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ON RADO CONDITIONS FOR NONLINEAR DIOPHANTINE EQUATIONS

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ABSTRACT. Building on previous work of Di Nasso and Luperi Baglini, we provide general necessary conditions for a Diophantine equation to be partition regular. These conditions are inspired by Rado’s characterization of partition regular linear homogeneous equations. We conjecture that these conditions are also sufficient for partition regularity, at least for equations whose corresponding monovariate polynomial is linear. This would provide a natural generalization of Rado’s theorem.

We verify that such a conjecture holds for the equations $x^2 - xy + ax + by + cz = 0$ and $x^2 - y^2 + ax + by + cz = 0$ for $a, b, c \in \mathbb{Z}$ such that $abc = 0$ or $a + b + c = 0$. To deal with these equations, we establish new results concerning the partition regularity of polynomial configurations in \mathbb{Z} such as $\{x, x + y, xy + x + y\}$, building on the recent result on the partition regularity of $\{x, x + y, xy\}$.

1. INTRODUCTION

Partition regularity is a central notion in Ramsey theory. Briefly, a family \mathcal{A} of subsets of \mathbb{N} is *partition regular* if every finite coloring of \mathbb{N} admits infinitely many monochromatic sets from \mathcal{A} (see Section 2 for precise definitions). A *configuration* is partition regular if the family of sets that realize that configuration is partition regular. For instance, the seminal Schur Lemma from 1916 [28] asserts that *Schur’s triples*, i.e. configurations $\{a, b, a + b\}$ for $a, b \in \mathbb{N}$, are partition regular. Such a result inspired and motivated one of the early gems of Ramsey theory: the celebrated van der Waerden theorem on arithmetic progressions [29], asserting that arithmetic progressions of arbitrary finite length are partition regular.

Schur’s triples can be seen as solutions to the Diophantine equation $z = x + y$. Similarly, arithmetic progressions can be seen as solutions to a suitable system of linear Diophantine equations. This perspective led Rado in 1933 [26] to provide a complete characterization of which systems of linear Diophantine equations are partition regular. An equation (or system of equations) $P(x_1, \dots, x_n) = 0$ is called partition regular if the family

$$\{\{a_1, \dots, a_n\} \subseteq \mathbb{N} : P(a_1, \dots, a_n) = 0\}$$

of its solutions is partition regular. Rado’s theorem provides a characterization of which linear Diophantine equations are partition regular in terms of a simple condition on their coefficients (Rado’s condition).

Since Rado’s theorem, progress in the study of partition regularity of Diophantine equations has been slower, as the nonlinear case has proved to be substantially more difficult. Important results were obtained by Bergelson [1, 2], Khalfalah and Szemerédi [20], Csikvári, Gyarmati, and

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Sárkőzy [9], Hindman [19], Frantzikinakis and Host [16], Green and Sanders [18], Bergelson and Moreira [7, 8], Moreira [25], and Bergelson, Moreira, and Johnson [4]. Recently, some general necessary conditions for partition regularity were obtained using nonstandard methods by Di Nasso and Lupieri Baglini [14], building on previous work of Di Nasso and Riggio [15], and Lupieri Baglini [22, 23].

Despite these efforts, the problem of determining which Diophantine equations are partition regular has been solved only in very special cases. A major problem in this area is to find explicit necessary and sufficient conditions for a Diophantine equation to be partition regular, generalizing Rado's theorem in the linear case. Incidentally, it should be noted that the problem of determining whether a Diophantine equation has *some* integer solution is undecidable (in the sense of computability theory), as shown by Yuri Matiyasevich—building on previous work of Martin Davis, Hilary Putnam, and Julia Robinson—in his solution of Hilbert's 10th problem [24].

In this work, we present general necessary conditions for partition regularity of Diophantine equations. We name them *Rado conditions*, as they are inspired by Rado's characterization of partition regular linear Diophantine equations, and indeed in the particular case of linear equations, they recover the conditions from Rado's theorem. They can also be seen as a strengthening of some conditions recently proved to be necessary by Di Nasso and Lupieri Baglini [14]; see also [13]. We show that the Rado conditions are indeed *sufficient* for partition regularity for the equations $x^2 - xy + ax + by + cz = 0$ and $x^2 - y^2 + ax + by + cz = 0$ for $a, b, c \in \mathbb{Z}$ such that $abc = 0$ or $a + b + c = 0$.

In order to verify that the Rado conditions are sufficient for the latter equation, we parametrize the solutions with polynomial configurations in \mathbb{Z} of the form

$$\{r + p(s), r + q(s), rs + r + ds\},$$

where $p(s), q(s)$ are polynomials with integer coefficients vanishing at 0, and $d \in \mathbb{Q}$. We show that such configurations are partition regular, building on the analogue result from [25] for the configurations $\{r + p(s), r + q(s), rs\}$.

We believe that the results of this paper are an important contributions to the study of partition regularity of Diophantine equations. Indeed, they provide general necessary conditions that can be used to rule out partition regularity in many cases, and can be seen as a natural generalization to the nonlinear case of the condition in Rado's theorem. This paper is divided in five sections, besides this introduction. Section 2 introduces the terminology to be used in the rest of the paper, and presents the statements of the main results. In Section 3 we obtain the necessity of the Rado conditions, and in Section 4 we establish that certain polynomial configurations in \mathbb{Z} are partition regular. These results are then applied in Section 5 to show that the Rado conditions are both necessary and sufficient in the case of the equations $x^2 - xy + ax + by + cz = 0$ and $x^2 - y^2 + ax + by + cz = 0$ where $a, b, c \in \mathbb{Z}$ are such that $a + b + c = 0$ or $abc = 0$.

2. MAIN RESULTS

We let \mathbb{N} be the set of *strictly positive* integers, and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A finite coloring of \mathbb{N} is a function $c : \mathbb{N} \rightarrow \{1, 2, \dots, k\}$ for some $k \in \mathbb{N}$. In this case, k is the number of colors, and the elements of $\{1, 2, \dots, k\}$ are the *colors*. A subset A of \mathbb{N} is *monochromatic* for the coloring c if there exists a color $i \in \{1, 2, \dots, k\}$ such that, for every $a \in A$, $c(a) = i$.

Definition 2.1. Let \mathcal{A} be a family of subsets of \mathbb{N} . We say that \mathcal{A} is partition regular if, for every finite coloring c of \mathbb{N} , there exist infinitely many $A \in \mathcal{A}$ that are monochromatic for c .

In the following we will use the same notation and terminology for polynomials as in [14, Section 3]. Particularly, a (multi)index is an n -tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$. We define the degree of α to be $|\alpha| := \alpha_1 + \dots + \alpha_n$, and the (Hamming) length $\ell(\alpha)$ to be $|\{i \in \{1, 2, \dots, n\} : \alpha_i > 0\}|$. If $x = (x_1, \dots, x_n)$ is an n -tuple of variables, x^α represents the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$. We can then represent a polynomial $P \in \mathbb{Z}[x_1, \dots, x_n]$ in the variables x_1, \dots, x_n as $\sum_{\alpha} c_{\alpha} x^{\alpha}$, where α ranges in \mathbb{N}_0^n and $c_{\alpha} = 0$ for all but finitely many α . We define $\text{Supp}(P) \subset \mathbb{N}_0^n$ to be the (finite) set of indices α such that c_{α} is nonzero. There is a natural partial order among indices, obtained by setting $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i$ for every $i \in \{1, 2, \dots, n\}$. An index α is *maximal* for P if it is a maximal element in $\text{Supp}(P)$ with respect to such an order, and *minimal* for P if it is a minimal element in $\text{Supp}(P)$. Accordingly, the term $c_{\alpha} x^{\alpha}$ in P is maximal (respectively, minimal) if α is a maximal (respectively, minimal) index for P . We say that a set J of indices is homogeneous of degree d if $|\alpha| = d$ for every $\alpha \in J$. Furthermore, for indices $\beta \leq \alpha$ we set

$$\alpha! = \prod_{i=1}^n (\alpha_i!) \quad \text{and} \quad \binom{\alpha}{\beta} = \prod_{i=1}^n \binom{\alpha_i}{\beta_i} = \frac{\alpha!}{\beta!(\alpha - \beta)!}.$$

For $P \in \mathbb{Z}[x_1, \dots, x_n]$ we define the partial derivatives

$$\frac{\partial^{\beta} P}{\partial x^{\beta}} := \frac{\partial^{|\beta|} P}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \dots \partial x_n^{\beta_n}}.$$

Note that

$$\frac{\partial^{\beta} x^{\alpha}}{\partial x^{\beta}} = \beta! \binom{\alpha}{\beta} x^{\alpha - \beta}$$

if $\beta \leq \alpha$ and it equals 0 otherwise. For $r \in \mathbb{Z}$, we consider the polynomial $P^{(r)}(x_1, \dots, x_n) = P(x_1 + r, \dots, x_n + r)$. One can express the coefficients of $P^{(r)}(x)$ in terms of the partial derivatives of P evaluated at r , as the following computation shows. Suppose that $P(x) = \sum_{\alpha} c_{\alpha} x^{\alpha}$. Then we have that

$$\begin{aligned} P^{(r)}(x) &= \sum_{\alpha} c_{\alpha} (x_1 + r)^{\alpha_1} \dots (x_n + r)^{\alpha_n} \\ &= \sum_{\alpha} \sum_{\beta \leq \alpha} c_{\alpha} \binom{\alpha}{\beta} r^{|\alpha| - |\beta|} x^{\beta} \\ &= \sum_{\beta} \left(\sum_{\alpha \geq \beta} c_{\alpha} \binom{\alpha}{\beta} r^{|\alpha| - |\beta|} \right) x^{\beta} \\ &= \sum_{\beta} \frac{1}{\beta!} \frac{\partial^{\beta} P}{\partial x^{\beta}}(r, \dots, r) x^{\beta}. \end{aligned}$$

If we let \mathbf{r} be the n -tuple (r, \dots, r) , then we can write

$$P^{(r)}(x) = \sum_{\beta} \frac{1}{\beta!} \frac{\partial^{\beta} P}{\partial x^{\beta}}(\mathbf{r}) x^{\beta}.$$

In particular, one has that

$$P(x) = \sum_{\beta} \frac{1}{\beta!} \frac{\partial^{\beta} P}{\partial x^{\beta}}(\mathbf{0}) x^{\beta}.$$

We denote by $\tilde{P}(w) \in \mathbb{Z}[w]$ the monovariate polynomial $\tilde{P}(w) = P(w, \dots, w)$.

For a prime $p \in \mathbb{N}$, we denote by \mathbb{Z}_p the ring of p -adic integers; see [27, Chapter 1, Section 1]. It is easy to see that, for $Q(w) \in \mathbb{Z}[w]$, the equation $Q(w) = 0$ has an invertible solution in

\mathbb{Z}_p if and only if, for every $n \in \mathbb{N}$, the equation $Q(w) = 0$ has an invertible solution in $\mathbb{Z}/p^n\mathbb{Z}$; see [27, Chapter 1, Section 6].

Definition 2.2. Fix $q \in \mathbb{N}$ with $q \geq 2$. Consider a function $f : \mathbb{N}^n \rightarrow \mathbb{Q}$. Define

$$\mathcal{A}_f = \{ \{a_1, \dots, a_n\} \subseteq \mathbb{N} : f(a_1, \dots, a_n) = 0 \}.$$

The equation $f(x_1, \dots, x_n) = 0$, and the function f , are called:

- partition regular if the family \mathcal{A}_f is partition regular;
- q -partition regular if, for every $k \in \mathbb{N}$, the family consisting of $\{a_1, \dots, a_n\} \in \mathcal{A}_f$ such that $a_1 \equiv \dots \equiv a_n \equiv 0 \pmod{q^k}$, is partition regular.

In particular, this definition applies when the function f is a polynomial with integer coefficients.

It is clear that a homogeneous polynomial is partition regular if and only if it is q -partition regular for every $q > 1$. We remark in passing that the converse does not hold, i.e., there are polynomials (e.g. $f(x, y, z) = xy - z$) which are q -partition regular for every $q > 1$ but are not homogenous. Nevertheless, a q -partition regular equation necessarily has zero constant term. On the other hand, a partition regular equation might have a nonzero constant term, such as $x - y + z^2 = 1$ (this follows from [6, Theorem 5.2] with $r = 1$ and $p_1(z) = z^2 - 1$).

Proposition 2.3. *If the equation $P(x_1, \dots, x_n) = 0$ is partition regular, then for every $r \in \mathbb{Z} \setminus \{0\}$, the equations $P(x_1 + r, \dots, x_n + r) = 0$ and $P(\frac{1}{r}x_1, \dots, \frac{1}{r}x_n) = 0$ are also partition regular. If the equation $P(x_1, \dots, x_n) = 0$ is q -partition regular for some $q \in \mathbb{N} \setminus \{1\}$, then the equation $P(qx_1, \dots, qx_n) = 0$ is also q -partition regular.*

Proof. We prove only the last assertion; the others can be established in a similar way. Suppose $c : \mathbb{N} \rightarrow \{1, \dots, k\}$ is an arbitrary coloring. Consider a new coloring $\tilde{c} : \mathbb{N} \rightarrow \{1, \dots, k+1\}$ defined as $\tilde{c}(i) = c(i/q)$ if $i \equiv 0 \pmod{q}$ and $\tilde{c}(i) = k+1$ otherwise. Let $(a_1, \dots, a_n) \in \mathbb{N}^n$ be a \tilde{c} -monochromatic solution to $P(x_1, \dots, x_n) = 0$ with all a_i multiples of q . Then $(\frac{1}{q}a_1, \dots, \frac{1}{q}a_n)$ is a c -monochromatic solution to $P(qx_1, \dots, qx_n) = 0$. \square

The classical characterization of partition regularity for homogeneous *linear* polynomials due to Rado [26] states that a (homogeneous) linear polynomial $P(x) = a_1x_1 + \dots + a_nx_n$ is partition regular if and only if there exists a set $J \subseteq \{1, \dots, n\}$ such that $\sum_{j \in J} a_j = 0$. This condition can be phrased without explicitly using the coefficients of P by saying that there exists a set $J \subseteq \text{Supp}(P)$ such that

$$\sum_{\alpha \in J} \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{0}) = 0. \quad (1)$$

For instance, the equation $x + y = 3z$ is not partition regular. Rado's condition can also be used to deduce that certain polynomial equations of higher degree are not partition regular. We illustrate this point in the following examples.

Example 2.4. The equation $x^2 + y^2 = 3z^2$ is not partition regular.

Indeed, given a coloring $c : \mathbb{N} \rightarrow \{1, \dots, k\}$, let $\tilde{c} : \mathbb{N} \rightarrow \{1, \dots, k\}$ be the coloring defined by $\tilde{c}(x) = c(x^2)$. Whenever (x, y, z) is a \tilde{c} -monochromatic solution to $x^2 + y^2 = 3z^2$, the triple (x^2, y^2, z^2) is a c -monochromatic solution to $x + y = 3z$. Choosing a coloring c which admits no monochromatic solution to $x + y = 3z$ we constructed a coloring \tilde{c} which admits no monochromatic solutions to $x^2 + y^2 = 3z^2$.

In fact, a variation of this idea can be extended to work for general homogeneous polynomials. This observation was first made by Lefmann in [21, Theorem 2.1], and was also used in [14, Corollary 3.9].

Example 2.5. The equation $xy = z^3$ is not partition regular.

Indeed, let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a map satisfying $f(xy) = f(x) + f(y)$ for all $x, y \geq 2$. Given a coloring $c : \mathbb{N} \rightarrow \{1, \dots, k\}$, let $\tilde{c} : \mathbb{N} \rightarrow \{1, \dots, k\}$ be the coloring defined by $\tilde{c}(x) = c(f(x))$. Whenever (x, y, z) is a \tilde{c} -monochromatic solution to $xy = z^3$, the triple $(f(x), f(y), f(z))$ is a c -monochromatic solution to $x + y = 3z$. Choosing a coloring c which admits no monochromatic solution to $x + y = 3z$ we constructed a coloring \tilde{c} which admits no monochromatic solutions to $xy = z^3$ other than the trivial solution $(1, 1, 1)$.

The idea used in Example 2.5 can also be adapted to work for more general monomials. In fact, it is not necessary to assume that the map f satisfies $f(xy) = f(x) + f(y)$: it suffices that $f(xy) - f(x) - f(y)$ is bounded. This will be the case when $f(x)$ is the number of digits of x in base q (for an arbitrary fixed q). The advantage of using such f is that the original proof of the necessity of the Rado condition also involves a coloring which looks at the base q expansion of x .

In what follows, we combine the two methods from Examples 2.4 and 2.5 to generate new necessary conditions for polynomial equations of higher degree to be partition regular. Before doing so we need some more definitions.

Definition 2.6. Let $\phi : \mathbb{Z}^n \rightarrow \mathbb{Z}$ be a positive linear map $(\alpha_1, \dots, \alpha_n) \mapsto t_1\alpha_1 + \dots + t_n\alpha_n$ for some $t_1, \dots, t_n \in \mathbb{N}_0$.

- If c is a finite coloring of \mathbb{N}_0 , then we say that ϕ is c -monochromatic if $\{t_1, \dots, t_n\}$ is c -monochromatic;
- If $P \in \mathbb{Z}[x_1, \dots, x_n]$, and (M_0, \dots, M_ℓ) is the increasing enumeration of $\phi(\text{Supp}(P))$, then the partition of $\text{Supp}(P)$ determined by ϕ is the ordered tuple (J_0, \dots, J_ℓ) , where $J_i = \{\alpha \in \text{Supp}(P) : \phi(\alpha) = M_i\}$.

Suppose that $P \in \mathbb{Z}[x_1, \dots, x_n]$ is a polynomial.

Definition 2.7. A *Rado partition* of P is an ordered tuple (J_0, \dots, J_ℓ) such that, for every finite coloring c of \mathbb{N}_0 , there exist infinitely many c -monochromatic positive linear maps $\phi : \mathbb{Z}^n \rightarrow \mathbb{Z}$ such that (J_0, \dots, J_ℓ) is the partition of $\text{Supp}(P)$ determined by ϕ .

Since ϕ must be a positive linear map, J_ℓ contains at least one maximal index, and J_0 contains at least one minimal index.

Definition 2.8. A *Rado set* for P is a set $J \subseteq \text{Supp}(P)$ such that there exists a Rado partition (J_0, \dots, J_ℓ) for P such that $J = J_i$ for some $i \in \{0, 1, \dots, \ell\}$.

A minimal (respectively, maximal) Rado set for P is a set $J \subseteq \text{Supp}(P)$ such that there exists a Rado partition (J_0, \dots, J_ℓ) for P with $J = J_0$ (respectively, $J = J_\ell$).

Example 2.9. Consider the polynomial $P(x_1, x_2, x_3, x_4) = -x_3 - x_4 + x_1x_2 + x_1x_2^2$. We have that

$$\text{Supp}(P) = \{(0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 0, 0), (1, 2, 0, 0)\}.$$

We claim that the partition $J = (J_0, J_1, J_2, J_3)$ defined by

$$\begin{aligned} J_0 &= \{(0, 0, 1, 0)\} \\ J_1 &= \{(1, 1, 0, 0)\} \\ J_2 &= \{(0, 0, 0, 1)\} \\ J_3 &= \{(1, 2, 0, 0)\} \end{aligned}$$

is a Rado partition for P . Indeed, suppose that c is a finite coloring of \mathbb{N} . By Brauer's extension of the van der Waerden theorem on arithmetic progressions, there exist $a, b \geq 3$

such that $\{a, b, a + b - 1, a + 2b - 2\}$ is c -monochromatic. We can consider the corresponding c -monochromatic linear map $\phi : \mathbb{Z}^4 \rightarrow \mathbb{Z}$,

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto a\alpha_1 + b\alpha_2 + (a + b - 1)\alpha_3 + (a + 2b - 2)\alpha_4.$$

Then

$$\phi(0, 0, 1, 0) < \phi(1, 1, 0, 0) < \phi(0, 0, 0, 1) < \phi(1, 2, 0, 0).$$

Therefore the partition of $\text{Supp}(P)$ determined by ϕ is J .

Notation 2.10. Consider variables z_1, \dots, z_n . For $\alpha \in \mathbb{Z}^n$ let $p_\alpha(z_1, \dots, z_n)$ be the linear polynomial $\alpha_1 z_1 + \dots + \alpha_n z_n$. For $J \subseteq \mathbb{N}_0^n$, let E_J be the \mathbb{Q} -linear span of $\{p_{\alpha-\beta} : \alpha, \beta \in J\}$.

Proposition 2.11. *Suppose that (J_0, \dots, J_ℓ) is a Rado partition for P . Then one has that, for every $i \in \{0, 1, \dots, \ell\}$:*

- (1) *The set J_i is convex, in the sense that if $\alpha^{(1)}, \dots, \alpha^{(m)} \in J_i$ and $\lambda_1, \dots, \lambda_m \in \mathbb{Q}$ are such that $\lambda_1 + \dots + \lambda_m = 1$, and $\lambda_1 \alpha^{(1)} + \dots + \lambda_m \alpha^{(m)} \in \text{Supp}(P)$, then $\lambda_1 \alpha^{(1)} + \dots + \lambda_m \alpha^{(m)} \in J_i$;*
- (2) *every nonzero polynomial $p(z_1, \dots, z_n) = a_1 z_1 + \dots + a_n z_n$ in E_{J_i} is partition regular, i.e. there exist $\ell \geq 2$ and $1 \leq i_1 < i_2 < \dots < i_\ell \leq n$ such that $a_{i_1}, \dots, a_{i_\ell}$ are nonzero and $a_{i_1} + \dots + a_{i_\ell} = 0$;*
- (3) *if $\alpha, \beta \in J_i$ and $\ell(\alpha + \beta) \leq 2$, then $|\alpha| = |\beta|$.*

Proof. (1) This follows from the definitions and the fact that linear maps preserve affine combinations.

- (2) By the definition of Rado partition, for every finite coloring c of \mathbb{N} there exists a positive linear map $\phi : \mathbb{Z}^n \rightarrow \mathbb{Z}$, say $\phi : (\alpha_1, \dots, \alpha_n) \mapsto t_1 \alpha_1 + \dots + t_n \alpha_n$, such that $\{t_1, \dots, t_n\}$ is c -monochromatic and $\phi(\alpha) = \phi(\beta)$ for every $\alpha, \beta \in J_i$. Therefore $p_{\alpha-\beta}(t_1, \dots, t_n) = 0$ whenever $\alpha, \beta \in J_i$, and hence $p(t_1, \dots, t_n) = 0$ whenever $p \in E_{J_i}$.
- (3) If $\ell(\alpha + \beta) \leq 2$, then the polynomial $p_{\alpha-\beta}$ depends on at most two variables, and hence if it is partition regular must take the form $p_{\alpha-\beta}(z_1, \dots, z_n) = \lambda(z_i - z_j)$ for some $\lambda \in \mathbb{N}_0$ and some $i \neq j$. Consequently, $|\alpha| - |\beta| = p_{\alpha-\beta}(1, \dots, 1) = 0$ as required. \square

Definition 2.12. A *lower Rado functional* of order $m \in \mathbb{N}_0$ for $P \in \mathbb{Z}[x_1, \dots, x_n]$ is a tuple $(J_0, \dots, J_\ell, d_1, \dots, d_m)$ for some $\ell \geq m$ and $d_1, \dots, d_m \in \mathbb{N}$ such that, for every finite coloring c of \mathbb{N} and for every $n \in \mathbb{N}$, there exist infinitely many c -monochromatic positive linear maps $\phi : \mathbb{Z}^n \rightarrow \mathbb{Z}$, $(\alpha_1, \dots, \alpha_n) \mapsto t_1 \alpha_1 + \dots + t_n \alpha_n$ such that (J_0, \dots, J_ℓ) is the partition of $\text{Supp}(P)$ determined by ϕ and, if (M_0, \dots, M_ℓ) is the increasing enumeration of $\phi(\text{Supp}(P))$, then $M_i - M_0 = d_i$ for $i \in \{1, 2, \dots, m\}$, and $M_{m+1} - M_m \geq n$.

Definition 2.13. An *upper Rado functional* of order $m \in \mathbb{N}_0$ for $P \in \mathbb{Z}[x_1, \dots, x_n]$ is a tuple $(J_0, \dots, J_\ell, d_0, \dots, d_{m-1})$ for some $\ell \geq m$ and $d_0, \dots, d_{m-1} \in \mathbb{N}$ such that, for every finite coloring c of \mathbb{N} and for every $n \in \mathbb{N}$, there exist infinitely many c -monochromatic positive linear maps $\phi : \mathbb{Z}^n \rightarrow \mathbb{Z}$, $(\alpha_1, \dots, \alpha_n) \mapsto t_1 \alpha_1 + \dots + t_n \alpha_n$ such that (J_ℓ, \dots, J_0) is the partition of $\text{Supp}(P)$ determined by ϕ and, if (M_ℓ, \dots, M_0) is the increasing enumeration of $\phi(\text{Supp}(P))$, then $M_i - M_m = d_i$ for $i \in \{0, 1, \dots, m-1\}$, and $M_m - M_{m+1} \geq n$.

In Remark 2.26 below we give a simpler description of Rado functionals in the language of non-standard analysis.

Example 2.14. Consider as in Example 2.9 the polynomial $P(x_1, x_2, x_3, x_4) = -x_3 - x_4 + x_1 x_2 + x_1 x_2^2$ and the Rado partition (J_0, J_1, J_2, J_3) defined there. We claim that $(J_3, J_2, J_1, J_0, 2)$ is an upper Rado functional of order 1 for P , and that $(J_0, J_1, J_2, J_3, 1)$ is a lower Rado functional of

order 1 for P . Indeed, suppose that c is a finite coloring of \mathbb{N} , and $n \in \mathbb{N}$. By Brauer's extension of the van der Waerden theorem on arithmetic progressions there exist $a, b \geq n + 2$ such that $\{a, b, a + b - 1, a + 2b - 2\}$ is c -monochromatic. Consider the linear functional $\phi : \mathbb{Z}^4 \rightarrow \mathbb{Z}$,

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto a\alpha_1 + b\alpha_2 + (a + b - 1)\alpha_3 + (a + 2b - 2)\alpha_4.$$

Then we have that

$$\phi(0, 0, 1, 0) < \phi(1, 1, 0, 0) < \phi(0, 0, 0, 1) < \phi(1, 2, 0, 0).$$

Furthermore,

$$\begin{aligned} \phi(1, 2, 0, 0) - \phi(0, 0, 0, 1) &= 2 \\ \phi(0, 0, 0, 1) - \phi(1, 1, 0, 0) &= b - 2 \geq n \\ \phi(1, 1, 0, 0) - \phi(0, 0, 1, 0) &= 1. \end{aligned}$$

This witnesses that $(J_3, J_2, J_1, J_0, 2)$ is an upper Rado functional of order 1 for P , and that $(J_0, J_1, J_2, J_3, 1)$ is a lower Rado functional of order 1 for P .

Example 2.15. Consider the polynomial $P(x, y, z, w) = xy^2 + yz + w$. Then we have that

$$\text{Supp}(P) = \{(1, 2, 0, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}.$$

We claim that $(J_0, J_1, J_2, 1, 2)$ is a lower Rado functional of order 2 for P , where

$$\begin{aligned} J_0 &= \{(0, 0, 0, 1)\} \\ J_1 &= \{(0, 1, 1, 0)\} \\ J_2 &= \{(1, 2, 0, 0)\}. \end{aligned}$$

Indeed, suppose that c is a finite coloring of \mathbb{N} . By Brauer's extension of the van der Waerden theorem there exist $a, b \geq n + 2$ such that $\{a, b, a + b - 1, a + 2b - 2\}$ is c -monochromatic. Consider the linear functional $\phi : \mathbb{Z}^4 \rightarrow \mathbb{Z}$,

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto a\alpha_1 + b\alpha_2 + (a + b - 1)\alpha_2 + (a + 2b - 2)\alpha_3.$$

Then we have that

$$\phi(0, 0, 0, 1) + 2 = \phi(0, 1, 1, 0) + 1 = \phi(1, 2, 0, 0) = a + 2b.$$

This witnesses that $(J_0, J_1, J_2, 1, 2)$ is a lower Rado functional of order 2 for P .

We consider the following generalizations of Rado's criterion of partition regularity.

Definition 2.16. A polynomial $P(x) = \sum_{\alpha} c_{\alpha} x^{\alpha} \in \mathbb{Z}[x_1, \dots, x_n]$ satisfies the *maximal Rado condition* if for every $q \in \mathbb{N} \setminus \{1\}$ there exists an upper Rado functional $(J_0, \dots, J_{\ell}, d_0, d_1, \dots, d_{m-1})$ for P such that, setting $d_m := 0$, the polynomial

$$w \mapsto \sum_{i=0}^m q^{d_i} \sum_{\alpha \in J_i} c_{\alpha} w^{|\alpha|}$$

has a root in $[1, q]$.

Definition 2.17. Let P be a polynomial such that \tilde{P} is nonzero and splits as a product of linear factors over \mathbb{Z} . We say that P satisfies the *minimal Rado condition* if for every prime $p \in \mathbb{N}$ there exists a root $a \in \mathbb{Z}$ of \tilde{P} and a minimal Rado functional $(J_0, \dots, J_{\ell}, d_1, \dots, d_m)$ for $P^{(a)}$ such that, setting $d_0 := 0$, the equation

$$\sum_{i=0}^m p^{d_i} \sum_{\alpha \in J_i} \frac{1}{\alpha!} \frac{\partial^{\alpha} P}{\partial x^{\alpha}}(\mathbf{a}) w^{|\alpha|} = 0$$

has an invertible solution in \mathbb{Z}_p .

Lemma 2.18. Let P be a polynomial such that \tilde{P} is nonzero and splits as a product of linear factors over \mathbb{Z} and P satisfies the minimal Rado condition. Then there exists a root $a \in \mathbb{Z}$ of \tilde{P} and a minimal Rado set J of $P^{(a)}$ such that the equation

$$\sum_{\alpha \in J} \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{a}) w^{|\alpha|} = 0$$

has a nonzero solution in $\mathbb{Z}/p\mathbb{Z}$ for infinitely many primes $p \in \mathbb{Z}$. If J is homogeneous, this implies that

$$\sum_{\alpha \in J} \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{a}) = 0.$$

Proof. Fix a prime number p . Then since P satisfies the minimal Rado condition, there exists a root $a \in \mathbb{Z}$ of \tilde{P} and a minimal Rado functional $(J_0, \dots, J_\ell, d_1, \dots, d_m)$ for $P^{(a)}$ such that, setting $d_0 := 0$, the equation

$$\sum_{i=0}^m p^{d_i} \sum_{\alpha \in J_i} \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{a}) w^{|\alpha|} = 0$$

has an invertible solution in \mathbb{Z}_p . Considering the canonical quotient map $\mathbb{Z}_p \rightarrow \mathbb{Z}/p\mathbb{Z}$, we obtain that the equation

$$\sum_{\alpha \in J_0} \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{a}) w^{|\alpha|} = 0$$

has a nonzero solution in $\mathbb{Z}/p\mathbb{Z}$. Notice that, by definition, J_0 is a minimal Rado set of $P^{(a)}$.

As \tilde{P} has finitely many roots, there exists a root a of \tilde{P} that is obtained as above from infinitely many primes p . For this root a , as $P^{(a)}$ has finitely many minimal Rado sets, there exists a minimal Rado set J_0 of $P^{(a)}$ that is obtained as above from infinitely many primes p . This concludes the proof of the first assertion.

In the case when J is homogeneous of degree d , we have that

$$\sum_{\alpha \in J_0} \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{a}) w^{|\alpha|} = \sum_{\alpha \in J_0} \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{a}) w^d.$$

Therefore, the equation

$$\sum_{\alpha \in J_0} \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{a}) w^{|\alpha|} = 0$$

has a nonzero solution in $\mathbb{Z}/p\mathbb{Z}$ if and only if

$$\sum_{\alpha \in J_0} \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{a}) \equiv 0 \pmod{p}.$$

If this holds for infinitely many primes p , this implies that

$$\sum_{\alpha \in J_0} \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{a}) = 0.$$

Thus, the second assertion follows from the first one. \square

In the next section, we will prove the following *necessary* conditions for partition regularity in terms of the Rado conditions.

Theorem 2.19. Fix $P \in \mathbb{Z}[x_1, \dots, x_n]$. Suppose that P is partition regular. Then:

- (1) P satisfies the maximal Rado condition;
- (2) if \tilde{P} is nonzero and splits over \mathbb{Z} as a product of linear factors, then P satisfies the minimal Rado condition.

The first assertion of Theorem 2.19 is proved in Theorem 3.1 below, while the second assertion of Theorem 2.19 is an immediate consequence of Proposition 3.2 and Theorem 3.3.

We observe that Part (2) of Theorem 2.19 recovers, in the particular case of homogeneous linear polynomial, one direction in the Rado characterization of partition regular homogeneous linear equations. Furthermore, Corollary 3.9 of [14] is the particular instance of Part (2) of Theorem 2.19 when P is homogeneous. In view of Corollary 2.27 below, Theorem 3.10 of [14] is the particular instance of Part (1) of Theorem 2.19 when $P(x_1, \dots, x_n)$ is of the form $P_1(x_1) + \dots + P_n(x_n)$ for some univariate polynomials P_1, \dots, P_n .

It would be interesting to know whether the Rado conditions provide *sufficient* criteria for partition regularity for more general Diophantine equations, at least for equations $P(x_1, \dots, x_n) = 0$ with infinitely many solutions in \mathbb{N} and such that the corresponding univariate polynomial \tilde{P} is a homogeneous *linear* polynomial. Such a result would be a natural generalization of Rado's characterization of homogeneous linear equations from [26].

Conjecture 2.20. Let $P \in \mathbb{Z}[x_1, \dots, x_n]$ be a polynomial such that $\tilde{P}(w) := P(w, \dots, w)$ is a nonzero homogeneous linear polynomial. Suppose that the equation $P(x_1, \dots, x_n) = 0$ has infinitely many solutions in \mathbb{N} . Then, the following assertions are equivalent:

- (1) P is partition regular;
- (2) P satisfies the maximal Rado condition and the minimal Rado condition.

We provide evidence towards Conjecture 2.20 by considering the families of polynomials $x^2 - xy + ax + by + cz$ and $x^2 - y^2 + ax + by + cz$ for $a, b, c, k \in \mathbb{Z}$. We establish that Conjecture 2.20 holds for such families, as long as either the product or the sum of the coefficients is zero.

Theorem 2.21. Fix $a, b, c \in \mathbb{Z}$. Assume that either $abc = 0$ or $a + b + c = 0$. Suppose that $P(x, y, z)$ is either the polynomial $x^2 - xy + ax + by + cz$, or the polynomial $x^2 - y^2 + ax + by + cz$. Then the following assertions are equivalent:

- (1) P is partition regular;
- (2) P satisfies the minimal Rado condition whenever \tilde{P} is nonzero.

Remark 2.22. Notice that the maximal Rado condition is always satisfied for a polynomial P as in Theorem 2.21.

The only case left out by Theorem 2.21 is when a, b, c , and $a + b + c$ are all nonzero. In this case, P satisfies the minimal Rado condition if and only if $0 \in \{a + b, a + c, b + c\}$. Hence, only the following six cases remain.

Problem 2.23. Fix $a, b \in \mathbb{Z} \setminus \{0\}$. Are the following polynomials partition regular?

- $x(x - y) - xy + a(x - y) + bz$
- $x(x - y) + a(y - z) + bx$
- $x(x - y) + a(x - z) + by$
- $(x + y)(x - y) + a(x - y) + bz$
- $(x + y)(x - y) + a(y - z) + bx$
- $(x + y)(x - y) + a(x - z) + by$

In order to prove Theorem 2.21, we will use the recent result from [25] asserting that the configuration $\{x + p(y), x + q(y), xy\}$ is partition regular whenever $p(y), q(y)$ are polynomials with integer coefficients vanishing at 0 (see Theorem 4.6 below). In other words, for every finite

coloring of \mathbb{N} there exist infinitely many $a, b \in \mathbb{N}$ such that the set $\{a + p(b), a + q(b), ab\}$ is monochromatic. In fact, we will need a similar result for the configurations

$$\{x + p(y), x + q(y), xy + x + dy\}$$

where $d \in \mathbb{Q}$; see Theorem 4.9.

Theorem 2.19 will be proved using the formalism of hypernatural numbers and nonstandard extensions. We will adopt the notation and terminology from [14]. Particularly, we will consider a \mathfrak{c}^+ -saturated nonstandard extension ${}^*\mathbb{N}$ of \mathbb{N} . Any finite coloring $c : \mathbb{N} \rightarrow \{1, 2, \dots, k\}$ extends to a coloring of ${}^*\mathbb{N}$, which we still denote by c . If $\xi, \eta \in {}^*\mathbb{N}$, we write $\xi \ll \eta$ and $\eta \gg \xi$ if $\eta - \xi$ is infinite. Two hypernatural numbers ξ, η are *indiscernible* (or *u-equivalent*) if $c(\xi) = c(\eta)$ for every finite coloring c of \mathbb{N} ; see [14, Definition 3.1]. This is equivalent to the assertion that, for every $A \subseteq \mathbb{N}$, $\xi \in {}^*A$ if and only if $\eta \in {}^*A$. To denote that ξ, η are indiscernible, we write $\xi \sim \eta$. Several properties of such relation are listed in [10–12]. In particular, we have the following:

- If $\xi, \eta \in {}^*\mathbb{N}$ are indiscernible and $\xi < \eta$, then $\xi \ll \eta$;
- If $\xi, \eta \in {}^*\mathbb{N}$ are indiscernible, and $f : \mathbb{N} \rightarrow \mathbb{N}$, then $f(\xi), f(\eta) \in {}^*\mathbb{N}$ are indiscernible;
- If $\xi \in {}^*\mathbb{N}$ and $f : \mathbb{N} \rightarrow \mathbb{N}$ are such that $\xi \sim f(\xi)$, then $\xi = f(\xi)$;
- If $\xi \in {}^*\mathbb{N}$ and $k \in \mathbb{N}$ are such that $\xi \sim k$, then $\xi = k$.

Fix $\xi \in {}^*\mathbb{N}$ and $q \in \mathbb{N}$ with $q \geq 2$. By transfer, ξ admits a unique *base q expansion*. This is an internal sequence $(a_i)_{i \in {}^*\mathbb{N}_0}$ in $\{0, 1, \dots, q-1\}$ such that

$$\text{Supp}_q(\xi) := \{i \in {}^*\mathbb{N}_0 : a_i \neq 0\}$$

is hyperfinite, and $\sum_{i \in {}^*\mathbb{N}} a_i q^i = \xi$. If σ is the least element of $\text{Supp}_q(\xi)$, and τ is the largest element of $\text{Supp}_q(\xi)$, then we call τ the position of the first nonzero digit in the base q expansion of ξ , and we call σ the position of the last nonzero digit of the base q decomposition of ξ . Accordingly, we call a_τ the first nonzero digit (in the base q expansion of ξ), and a_σ the last nonzero digit (in the base q decomposition of ξ). Similarly, if $i \in \{1, 2, \dots, \tau\}$, we call $a_{\tau-i}$ the i -th digit, and $a_{\sigma+i}$ the i -th to last digit.

It follows from the properties of indiscernible pairs that, when $q \in \mathbb{N} \setminus \{1\}$ and $\xi, \xi' \in {}^*\mathbb{N}$ are indiscernible, if σ, σ' are the positions of the last nonzero digits of ξ, ξ' , respectively, then σ, σ' are indiscernible. The same applies to the positions τ, τ' of the first nonzero digits of ξ, ξ' . Furthermore, we have that the last nonzero digits of ξ, ξ' are equal, and the first nonzero digits of ξ, ξ' are equal as well. More generally, for every $i \in \mathbb{N}$, the i -th digits of ξ, ξ' are equal, and the i -th to last digits of ξ, ξ' are equal. It follows by overspill that there exists $\nu \in {}^*\mathbb{N}$ infinite such that, for every $i \leq \nu$, the i -th digits of ξ, ξ' are equal, and the i -th to last digits of ξ, ξ' are equal. If $\xi \in {}^*\mathbb{N}$ and $q \in \mathbb{N} \setminus \{1\}$, then we say that ξ has no finite tail in base q if the position of its last nonzero digit in the base q expansion of ξ is infinite. This is equivalent to the assertion that $\xi \equiv 0 \pmod{q^n}$ for every $n \in \mathbb{N}$.

The following characterization of partition regularity is well-known; see for example [14, Proposition 3.2].

Remark 2.24. The equation $P(x_1, \dots, x_n) = 0$ is partition regular if and only if there exist $\xi_1, \dots, \xi_n \in {}^*\mathbb{N}$ which are infinite, indiscernible, and such that $P(\xi_1, \dots, \xi_n) = 0$.

In the same way, one can prove the following characterization of q -partition regular equations.

Remark 2.25. Fix $q \in \mathbb{N} \setminus \{1\}$. The equation $P(x_1, \dots, x_n) = 0$ is q -partition regular if and only if there exist $\xi_1, \dots, \xi_n \in {}^*\mathbb{N}$ which are infinite, indiscernible, with no finite tail in base q , and such that $P(\xi_1, \dots, \xi_n) = 0$.

One can also characterize upper and lower Rado functionals, as follows.

Remark 2.26. Let $P \in \mathbb{Z}[x_1, \dots, x_n]$ be a polynomial. Then $(J_0, \dots, J_\ell, d_0, \dots, d_{m-1})$ is an upper Rado functional for P if and only if there exist infinite, indiscernible $\tau_1, \dots, \tau_n \in {}^*\mathbb{N}$ and $M_0, \dots, M_\ell \in {}^*\mathbb{N}$ such that $M_\ell < M_{\ell-1} < \dots < M_0$,

$$J_i = \{\alpha \in \text{Supp}(P) : \alpha_1 \tau_1 + \dots + \alpha_n \tau_n = M_i\}$$

for $i \in \{0, 1, \dots, \ell\}$,

$$d_i = M_i - M_m$$

for $i \in \{0, 1, \dots, m-1\}$, and $M_m - M_i$ is infinite for $i \in \{m+1, \dots, \ell\}$.

Similarly, $(J_0, \dots, J_\ell, d_0, \dots, d_{m-1})$ is a lower Rado functional if and only if there exist infinite, indiscernible $\sigma_1, \dots, \sigma_n \in {}^*\mathbb{N}$ and $M_0, \dots, M_\ell \in {}^*\mathbb{N}$ such that $M_0 < M_1 < \dots < M_\ell$,

$$J_i = \{\alpha \in \text{Supp}(P) : \alpha_1 \sigma_1 + \dots + \alpha_n \sigma_n = M_i\}$$

for $i \in \{0, 1, \dots, \ell\}$,

$$d_i = M_i - M_0$$

for $i \in \{1, \dots, m\}$, and $M_i - M_m$ is infinite for $i \in \{m+1, \dots, \ell\}$.

Corollary 2.27. If $(J_0, \dots, J_\ell, d_0, d_1, \dots, d_m)$ is an upper or lower Rado functional, $i, j \in \{0, 1, \dots, m\}$, $\alpha \in J_i$ and $\beta \in J_j$ are such that $\ell(\alpha + \beta) \leq 2$, then $i = j$ and $|\alpha| = |\beta|$.

Proof. We consider the case when $(J_0, \dots, J_\ell, d_0, d_1, \dots, d_m)$ is an upper Rado functional, as the other case is analogous. Suppose that $\tau_1, \dots, \tau_n \in {}^*\mathbb{N}$ and $M_\ell < \dots < M_0$ are obtained from $(J_0, \dots, J_\ell, d_0, d_1, \dots, d_m)$ as in the previous remark. Without loss of generality, we can assume that $i \leq j$, $\alpha = (\alpha_1, \alpha_2, 0, \dots, 0)$ and $\beta = (\beta_1, \beta_2, 0, \dots, 0)$. Then we have that

$$\alpha_1 \tau_1 + \alpha_2 \tau_2 = M_i$$

and

$$\beta_1 \tau_1 + \beta_2 \tau_2 = M_j.$$

Observe that $M_i - M_j$ is finite. Define $\gamma_1 := \alpha_1 - \beta_1$ and $\gamma_2 := \alpha_2 - \beta_2$. Thus,

$$M_i - M_j = \gamma_1 \tau_1 + \gamma_2 \tau_2$$

and

$$\gamma_1 \tau_1 = -\gamma_2 \tau_2 + M_i - M_j \sim -\gamma_2 \tau_1 + M_i - M_j.$$

It follows that

$$\gamma_1 \tau_1 = -\gamma_2 \tau_1 + M_i - M_j$$

and hence

$$(\gamma_1 + \gamma_2) \tau_1 = M_i - M_j.$$

Since τ_1 is infinite, we have that $\gamma_1 = -\gamma_2$, $M_i = M_j$, and $i = j$. This concludes the proof. \square

3. NECESSITY OF THE RADO CONDITIONS

In this section, we assume that $P(x_1, \dots, x_n) = \sum_{\alpha} c_{\alpha} x^{\alpha} \in \mathbb{Z}[x_1, \dots, x_n]$ is a polynomial with integer coefficients.

Theorem 3.1. *Suppose that P is partition regular. Then P satisfies the maximal Rado condition.*

Proof. For $\xi, \eta \in {}^*\mathbb{R}$ finite, we write $\xi \approx \eta$ if $\xi - \eta$ is infinitesimal. We also let $\text{st}(\xi)$ be the standard part of ξ . Fix $q \in \mathbb{N} \setminus \{1\}$. Set $c_\alpha := \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{0})$ for every index α . By assumption there exist $\xi_1, \dots, \xi_n \in {}^*\mathbb{N}$ such that ξ_1, \dots, ξ_n are indiscernible, infinite, and $P(\xi_1, \dots, \xi_n) = 0$. For $i \in \{1, 2, \dots, n\}$, let τ_i be the position of the first nonzero digit in the base q expansion of ξ_i . Observe that τ_1, \dots, τ_n are infinite and indiscernible. Set $\tau := (\tau_1, \dots, \tau_n)$. For an index α , we define $\alpha \cdot \tau$ to be $\alpha_1 \tau_1 + \dots + \alpha_n \tau_n$. Let (M_ℓ, \dots, M_0) be the increasing enumeration of $\{\alpha \cdot \tau : \alpha \in \text{Supp}(P)\}$. Define m to be the least element of the set $\{t \in \{0, 1, \dots, \ell\} : M_{t+1} \ll M_0\}$. For $t \in \{0, 1, \dots, \ell\}$, define

$$J_t = \{\alpha \in \text{Supp}(P) : \alpha \cdot \tau = M_t\}.$$

For $i \in \{0, 1, \dots, m\}$, define $d_i := M_i - M_m$. Observe that $(J_0, \dots, J_\ell, d_0, \dots, d_m)$ is an upper Rado functional for P , as witnessed by (τ_1, \dots, τ_n) ; see Remark 2.26.

Fix now an infinite $\nu \in {}^*\mathbb{N}$ such that, for $i \leq \nu$, the i -th digits of ξ_1, \dots, ξ_n in the base q expansion are equal. We can write

$$\xi_i = q^{\tau_i} (\sigma + q^{-\nu} \rho_i)$$

where $1 \leq \sigma < q$ and $\rho_1, \dots, \rho_n \in {}^*\mathbb{R}$ are such that $0 \leq \rho_i \leq 1$ for every $i \in \{1, 2, \dots, n\}$. Define $\zeta_i := \sigma + q^{-\nu} \rho_i$ for $i \in \{1, 2, \dots, n\}$ and, for $t \in \{0, 1, \dots, \ell\}$,

$$Q_t(x) := \sum_{\alpha \in J_t} c_\alpha x^\alpha.$$

Set also $w = \text{st}(\sigma) \in \mathbb{R}$ and observe that $1 \leq w \leq q$. Then we have that

$$0 = P(\xi) = \sum_{t=0}^{\ell} Q_t(\xi) = \sum_{t=0}^{\ell} q^{M_t} Q_t(\zeta_1, \dots, \zeta_n).$$

Thus

$$\sum_{t=0}^m q^{d_t} Q_t(\zeta_1, \dots, \zeta_n) = - \sum_{t=m+1}^{\ell} \frac{1}{q^{M_m - M_t}} Q_t(\zeta_1, \dots, \zeta_n) \approx 0$$

is infinitesimal. Set $\rho := (\rho_1, \dots, \rho_n)$. For $t \in \{\ell - m, \dots, \ell\}$, we have

$$\begin{aligned} Q_t(\zeta_1, \dots, \zeta_n) &= Q_t(\sigma + q^{-\nu} \rho_1, \dots, \sigma + q^{-\nu} \rho_n) \\ &= \sum_{\alpha \in J_t} \sum_{\beta \leq \alpha} c_\alpha \sigma^{|\alpha| - |\beta|} q^{-\nu|\beta|} \rho^\beta \\ &= \sum_{\alpha \in J_t} c_\alpha \sigma^{|\alpha|} + \sum_{\substack{\alpha \in J_t \\ \beta \leq \alpha \\ \beta \neq \mathbf{0}}} c_\alpha \sigma^{|\alpha| - |\beta|} q^{-\nu|\beta|} \rho^\beta. \end{aligned}$$

Observe that

$$\sum_{\substack{\alpha \in J_t \\ \beta \leq \alpha \\ \beta \neq \mathbf{0}}} c_\alpha \sigma^{|\alpha| - |\beta|} q^{-\nu|\beta|} \rho^\beta \approx 0$$

is infinitesimal. Therefore, we have that

$$Q_t(\zeta_1, \dots, \zeta_n) \approx \sum_{\alpha \in J_t} c_\alpha \sigma^{|\alpha|}.$$

Thus

$$0 \approx \sum_{t=0}^m q^{d_t} Q_t(\zeta_1, \dots, \zeta_n) \approx \sum_{t=0}^m q^{d_t} \sum_{\alpha \in J_t} c_\alpha \sigma^{|\alpha|}.$$

Considering the standard part, we have

$$0 = \sum_{t=0}^m q^{d_t} \sum_{\alpha \in J_t} c_\alpha w^{|\alpha|}.$$

Since $w \in [1, q]$, this concludes the proof. \square

Proposition 3.2. *Suppose that the monovariate polynomial \tilde{P} associated with P is nonzero and splits over \mathbb{Z} as a product of linear factors. If P is partition regular, then for every prime $p \in \mathbb{N}$ there exists a root a of \tilde{P} such that $P^{(a)}$ is p -partition regular.*

Proof. Define ℓ to be the degree of \tilde{P} , and suppose that $a_1, \dots, a_\ell \in \mathbb{Z}$ are the roots of \tilde{P} . Suppose that P is partition regular. Then there exist infinite, indiscernible $\xi_1, \dots, \xi_n \in {}^*\mathbb{N}$ such that $P(\xi_1, \dots, \xi_n) = 0$. Fix an infinite $\nu \in {}^*\mathbb{N}$ and $\rho \in \{0, 1, \dots, p^{\ell\nu} - 1\}$ such that

$$\xi_1 \equiv \dots \equiv \xi_n \equiv \rho \pmod{p^{\ell\nu}},$$

where $\{0, 1, \dots, p^{\ell\nu} - 1\}$ denotes the set of $x \in {}^*\mathbb{N}_0$ such that $x < p^{\ell\nu}$. Thus, we have that $\tilde{P}(\rho) \equiv 0 \pmod{p^{\ell\nu}}$, which implies that $p^{\ell\nu}$ divides $(\rho - a_1) \cdots (\rho - a_\ell)$. Hence, there exists a root a of \tilde{P} such that p^ν divides $\rho - a$. This implies that

$$\xi_1 - a \equiv \dots \equiv \xi_n - a \equiv \rho - a \equiv 0 \pmod{p^\nu}.$$

Define now $\eta_i := \xi_i - a$ for $i \in \{1, 2, \dots, n\}$. Observe that η_1, \dots, η_n are indiscernible, infinite, with no finite tail in base p , and such that $P^{(a)}(\eta_1, \dots, \eta_n) = 0$. The conclusion thus follows from Remark 2.25. \square

Theorem 3.3. *Suppose that $p \in \mathbb{N}$ is a prime. If P is p -partition regular, then there exists a minimal Rado functional $(J_0, \dots, J_\ell, d_1, \dots, d_m)$ for P such that, setting $d_0 = 0$, the equation*

$$\sum_{i=0}^m p^{d_i} \sum_{\alpha \in J_i} \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{0}) w^{|\alpha|} = 0$$

has an invertible solution in the ring \mathbb{Z}_p of p -adic integers.

Proof. Suppose that P is p -partition regular. Thus, there exist infinite, indiscernible $\xi_1, \dots, \xi_n \in {}^*\mathbb{N}$ with no finite tail in base p such that $P(\xi_1, \dots, \xi_n) = 0$. For $i \in \{1, 2, \dots, n\}$, let σ_i be the position of the last nonzero digit in the base p expansion of ξ_i . Observe that $\sigma_1, \dots, \sigma_n$ are infinite and indiscernible. Set $\sigma := (\sigma_1, \dots, \sigma_n)$. For an index α , define $\alpha \cdot \sigma$ to be $\alpha_1 \sigma_1 + \dots + \alpha_n \sigma_n$. Let (M_0, \dots, M_ℓ) be the increasing enumeration of $\{\alpha \cdot \sigma : \alpha \in \text{Supp}(P)\}$. Define m be the least element of the set $\{i \in \{0, 1, \dots, \ell\} : M_0 \ll M_{i+1}\}$. For $t \in \{0, 1, \dots, \ell\}$, define

$$J_t = \{\alpha \in \text{Supp}(P) : \alpha \cdot \sigma = M_t\}.$$

For $i \in \{0, 1, \dots, m\}$, define $d_i = M_i - M_0$. Observe that $(J_0, \dots, J_\ell, d_0, \dots, d_m)$ is a lower Rado functional for P , as witnessed by $(\sigma_1, \dots, \sigma_n)$. For $i \in \{1, 2, \dots, n\}$, let $\zeta_i \in {}^*\mathbb{N}$ be such that

$$\xi_i = p^{\sigma_i} \zeta_i.$$

Define $\zeta = (\zeta_1, \dots, \zeta_n)$. Observe that ζ_1, \dots, ζ_n are indiscernible and, for $\alpha \in J_t$, we have that

$$\xi^\alpha = p^{M_t} \zeta^\alpha.$$

Fix $\nu \in {}^*\mathbb{N}$ infinite such that $\zeta_1 \equiv \dots \equiv \zeta_n \pmod{p^\nu}$ and $M_m + \nu \ll M_{m+1}$. Let $\rho \in \{1, \dots, p^\nu - 1\}$ be such that $\zeta_1 \equiv \rho \pmod{p^\nu}$, where as before $\{1, \dots, p^\nu - 1\}$ denotes the set of

$x \in {}^*\mathbb{N}$ such that $x < p^\nu$. Observe that, by definition of ζ_1 and ρ , p does not divide ρ . To ease the notation, let $c_\alpha := \frac{1}{\alpha!} \frac{\partial^\alpha P}{\partial x^\alpha}(\mathbf{0})$. We have that

$$0 = \sum_{\alpha} c_\alpha \zeta^\alpha \equiv \sum_{i=0}^m p^{M_i} \sum_{\alpha \in J_i} c_\alpha \zeta^\alpha = p^{M_0} \sum_{i=0}^m p^{M_i - M_0} \sum_{\alpha \in J_i} c_\alpha \zeta^\alpha \pmod{p^{M_m + \nu}}.$$

Hence,

$$0 \equiv \sum_{i=0}^m p^{M_i - M_0} \sum_{\alpha \in J_i} c_\alpha \zeta^\alpha \equiv \sum_{i=0}^m p^{d_i} \sum_{\alpha \in J_i} c_\alpha \rho^{|\alpha|} \pmod{p^\nu}.$$

In particular, for every $k \in \mathbb{N}$,

$$\sum_{i=0}^m p^{d_i} \sum_{\alpha \in J_i} c_\alpha \rho^{|\alpha|} \equiv 0 \pmod{p^k}.$$

Since p does not divide ρ , this witnesses that the equation $\sum_{i=0}^m p^{d_i} \sum_{\alpha \in J_i} c_\alpha w^{|\alpha|} \equiv 0 \pmod{p^k}$ has an invertible solution in $\mathbb{Z}/p^k\mathbb{Z}$. Since this holds for every $k \in \mathbb{N}$, we have that the equation

$$\sum_{i=0}^m p^{d_i} \sum_{\alpha \in J_i} c_\alpha w^{|\alpha|} = 0$$

has an invertible solution in \mathbb{Z}_p . □

Example 3.4. Consider the quadratic polynomial

$$P(x, y) = x^2 + y^2 - xy - ax - by + ab$$

with $a, b \in \mathbb{Z}$. We claim that the equation $P(x, y) = 0$ has only finitely many integer solutions when $a = b$, and it does not satisfy the minimal Rado condition when $a \neq b$. In particular, P is not partition regular.

Suppose that $a = b$, in which case

$$P(x, y) = x^2 + y^2 - xy - ax - ay + a^2.$$

Set

$$r = x - y - a$$

and observe that

$$\begin{aligned} P(x, y) &= x^2 + y^2 - xy - ax - ay + a^2 \\ &= r^2 + xy + ax - 3ay \\ &= y^2 + y(r - a) + a^2 + ar + r^2. \end{aligned}$$

If x, y is an integer solution to $P(x, y) = 0$, we have necessarily that $(r - a)^2 \geq 4(a^2 + ar + r^2)$ and hence $a + r = 0$. Therefore $x = y$ and

$$P(x, y) = (x - a)^2.$$

Thus the only integer solution to $P(x, y) = 0$ is (a, a) when $a = b$.

Consider the case when $a \neq b$. In this case we have

$$\tilde{P}(w) = (w - a)(w - b).$$

Furthermore

$$\begin{aligned}
& P^{(a)}(x, y) \\
&= (x+a)^2 + (y+a)^2 - (x+a)(y+a) - a(x+a) - b(y+a) + ab \\
&= x^2 + y^2 - xy + 2ax + 2ay - ax - ay - ax - by \\
&= x^2 + y^2 - xy + (a-b)y.
\end{aligned}$$

Similarly, we have that

$$P^{(b)}(x, y) = x^2 + y^2 - xy + (b-a)x.$$

As $a \neq b$, we have that $(a-b)y$ and x^2 are the only minimal terms in $P^{(a)}$. Therefore, all minimal Rado sets for $P^{(a)}$ are singletons. Similarly, all minimal Rado sets for $P^{(b)}$ are singletons. By Lemma 2.18, this shows that P does not satisfy the minimal Rado condition when $a \neq b$.

Example 3.5. Consider the polynomial

$$P(x, y) = x^2 - y^2 + xy - ax - by + ab$$

with $a, b \in \mathbb{Z}$. We claim that P does not satisfy the minimal Rado condition when $a \neq b$. In particular, P is not partition regular.

We have that

$$\tilde{P}(w) = (w-a)(w-b).$$

Furthermore,

$$\begin{aligned}
& P^{(a)}(x, y) \\
&= (x+a)^2 - (y+a)^2 + (x+a)(y+a) - a(x+a) - b(y+a) + ab \\
&= x^2 - y^2 + xy + 2ax - (a+b)y
\end{aligned}$$

and

$$\begin{aligned}
& P^{(b)}(x, y) \\
&= (x+b)^2 - (y+b)^2 + (x+b)(y+b) - a(x+b) - b(y+b) + ab \\
&= x^2 - y^2 + xy + (3b-a)x - 2by.
\end{aligned}$$

Since either a, b is nonzero, at least one between $2a$ and $a+b$ is nonzero. Hence the minimal terms $P^{(a)}(x, y)$ are contained in the set $\{2ax, -(a+b)y\}$. Hence, all the minimal Rado sets for $P^{(a)}$ are homogeneous. Similarly, the terms of minimal indices for $P^{(b)}$ are contained in $\{(3b-a)x, (-2b)y\}$, and all the minimal Rado sets for $P^{(b)}$ are homogeneous. Since we have

$$\begin{aligned}
2a - (a+b) &= a - b \neq 0 \\
(3b-a) - 2b &= b - a \neq 0
\end{aligned}$$

we conclude by Lemma 2.18 that P does not satisfy the minimal Rado condition.

4. PARTITION REGULARITY OF POLYNOMIAL CONFIGURATIONS

In this section we show, building on results from [25], that certain polynomial configurations in \mathbb{N} are partition regular. Let us define the following notation:

Notation 4.1. Let \mathbb{Q}_+ be the set of nonnegative rational numbers. Given $d \in \mathbb{Q}$ and a subset $E \subseteq \mathbb{Q}_+$, we define

$$E + d = \{x \in \mathbb{Q}_+ \mid x - d \in E\}$$

and

$$E/d = \{x \in \mathbb{Q}_+ \mid |dx| \in E\}.$$

Definition 4.2. Fix $d \in \mathbb{Q}$. A subset $A \subseteq \mathbb{N} + d$ is *piecewise syndetic* if there exists some fixed $k \in \mathbb{N}$ such that A contains arbitrarily long increasing sequences $a_1 < \dots < a_n$ satisfying $a_{i+1} - a_i \leq k$, for all $i \in \{1, \dots, n-1\}$.

Definition 4.3. A subset A of \mathbb{N} is:

- *IP-set* if there exists an infinite increasing sequence $(a_k)_{k \in \mathbb{N}}$ such that, for every increasing sequence of indices $k_1 < k_2 < \dots < k_n$ in \mathbb{N} , $a_{k_1} + a_{k_2} + \dots + a_{k_n} \in A$;
- an *IP*-set* if, for every IP-set B , $A \cap B \neq \emptyset$.

We will make use of the following well-known property of piecewise syndetic subsets of \mathbb{N} : if $E \subseteq \mathbb{N}$ is piecewise syndetic and c is a finite coloring of \mathbb{N} , then there exists a c -monochromatic piecewise syndetic subset of E (see, for instance, [17, Theorem 1.24]).

A polynomial generalization of the classical van der Waerden theorem on arithmetic progressions was established by Bergelson and Leibman [5]. We will use the following reformulation; see for example [3, Theorem 4.5].

Theorem 4.4 (Bergelson and Leibman). *Fix $d \in \mathbb{Q}$. Let $E \subseteq \mathbb{N} + d$ be piecewise syndetic, and let $F \subseteq \mathbb{Z}[x]$ be a finite set of integer-valued polynomials such that $f(0) = 0$ for every $f \in F$. Then the set*

$$\left\{ n \in \mathbb{N} : E \cap \bigcap_{f \in F} (E - f(n)) \text{ is piecewise syndetic} \right\}$$

is *IP**.

By replacing the function $n \mapsto f(n)$ with the function $n \mapsto f(n+1)$, we immediately obtain the following consequence.

Corollary 4.5. Fix $d \in \mathbb{Q}$. Let $E \subseteq \mathbb{N} + d$ be piecewise syndetic, and let $F \subseteq \mathbb{Z}[x]$ be a finite set of integer-valued polynomials such that $f(1) = 0$ for every $f \in F$. Then the set

$$\left\{ n \in \mathbb{N} : E \cap \bigcap_{f \in F} (E - f(n)) \text{ is piecewise syndetic} \right\} - 1$$

is *IP**.

Using Theorem 4.4, the following partition regularity result was established in [25].

Theorem 4.6. *Let c be a finite coloring of \mathbb{N} . Let $s \in \mathbb{N}$ and, for each $i \in \{1, \dots, s\}$, let F_i be a finite set of functions $\mathbb{N}^i \rightarrow \mathbb{Z}$ such that for all $f \in F_i$ and any $x_1, \dots, x_{i-1} \in \mathbb{N}$, the function $x \mapsto f(x_1, \dots, x_{i-1}, x)$ is a polynomial with integer coefficients vanishing at 0. Then, there exist infinitely many $x_0, \dots, x_s \in \mathbb{N}$ such that the set*

$$\{x_0 \cdots x_j : 0 \leq j \leq s\} \cup \{x_0 \cdots x_j + f(x_{j+1}, \dots, x_i) \mid 0 \leq j < i \leq s, f \in F_{i-j}\}$$

is *monochromatic* for c .

The following corollary is an immediate consequence of Theorem 4.6 and saturation.

Corollary 4.7. Fix $s \in \mathbb{N}$. There exist infinite $\xi_0, \dots, \xi_s \in {}^*\mathbb{N}$ such that, for every $0 \leq j < i \leq s$, and for every function $f : \mathbb{N}^{i-j} \rightarrow \mathbb{Z}$ such that, for every $x_{j+1}, \dots, x_{i-1} \in \mathbb{N}$, the function $x \mapsto f(x_{j+1}, \dots, x_{i-1}, x)$ is a polynomial with integer coefficients vanishing at 0, one has that

$$\xi_0 \sim \xi_0 \cdots \xi_i \sim \xi_0 \cdots \xi_j + f(\xi_{j+1}, \dots, \xi_i).$$

Remark 4.8. If $\xi_0, \dots, \xi_s \in {}^*\mathbb{N}$ are as in Corollary 4.7, then we have that $\xi_0 \equiv \dots \equiv \xi_s \equiv 0 \pmod{q}$ for every $q \in \mathbb{N}$. Indeed, we have that, for $0 < i \leq s$,

$$\xi_0 \cdots \xi_{i-1} + \xi_i \equiv \xi_0 \cdots \xi_{i-1} \pmod{q}$$

and hence $\xi_i \equiv 0 \pmod{q}$. It follows that $\xi_0 \equiv \xi_0 \xi_1 \equiv 0 \pmod{q}$.

The following result is the analogue of Theorem 4.6 for polynomials vanishing at 1. While the proof of Theorem 4.9 bears some resemblance to the proof of Theorem 4.6, some modifications are needed to deal with polynomials vanishing at 1. It is unclear how to obtain an analogue of such results for polynomials vanishing at a point other than 0 or 1. Recall that we let \mathbb{N} be the set of integer that are greater than or equal to 1, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Notice that, according to Notation 4.1, $\mathbb{N} \cap (\mathbb{N}_0/d + 1) = \mathbb{N}$ when $d = 0$.

Theorem 4.9. *Fix $d \in \mathbb{Q}$. Let c be a finite coloring of $\mathbb{N} + d$. Let $s \in \mathbb{N}$ and, for each $i \in \{1, \dots, s\}$, let F_i be a finite set of functions $\mathbb{N}^i \rightarrow \mathbb{Z}$ such that for all $f \in F_i$ and any $x_1, \dots, x_{i-1} \in \mathbb{N}$, the function $x \mapsto f(x_1, \dots, x_{i-1}, x)$ is a polynomial with integer coefficients vanishing at 1. Then there exist infinitely many $x_0 \in \mathbb{N} + d$ and $x_1, \dots, x_s \in \mathbb{N} \cap (\mathbb{N}_0/d + 1)$ such that the set*

$$\{x_0 \cdots x_j : 0 \leq j \leq s\} \cup \{x_0 \cdots x_j + f(x_{j+1}, \dots, x_i) \mid 0 \leq j < i \leq s, f \in F_{i-j}\}$$

is contained in $\mathbb{N} + d$ and monochromatic for c .

Proof. Observe that we have that

$$\{x_0 \cdots x_j : 0 \leq j \leq s\}$$

is contained in $\mathbb{N} + d$, as long as $x_0 \in \mathbb{N} + d$ and $x_1, \dots, x_s \in \mathbb{N} \cap (\mathbb{N}_0/d + 1)$.

Fix a finite coloring $c : \mathbb{N} + d \rightarrow \{1, 2, \dots, r\}$, and set $C_i = \{n \in \mathbb{N} + d : c(n) = i\}$ for $i \in \{1, 2, \dots, r\}$. We will recursively construct four sequences:

- A sequence $(y_i)_{i=1}^\infty$ in $\mathbb{N} \cap (\mathbb{N}_0/d + 1)$;
- Two sequences $(B_i)_{i=0}^\infty$ and $(D_i)_{i=1}^\infty$ of piecewise syndetic subsets of $\mathbb{N} + d$;
- A sequence $(t_i)_{i=0}^\infty$ of elements of $\{1, \dots, r\}$;

such that $B_i \subseteq C_{t_i}$ for every $i \geq 0$.

We begin by choosing $t_0 \in \{1, \dots, r\}$ such that $C_{t_0} \cap (\mathbb{N} + d)$ is piecewise syndetic (which is always possible since $\mathbb{N} + d$ itself is piecewise syndetic), and let $B_0 := C_{t_0}$. Now, assume that $n \geq 1$ and $(y_j)_{j=1}^{n-1}, (B_j)_{j=0}^{n-1}, (D_j)_{j=1}^{n-1}$ and $(t_j)_{j=0}^{n-1}$ have already been defined. For $i, j \in \mathbb{N}$ with $i \leq j$, we set $y_{i,j} := y_i y_{i+1} \cdots y_j$, whenever these have been defined.

For each $k \in \{1, \dots, s\}$ and every $f \in F_k$, we define the collection $G_n(f)$ of all functions $g : \mathbb{Z} \rightarrow \mathbb{Z}$ of the form

$$g(z) = y_{m_1+1, n-1} (f(y_{m_1+1, m_2}, y_{m_2+1, m_3}, \dots, y_{m_k+1, n-1} z)) - f(y_{m_1+1, m_2}, y_{m_2+1, m_3}, \dots, y_{m_k+1, n-1}) \quad (2)$$

for any possible choice $0 \leq m_1 < m_2 < \dots < m_k < n$. In (2), we adopt the convention that an empty product (such as $y_{m_k+1} \cdots y_{n-1}$ when $m_k+1 = n-1$) is equal to 1. If $k > n$, then we set $G_n(f)$ to be empty. Observe that each $g \in G_n(f)$ is a polynomial with integer coefficients vanishing at 1. Notice that $\mathbb{N} \cap \mathbb{N}_0/d$ is an IP-set. By Corollary 4.5, we can find some $y_n \in \mathbb{N} \cap (\mathbb{N}_0/d + 1)$ such that the intersection

$$D_n := B_{n-1} \cap \bigcap_{k=1}^s \bigcap_{f \in F_k} \bigcap_{g \in G_n(f)} (B_{n-1} - g(y_n)) \quad (3)$$

is piecewise syndetic. Therefore, $y_n D_n \subseteq \mathbb{N} + d$ is also piecewise syndetic. (The fact that $y_n \in \mathbb{N} \cap (\mathbb{N}_0/d + 1)$ and $D_n \subseteq \mathbb{N} + d$ guarantee that $y_n D_n \subseteq \mathbb{N} + d$.) We then define B_n to be a c -monochromatic piecewise syndetic subset of $y_n D_n$, and let $t_n \in \{1, 2, \dots, r\}$ be such that $B_n \subseteq C_{t_n}$. This concludes the recursive construction.

We note that, for every $i \in \mathbb{N}$, $B_i \subseteq y_i D_i \subseteq y_i B_{i-1}$. This implies that, for every $j < i$, $B_i \subseteq y_i y_{i-1} \cdots y_{j+1} B_j$. Since $(t_i)_{i=0}^\infty$ takes only finitely many values, there exists $t \in \{1, \dots, r\}$ and infinitely many tuples of natural numbers $n_0 < \cdots < n_s$ such that $t_{n_0} = t_{n_1} = \cdots = t_{n_s} = t$. Consider the elements $x_1 := y_{n_0+1, n_1}$, $x_2 := y_{n_1+1, n_2}$, \dots , $x_s := y_{n_{s-1}+1, n_s}$ of $\mathbb{N} + 1$. Pick then $\tilde{x} \in B_{n_s}$, and observe that

$$\tilde{x} \in B_{n_s} \subseteq x_s B_{n_{s-1}} \subseteq x_s x_{s-1} B_{n_{s-2}} \subseteq \cdots \subseteq x_s x_{s-1} \cdots x_1 B_{n_0},$$

and hence there is $x_0 \in B_{n_0} \subseteq \mathbb{N} + d$ such that $x_s x_{s-1} \cdots x_0 = \tilde{x}$ and $x_j x_{j-1} \cdots x_0 \in B_{n_j} \subseteq C_t$ for $0 \leq j \leq s$. For $0 \leq j \leq i \leq s$, set $x_{j,i} := x_j x_{j+1} \cdots x_i = y_{n_{j-1}+1, n_i}$. We need to show that the set

$$A := \{x_{0,s}\} \cup \{x_{0,j} + f(x_{j+1}, \dots, x_i) \mid 0 \leq j < i \leq s, f \in F_{i-j}\}$$

is contained in C_t . Fix $0 \leq j < i \leq s$ and $f \in F_{i-j}$. We claim that $x_0 x_1 \cdots x_j + f(x_{j+1}, \dots, x_i) \in B_{n_j}$. We have that

$$\begin{aligned} x_{0,i} + x_{j+1,i} f(x_{j+1}, \dots, x_i) &\in B_{n_i} + x_{j+1,i} f(x_{j+1}, \dots, x_i) \\ &\subseteq y_{n_i} D_{n_i} + x_{j+1,i} f(x_{j+1}, \dots, x_i) \\ &= y_{n_i} (D_{n_i} + x_{j+1,i-1} y_{n_{i-1}+1, n_i-1} f(x_{j+1}, \dots, x_i)) \end{aligned}$$

By the definition of D_{n_i} , we have that

$$D_{n_i} + x_{j+1,i-1} y_{n_{i-1}+1, n_i-1} (f(x_{j+1}, \dots, x_i) - f(x_{j+1}, \dots, x_{i-1}, y_{n_{i-1}+1} \cdots y_{n_i-1})) \subseteq B_{n_{i-1}}$$

Hence,

$$x_{0,i} + x_{j+1,i} f(x_{j+1}, \dots, x_i) \in y_{n_i} (B_{n_{i-1}} + x_{j+1,i-1} y_{n_{i-1}+1, n_i-1} f(x_{j+1}, \dots, x_{i-1}, y_{n_{i-1}+1} \cdots y_{n_i-1}))$$

Proceeding in this fashion, after $n_i - n_{i-1}$ steps one obtains

$$x_{0,i} + x_{j+1,i} f(x_{j+1}, \dots, x_i) \in x_i (B_{n_{i-1}} + f(x_{j+1}, \dots, x_{i-1}, 1)).$$

Since by assumption $f(x_{j+1}, \dots, x_{i-1}, 1) = 0$, we have that

$$x_{0,i} + x_{j+1,i} f(x_{j+1}, \dots, x_i) \in x_i B_{n_{i-1}}.$$

Dividing by x_i we deduce that

$$x_{0,i-1} + x_{j+1,i-1} f(x_{j+1}, \dots, x_i) \in B_{n_{i-1}} \subset x_{j+1,i-1} B_{n_j}.$$

Therefore, dividing by $x_{j+1,i-1}$ it follows that

$$x_{0,j} + f(x_{j+1}, \dots, x_i) \in B_{n_j} \subseteq C_t$$

as claimed. This concludes the proof that $A \subseteq C_t$. \square

Corollary 4.10. Fix $d \in \mathbb{Q}$, and $s \in \mathbb{N}$. Then there exist infinite $\xi_0 \in {}^*\mathbb{N} + d$ and $\xi_1, \dots, \xi_s \in {}^*\mathbb{N}_0/d + 1$ such that, for every function $f : \mathbb{N}^i \rightarrow \mathbb{Z}$ such that, for every $x_1, \dots, x_{i-1} \in \mathbb{N}$, the function $x \mapsto f(x_1, \dots, x_{i-1}, x)$ is a polynomial with integer coefficients vanishing at 1, one has that

$$\xi_0 \sim \xi_0 \cdots \xi_i \sim \xi_0 \cdots \xi_j + f(\xi_{j+1}, \dots, \xi_i)$$

for $0 \leq j < i \leq s$.

Corollary 4.11. Fix $d \in \mathbb{Q}$. There exist infinite $r \in {}^*\mathbb{N}$ and $s \in {}^*\mathbb{N}_0/d$ such that for every polynomial with integer coefficients $p(w)$ vanishing at 0 one has that

$$r \sim r + p(s) \sim rs + r + ds.$$

Proof. By Corollary 4.10 there exist infinite $r_0 \in {}^*\mathbb{N} + d$ and $s_0 \in {}^*\mathbb{N}_0/d + 1$ such that

$$r_0 \sim r_0 + p(s_0 - 1) \sim r_0 s_0$$

for every polynomial with integer coefficients $p(w)$ vanishing at 0. Set now $r := r_0 - d$ and $s := s_0 - 1$. Observe that

$$r + d \sim r + d + p(s) \sim rs + r + ds + d$$

and hence

$$r \sim r + p(s) \sim rs + r + ds$$

for every polynomial with integer coefficients $p(w)$ vanishing at 0. \square

As an application of Theorem 4.9 and Theorem 4.6, we establish the partition regularity of certain Diophantine equations.

Example 4.12. Let

$$p(x, z) = a_0 x^d + a_1 x^{d-1} z + \cdots + a_{d-1} x z^{d-1} + a_d z^d$$

be a homogeneous polynomial of degree $d \geq 1$ in the variables x and z . Set

$$P(x, y, z) = x^d(x - y) + p(x, z).$$

The solutions to the equation $P(x, y, z) = 0$ are parametrized by

$$\begin{cases} x = r \\ y = r + p(1, s) \\ z = rs \end{cases}.$$

Notice that $p(1, 0) = a_0$ and $p(1, 1) = \sum_{i=0}^d a_i$. Therefore Corollary 4.7 proves that the equation $P(x, y, z) = 0$ is partition regular when $a_0 = 0$, and Corollary 4.10 proves that the equation $P(x, y, z) = 0$ is partition regular when $\sum_{i=0}^d a_i = 0$.

5. APPLICATIONS

The goal of this section is to use Theorem 2.19 to establish Theorem 2.21. Fix $a, b, c \in \mathbb{Z}$ not all zero such that $abc = 0$ or $a + b + c = 0$. We start by considering the polynomial $P(x, y, z) = x^2 - xy + ax + by + cz = 0$. Observe that $\tilde{P}(w) = (a + b + c)w$.

Lemma 5.1. If $a + b + c \neq 0$ and $P(x, y, z) = x^2 - xy + ax + by + cz$ satisfies the minimal Rado condition, then one of the following holds:

- (1) $a = b = 0$,
- (2) at most one among a, b, c is zero, and $0 \in \{a + b, a + c, b + c\}$.

Proof. Suppose that P satisfies the minimal Rado condition, and assume that at least one between a, b is nonzero. We have that the only root of \tilde{P} is 0, and the only minimal indices for P have degree 1. In particular, any minimal Rado set for P is homogeneous of degree 1. This implies that at most one among a, b, c is zero, and $0 \in \{a + b, a + c, b + c\}$. \square

Lemma 5.2. Consider the polynomial $P(x, y, z) = x^2 - xy + ax + by + cz$ for $a, b, c \in \mathbb{Z}$ such that $abc = 0$ or $a + b + c = 0$. If $\tilde{P} = 0$ or P satisfies the minimal Rado condition, then P is partition regular.

Proof. By Lemma 5.1, either $a = b = 0$ or at most one among a, b, c is zero, and $0 \in \{a + b, a + c, b + c, a + b + c\}$.

When $a = b = 0$, P reduces to $x(x - y) + cz$. Such a polynomial is partition regular by Example 4.12.

We now consider the case when exactly one among a, b, c is zero. When $c = 0$ and $a + b = 0$, the equation reduces to $(x + a)(x - y) = 0$, which admits the constant solutions $x = y$, and hence it is (trivially) partition regular. When $b = 0$ and $a + c = 0$, the equation reduces to $x(x - y) + a(x - z) = 0$, which is partition regular by Example 4.12. When $a = 0$ and $b + c = 0$, the polynomial reduces to $P(x, y, z) = x(x - y) + b(y - z)$. Observe now that $P^{(b)}(x, y) = x(x - y) + b(x - z)$, which has just been shown to be partition regular.

Similarly, when a, b, c are all nonzero, and $a + b + c = 0$, one has that $P^{(b)}(x, y, z) = x(x - y) + (a + b)x + cz$. Thus, $P^{(b)}$ is partition regular, as shown above. This concludes the proof. \square

We now consider the polynomial $P(x, y, z) = x^2 - y^2 + ax + by + cz$. Observe that $\tilde{P}(w) = (a + b + c)w$.

Lemma 5.3. Suppose that $a + b + c \neq 0$ and $P(x, y, z) = x^2 - y^2 + ax + by + cz$ satisfies the minimal Rado condition. Then at most one among a, b, c is zero, and $0 \in \{a + b, a + c, b + c\}$.

Proof. Suppose that P satisfies the minimal Rado condition. We have that the only root of \tilde{P} is 0. By Corollary 2.27, we have that any minimal Rado set for P is homogeneous of degree 1. This implies that at most one among a, b, c is zero, and $0 \in \{a + b, a + c, b + c\}$. \square

Lemma 5.4. Suppose that $a, b, c \in \mathbb{Z}$ are such that either $a + b + c = 0$ or $abc = 0$. Set $P(x, y, z) = x^2 - y^2 + ax + by + cz$. If $\tilde{P} = 0$ or P satisfies the minimal Rado condition, then P is partition regular.

Proof. By Lemma 5.3, either $a = b = c = 0$ or at most one among a, b, c is zero, and $0 \in \{a + b, a + c, b + c, a + b + c\}$.

When $a + b = 0$, $P(x, y, z) = 0$ reduces to $(x - y)(x + y + a) = 0$, which admits the constant solutions $x = y$, and hence it is (trivially) partition regular.

We consider now the case when $a + b \neq 0$ and $a + b + c = 0$. By Proposition 2.3, it suffices to show that the polynomial $P(x, y, z) = \frac{1}{4}(x^2 - y^2) + ax + by + cz$ is partition regular. Observe that setting $x = r + s$ and $y = r - s$ we have that $P(x, y, z) = rs + (a + b)r + (a - b)s - (a + b)z$. Therefore, we have that $P(x, y, z)$ is partition regular if and only if there exist $r, s \in {}^*\mathbb{N}$ such that $a + b$ divides $rs + (a - b)s$ and $r + s \sim r - s \sim \frac{1}{a+b}rs + r + \frac{(a-b)}{a+b}s$. For this, it suffices to find $r_0, s_0 \in {}^*\mathbb{N}$ such that $\frac{s_0}{a+b} \in {}^*\mathbb{N}$ and $r_0 + s_0 \sim r_0 - s_0 \sim r_0 s_0 + r_0 + \frac{a-b}{a+b}s_0$, as then one can set $r := (a + b)r_0$ and $s := (a + b)s_0$. We have that such r_0, s_0 exist by Corollary 4.11.

We now consider the case when $a + b \neq 0$ and $a + b + c \neq 0$. In this case, we have that P does not satisfy the minimal Rado condition, and hence there is nothing to prove. To see this, notice that if P satisfies the minimal Rado condition, then by Lemma 5.3 at most one among a, b, c is zero, and $0 \in \{a + b, a + c, b + c\}$. Since by hypothesis $abc = 0$ and $a + b \neq 0$, it follows that $c = 0$. This gives the contradiction that $0 \in \{a, b\}$, concluding the proof. \square

Theorem 2.21 is an immediate consequence of the lemmas in this section and Theorem 2.19.

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