi_1 \rangle \Psi_1. \end{cases}$$

Hence we need to compute the dimension of

$$\Lambda = \left\{ \eta \in H^{\frac{1}{2}}(\Sigma \mathbb{S}^3) : D_{g_{\mathbb{S}^3}} \eta = \frac{3}{2} \eta + \frac{3}{2} \langle \eta, \Psi_1 \rangle \Psi_1 \right\}.$$

This computation has been carried out by Isobe in [12] for general dimensions of the sphere \mathbb{S}^m , so in our situation it suffices to take $m = 3$ in [12, Lemma 5.1] to get $\dim(\Lambda) = 7$ as desired. \square

Now we will focus on the reduced functional. For a fixed $a \in \Sigma \mathbb{R}^3$, with $|a| = 1$, we set $V_{\lambda, \xi} = |U_{\lambda, \xi}|^2 |\Phi_{\lambda, \xi, a}|^2$, so that $V_{\lambda, \xi}(x) = \frac{1}{\lambda^3} V_{1,0}(\frac{1}{\lambda}(x - \xi))$ and let

$$\Gamma(\lambda, \xi) = \frac{1}{2} \int_{\mathbb{R}^3} h(x) V_{\lambda, \xi}(x) dx$$

for $(\lambda, \xi) \in (0, +\infty) \times \mathbb{R}^3$. Then we have the following:

Proposition 3.4. *The function Γ is of class C^2 on $(0, +\infty) \times \mathbb{R}^3$ and it can be extended to a C^1 function at $\lambda = 0$ by*

$$\Gamma(0, \xi) = c_0 h(\xi), \quad c_0 = \frac{1}{2} \int_{\mathbb{R}^3} V_{1,0}(x) dx.$$

Also,

$$\lim_{\lambda \rightarrow 0} \nabla_{\xi}^2 \Gamma(\lambda, \xi) = c_0 \nabla^2 h(\xi),$$

uniformly on every compact of \mathbb{R}^3 . Moreover, for any compact set $\Sigma \subset \mathbb{R}^3$, there exists a constant $C = C_{\Sigma}$ such that

$$|\partial_{\lambda} \Gamma(\lambda, \xi) - c_1 \lambda \Delta h(\xi)| \leq C_{\Sigma} \lambda^2$$

for all $\lambda > 0$ and all $\xi \in \Sigma$, being

$$c_1 = \int_{\mathbb{R}^3} |y|^2 V_{1,0}(y) dy.$$

Proof. We have by a change of variable that

$$\Gamma(\lambda, \xi) = \int_{\mathbb{R}^3} h(\lambda x + \xi) V_{1,0}(x) \, dx.$$

Using the smoothness of h and the dominated convergence, we have that

$$\lim_{\lambda \rightarrow 0} \Gamma(\lambda, \xi) = c_0 h(\xi).$$

The same reasoning applies to show that one has

$$\nabla_{\xi} \Gamma(0, \xi) = c_0 \nabla h(\xi), \quad \nabla_{\xi}^2 \Gamma(0, \xi) = c_0 \nabla^2 h(\xi) \quad \text{and} \quad \nabla_{\lambda} \Gamma(0, \xi) = 0.$$

The last equality follows from the oddness of the integral, that is,

$$\int_{\mathbb{R}^3} x_i V_{1,0}(x) \, dx = 0, \quad i = 1, 2, 3.$$

We fix now a compact set Σ ; then by Taylor expansion of $y \mapsto h(y + \xi)$, we have

$$\left| \partial_{\xi_i} h(y + \xi) - \partial_{\xi_i} h(\xi) - \sum_{j=1}^3 \partial_{\xi_i \xi_j}^2 h(\xi) y_j \right| \leq C_{\Sigma} |y|^2.$$

Also, notice that since

$$\int_{\mathbb{R}^3} y_i y_j V_{1,0}(y) \, dy = 0 \quad \text{if } i \neq j,$$

we have for our choice of c_1 ,

$$c_1 \lambda \Delta h(\xi) = \int_{\mathbb{R}^3} \sum_{i=1}^3 \left(\partial_{\xi_i} h(\xi) + \sum_{j=1}^3 \partial_{\xi_i \xi_j}^2 h(\xi) \lambda y_j \right) y_i V_{1,0}(y) \, dy.$$

Therefore,

$$|\partial_{\lambda} \Gamma(\lambda, \xi) - c_1 \lambda \Delta h(\xi)| \leq C_{\Sigma} \lambda^2. \quad \square$$

Proposition 3.5. *Let k and h be functions as in Theorem 1.1. Then there exists an open set $\Omega \subset (0, +\infty) \times \mathbb{R}^3$ such that $\nabla \Gamma \neq 0$ on $\partial \Omega$ and*

$$\deg(\nabla \Gamma, \Omega, 0) = \sum_{\substack{\xi \in \text{crit}[h] \\ \Delta h(\xi) < 0}} (-1)^{m(h, \xi)} + 1.$$

Proof. Let $s > 0$; we consider the set

$$\mathcal{B}_s = \left\{ (\lambda, \xi) \in (0, +\infty) \times \mathbb{R}^3 : |(\lambda, \xi) - (s, 0)| \leq s - \frac{1}{s} \right\}.$$

We will show that for s large enough, we can choose $\Omega = \mathcal{B}_s$. First, we set

$$\text{crit}[h] = \{\xi^1, \xi^2, \dots, \xi^l\}$$

for some $l \in \mathbb{N}$. Since the south pole is not a critical point for k , we have that for r large enough,

$$\text{crit}[h] \subset A_r = \{\xi \in \mathbb{R}^3 : |\xi| \leq r\}.$$

Since h is a Morse function (as well as k), it follows from the non-degeneracy condition (i) that there exist constants $\mu \in (0, r)$ and $\delta > 0$ such that

$$|\Delta h(\xi)| > \delta \quad \text{for all } \xi \in \bigcup_{i=1}^l B_{\mu}(\xi^i),$$

where $B_\mu(\xi^i)$ denote as usual the balls of centers ξ^i and radius μ . By using Proposition 3.4, we have that for s sufficiently large and μ even smaller if necessary,

$$\partial_\lambda \Gamma(\lambda, \xi) \neq 0 \quad \text{in } \partial \mathcal{B}_s \cap \left((0, \mu) \times \bigcup_{i=1}^l B_\mu(\xi^i) \right).$$

Hence,

$$\nabla \Gamma \neq 0 \quad \text{in } \partial \mathcal{B}_s \cap \left((0, \mu) \times \bigcup_{i=1}^l B_\mu(\xi^i) \right).$$

Again, by Proposition 3.4, since Γ extends to a C^1 function at $\lambda = 0$ and $\nabla_\xi \Gamma(0, \xi) = c_0 \nabla h(\xi)$, we have that

$$\nabla \Gamma \neq 0 \quad \text{in } \partial \mathcal{B}_s \cap \left((0, \mu) \times A_{2r} \setminus \bigcup_{i=1}^l B_\mu(\xi^i) \right).$$

Hence,

$$\nabla \Gamma \neq 0 \quad \text{in } \partial \mathcal{B}_s \cap ((0, \mu) \times A_{2r}).$$

So it remains to study Γ on the component of $\partial \mathcal{B}_s$ outside $(0, \mu) \times A_{2r}$. So we consider the Kelvin reflection

$$\tau : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{0\}, \quad \tau(x) = \frac{x}{|x|^2}.$$

We notice that

$$\tau^*(g_{\mathbb{R}^3}) = \frac{1}{|x|^4} g_{\mathbb{R}^3}.$$

Hence, for all $F \in L^6(\mathbb{R}^3)$, by putting $y = \tau(x)$, we have

$$\int_{\mathbb{R}^3} h(y) |F(y)|^6 dy = \int_{\mathbb{R}^3} h(\tau(x)) |F(\tau(x))|^6 f^3(x) dx = \int_{\mathbb{R}^3} h(\tau(x)) |F^*(x)|^6 dx,$$

where

$$F^*(x) = \frac{1}{|x|^2} F\left(\frac{x}{|x|^2}\right).$$

In particular, if we set

$$\tilde{\lambda} = \frac{\lambda}{\lambda^2 + |\xi|^2}, \quad \tilde{\xi} = \frac{\xi}{\lambda^2 + |\xi|^2},$$

we have that

$$|V_{\lambda, \xi}^*(x)| = |V_{\tilde{\lambda}, \tilde{\xi}}(x)|.$$

So we define

$$\tilde{\Gamma} = \frac{1}{2} \int_{\mathbb{R}^3} h(\tau(x)) |V_{\lambda, \xi}(x)| dx,$$

and we have that

$$\Gamma(\lambda, \xi) = \tilde{\Gamma}(\tilde{\lambda}, \tilde{\xi}).$$

Once again, by using Proposition 3.4, we have that $\tilde{\Gamma}$ can be extended to a C^1 function up to the origin $(0, 0) \in [0, \infty) \times \mathbb{R}^3$. Since $(\lambda, \xi) \mapsto (\tilde{\lambda}, \tilde{\xi})$ is a diffeomorphism, we have $\nabla \Gamma(\lambda, \xi) = 0$ if and only if $\tilde{\Gamma}(\tilde{\lambda}, \tilde{\xi}) = 0$. But by assumption, the south pole is not a critical point of h , hence 0 is not a critical point of $h(\tau(x))$. Therefore, $\nabla \tilde{\Gamma} \neq 0$ in a neighborhood of the origin and so $\nabla \Gamma \neq 0$ in a neighborhood of infinity. Finally, we have that for r and s large enough,

$$\nabla \Gamma \neq 0 \quad \text{on } \partial \mathcal{B}_s \setminus ((0, \mu) \times \bar{A}_{2r}).$$

The degree computation is by now standard and it follows for instance as in [11]. \square

Remark 3.6. We want explicitly to notice that at this point we cannot directly conclude as in the classical cases (see for instance [2, 21]), since the critical points of Γ on Z are degenerate: this is due to the invariance of the functional with respect to the parameters a and this degeneracy causes the degree to vanish.

We recall that Z is a non-degenerate manifold of critical points of J_0 and J_0'' is Fredholm of index zero, therefore we have that there exists $\varepsilon > 0$ such that for all $z \in Z_c \subset Z$ with Z_c compact, there exists a unique $w(z) \in T_z Z^\perp$ such that

$$PJ'_\varepsilon(z + w(z)) = 0,$$

where $P : A \rightarrow T_z Z^\perp$ is the orthogonal projection. Now, to find a solution to our problem, it is enough to find a critical point for the function $\Phi_\varepsilon : Z \rightarrow \mathbb{R}$ defined by

$$\Phi_\varepsilon(z) = J_\varepsilon(z + w(z)).$$

In order to do this, we will consider the set of the parameters a

$$\{a \in \Sigma\mathbb{R}^3 : |a| = 1\} \simeq \mathbb{S}^3$$

as a Lie group. Hence, we will consider the natural action of \mathbb{S}^3 on $Z \simeq (0, +\infty) \times \mathbb{R}^3 \times \mathbb{S}^3$, being Z parameterized by (λ, ξ, a) . Also, we notice that $(J_0)|_Z$ and $G|_Z$ are invariant under this action: then we need to extend the action to the whole space $D^{\frac{1}{2}}(\Sigma\mathbb{R}^3)$. In order to do this, we recall that the spinor bundle of \mathbb{R}^3 can be trivialized by Killing spinors that are either constant (parallel spinors) or spinors of the form $x \cdot \phi$ with ϕ constant. So we fix an orthonormal basis of $\Sigma\mathbb{R}^3$ of the form $\{a_1, a_2, x \cdot a_1, x \cdot a_2\}$, where a_1, a_2 are (distinct) constant spinors with $|a_1| = |a_2| = 1$. Hence, if $\phi \in D^{\frac{1}{2}}(\Sigma\mathbb{R}^3)$, there exist f_1, f_2, g_1, g_2 such that

$$\phi(x) = (f_1(x) + g_1(x)x) \cdot a_1 + (f_2(x) + g_2(x)x) \cdot a_2.$$

Since a_1 and a_2 can be seen as elements in \mathbb{S}^3 , we can define the action for a general $w \in \mathbb{S}^3$ and $\phi \in D^{\frac{1}{2}}(\Sigma\mathbb{R}^3)$ by

$$w\phi = (f_1(x) + g_1(x)x) \cdot wa_1 + (f_2(x) + g_2(x)x) \cdot wa_2.$$

In this way, this last action extends the one previously defined on Z and in addition both J_0 and G are invariant under this action. Therefore, Φ_ε descends to a C^1 function $\tilde{\Phi}_\varepsilon$ defined on the quotient

$$Z/\mathbb{S}^3 \simeq (0, \infty) \times \mathbb{R}^3.$$

The same argument works for Γ ; therefore for ε small enough, we have that

$$\tilde{\Phi}'_\varepsilon = \varepsilon\Gamma + o(\varepsilon).$$

At this point, from the invariance of the degree by homotopy, we have that

$$\deg(\tilde{\Phi}'_\varepsilon, \mathcal{B}_s, 0) = \sum_{\substack{\xi \in \text{crit}[h] \\ \Delta h(\xi) < 0}} (-1)^{m(h, \xi)} + 1.$$

Finally, by assumption (ii), by contradiction (for the argument, see for instance [1, 2]) if

$$\sum_{\substack{\xi \in \text{crit}[h] \\ \Delta h(\xi) < 0}} (-1)^{m(h, \xi)} \neq -1,$$

then $\tilde{\Phi}_\varepsilon$ has a critical point that can be lifted as a critical orbit of Φ_ε , which in turn ends the proof of the Main Theorem 1.1.

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