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Quantified Modal Logics: One Approach to Rule (Almost) them All!

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Abstract
We present a general approach to quantified modal logics that can simulate most other approaches. The language is based on operators indexed by terms which allow to express *de re* modalities and to control the interaction of modalities with the first-order machinery and with non-rigid designators. The semantics is based on a primitive counterpart relation holding between n-tuples of objects inhabiting possible worlds. This allows an object to be represented by one, many, or no object in an accessible world. Moreover by taking as primitive a relation between n-tuples we avoid some shortcoming of standard individual counterparts. Finally, we use cut-free labelled sequent calculi to give a proof-theoretic characterisation of the quantified extensions of each first-order definable propositional modal logic. In this way we show how to complete many axiomatically incomplete quantified modal logics.

Keywords Quantified modal logics · Counterparts · Completeness · Sequent calculi · Structural rules

1 Introduction

It is well known that the interaction of standard modal operators with the first-order machinery generates many possibilities and difficulties, cf. [6, 17, 25]. Semantically we have to choose whether the domains of quantification are the same, grow, shrink, or vary in moving from one world to another; moreover, we have to choose whether terms have the same denotation at all worlds or not—i.e., whether they are *rigid* or *non-rigid* designators. These different choices have consequences with respect to proof-theoretic characterisations of quantified modal logics (QML). To illustrate, if we extend the axiomatisation of the propositional modal logic (PML) **K** with the standard

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axioms and rules for the quantifiers and identity, let's call this system $\mathbf{Q}.\mathbf{K}^-$,[1] then the Converse Barcan Formula

$$\Box \forall x A \supset \forall x \Box A \tag{CBF}$$

and the necessity of identity

$$s = t \supset \Box s = t \tag{NI}$$

are theorems. This notwithstanding, the first is not valid when quantifier domains vary and the latter is not valid if terms are non-rigid. More problematically, the combination of an axiomatisation of a complete PML with that of the first-order machinery usually determines an incomplete system—*i.e.*, the axiomatic calculus does not have as theorems all formulas that are valid over the intended class of Kripke frames. In some cases we have *completable* logics—e.g., $\mathbf{Q}.\mathbf{K}^-$ is completable by adding the necessity of distinctness

$$s \neq t \supset \Box s \neq t \tag{ND}$$

to our set of axioms, $\mathbf{Q}.\mathbf{S4M}$ by adding $\Diamond \forall \vec{x}(A \supset \Box A)$ [11], and $\mathbf{Q}.\mathbf{K2}.\mathbf{BF}$ by adding some presently unknown axiom [10].[2] In many other cases we have *incompletable* logics—e.g., $\mathbf{Q}.\mathbf{GL}$ is not recursively axiomatisable [31].

As Garson puts it, the main problem of QML is that

> [t]he lack of a unified framework in which to view QML and their completeness properties puts pressure on those who develop, apply, and teach QML to work with the (allegedly) simplest systems, namely those that adopt the Barcan Formulas and predicate logic rules for the quantifiers. [19, p. 621]

In order to develop such a unified framework it is necessary to overcome the limitations of Kripke semantics for QML, and this has been done in two different ways. Some proposals—e.g., [15, 19, 28]—are based on replacing quantification over world-bound objects with quantification over individual concepts, where an individual concept is a function from worlds to objects. Some other proposals—e.g., [2, 3, 7, 8, 18, 21, 22, 27, 39]—are based on keeping quantification over world-bound objects, as in Kripke semantics, while at the same time enriching the semantics with (functions or) relations between these objects, either defined in categorial terms or as a primitive counterpart relation. These additional (functions or) relations are then used to evaluate modal formulas involving free variables.

In this paper we present an approach to QML—where a QML is defined semantically as the set of formulas that are valid in some class of structures—that follows the second strategy.[3] Our proposal generalises the counterpart-theoretic semantics studied in [2, 3, 7, 8] by replacing (the accessibility relation and) the two-places counterpart relation between objects with a primitive two-place relation between $n+1$-tuples

[1] $\mathbf{Q}.\mathbf{K}^-$, is the calculus called $\mathbf{QK}_=$ in [6, pag. 555] that is obtained by extending a calculus for $\mathbf{K}$ with axioms and rules for classical quantifiers and identity.

[2] The paper [10] shows the incompleteness of $\mathbf{Q}.\mathbf{K2}.\mathbf{BF}$ by presenting a formula that is consistent with the axiomatic calculus while at the same time being not satisfiable in the corresponding class of Kripke-frames. On the other hand, we know that the set of formulas valid over that class of Kripke-frames is recursively enumerable, and hence the logic is completable, because that class is first-order definable and its validities can be embedded in classical first-order logic by means of the standard translation, see [18].

[3] We refer the reader to [15, 28] for comparisons between the two strategies.

composed of one world and n objects inhabiting that world. As we will show, this move allows to cover a wider range of modal contexts. Moreover, we replace the axiomatic calculi considered in [7, 8] with labelled sequent calculi in the style of [33]. These calculi provide a cut-free proof-theoretic characterisation of a wide class of QML: each quantified extension of a first-order definable PML has a complete proof system and the completeness proof is modular. We hope in this way to dispel some *locus communis* on the intractability of QML. The rest of this introduction explains the main ingredients of our proposal.

Following a proposal made in [7, 8], standard modal operators—$\Box$ and $\Diamond$—are replaced by *indexed modal operators*: modalities indexed by a set of pairs made of a term and a variable. Indexed modal operators allow for a fine-grained distinction between formulas expressing *de re* modal claims—*i.e.*, formulas expressing that some object has some modal property as in 'the number of planets is such that it is necessarily even'—and formulas expressing *de dicto* modal claims—*i.e.*, formulas expressing that some sentence is modally true as in 'it is necessary that the number of planets is even'. Roughly, the indexed modal language is obtained from a standard modal language by replacing formulas of shape $\Box A$ with formulas of shape $[{}^{\vec{t}}_{\vec{x}}]A$, where $\vec{x}$ is a sequence of (pairwise disjoint) variables containing all variables free in A—to be denoted by $\mathtt{FV}(A)$— and $\vec{t}$ is a sequence of terms having the same length of $\vec{x}$, see Section 3 for the details. We have indexed modal sentences such has[4]

$$[{}^{t_1}_{x_1} \dots {}^{t_n}_{x_n}]P(x_1, \dots, x_n) \tag{1}$$

expressing the following *de re* claim

It is necessary/known of $t_1, \dots, t_n$ that they have property P.

The sentence (1) has to be contrasted with

$$[{}^{\emptyset}_{\emptyset}]P(t_1, \dots t_n) \tag{2}$$

expressing the *de dicto* modal claim

It is necessary/known that $P(t_1, \dots, t_n)$.

Indexed modalities provides a scoping mechanism for non-rigid terms that is equivalent to λ-abstraction [17] (cf. Section 2.1). Moreover, in the given counterpart-theoretic setting they are needed to validate all *de re* instances of some theorem of PML (cf. Section 2.4).

The semantics we adopt is a generalisation of the counterpart-theoretic semantics considered in [2, 3, 7, 8]: a *skeleton* will be defined as a triple made of a set of worlds, a set of world-dependent domains, and a binary *transition relation* between $n+1$-tuples made of one world and n objects inhabiting that world. This generalisation of individual's counterparts—which resembles the move from *presheaf semantics* [6, 21,

[4] We are assuming that $t_1, \dots, t_n$ are closed terms, and, for brevity, we consider only the alethic and the epistemic reading of the modalities.

22] to *metaframes* [18, 39] in categorial semantics for QML—is needed to avoid the validity of (CBF) without having to weaken the classical theory of quantification and to avoid problems with essential relational properties. As we shall see, all results given in [7, 8] are easily generalisable to the present setting.

Indexed modal logics (IML) will be defined as the sets of indexed modal formulas that are valid on some (first-order definable) class of skeletons. In giving a proof-theoretic characterisation of IML we avoid the use of axiomatic calculi since they make completeness results hard to find [10, 11]. We will instead opt for labelled sequent calculi that allow to internalise the semantics into the syntax of the calculus. This approach has already shown to provide a well-behaved proof-theoretic characterisation of the PML in the cube of normal modalities [33], of their standard quantified extensions [36, 37], of their epistemic indexed extensions [9], and of many non-normal QML [35]. In these calculi all rules are height-preserving invertible, weakening and contraction are height-preserving admissible, and cut is syntactically admissible. Moreover each calculus is sound and complete with respect to its intended semantics. In particular, the completeness proof is a direct Tait-Schütte-Takeuti-style one: a procedure that takes as inputs a calculus and a sequent and outputs a countermodel (in the appropriate class) for each underivable sequent. As a consequence we also have a semantic proof of the admissibility of cut. In this paper we show these results holds also in the present setting. Moreover, by applying the technique of *semidefinitional extensions* introduced in [12], we extend these results to calculi capturing the indexed quantified extensions of each first-order definable PML. In this way we obtain a general completeness result that is stronger than other results obtained in the context of counterpart-theoretic or categorial semantics, since they cover at most the quantified extension of canonical PML [39].[5] The present approach to the (in)completeness phenomenon in QML can be seen as the opposite of that in Goldblatt's [23]: whereas Goldblatt introduces a general frames-style semantics validating only the formulas that are derivable, we consider stronger calculi which derive all valid formulas.

The rest of the paper is organised as follows. Section 2 explains the need for indexed modalities and for a semantics based on a transition relation between $n+1$-tuples. Sections 3 and 4 introduce the syntax and the semantics of IML. Section 5 shows that this approach can simulate QML based on Kripke semantics. Section 6 introduces the minimal labelled sequent calculus **GIM**.**K** and Section 7 shows how to define a calculus for the quantified extensions of each first-order definable PML. Section 8 shows that these calculi have the good structural properties of **G3**-style calculi. Section 9 proves that each calculus is sound and complete with respect to its intended semantics. Finally, Section 10 presents some conclusive remarks and some open questions.

[5] It is also more general then the completeness result with respect a semantics based on individual concepts given in [19] since the latter (i) does not cover classes of frames defined by some natural first-order existential property such as weak convergence and (ii) it does not cover QML based on classical quantification where the Barcan Formula ($\forall x \Box A \supset \Box \forall x A$) fails.

2 Motivating Indexed Modalities

2.1 Non-rigid Terms

Let us consider QML based on the standard quantified modal language—i.e., the language ($\mathcal{L}^{\Box}$) of Definition 10—that are based on Kripke frames with constant domains: a *frame* is a triple made of a non-empty set of worlds $\mathcal{W}$, a binary accessibility relation $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$, and a non-empty domain of objects $\mathcal{D}$. A model is a frame together with a function $\mathcal{V}$ mapping each $w \in \mathcal{W}$ to a classical first-order interpretation V_w of the signature over $\mathcal{D}$, a modal sentence $\Box A$ is true at a world w when A is true at all worlds accessible from w; finally, a sentence is valid in a class of frames when it is true at all worlds of each model based on some frame in that class. See Section 5 for the details.

In this setting it is usual to assume that terms are rigid designators: if c is an individual constant denoting a given object in a world w of a model then it has to denote the same object in each world (accessible from w) of that model. Nonetheless, for many interpretations of modalities terms should be non-rigid. Rigid terms make the necessity of identity (NI) valid and, therefore, under an epistemic reading of $\Box$, the fact that the names ‘Hesperus’ and ‘Phosphorus’ denote the planet Venus entails that the ancient Babylonians knew that Hesperus is Phosphorus. Non-rigid terms avoid this problem, but present some technical difficulties. Once we allow for non-rigid terms the modal sentence

$$\Box P(t) \tag{3}$$

becomes semantically ambiguous: it might mean that $P(t)$ is true at all accessible worlds for the denotation of t in those worlds—this is a *de dicto* reading—or it might mean that the object actually denoted by t has property P in all accessible worlds—this is a *de re* reading. It is immediate to realise that if t is non-rigid then the two readings can have distinct semantic values. This is illustrated by the model depicted in Fig. 1: $\Box P(t)$ is true at w under the *de dicto* reading but not under the *de re* one. The standard modal language cannot disambiguate the two readings, thus we have either to read all formulas under one of the two or to supplement the standard modal language.

In a serie of works [14, 15, 17] Fitting proposes to disambiguate the modal language by supplementing it with λ-abstraction. The idea, roughly, is that λ abstracts the predicate $\langle \lambda x.A \rangle$ from the formula A (having x free) and that a non-rigid individual constant t is an expression that can be applied to a predicate (abstracted from a formula) in such a way that we know whether it has narrow or broad scope with respect to a

Fig. 1 A Kripke model where t is non-rigid

modal operator. To illustrate, the *de dicto* sentence $\Box\langle\lambda x.P(x)\rangle(t)$ is evaluated at a world w by first moving to worlds accessible from w and then by seeing whether the object denoted by t in these worlds satisfies $P(x)$ therein. This has to be contrasted with the *de re* sentence $\langle\lambda x.\Box P(x)\rangle(t)$ that is evaluated by first picking the object o denoted by t in the actual world w and then by seeing whether the formula $P(x)$ is true of the object o at all worlds accessible from w. In the model in Fig. 1 the sentence $\Box\langle\lambda x.P(x)\rangle(t)$ is true at w, since $P(t)$ is true at v, and $\langle\lambda x.\Box P(x)\rangle(t)$ is false, since o_1 does not satisfy $P(x)$ at v.

Indexed modalities are an alternative way to obtain the same result: we have the *de dicto* sentence $[{}^{\emptyset}_{\emptyset}]P(t)$ and the *de re* one $[{}^{t}_{x}]P(x)$. As we will show in Section 5, IML can simulate the QML with λ-abstraction for non-rigid designators. Technically, the indexed operator occurring in $[{}^{t}_{x}]P(x)$ binds the occurrence of x in $P(x)$ and applies the term t to the modal predicate 'being necessarily P' (like λ in $\langle\lambda x.\Box P(x)\rangle(t)$). Thus, indexed operators provide a scoping mechanism for non-rigid terms in modal contexts that is alternative to λ-abstraction.[6] They also allow to control substitutions in modal contexts: in applying the substitution $[s/x]$ to the formula $[{}^{t}_{x}]P(x)$ we will substitute instances of x with instances of s in the term t occurring in $[{}^{t}_{x}]$, and not in the (atomic) formula $P(x)$ occurring in the scope of the indexed operator.[7] Among other things, this is needed to make first-order principles such as

$$\forall x A \supset A[t/x] \tag{UI}$$

and

$$s = t \supset (A[s/x] \supset A[t/x]) \tag{Repl}$$

valid when A is a formula involving modalities. If, e.g., some non-modal property $\langle\lambda x.P(x)\rangle$ is necessary of all objects then it is necessary of t, without further implying that the sentence $P(t)$ is necessary. To sum up, indexed modalities, like λ-abstraction, give us control over non-rigid designators and their interaction with the first-order machinery.

2.2 Trans-world Identity and *de re* Modalities

Both Fitting's proposal and QML based on the indexed language can invalidate the *de dicto* formula expressing that the ancient Babylonians knew that Hesperus is Phosphorus: if we consider the model given in Fig. 1 and we impose that $V_w(h) = V_w(p) = o_1$, $V_v(h) = o_1$, and $V_v(p) = o_2$, then the *de dicto* version of (NI)—i.e., the formula $\langle\lambda x.\langle\lambda y.x = y\rangle(p)\rangle(h) \supset \Box\langle\lambda x.\langle\lambda y.x = y\rangle(p)\rangle(h)$—is false at w. One limitation of Fitting's proposal is that *de re* modalities are evaluated by means of trans-world identity: to see if an object o necessarily has property $\langle\lambda x.P(x)\rangle$ at a world w, we have to check if o satisfies $P(x)$ at all worlds accessible from w. Moreover, to avoid truth value-gaps or other complications, Fitting's proposal, like most

6 In this respect indexed operator behave like the binding modalities studied in [16].

7 Indexed modalities make modal context opaque as advocated by Kaplan [26].

approaches based on trans-world identity,[8] requires that the object o exists at all worlds accessible from w. As a consequence the *de re* version of (NI)—i.e., the formula $\langle\lambda x.\langle\lambda y.x = y\rangle(p)\rangle(h) \supset \langle\lambda x.\langle\lambda y.\Box x = y\rangle(p)\rangle(h)$— is valid since if two terms denote the same object in the actual world then that object is self-identical at each accessible world. One famous counterexample to the validity of the *de re* version of (NI) has been provided in [20] by Gibbard: consider a statue and the piece of clay from which it is composed. Even if the two are identical, they might have been distinct, e.g., that piece of clay might have been used to make a vase instead of that statue.

Another unwelcome consequence of the adoption of trans-world identity is that the Ghilardi Formula

$$\exists x \Box A \supset \Box \exists x A \tag{GF}$$

is valid when we adopt the classical theory of quantification: if at w there is an object o having the modal property $\langle\lambda x.\Box A\rangle$, then o has the property $\langle\lambda x.A\rangle$ at each world accessible from w. Nonetheless, (GF) is disputable under many interpretations of modalities—e.g., the fact that actually there is something that is necessarily a human being would entail that it is necessary that there are human beings. Obviously this last claim is highly controversial: the mass extinctions of human beings is a sad but real possibility! To invalidate (GF) trans-world identity must be coupled with a varying domains semantics and, consequently, a weaker theory of quantification where universal instantiation (UI) and existential generalisation (EG) have to be restricted. Nonetheless, there might be modal contexts where (GF) fails that require classical quantification.

2.3 Adding a Primitive Counterpart Relation

One well-know way to avoid these problems is to evaluate *de re* formulas with respect to a primitive counterpart relation holding between objects inhabiting possible worlds instead of relying on trans-world identity, see [2, 3, 6–8, 27]. One way to sketch the counterpart-theoretic semantics is the following: a *C-frame* is a tuple $\langle\mathcal{W}, \mathcal{R}, \mathcal{D}, \mathcal{C}\rangle$ where $\mathcal{W}$ and $\mathcal{R}$ are as for Kripke frames; $\mathcal{D}$ contains, for each $w \in \mathcal{W}$, a non-empty set of objects D_w, where $w \neq v$ implies $D_w \cap D_v = \varnothing$; $\mathcal{C} \subseteq \mathcal{D} \times \mathcal{D}$ is the binary counterpart relation holding between objects. It is also imposed that $o_1\mathcal{C}o_2$, $o_1 \in D_w$, and $o_2 \in D_v$ jointly imply that $w\mathcal{R}v$. A *C-model* is a C-frame augmented with a function $\mathcal{V}$ mapping each $w \in \mathcal{W}$ to a classical first-order interpretation V_w of the signature over D_w. A *de dicto* modal formula is evaluated as in standard QML. A *de re* modal formula, instead, is evaluated by means of the counterpart relation: $\Box A$ is true of the objects $o_1, \ldots, o_n$ at the world w they inhabit when A is true at each world v accessible from w with respect to all tuples $o_1^*, \ldots, o_n^*$ of objects inhabiting v and such that $o_i\mathcal{C}o_i^*$, for all $i \in \{1, \ldots, n\}$. C-validity is defined as truth at all worlds of each model in the given class of C-frames.

There are two main motivations for adopting a counterpart-theoretic semantics: one is mathematical and the other philosophical. Mathematically, it provides a simple relational version of some of the categorial generalisations of Kripke semantics

[8] See [4] for an exception.

considered in [6, 18, 21, 22, 39]. These semantics are defined by taking as primitive a family of first-order domains and a family of functions or relations between these domains, which are then used to define the accessibility relation. The advantage of these categorial semantics is that they have enough mathematical structure to understand and overcome some intrinsic limitation of Kripke semantics for QML. Presheaf semantics has been used in [21] to prove the following general incompleteness result: if **L** is a PML such that **S4.3**$\subset$**L**$\subset$**S5**, then the axiomatic calculus **Q.L** for the (standard) quantified extension of **L** is incomplete with respect to Kripke frames. On the other hand, categorial semantics allow for general completeness results:

Theorem 1 *If* **L** *is a PML such that (i)* **L**$\supseteq$**S4** *and (ii)* **L** *is canonical, then*

1. **Q.L** *is complete with respect to metaframe semantics* [39]*;*
2. *If* **L** *is also closed under a 'cluster expansion' condition then* **Q.L** *is complete with respect to presheaf semantics* [22].

As convincingly argued in [27], a counterpart-theoretic semantics has two main advantages with respect to its categorial twin: it weakens the base PML from **S4** to **K** and it is mathematically simpler, thus providing 'a useful tool for philosophers, linguists or people working in AI' [27, pag. 293]. In particular, C-frames are twin to presheaves and they allow to extend Theorem 1.2 to the quantified extension of PML below **S4**, cf. [6, 27].

From the philosophical perspective, already David Lewis [30] proposed to replace trans-world identity with a counterpart relation in order to avoid validating some problematic *de re* modal formulas. The adoption of a counterpart relation between objects to evaluate *de re* modalities provides a way to falsify the indexed version of (GF):[9]

$$\exists y[\vec{x}\ y]A \supset [\vec{x}]\exists y A \tag{GF}$$

that is compatible with the classical theory of quantification. In a C-frame an object can have one, many, or no counterpart in an accessible world. A *de re* claim about an object o inhabiting w is true therein if at each world accessible from w where there are counterparts of o, all these counterparts have the property under consideration. In particular, even if it is necessary of me, and thus of someone, that I am an human being, there might be accessible worlds where I have no counterpart, and the same holds for all other human beings. A *de dicto* existential claim, instead, is true at w if at each accessible world there is an object satisfying that property. The adoption of a counterpart-theoretic semantics makes *de re* and *de dicto* claims mutually independent. To illustrate, in the C-frame depicted in Fig. 2 every property is necessary of b (and thus of something), but this does not entail any *de dicto* claim. In particular, b falsifies the following instance of (GF): $\exists x([x]x \neq x) \supset [\emptyset]\exists x(x \neq x)$. $\exists x([x]x \neq x)$ is true at w because b has no counterpart and, hence, the formula $[x]x \neq x$ is vacuously true of b. On the other hand, $[\emptyset]\exists x(x \neq x)$ is false at w since $\exists x(x \neq x)$ is false at each world of every model and v is accessible from w.

[9] We use the same name for formulas over the standard modal language and their indexed version and we rely on the context to disambiguate. Moreover, we use $[\vec{x}]$ as short for $[^{\vec{x}}_{\vec{x}}]$.

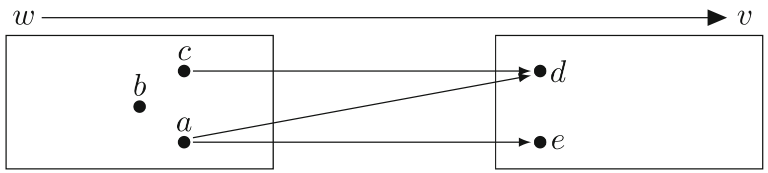

Fig. 2 A C-frame falsifying GF, NI, and ND

It is thus helpful to accept a counterpart-theoretic relation where an object might have no counterpart at some accessible world, as this gives us a way to falsify (GF) without having to weaken the classical theory of quantification.[10] It is also helpful to accept that an object can have more than one counterpart in one and the same accessible world: this vindicates Gibbard's [20] claim that even if a given statue and a given piece of clay are identical they might have been distinct. If we agree with Gibbard then we must concede that one and the same object can be represented by more than one object in some accessible world: there are *de re* instances of (NI) that could be false. Moreover, it is easy to modify Gibbard's example to argue that two distinct actual objects can be represented by a singe object in an accessible world: there are *de re* instances of (ND) that could be false. In IML neither $\forall x, y(x = y \supset [\,x\ y]x = y)$ nor $\forall x, y(x \neq y \supset [\,x\ y]x \neq y)$ are valid as shown, respectively, by objects a and d in Fig. 2.

Observe that the failure of (NI) does not involve the need to restrict Leibniz's law of replacement (Repl). To obtain (NI) from the instance of Repl

$$s = t \supset ([\,{}^{s\ z[s/z]}_{x\ y}\,]x = y \supset [\,{}^{s\ z[t/z]}_{x\ y}\,]x = y)$$

we need $[\,{}^{s\ z[s/z]}_{x\ y}\,]x = y$ to be C-valid, but this holds only when we impose that the counterpart relation is functional—i.e., that each object has at most one counterpart in each accessible world, see Theorem 8. In the present context, the standard axioms of identity entail only the following innocuous modal claim about self-identity: $\forall x[x](x = x)$—i.e., it is necessary of each object that it is self-identical.

2.4 Counterparts and Box-distribution

By replacing trans-world identity with a counterpart relation we avoid the validity of many formulas that are disputable under some interpretation of the modalities. Now we show that a counterpart-theoretic semantics has to be backed up by indexed modalities in order to validate some acceptable principle. In C-frames an object can be represented by no object in some accessible world, but this pose a problem with respect to the validity of Box-distribution

$$\Box(A \supset B) \supset (\Box A \supset \Box B) \tag{K}$$

[10] If, as in [27], it is assumed that the counterpart relation is total, then the only way to falsify (GF) is to weaken the classical theory of quantification as in approaches based on Kripke semantics.

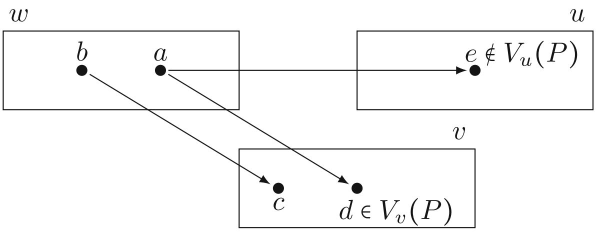

Fig. 3 A problematic C-model for axiom (K)

In general a formula $\Box C$ where the objects (denoted by) $t_1, \ldots, t_n$ occur *de re* is true at a world only if at each accessible world where counterparts $o_1, \ldots o_n$ of these object exist, the formula C is true of $o_1, \ldots o_n$. Let us assume that A is the atomic formula $x \neq y$ and B is the atomic formula $P(x)$, and let us consider the model in Fig. 3. $\Box(A \supset B)$ is true at world w if we interpret x over a and y over b: v is the only world where both a and b have counterparts and their counterparts make true both A and B. As a consequence, also $\Box A$ is true at w. Nonetheless, it is not the case that B is true at all worlds having counterparts of a—it is false at u—and, hence, $\Box B$ is false at w. We have a counterexample to axiom (K). The problem is that we are distributing a modal operator over two formulas not sharing the same free variables and, hence, we are quantifying over different sets of worlds. [11] To solve this problem we have to ensure that we can distribute modalities only over formulas having the same set of free variables. This can be achieved by keeping track of the *de re* terms under consideration also in formulas where they do not occur. In the context of IML it is imposed that $[\frac{\vec{t}}{\vec{x}}]C$ is well-formed only if all the variables free in C are contained in $\vec{x}$ and it is possible to distribute only modalities that are indexed by the same sets of terms. The first things is needed because the free variables in the scope of an indexed modalities represent the objects about which we are making a *de re* claim. The second one to make axiom (K) C-valid. If we reconsider the example above with indexed modalities in place of $\Box$, we have the following true (and C-valid) instance of axiom (K):

$$[x\ y](x \neq y \supset P(x)) \supset ([x\ y]x \neq y \supset [x\ y]P(x))$$

that, as shown by the model in Fig. 3, becomes false if we replace $[x, y]P(x)$ with $[x]P(x)$.

To sum up, indexed modalities are needed to keep track of the *de re* objects in modal formulas and to discriminate between 'good' and 'bad' instances of distribution.[12]

[11] This also shows that (K) is not valid in the semantics for QML considered by van Benthem in [4] where trans-world identity is a partial function.

[12] It is possible to obtain the same result while keeping the standard modal operators by adopting typed languages and using finitary assignments in the semantics as it is done in [2, 6]. Nonetheless, we believe the present approach is simpler. Another possibility is to assume that the counterpart relation is total—i.e., each object inhabiting a world w has at least one counterpart in all worlds accessible from w. This solution has been adopted in [27], but we agree with [2] that this assumption is rather strong and philosophically unmotivated.

$$\dfrac{\dfrac{\dfrac{}{A \supset \exists y A}\,EG}{[\,\vec{x}\,y]A \supset [\,\vec{x}\,y]\exists y A}\,Nec+K \qquad \dfrac{}{[\,\vec{x}\,y]\exists y A \supset [\,\vec{x}\,]\exists y A}\,SHRT}{\exists y[\,\vec{x}\,y]A \supset [\,\vec{x}\,]\exists y A}\,Trans$$

Fig. 4 Axiomatic derivation of GF [7]

Moreover, they correct the proof-theoretic interaction between (GF) and (K): (GF) is derivable by axiom (K) together with the classical axiom EG: $A[y/x] \supset \exists x A$, cf. [4]. With indexed modalities (GF) is not derivable by axioms (K) and EG alone: the operator in its consequent is indexed by $\vec{x}$ and we can distribute only with respect to $\vec{x}$, y. As shown by the axiomatic derivation in Fig. 4, to make (GF) derivable we need also

$$[\,\vec{x}\;y]A \supset [\,\vec{x}\,]A, \quad \text{where } \mathtt{FV}(A) \subseteq \{\vec{x}\} \tag{SHRT}$$

which, like (GF), is C-valid only if the counterpart relation is total [7].

2.5 Transitions of Tuples

In C-frames the counterparts of a n-tuple are obtained by composing the counterparts of its individual members: the pair $\langle c, d\rangle$ is a counterpart of $\langle a, b\rangle$ if and only if c is a counterpart of a and d of b, we'll call this the *composition approach*. As a consequence, the formulas

$$[\,\vec{x}\,]A \supset [\,\vec{x}\;y]A \tag{LNGT}$$

and

$$[\,\vec{x}\,]\forall y A \supset \forall y[\,\vec{x}\;y]A \tag{CBF}$$

are C-valid. Nonetheless, there are arguments against the composition approach as well as against the validity of (CBF). We address these issue in order.

As convincingly argued by Hazen [24], the composition approach is in tension with the assumption that there are essential relational properties. This can be illustrated with the following example taken from [29]. Let us assume with D. Lewis [30] that the counterpart relation is a relation of qualitative similarity, that modalities express metaphysical necessity, and that the father-daughter relation is an essential relation—i.e., a relation that if true, is necessarily true. If Adam is Berta's father then the following sentence has to be true

It is necessary of Adam and Berta that he is her father. (4)

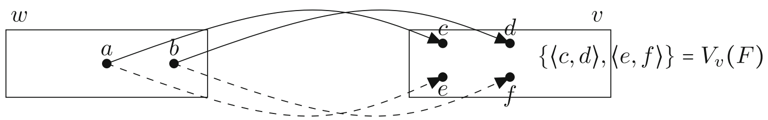

Fig. 5 Fatherhood in C-models

$$\dfrac{\dfrac{}{[\,\vec{x}\,]\forall yA \supset [\,\vec{x}\,y\,]\forall yA}\;LNGT \qquad \dfrac{\dfrac{}{\forall yA \supset A}\;UI}{[\,\vec{x}\,y\,]\forall yA \supset \forall y[\,\vec{x}, y\,]A}\;Nec+K}{[\,\vec{x}\,]\forall yA \supset \forall y[\,\vec{x}\,y\,]A}\;Trans$$

Fig. 6 Axiomatic derivation of CBF [7]

Nonetheless, this sentence is false under the composition approach if there is an accessible world where each object has an indistinguishable duplicate. This is shown by the C-model in Fig. 5: at v the pair $\langle e, f\rangle$ is a duplicate of $\langle c, d\rangle$, which is a counterpart of the F-pair $\langle a, b\rangle$. Hence the relation F holds for the pairs $\langle c, d\rangle$ and $\langle e, f\rangle$. The problem is that F does not hold for the 'mixed' pairs $\langle c, f\rangle$ and $\langle e, d\rangle$ even if these pairs are counterparts of $\langle a, b\rangle$ (this follows from the composition approach). The formula $[{}^{a\,b}_{x\,y}]xFy$, expressing (4), is false at w: $\langle c, f\rangle$ is a counterpart of $\langle a, b\rangle$ and it does not satisfy F. The problem, roughly, is that even if e and f are indistinguishable from c and d, respectively, the 'mixed' pairs $\langle c, f\rangle$ and $\langle e, d\rangle$ are distinguishable from $\langle c, d\rangle$ and $\langle e, f\rangle$.

To avoid this problem we generalise the counterpart relation into a *transition relation* $\mathcal{T}$ holding between n-tuples of objects inhabiting worlds. In doing so we follow Hazen [24] and metaframe semantics [18, 29, 39]. In this way c can be a counterpart of a and f of b without $\langle c, f\rangle$ being a counterpart of $\langle a, b\rangle$—or *vice versa*.[13] If the transition relation expresses a relation of qualitative similarity, then it has to be defined for n-tuples by considering similarity with respect to predicates of length n. In particular, in the model in Fig. 5 the transitions holding between pairs are represented by the solid and the dotted lines and, hence, the formula $[{}^{a\,b}_{x\,y}]xFy$ is true at w. This shows that a primitive notion of transition of tuples solves the problem with essential relations.

Now we consider (CBF). Under an epistemic reading of modalities, it entails that if an agent knows that all objects of a given domain have some property then she knows of each of them that it has that property, but this is acceptable only if she knows the composition of that domain. To illustrate, an agent might know that all real numbers are expressible as decimal expansions without knowing of each given real number that it is expressible as a decimal expansion: she might not know whether some given number is real or imaginary. (CBF) is disputable under an epistemic reading. Nonetheless it (and (LNGT)) is C-valid, cf. [7]. Axiomatically (CBF) is derivable in standard QML from the classical axiom (UI): $\forall xA \supset A[y/x]$ and axiom (K). As shown in Fig. 6, in IML we need also (LNGT) because we are distributing a modality indexed by $\vec{x}$, y and we have to pass from $[\vec{x}\ y]\forall yA$ to $[\vec{x}]\forall yA$ (this case is dual to that of (GF)). This move

[13] In [24] and in metaframe semantics it is assumed that transitions can be decomposed—i.e., if $\langle c, d\rangle$ is a transition of $\langle a, b\rangle$ then c is a transition of a and d of b. For the sake of generality, we will take transition of a n-tuples to be independent from transitions of m-tuples (for $n \neq m$). This is feasible because we are not assuming that the transition relation expresses qualitative similarity and we don't want to have (CBF) unrestrictedly valid.

is feasible under the composition approach: the worlds where there are counterparts of $\vec{x}, y$ are a subset of those where there are counterparts of $\vec{x}$ and the formula $\forall y A$, having free only variables in $\vec{x}$, is evaluated over the same objects in these two sets of worlds. Things are different if we replace individual counterparts with transitions of n-tuples. The transitions of $\vec{x}, y$ and those of $\vec{x}$ are mutually independent: there might be worlds where there are transitions of only one of these two sets and they might select different objects for the variables in $\vec{x}$ in a world where both are defined. As a consequence (LNGT) and (CBF) are valid only if we assume that the transitions of an n-tuple are decomposable into their subsets. Thus, a primitive non-decomposable notion of transition of tuples allows to make (CBF) invalid without having to weaken the classical theory of quantification (or axiom (K)).[14]

3 Syntax

Let us consider a signature Σ containing, for each $n \in \mathbb{N}$, an at most countable set of n-ary relational symbols Rel^n and an at most countable set of n-ary functional symbols Fun^n. Let, moreover, Var be a denumerable set of individual variables. The set Ter of terms is generated by:

$$t \ ::= \ x \mid f^n(t_1, \dots, t_n) \qquad (Ter)$$

where $x \in Var$ and $f^n \in Fun^n$. Variables will be denoted by x, y, z; 0-ary functions (individual constants) by a, b, c; and terms by t, s, r. Sequences of terms and variables are denoted using the vector arrow. Moreover, by $\vec{t}^{\pi}$ we denote some fixed permutation of the sequence $\vec{t}$.

The logical symbols are $=, \bot, \top, \wedge, \vee, \supset, \forall, \exists, [\,\dot{:}\,]$, and $\langle\dot{:}\rangle$.[15] Formulas of the language $\mathcal{L}^{\Sigma}$ and their free variables are defined simultaneously as follows:

Definition 2 (Language and free variables).

- If $P^n \in Rel^n$ then $P^n(t_1, \dots, t_n) \in \mathcal{L}^{\Sigma}$ and its free variables are all variables in $\{t_1, \dots, t_n\}$;
- $t = s \in \mathcal{L}^{\Sigma}$ and its free variables are all variables in $\{t, s\}$;
- $\bot, \top \in \mathcal{L}^{\Sigma}$ and they contain no free variable;
- If $A, B \in \mathcal{L}^{\Sigma}$ and $\circ \in \{\wedge, \vee, \supset\}$ then $A \circ B \in \mathcal{L}^{\Sigma}$ and $\mathrm{FV}(A \circ B) = \mathrm{FV}(A) \cup \mathrm{FV}(B)$;
- If $A \in \mathcal{L}^{\Sigma}$ and $\mathcal{Q} \in \{\forall, \exists\}$ then $\mathcal{Q}x A \in \mathcal{L}^{\Sigma}$ and $\mathrm{FV}(\mathcal{Q}x A) = \mathrm{FV}(A) - x$;
- If $A \in \mathcal{L}^{\Sigma}$ and $\mathrm{FV}(A) \subseteq \{x_1, \dots, x_n\}$ then $[\frac{t_1}{x_1} \dots \frac{t_n}{x_n}]A$ and $\langle\frac{t_1}{x_1} \dots \frac{t_n}{x_n}\rangle A$ are in $\mathcal{L}^{\Sigma}$ and their free variables are all variables in $\{t_1, \dots, t_n\}$.

The terms inside an indexed modal operator $[\frac{\vec{s}}{\vec{x}}]$ are a sequence of pairs $\langle s_i, x_i\rangle$. The sequence $\vec{s}$ ($\vec{x}$) is called the *numerator* (*denominator*) of $[\frac{\vec{s}}{\vec{x}}]$. Observe that we have

[14] Observe that the problem of essential relations and the validity of CBF are dual to each other in the following sense: the first problem is avoided by imposing that transitions of n-tuples cannot be composed and the latter by imposing that they cannot be decomposed. Thus, e.g., the metaframes semantics considerted in [29] validates CBF and avoids the problem of essential relations.

[15] The context disambiguate whether '$\langle\cdot\rangle$' is a diamond-operator or an ordered tuple.

defined formulas and their free variables simultaneously to impose that the denominator of an indexed operator contains (at least) all the variables occurring free in its scope. This is needed because the variables free in the scope of an indexed operator denote the objects about which we are expressing a *de re* modal claim and we want *de re* instances of (K) to be valid.

We follow the usual conventions for parentheses and we call *bound* any occurrence of a variable that is not free. A *sentence* is a formula without free occurrences of variables. A formula is *atomic* if no logical operator other than $=$ occurs in it. We use A, B, C for arbitrary formulas and P, Q for atomic ones. We use $A(\vec{x})$ to highlight that the free variables of A are among $\vec{x}$. The formulas $\neg A$, $s \neq t$, and $A \supset\subset B$ are defined as usual. $[\vec{x}]A$ is short for $[{}^{\vec{x}}_{\vec{x}}]A$ and $\Box A$ is short for $[\emptyset]A$ (we use analogous abbreviations for diamonds). The *weight* of a formula A, $\mathtt{w}(A)$, is the number of occurrences of operators that are not $=, \bot, \top$ in A.

Substitution of terms for free variables in terms is defined as expected and for formulas it is defined as follows (where $\equiv$ stands for syntactical identity):

Definition 3 (Substitution).

$$
\begin{array}{lcl}
(P^n(s_1, \dots, s_n))[t/x] & \equiv & P^n(s_1[t/x], \dots, s_n[t/x]) \\
(s_1 = s_2)[t/x] & \equiv & s_1[t/x] = s_2[t/x] \\
\dagger[t/x] & \equiv & \dagger, \quad \text{for } \dagger \in \{\bot, \top\} \\
(B \circ C)[t/x] & \equiv & B[t/x] \circ C[t/x], \quad \text{for } \circ \in \{\wedge, \vee, \supset\} \\
\begin{array}{l}(\mathcal{Q}yB)[t/x] \\ \text{for } \mathcal{Q} \in \{\forall, \exists\}\end{array} & \equiv & \begin{cases} \mathcal{Q}yB & \text{if } y \equiv x, \\ \mathcal{Q}z((B[z/y])[t/x]) & \text{if } y \not\equiv x \text{ and } y \in t \\ z \notin \{B, x, t\} & \\ \mathcal{Q}y(B[t/x]) & \text{if } y \not\equiv x \text{ and } y \notin t \end{cases} \\
([{}^{s_1}_{x_1} \dots {}^{s_n}_{x_n}]B)[t/x] & \equiv & [{}^{s_1[t/x]}_{x_1} \dots {}^{s_n[t/x]}_{x_n}]B \\
(\langle{}^{s_1}_{x_1} \dots {}^{s_n}_{x_n}\rangle B)[t/x] & \equiv & \langle{}^{s_1[t/x]}_{x_1} \dots {}^{s_n[t/x]}_{x_n}\rangle B
\end{array}
$$

Observe that substitution is always defined: for quantifiers it involves α-conversion to avoid capturing free variables. Moreover, substitution is performed inside indexed operators instead of permuting with them, as it happens for standard modal operators. To illustrate, $([x]P(x))[t/x]$ stands for $[{}^{t}_{x}]P(x)$ and not for $[x]P(t)$. We will also make extensive use of simultaneous substitution: $A[\vec{s}/\vec{x}]$, where the x_i are pairwise disjoint, is the formula obtained by replacing each free occurrence of x_i with s_i.

4 Semantics

Definition 4 A *skeleton* is a tuple $\mathcal{S} = \langle \mathcal{W}, \mathcal{D}, \mathcal{T} \rangle$, where:

- $\mathcal{W}$ is a non-empty set of *(possible) worlds*;
- $\mathcal{D} = \bigcup\{D_w : w \in \mathcal{W}\}$ is the union of a family of non-empty and pairwise disjoint sets (the *domains*) indexed by possible worlds;

- $\mathcal{T} \subseteq \bigcup\{(\mathcal{W} * (\mathcal{D})^n) \times (\mathcal{W} * (\mathcal{D})^n) \; : \; n \in \mathbb{N}\}$ is the *transition relation*; we impose that $\mathcal{T}$ is closed under permutations: $\langle w, \vec{o_1}\rangle \mathcal{T} \langle v, \vec{o_2}\rangle$ implies $\langle w, \vec{o_1}^{\pi}\rangle \mathcal{T} \langle v, \vec{o_2}^{\pi}\rangle$ for an arbitrary permutation π.
  If $\langle w, \vec{o_1}\rangle \mathcal{T} \langle v, \vec{o_2}\rangle$ we say that $\vec{o_2}$ is a *v-w-counterpart* of $\vec{o_1}$.[16]

A *model* $\mathcal{M}$ is a skeleton augmented with a function $\mathcal{V}$ mapping each world w to a classical valuation V_w interpreting symbols in Σ over D_w. We say that $\mathcal{M}$ is based on $\mathcal{S}$ if the latter is contained in the former.

Given a world w of a model $\mathcal{M}$, a *w-assignment* is a function σ mapping variables to objects in D_w. If σ is a w-assignment, we use $\sigma^{x \rhd o}$ for the w-assignment mapping x to $o \in D_w$ and behaving like σ for all other variables. If $\mathcal{M} = \langle \mathcal{W}, \mathcal{D}, \mathcal{T}, \mathcal{V}\rangle$ and σ is a w-assignment, for some $w \in \mathcal{W}$, then we use $\sigma(t)$ as shorthand for $V_w(t)$ when t is not a variable; $\sigma(\vec{s})$ stands for $\langle \sigma(s_1), \ldots \sigma(s_n)\rangle$.

Now we have all elements to define the notion of satisfaction.

Definition 5 (Satisfaction). The notion of *satisfaction* of a formula A at a point w of a model $\mathcal{M}$ under a w-assignment σ—$\sigma \models^{\mathcal{M}}_{w} A$, omitting $\mathcal{M}$ whenever convenient—is defined in the standard way for the extensional operators (with quantifiers ranging over D_w) and as follows for indexed modalities:

$\sigma \models^{\mathcal{M}}_{w} [\frac{\vec{s}}{\vec{x}}]B$	iff	for all worlds v and all v-assignment τ, if $\langle w, \sigma(\vec{s})\rangle \mathcal{T} \langle v, \tau(\vec{x})\rangle$ then $\tau \models^{\mathcal{M}}_{v} B$
$\sigma \models^{\mathcal{M}}_{w} \langle\frac{\vec{s}}{\vec{x}}\rangle B$	iff	there are a world v and a v-assignment τ s.t. $\langle w, \sigma(\vec{s})\rangle \mathcal{T} \langle v, \tau(\vec{x})\rangle$ and $\tau \models^{\mathcal{M}}_{v} B$

Observe that the assumption that $\mathcal{T}$ is closed under permutations implies that the sequences in $[\frac{\vec{s}}{\vec{x}}]B$ and $\langle\frac{\vec{s}}{\vec{x}}\rangle B$ behave semantically as sets of pairs $\langle s_i, x_i\rangle$ and not as a sequences thereof.[17] We have defined them syntactically as sequences for the sake of precision, cf. [16, p. 151].

The notions of *truth in a world*, $\models^{\mathcal{M}}_{w} A$, *truth in a model*, $\models^{\mathcal{M}} A$, and *validity in a class $\mathcal{C}$ of skeletons*, $\mathcal{C} \models A$, are defined in the standard way. An IML is the set of all $\mathcal{L}^{\Sigma}$-formulas that are valid in a given class of skeletons.

We present here a standard lemma relating assignments and substitutions that will be useful later on.

Lemma 6 *Let w be a world of a model $\mathcal{M}$, σ a w-assignment, t a term, and A an $\mathcal{L}^{\Sigma}$-formula,*

$$\sigma^{x \rhd \sigma(t)} \models^{\mathcal{M}}_{w} A \quad \textit{iff} \quad \sigma \models^{\mathcal{M}}_{w} A[t/x]$$

Proof A straightforward induction on the weight of A. □

Transition semantics has a very rich correspondence theory. The first class of correspondence results are extensions of the analogous results for PML [5]. The language of PML is defined by the following grammar (for $i \in \mathbb{N}$):

$$\phi \;::=\; p_i \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \supset \phi \mid \Box\phi \mid \Diamond\phi \qquad (\mathcal{L}^{\Box}_{p})$$

[16] For simplicity, we can assume that $\langle w, \vec{o_1}\rangle \mathcal{T} \langle v, \vec{o_2}\rangle$ implies $\vec{o_1}$ ($\vec{o_2}$) is made of objects from D_w (D_v). The limit case $w\mathcal{T}v$ is (a notational variant of) the propositional accessibility relation $w\mathcal{R}v$.

[17] To be fully precise, they behave as multisets without repetitions: the pairs are distinct since the denominator of an indexed modal operator is made of distinct variables.

If ϕ is a propositional modal axiom having ψ as only schematic letter, then $(\phi)^n$, for $n \in \mathbb{N}$, stands for the $\mathcal{L}^{\Sigma}$-formula obtained by replacing each instance of ψ with $A(x_1, \ldots, x_n)$ and each instance of $\Box$ ($\Diamond$) with an instance of $[x_1, \ldots x_n]$ (respectively, $\langle x_1 \ldots x_n \rangle$).[18] We can extend correspondence results in PML to the indexed case as follows:

Theorem 7 (Propositional correspondence). *If the propositional axiom ϕ corresponds to the first-order formula α (over the signature $\{\mathcal{R}\}$), then $(\phi)^n$, for $n \in \mathbb{N}$, is valid over the class of all skeletons satisfying the formula $(\alpha)^{\mathcal{T}_n}$; where $(\alpha)^{\mathcal{T}_n}$ is the (two-sorted) first-order formula obtained from α by replacing occurrences of $\mathcal{R}$ with $(n+1) \times (n+1)$-ary occurrences of $\mathcal{T}$ and quantifiers (identities) ranging over worlds in $\mathcal{W}$ with quantifiers (identities) ranging over $n+1$-tuples made of a world $w \in \mathcal{W}$ and n objects in D_w.*

Proof *(sketch).* Assume that $\phi (\in \mathcal{L}_p^{\Box})$ corresponds to α. We show by induction on $n \in \mathbb{N}$ that for all $n \in \mathbb{N}$, $(\phi)^n$ corresponds to $(\alpha)^{\mathcal{T}_n}$. If $n = 0$ there is nothing to prove. Suppose that $n = m + 1$ and that the claim holds for m, the claim holds also for n since indexes of length m and n behave analogously. $\Box$

Example 1 We present here some propositional correspondences ('$\forall \mathbf{x} \in D_w$' expresses 'for all $n \in \mathbb{N}$ and all $n+1$-tuple made of one $w \in \mathcal{W}$ and n objects from D_w').

T. $[\vec{x}]A \supset A$ corresponds to the reflexivity of $\mathcal{T}$:
$$\forall \mathbf{x} \in D_w (\mathbf{x}\mathcal{T}\mathbf{x})$$
4. $[\vec{x}]A \supset [\vec{x}][\vec{x}]A$ corresponds to the transitivity of $\mathcal{T}$:
$$\forall \mathbf{x} \in D_w, \mathbf{y} \in D_v, \mathbf{z} \in D_u (\mathbf{x}\mathcal{T}\mathbf{y} \wedge \mathbf{y}\mathcal{T}\mathbf{z} \supset \mathbf{x}\mathcal{T}\mathbf{z})$$
2. $\langle\vec{x}\rangle[\vec{x}]A \supset [\vec{x}]\langle\vec{x}\rangle A$ corresponds to the weak convergence of $\mathcal{T}$:
$$\forall \mathbf{x} \in D_w, \mathbf{y} \in D_v, \mathbf{z} \in D_u (\mathbf{x}\mathcal{T}\mathbf{y} \wedge \mathbf{x}\mathcal{T}\mathbf{z} \supset \exists \mathbf{t} \in D_{w'} (\mathbf{y}\mathcal{T}\mathbf{t} \wedge \mathbf{z}\mathcal{T}\mathbf{t}))$$
M. Over reflexive and transitive skeletons, $[\vec{x}]\langle\vec{x}\rangle A \supset \langle\vec{x}\rangle[\vec{x}]A$ corresponds to the finality of $\mathcal{T}$:[19]
$$\forall \mathbf{x} \in D_w, \exists \mathbf{y} \in D_v (\mathbf{x}\mathcal{T}\mathbf{y} \supset \forall \mathbf{z} \in D_u (\mathbf{y}\mathcal{T}\mathbf{z} \supset \mathbf{y} = \mathbf{z}))$$

Observe that in the proof of Theorem 7 and in Example 1 we have considered classes of skeletons satisfying $(\alpha)^{\mathcal{T}_n}$ for all $n \in \mathbb{N}$, and not those satisfying $(\alpha)^{\mathcal{T}_k}$ for some particular k. The reason for doing so is that we are interested in IML closed under second-order substitution—i.e., substitution of arbitrary formulas for atomic ones—and not in IML closed under first-order substitution only, see [1, 27] for this distinction in counterpart-theoretic semantics. If, e.g., we had considered the class of all skeletons satisfying $(\forall w(w\mathcal{R}w))^{\mathcal{T}_0}$ but not $(\forall w(w\mathcal{R}w))^{\mathcal{T}_m}$ for $m > 0$, we would have $\Box A \supset A$

[18] For simplicity we consider correspondence axioms involving a single fixed schematic letter ψ. For an axiom ϕ involving more than one schematic letter, such as 3: $\Box(\Box\psi \wedge \psi \supset \xi) \vee \Box(\Box\xi \wedge \xi \supset \psi)$, the procedure is analogous, but each modal operator has to be indexed by all variables occurring free in in $(\phi)^n$—e.g., $(3)^n$ is the following indexed formula $[\vec{x}]([\vec{x}]A \wedge A \supset B) \vee [\vec{x}]([\vec{x}]B \wedge B \supset A)$. We are grateful to Melvin Fitting for having pressed us to clarify this issue.

[19] Where $\mathbf{y} = \mathbf{z}$ is short for $\bigwedge\{y_i = z_i \ : \ y_i \in \mathbf{y} \text{ and } z_i \in \mathbf{z}\} \wedge v = u$.

valid, but $[\vec{x}]B \supset B$ not valid when $\vec{x}$ is not empty. Thus, we restrict ourselves to skeletons satisfying $(\alpha)^{\mathcal{T}_n}$ for all $n \in \mathbb{N}$.

Next, we present correspondence results relating indexed modalities with quantifiers and identity.

Theorem 8 (Transitional correspondence).

CBF. $[\vec{x}]\forall y A \supset \forall y[\vec{x}\ y]A$ *corresponds to the* decomposability *of* $\mathcal{T}$*:*
$\forall w, v \in \mathcal{W}, \forall \vec{x}_1, \vec{x}_2 \in D_w, \forall \vec{y}_1, \vec{y}_2 \in D_v(\langle w, \vec{x}_1, \vec{x}_2\rangle \mathcal{T} \langle v, \vec{y}_1, \vec{y}_2\rangle \supset \langle w, \vec{x}_1\rangle \mathcal{T} \langle v, \vec{y}_1\rangle)$
GF. $\exists y[\vec{x}\ y]A \supset [\vec{x}]\exists y A$ *corresponds to the* totality *of* $\mathcal{T}$*:*
$\forall w, v \in \mathcal{W}, \forall \vec{x}_1, \vec{x}_2 \in D_w, \forall \vec{y} \in D_v(\langle w, \vec{x}_1\rangle \mathcal{T} \langle v, \vec{y}\rangle \supset \exists \vec{z} \in D_v(\langle w, \vec{x}_1, \vec{x}_2\rangle \mathcal{T} \langle v, \vec{y}, \vec{z}\rangle))$
BF. $\forall y[\vec{x}\ y]A \supset [\vec{x}]\forall y A$ *corresponds to the* surjectivity *of* $\mathcal{T}$*:*
$\forall w, v \in \mathcal{W}, \forall \vec{x}_1, \vec{x}_2 \in D_v, \forall \vec{y} \in D_w(\langle w, \vec{y}\rangle \mathcal{T} \langle v, \vec{x}_1\rangle \supset \exists \vec{z} \in D_w(\langle w, \vec{y}, \vec{z}\rangle \mathcal{T} \langle v, \vec{x}_1, \vec{x}_2\rangle))$
NI. $x = y \supset [x\ y]x = y$ *corresponds to the* functionality *of* $\mathcal{T}$*:*
$\forall w, v \in \mathcal{W}, \forall x \in U_w \forall y, z \in U_v(\langle w, x, x\rangle \mathcal{T} \langle v, y, z\rangle \supset y = z)$
ND. $x \neq y \supset [x\ y]x \neq y$ *corresponds to the* injectivity *of* $\mathcal{T}$*:*
$\forall w, v \in \mathcal{W}, \forall x, y \in D_w \forall z \in D_v(\langle w, x, y\rangle \mathcal{T} \langle v, z, z\rangle \supset x = y)$

Proof Some cases of the right-to-left direction are given in Example 2. The proofs of the left-to-right direction are standard. We detail only the case of CBF. Assume that $\mathcal{S}$ is a skeleton such that $(\star)$ $\langle w, \vec{o}_1, \vec{o}_2\rangle \mathcal{T} \langle v, \vec{p}_1, \vec{p}_2\rangle$ and consider a w-assignment σ such that $\sigma(\vec{x}) = \vec{o}_1$. Let $V_u(P)$, for $u \in \mathcal{W}$, be such that $\langle \vec{q}_1, \vec{q}_2\rangle \in V_u(P)$ iff $\langle w, \sigma_w(\vec{x})\rangle \mathcal{T} \langle u, \vec{q}_1\rangle$. By construction we have that $\sigma \models_w [\vec{x}]\forall \vec{y} P(\vec{x}, \vec{y})$ and, thanks to CBF, $\sigma \models_w \forall \vec{y}[\vec{x}\ \vec{y}]P(\vec{x}, \vec{y})$. Given $(\star)$, this means that $\langle \vec{p}_1, \vec{p}_2\rangle \in V_v(P)$ and, thanks to definition of $V_u(P)$, we can conclude that $\langle w, \vec{o}_1\rangle \mathcal{T} \langle v, \vec{p}_1\rangle$. □

Finally, we present two correspondence results allowing to import and export of substitution in modal contexts. Taken together these two results make closed terms behave like rigid terms in standard modal logic.[20] These results pertain to classes of models and not of skeletons, but we disregard this difference for the sake of simplicity.

Theorem 9 (Rigidity).

RG. $[\ldots {}^{t}_{x} \ldots]A \supset [\ldots]A[t/x]$, *for t a closed term, is true in a model*
iff *for each* $\langle w, \vec{o}_1\rangle$ *and* $\langle v, \vec{o}_2\rangle$,
if $\langle w, \vec{o}_1\rangle \mathcal{T} \langle v, \vec{o}_2\rangle$ *then* $\langle w, \vec{o}_1, \mathcal{V}_w(t)\rangle \mathcal{T} \langle v, \vec{o}_2, \mathcal{V}_v(t)\rangle$.
RG$^-$. $[\ldots]A[t/x] \supset [\ldots {}^{t}_{x} \ldots]A$, *for t a closed term, is true in a model*
iff *for each* $\langle w, \vec{o}_1\rangle$ *and* $\langle v, \vec{o}_2\rangle$,
if $\langle w, \vec{o}_1, \mathcal{V}_w(t)\rangle \mathcal{T} \langle v, \vec{o}_2, o_3\rangle$ *then* $\langle w, \vec{o}_1\rangle \mathcal{T} \langle v, \vec{o}_2\rangle$ *and* $\mathcal{V}_v(t) = o_3$.

Proof The right-to-left direction of RG is proved in Example 2. □

20 Indexed modalities behave better than Fitting's λ-formulas in that only the former make importation and exportation mutually independent, cf. [17, Proposition 10.2.4].

5 Simulation of Standard QML

In this section we show that IML can simulate QML based on Kripke semantics, including those with λ-abstraction for non-rigid terms.[21]

We introduce here two languages based on the signature Σ (without function symbols of arity grater than 0, for simplicity). The first language ($\mathcal{L}^{\Box}$) is a standard language for QML with only variables as terms. The language ($\mathcal{L}^{\lambda}$) is ($\mathcal{L}^{\Box}$) together with non-rigid individual constants that can be applied to formulas by means of λ-abstraction as in Fitting's approach.[22]

Definition 10

$$\phi \ ::= \ P(\vec{x}) \,|\, x = y \,|\, \bot \,|\, \top \,|\, \phi \wedge \phi \,|\, \phi \vee \phi \,|\, \phi \supset \phi \,|\, \forall x \phi \,|\, \exists x \phi \,|\, \Box \phi \,|\, \Diamond \phi \qquad (\mathcal{L}^{\Box})$$

$$\phi \ ::= \ P(\vec{x}) \,|\, x = y \,|\, \bot \,|\, \top \,|\, \phi \wedge \phi \,|\, \phi \vee \phi \,|\, \phi \supset \phi \,|\, \forall x \phi \,|\, \exists x \phi \,|\, \Box \phi \,|\, \Diamond \phi \,|\, \langle \lambda x.\phi \rangle (t) \qquad (\mathcal{L}^{\lambda})$$

We introduce two classes of frames for defining logics over these languages: increasing and constant domains ones.[23]

Definition 11 (Kripke frames). A (first-order) *Kripke frame* is a triple $\mathcal{F} = \langle \mathcal{W}^{\mathcal{F}}, \mathcal{R}, \mathcal{D}^{\mathcal{F}} \rangle$ where $\mathcal{W}$ is a non-empty set of worlds; $\mathcal{R}$ is a binary accessibility relation over $\mathcal{W}$; $\mathcal{D}^{\mathcal{F}}$ is a function mapping each $w \in \mathcal{W}$ to a non empty set of objects $D^{\mathcal{F}}_w$ satisfying the following condition: $w\mathcal{R}v$ implies $D^{\mathcal{F}}_w \subseteq D^{\mathcal{F}}_v$.

A frame has *constant domains* if $D^{\mathcal{F}}_w = D^{\mathcal{F}}_v$ for all $w, v \in \mathcal{W}$; else it has *increasing domains*.

A *Kripke model* **M** is a Kripke frame augmented with a valuation function $\mathcal{V}^{\mathcal{F}}$ mapping each world w of the model to an interpretation ($V^{\mathcal{F}}_w$) of the symbols in Σ over $D^{\mathcal{F}}_w$. A *model with rigid terms* is a Kripke model where $\mathcal{V}^{\mathcal{F}}$ is such that if $w\mathcal{R}v$ then $V^{\mathcal{F}}_v(c) = V^{\mathcal{F}}_w(c)$ for each individual constant c.

A w-assignment is a function σ mapping variables to objects in D_w. Observe that if $w\mathcal{R}v$ then a w-assignment is also a v-assignment since $w\mathcal{R}v$ implies $D^{\mathcal{F}}_w \subseteq D^{\mathcal{F}}_v$. We make use of all the (relevant) definitions and abbreviations that have been introduced in Sections 3 and 4.

We are now ready to introduced the notion of satisfaction.

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[21] An analogous result is given in [2] with respect to C-frames with finitary assignments and a typed language with modal operators □ and ♢ (but without λ-abstraction and without individual constants). C-frames can be easily simulated by skeletons where $\mathcal{T}$ is decomposable, and typed languages and finitary assignments can be simulated by indexed modal operators. See also [3] for a similar result for QML based on a varying domain Kripke semantics with respect to C-frames and the indexed language (but without identity, λ-abstraction, and individual constants).

[22] Indexed modal operators are weaker than λ-abstraction in that they cannot deal with non-denoting terms in the style of [17].

[23] Varying (and decreasing) domains frames can be simulated by extending the language with a unary *existence predicate* $\mathcal{E}$ and by introducing quantifiers restricted by $\mathcal{E}$, cf. [25].

Definition 12 (Satisfaction). Satisfaction is defined as in Def. 5 for the extensional operator and as follows for $\Box$ and λ:

$$\sigma \models^{\mathbf{M}}_{w} \Box\phi \quad \text{iff} \quad \text{for all } v \in \mathcal{W}^{\mathcal{F}},\ w\mathcal{R}v \text{ implies } \sigma \models^{\mathbf{M}}_{v} \phi$$

$$\sigma \models^{\mathbf{M}}_{w} \langle\lambda x.\phi\rangle(t) \quad \text{iff} \quad \sigma^{x \rhd \sigma(t)} \models^{\mathbf{M}}_{w} \phi$$

Validity and logics are defined as in Section 4. In particular, it is well known that (CBF) is valid over all frames and BF—$\forall x\Box\phi \supset \Box\forall x\phi$— over constant domain ones. Moreover, frames with rigid terms validate (NI) and (ND), as well as the full substitutivity of terms—$\Box\langle\lambda x.\phi\rangle(c) \supset\subset \langle\lambda x.\Box\phi\rangle(c)$.

We can now show that QML based on Kripke frames be simulated by IML. More precisely, we will introduce two truth-preserving translations: $\circ$ mapping Kripke models to transition models where $\mathcal{T}$ is *Kripkean*: it is decomposable, total, functional, and injective; and $\bullet$ that does the inverse.

Definition 13 The translation function $\circ : \mathcal{L}^{\alpha\in\{\Box,\lambda\}} \longrightarrow \mathcal{L}^{\Sigma}$ is defined as follows:

$(P)^{\circ} \equiv P$	$(\dagger)^{\circ} \equiv \dagger,\ \dagger \in \{\bot, \top\}$	$(x = y)^{\circ} \equiv x = y$
$(\phi \triangle \psi)^{\circ} \equiv (\phi)^{\circ}\triangle(\psi)^{\circ}$	$(\mathcal{Q}x\phi)^{\circ} \equiv \mathcal{Q}x(\phi)^{\circ}$	$(\Box\phi(\vec{x}))^{\circ} \equiv [\vec{x}](\phi)^{\circ}$
where $\triangle \in \{\wedge, \vee \supset\}$	where $\mathcal{Q} \in \{\forall, \exists\}$	$(\Diamond\phi(\vec{x}))^{\circ} \equiv \langle\vec{x}\rangle(\phi)^{\circ}$
$(\langle\lambda x.\phi\rangle(t))^{\circ} \equiv (\phi)^{\circ}[t/x]$		

Definition 14 Given a Kripke model $\mathbf{M} = \langle \mathcal{W}^{\mathcal{F}}, \mathcal{R}, \mathcal{D}^{\mathcal{F}}, \mathcal{V}^{\mathcal{F}}\rangle$, its derived (indexed) model $\mathcal{M}^{\mathbf{M}} = \langle \mathcal{W}, \mathcal{D}, \mathcal{T}, \mathcal{V}\rangle$ is defined as follows:

- $\mathcal{W} = \mathcal{W}^{\mathcal{F}}$;
- $\mathcal{D}$ is the union of all sets D_w where $D_w = \{o_w : o \in D^{\mathcal{F}}_{w}\}$ and w ranges in $\mathcal{W}$;
- $\mathcal{T}$ is a Kripkean transition relation such that: $\langle w, \vec{o}_w\rangle\mathcal{T}\langle v, \vec{o}_v\rangle$ iff $w\mathcal{R}v$;
- $\mathcal{V}$ is thus defined: $V_w(c) = (o_w)$ iff $V^{\mathcal{F}}_{w}(c) = o$ and $\vec{o}_w \in V_w(P)$ iff $\vec{o} \in V^{\mathcal{F}}_{w}(P)$.

Moreover, if $\mathbf{M}$ has constant domain then $\mathcal{T}$ is surjective and if $\mathbf{M}$ is rigid then $\mathcal{T}$ and $\mathcal{V}$ satisfy the conditions given in Theorem 9.

Additionally, given a w-assignment σ (over $\mathbf{M}$), the assignment σ_v (over $\mathcal{M}^{\mathbf{M}}$), with $v = w$ or $w\mathcal{R}v$, is such that if $\sigma(x) = o$ then $\sigma_v(x) = o_v$.[24]

Lemma 15 *Let $\mathbf{M}$ be a Kripke model and let $\mathcal{M}^{\mathbf{M}}$ be its derived indexed model. For each world w of $\mathbf{M}$, each w-assignment σ, and each formula ϕ of the language $\mathcal{L}^{\lambda}$ (or $\mathcal{L}^{\Box}$) we have that:*

$$\sigma \models^{\mathbf{M}}_{w} \phi \quad \text{iff} \quad \sigma_w \models^{\mathcal{M}^{\mathbf{M}}}_{w} (\phi)^{\circ}$$

Proof By induction on the weight of ϕ. The basic cases hold by construction and the extensional cases are straightforward.

Let ϕ be $\Box\psi$ and let its free variables be $\vec{x}$. $\sigma \models^{\mathbf{M}}_{w} \Box\psi$ is equivalent to $\forall v \in \mathcal{W}^{\mathcal{F}}(w\mathcal{R}v \text{ implies } \sigma \models^{\mathbf{M}}_{v} \psi)$ which, by construction of $\mathcal{M}^{\mathbf{M}}$ and by induction, is equivalent to $\forall v \in \mathcal{W}(\langle w, \sigma_w(\vec{x})\rangle\mathcal{T}\langle v, \sigma_v(\vec{x})\rangle \supset \sigma_v \models^{\mathcal{M}^{\mathbf{M}}}_{v} (\psi)^{\circ})$. Given that $\mathcal{T}$ is

[24] This is needed to make variable behave like rigid designators in transition semantics.

Kripkean, the last fact is equivalent to $\forall v, \forall \tau_v(\langle w, \sigma_w(\vec{x})\rangle \mathcal{T} \langle v, \tau_v(\vec{x})\rangle \supset \tau_v \models_v^{\mathcal{M}^{\mathbf{M}}} (\psi)^\circ)$, which is the same as $\sigma_w \models_w^{\mathcal{M}^{\mathbf{M}}} [\vec{x}](\psi)^\circ$. If $\phi \equiv \Diamond\psi$, the proof is similar.

If $\phi \equiv \langle\lambda x.\psi\rangle(t)$, then $\sigma \models_w^{\mathbf{M}} \phi$ is equivalent to $\sigma^{x \triangleright \sigma(t)} \models_w^{\mathbf{M}} \psi$. By induction $\sigma^{x \triangleright \sigma(t)} \models_w^{\mathbf{M}} \psi$ is equivalent to $\sigma_w^{x \triangleright \sigma_w(t)} \models_w^{\mathcal{M}^{\mathbf{M}}} (\psi)^\circ$ which, by Lemma 6, is equivalent to $\sigma_w \models_w^{\mathcal{M}^{\mathbf{M}}} (\psi)^\circ[t/x]$ □

Next we show that IML based on Kripkean skeletons can be simulated by QML based on the language $\mathcal{L}^\lambda$.

Definition 16 The (back-)translation function $\bullet : \mathcal{L}^\Sigma \longrightarrow \mathcal{L}^\lambda$ is defined as follows:

$(P)^\bullet \equiv P$	$(\dagger)^\bullet \equiv \dagger,\ \ \dagger \in \{\bot, \top\}$	$(x = y)^\bullet \equiv x = y$
$(A \triangle B)^\bullet \equiv (A)^\bullet \triangle (B)^\bullet$	$(\mathcal{Q}x A)^\bullet \equiv \mathcal{Q}x(A)^\bullet$	$([\frac{\vec{t}}{\vec{x}}]A)^\bullet \equiv \langle\lambda\vec{x}.\Box(A)^\bullet\rangle(\vec{t})$
where $\triangle \in \{\wedge, \vee \supset\}$	where $\mathcal{Q} \in \{\forall, \exists\}$	$(\langle\frac{\vec{t}}{\vec{x}}\rangle A)^\bullet \equiv \langle\lambda\vec{x}.\Diamond(A)^\bullet\rangle(\vec{t})$

Definition 17 Given a Kripkean transition model $\mathcal{M} = \langle\mathcal{W}, \mathcal{D}, \mathcal{T}, \mathcal{V}\rangle$, its derived Kripke model $\mathbf{M}^{\mathcal{M}} = \langle\mathcal{W}^{\mathcal{F}}, \mathcal{R}, \mathcal{D}^{\mathcal{F}}, \mathcal{V}^{\mathcal{F}}\rangle$ is defined as follows:

- $\mathcal{W}^{\mathcal{F}} = \mathcal{W}$;
- $w\mathcal{R}v$ iff $w\mathcal{T}v$:
- $\mathcal{D}^{\mathcal{F}}$ maps $w \in \mathcal{W}^{\mathcal{F}}$ to $\{D_w^{\mathcal{F}} : \begin{cases} o \in D_w^{\mathcal{F}} & \text{if } o \in D_v \text{ and } \exists o'(\langle v, o\rangle \mathcal{T} \langle w, o'\rangle) \\ o' \in D_w^{\mathcal{F}} & \text{if } o' \in D_w \text{ and } \neg\exists v, o(\langle v, o\rangle \mathcal{T} \langle w, o'\rangle) \end{cases}\}$;
- $\mathcal{V}^{\mathcal{F}}$ is such that:
  - $V_w^{\mathcal{F}}(c) = \begin{cases} V_v(c) & \text{if } \exists v, o(\langle v, \mathcal{V}_v(c)\rangle \mathcal{T} \langle w, o\rangle) \\ V_w(c) & \text{else} \end{cases}$;
  - $\vec{o}\,' \in V_w^{\mathcal{F}}(P)$ iff $\vec{o} \in V_w(P)$, for $o_i \in \vec{o}$ s.t.: $\begin{cases} o_i = o'' & \text{if } o'' \in D_v \text{ and } \langle v, o''\rangle \mathcal{T} \langle w, o'\rangle \\ o_i = o_i' & \text{if } \neg\exists v, o''(\langle v, o''\rangle \mathcal{T} \langle w, o'\rangle) \end{cases}$.

It is easy to see that $\mathbf{M}^{\mathcal{M}}$ is well-defined. Given that $\mathcal{T}$ is decomposable $\langle w, \vec{o}_1\rangle \mathcal{T} \langle v, \vec{o}_2\rangle$ implies $w\mathcal{T}v$. Given that $\mathcal{T}$ is total and functional we know that $w\mathcal{T}v$ implies $\forall x \in \mathcal{D}_w \exists! y \in \mathcal{D}_v \langle w, x\rangle \mathcal{T} \langle v, y\rangle$ and, by injectivity, if $\langle w, x_1\rangle \mathcal{T} \langle v, y_1\rangle$ and $\langle w, x_2\rangle \mathcal{T} \langle v, y_2\rangle$ and $x_1 \neq x_2$, then $y_1 \neq y_2$. Thus, each $\mathcal{D}_w^{\mathcal{F}}$ is well-defined and $w\mathcal{R}v$ implies $\mathcal{D}_w^{\mathcal{F}} \subseteq \mathcal{D}_v^{\mathcal{F}}$. Moreover, if $\mathcal{T}$ is surjective, $w\mathcal{R}v$ implies $\mathcal{D}_w^{\mathcal{F}} = \mathcal{D}_v^{\mathcal{F}}$. Finally, $\mathcal{V}^{\mathcal{F}}$ is well-defined and if $\mathcal{M}$ satisfies one of the two condition of Theorem 9 then it satisfies also the other and in $\mathbf{M}^{\mathcal{M}}$ individual constants are rigid designators.

Lemma 18 *Let $\mathcal{M}$ be a Kripkean indexed model and $\mathbf{M}^{\mathcal{M}}$ its derived Kripke model. For each world w of $\mathcal{M}$, each w-assignment σ, and each formula A of the indexed language we have that:*

$$\sigma \models_w^{\mathcal{M}} A \quad \text{iff} \quad \sigma \models_w^{\mathbf{M}^{\mathcal{M}}} (A)^\bullet$$

Proof By induction on the weight of ϕ. We consider only the case where $A \equiv [\frac{\vec{t}}{\vec{x}}]B$. $\sigma \models_w^{\mathcal{M}} [\frac{\vec{t}}{\vec{x}}]B$ is equivalent to $\forall v, \tau(\langle w, \sigma(\vec{t})\rangle \mathcal{T} \langle v, \tau(\vec{x})\rangle \supset \tau \models_v^{\mathcal{M}} B)$. By induction this is equivalent to $\forall v, \tau(\langle w, \sigma(\vec{t})\rangle \mathcal{T} \langle v, \tau(\vec{x})\rangle \supset \tau \models_v^{\mathbf{M}^{\mathcal{M}}} (B)^\bullet)$. By the properties of $\mathcal{T}$ and the definition of $\mathcal{R}$, we rewrite this as $\forall v(w\mathcal{R}v \supset \sigma^{\vec{x} \triangleright \sigma(\vec{t})} \models_v^{\mathbf{M}^{\mathcal{M}}} (B)^\bullet)$. This is the same as $\sigma^{\vec{x} \triangleright \sigma(\vec{t})} \models_w^{\mathbf{M}^{\mathcal{M}}} \Box(B)^\bullet$, which is equivalent to $\sigma \models_w^{\mathbf{M}^{\mathcal{M}}} \langle\lambda\vec{x}.\Box(B)^\bullet\rangle(\vec{t})$. □

Theorem 19 *For each FO-definable class of Kripke frames (with or without rigid designators) there is a class of skeletons that is equivalent to it with respect to validity.*

Proof An immediate consequence of Lemmas 15 and 18 since $\mathcal{T}$ in $\mathcal{M}^{\mathbf{M}}$ satisfies all the first-order properties satisfied by $\mathcal{R}$ in $\mathbf{M}$ and the same holds for $\mathcal{M}$ and $\mathbf{M}^{\mathcal{M}}$. □

6 The Basic Labelled Calculus GIM.K

Labelled sequent calculi for first-order $\mathscr{L}^{\Box}$-logics have been considered in [35–37] and for some indexed epistemic logic in [9]. These calculi are based on extending the modal language in order to internalise the semantics. The systems introduced here are similar to those in [9].

First of all, we introduce a set LAB of fresh variables, called *labels*. Labels will be denoted by $w, v, u, \ldots$ and will represent worlds. Next, we extend the set of formulas by adding formulas of shape $(w, \vec{t})\mathscr{R}(v, \vec{s})$— the so-called *transition atoms* expressing that $\vec{s}$ is v-w-counterpart of $\vec{t}$—and formulas of shape $w \approx v$—the *l-identities* expressing that w and v represent the same world. Lastly, we replace each $\mathscr{L}^{\Sigma}$-formula A with the *labelled formula* $w : A$—expressing that A holds at w. The weight of a transition atom or of an l-identity is 0 and the weight of $w : A$ is the same as that of A. A *labelled sequent* is an expression:

$$\Gamma \Rightarrow \Delta$$

where Γ is a multiset composed of transition atoms, l-identities, and transition atoms; and Δ is a multiset of labelled formulas. Substitution is made sensitive to labels as follows:

- $(w : A)[v : t/x] \equiv \begin{cases} w : A[t/x] & \text{if } w \equiv v, \\ w : A & \text{else} \end{cases}$
- $((w, \vec{t})\mathscr{R}(v, \vec{s}))[u : r/x] \equiv \begin{cases} (w, \vec{t}[r/x])\mathscr{R}(v, \vec{s}) & \text{if } w \equiv u \text{ and } v \not\equiv u, \\ (w, \vec{t})\mathscr{R}(v, \vec{s}[r/x]) & \text{if } w \not\equiv u \text{ and } v \equiv u, \\ (w, \vec{t}[r/x])\mathscr{R}(v, \vec{s}[r/x]) & \text{if } w \equiv u \text{ and } v \equiv u, \\ (w, \vec{t})\mathscr{R}(v, \vec{s}) & \text{else} \end{cases}$

Given a formula E of this extended language, $E[w/v]$ is the formula obtained by substituting each occurrence of v in E with an occurrence of w. Notice that the substitution of labels works also on the label occurring in a label-sensitive substitution—e.g., $((w : A)[v : t/x])[u_1/u] \equiv (w[u_1/u] : A)[v[u_1/u] : t/x]$. Substitutions are extended to multisets by applying them componentwise.

Given a calculus $\mathbf{L}$, a $\mathbf{L}$-*derivation* of a sequent $\Gamma \Rightarrow \Delta$ is a tree of sequents, whose leaves are initial sequents, whose root is $\Gamma \Rightarrow \Delta$, and which grows according to the rules of $\mathbf{L}$. We consider only derivations of *pure sequents*—i.e., sequents where no variable has both free and bound occurrences and where all *eigenvariables* are distinct. The *height* of a $\mathbf{L}$-derivation is the number of nodes of its longest branch. We say that $\Gamma \Rightarrow \Delta$ is $\mathbf{L}$-derivable (with height n), and we write $\mathbf{L} \vdash^{(n)} \Gamma \Rightarrow \Delta$, if there is an $\mathbf{L}$-derivation (of height at most n) of $\Gamma \Rightarrow \Delta$ or of an alphabetic variant

Table 1 Rules of **GIM.K**

Initial sequents: $w : P, \Gamma \Rightarrow \Delta, w : P$, with P atomic

Logical rules:

$$\frac{}{w : \bot, \Gamma \Rightarrow \Delta}\ L\bot \qquad \frac{}{\Gamma \Rightarrow \Delta, w : \top}\ R\top$$

$$\frac{w : A, w : B, \Gamma \Rightarrow \Delta}{w : A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, w : A \quad \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \wedge B}\ R\wedge$$

$$\frac{w : A, \Gamma \Rightarrow \Delta \quad w : B, \Gamma \Rightarrow \Delta}{w : A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, w : A, w : B}{\Gamma \Rightarrow \Delta, w : A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, w : A \quad w : B, \Gamma \Rightarrow \Delta}{w : A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad \frac{w : A, \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \supset B}\ R\supset$$

$$\frac{w : A[t/x], w : \forall x A, \Gamma \Rightarrow \Delta}{w : \forall x A, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow \Delta, w : A[y/x]}{\Gamma \Rightarrow \Delta, w : \forall x A}\ R\forall,\ y \text{ fresh}$$

$$\frac{w : A[y/x], \Gamma \Rightarrow \Delta}{w : \exists x A, \Gamma \Rightarrow \Delta}\ L\exists,\ y \text{ fresh} \qquad \frac{\Gamma \Rightarrow \Delta, w : \exists x A, w : A[t/x]}{\Gamma \Rightarrow \Delta, w : \exists x A}\ R\exists$$

$$\frac{v : A[\vec{s}/\vec{x}], (w, \vec{t})\mathscr{R}(v, \vec{s}), w : [\tfrac{\vec{t}}{\vec{x}}]A, \Gamma \Rightarrow \Delta}{(w, \vec{t})\mathscr{R}(v, \vec{s}), w : [\tfrac{\vec{t}}{\vec{x}}]A, \Gamma \Rightarrow \Delta}\ L\Box$$

$$\frac{(w, \vec{t})\mathscr{R}(u, \vec{y}), \Gamma \Rightarrow \Delta, u : A[\vec{y}/\vec{x}]}{\Gamma \Rightarrow \Delta, w : [\tfrac{\vec{t}}{\vec{x}}]A}\ R\Box,\ u, \vec{y} \text{ fresh}$$

$$\frac{(w, \vec{t})\mathscr{R}(u, \vec{y}), u : A[\vec{y}/\vec{x}], \Gamma \Rightarrow \Delta}{w : \langle\tfrac{\vec{t}}{\vec{x}}\rangle A, \Gamma \Rightarrow \Delta}\ L\Diamond,\ u, \vec{y} \text{ fresh}$$

$$\frac{(w, \vec{t})\mathscr{R}(v, \vec{s}), \Gamma \Rightarrow \Delta, w : \langle\tfrac{\vec{t}}{\vec{x}}\rangle A, v : A[\vec{s}/\vec{x}]}{(w, \vec{t})\mathscr{R}(v, \vec{s}), \Gamma \Rightarrow \Delta, w : \langle\tfrac{\vec{t}}{\vec{x}}\rangle A}\ R\Diamond$$

Coherent rules: E is a transitional atom or an atomic formula

$$\frac{w : t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref \qquad \frac{E[w : t/x], w : s = t, E[w : s/x], \Gamma \Rightarrow \Delta}{w : s = t, E[w : s/x], \Gamma \Rightarrow \Delta}\ Repl$$

$$\frac{w : t = f(\dots, f(\dots, t, \dots), \dots), w : t = f(\dots, t, \dots), \Gamma \Rightarrow \Delta}{w : t = f(\dots, t, \dots), \Gamma \Rightarrow \Delta}\ Repl^c$$

$$\frac{w \approx w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref_{\approx} \qquad \frac{F[v/u], w \approx v, F[w/u], \Gamma \Rightarrow \Delta}{w \approx v, F[w/u], \Gamma \Rightarrow \Delta}\ Repl_{\approx}$$

F is a transitional atom, an l-identity or an atomic formula

$$\frac{(w, \vec{t}^{\pi})\mathscr{R}(v, \vec{s}^{\pi}), (w, \vec{t})\mathscr{R}(v, \vec{s}), \Gamma \Rightarrow \Delta}{(w, \vec{t})\mathscr{R}(v, \vec{s}), \Gamma \Rightarrow \Delta}\ Perm$$

of $\Gamma \Rightarrow \Delta$. A rule is said to be *(height-preserving) admissible* in **L**, if, whenever its premisses are **L**-derivable (with height at most n), also its conclusion is **L**-derivable (with height at most n). In each rule displayed in Tables 1 and 2, the multisets Γ and Δ are called *contexts*, the formulas occurring in the conclusion are called *principal*, and the formulas occurring in the premiss(es) only are called *active*.

The rules of the calculus **GIM.K** are given in Table 1, where the rules for identity, for $\approx$, and the rule *Perm*—which ensures that the indexes of an indexed operator are sequences of pairs where the order is irrelevant—are left coherent rules that will be discussed in the next section. Observe that l-identities are not needed in **GIM.K**,

Table 2 Some coherent rules

Propositional rules:

$$\frac{(w,\vec{t})\mathscr{R}(w,\vec{t}),\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}\ Ref_{\mathscr{R}}$$

$$\frac{(w,\vec{t})\mathscr{R}(u,\vec{r}),(w,\vec{t})\mathscr{R}(v,\vec{s}),(v,\vec{s})\mathscr{R}(u,\vec{r}),\Gamma\Rightarrow\Delta}{(w,\vec{t})\mathscr{R}(v,\vec{s}),(v,\vec{s})\mathscr{R}(u,\vec{r}),\Gamma\Rightarrow\Delta}\ Trans$$

$$\frac{(u,\vec{r})\mathscr{R}(u_1,\vec{x}),(v,\vec{s})\mathscr{R}(u_1,\vec{x}),(w,\vec{t})\mathscr{R}(v,\vec{s}),(w,\vec{t})\mathscr{R}(u,\vec{r}),\Gamma\Rightarrow\Delta}{(w,\vec{t})\mathscr{R}(v,\vec{s}),(w,\vec{t})\mathscr{R}(u,\vec{r}),\Gamma\Rightarrow\Delta}\ Conv,\ u_1,\vec{x}\text{ fresh}$$

$$\frac{(w,\vec{t})\mathscr{R}(v,\vec{s}),(v,\vec{s})\mathscr{R}(u_1,\vec{x}),\Gamma\Rightarrow\Delta}{(w,\vec{t})\mathscr{R}(v,\vec{s}),\Gamma\Rightarrow\Delta}\ Conv^c,\ u_1,\vec{x}\text{ fresh}$$

Transitional rules:

$$\frac{(w,\vec{s}_1)\mathscr{R}(v,\vec{t}_1),(w,\vec{s}_1,\vec{s}_2)\mathscr{R}(v,\vec{t}_1,\vec{t}_2),\Gamma\Rightarrow\Delta}{(w,\vec{s}_1,\vec{s}_2)\mathscr{R}(v,\vec{t}_1,\vec{t}_2),\Gamma\Rightarrow\Delta}\ Dec$$

$$\frac{(w,\vec{s}_1,\vec{s}_2)\mathscr{R}(v,\vec{r},\vec{y}),(w,\vec{s}_1)\mathscr{R}(v,\vec{r}),\Gamma\Rightarrow\Delta}{(w,\vec{s}_1)\mathscr{R}(v,\vec{r}),\Gamma\Rightarrow\Delta}\ Tot,\ \vec{y}\text{ fresh}$$

$$\frac{(w,\vec{s},\vec{y})\mathscr{R}(v,\vec{r}_1,\vec{r}_2),(w,\vec{s})\mathscr{R}(v,\vec{r}_1),\Gamma\Rightarrow\Delta}{(w,\vec{s})\mathscr{R}(v,\vec{r}_1),\Gamma\Rightarrow\Delta}\ Surj,\ \vec{y}\text{ fresh}$$

$$\frac{v:t=r,(w,s,s)\mathscr{R}(v,t,r),\Gamma\Rightarrow\Delta}{(w,s,s)\mathscr{R}(v,t,r),\Gamma\Rightarrow\Delta}\ Func \qquad \frac{w:t=r,(w,t,r)\mathscr{R}(v,s,s),\Gamma\Rightarrow\Delta}{(w,t,r)\mathscr{R}(v,s,s),\Gamma\Rightarrow\Delta}\ Inj$$

Rules for rigidity (t closed):

$$\frac{(w,\vec{s},t)\mathscr{R}(v,\vec{r},t),(w,\vec{s})\mathscr{R}(v,\vec{r}),\Gamma\Rightarrow\Delta}{(w,\vec{s})\mathscr{R}(v,\vec{r}),\Gamma\Rightarrow\Delta}\ Rig$$

$$\frac{v:t=r,(w,\vec{s})\mathscr{R}(v,\vec{r}),(w,\vec{s},t)\mathscr{R}(v,\vec{r},r),\Gamma\Rightarrow\Delta}{(w,\vec{s},t)\mathscr{R}(v,\vec{r},r),\Gamma\Rightarrow\Delta}\ Rig^-$$

nevertheless they will be essential to give a calculus for extensions of **K** that are defined by properties of $\mathcal{T}$ involving identities between worlds.

7 Calculi extending GIM.K

In order to define calculi for the IML extending **K**, we will express the properties of $\mathcal{T}$ defining these logics as left coherent rules [32]. In order to do so we will use the algorithm presented in [12] to convert any first-order formula into an equivalent (for our purposes) coherent first-order formula.

A *coherent implication* is the universal closure of an extensional first-order implicative formula whose antecedent and consequent are positive formulas.

Definition 20 (Coherent implication).

- A formula is *Horn* iff it is built from atoms and $\top$ using only $\wedge$;
- A formula is *coherent* iff it is built from atoms, $\top$, and $\bot$ using only $\wedge$, $\vee$, and $\exists$;
- A sentence is a *coherent implication* iff it is of the form $\forall\vec{x}(A\supset B)$ where A and B are coherent formulas.

Coherent formulas can be written in normal form as follows:

Theorem 21 (Coherent normal form (CNF)). *Any coherent implication is equivalent to a conjunction of sentences of shape*

$$\forall \vec{x}(A \supset B) \tag{CNF}$$

where A is Horn and B is a disjunction of existentially quantified Horn formulas.[25]

As explained in [32], from a sentence C in (CNF) we can extract a *coherent rule* L_C that can be added to a calculus without altering its structural properties. To illustrate, let us consider the following sentence in (CNF):

$$\forall \vec{x}(P_1(\vec{x}) \wedge \cdots \wedge P_k(\vec{x}) \supset \bigvee_{n=0}^{\ell} \exists \vec{y}(Q_{n_1}(\vec{x}, \vec{y}) \wedge \cdots \wedge Q_{n_m}(\vec{x}, \vec{y}))) \tag{C}$$

Such a sentence determines the following *coherent rule*, having one premiss for each disjunct in $\bigvee_{n=0}^{\ell} \exists \vec{y}(Q_{n_1}(\vec{x}, \vec{y}) \wedge \cdots \wedge Q_{n_m}(\vec{x}, \vec{y}))$:

$$\frac{\ldots \qquad Q_{n_1}(\vec{x}, \vec{y}), \ldots, Q_{n_m}(\vec{x}, \vec{y}), P_1(\vec{x}), \ldots, P_k(\vec{x}), \Gamma \Rightarrow \Delta \qquad \ldots}{P_1(\vec{x}), \ldots, P_k(\vec{x}), \Gamma \Rightarrow \Delta} \; L_C$$

The variables in $\vec{x}$ may be instantiated with arbitrary terms. The variables in $\vec{y}$, instead, are chosen to be *fresh* with respect to the conclusion of the rule, since they are existentially quantified in (C). Henceforth we shall omit mention of the variables.

To ensure the height-preserving admissibility of contraction, we impose the following condition:

Definition 22 (Closure condition). If a calculus contains a coherent rule with an instance with repetition of some principal formula such as:

$$\frac{\ldots \qquad Q_1, \ldots, Q_n, P_1, \ldots, P_{k-2}, P, P, \Gamma \Rightarrow \Delta \qquad \ldots}{P_1, \ldots, P_{k-2}, P, P, \Gamma \Rightarrow \Delta} \; L_C$$

then also the *contracted instance*

$$\frac{\ldots \qquad Q_1, \ldots, Q_n, P_1, \ldots, P_{k-2}, P, \Gamma \Rightarrow \Delta \qquad \ldots}{P_1, \ldots, P_{k-2}, P, \Gamma \Rightarrow \Delta} \; L_C^c$$

has to be included in the calculus.

The condition is unproblematic: since each atomic formula contains only a finite number of variables, the number of rules that have to be added for each coherent rule is finite. Moreover, in many cases contracted instances need not be added since they are already admissible in the calculus. To wit, in the coherent rules for identity given in Table 1 we have added contracted instances of *Repl* only for the case where E

[25] For brevity, we use B as short for $\top \supset B$.

is an identity and one of the two identical terms is a functional symbol: contracted instances of rule *Repl* may arise only if E is an identity $w : s = t$; moreover, if s and t are variables or individual constants, the calculus is already closed under contracted instances, see [13, Section 3].

By itself the method of left coherent rules is sufficient to define a calculus with good structural properties for all logics defined by coherent properties of $\mathcal{T}$. This class covers all transitional correspondence results given in Theorem 8 as well as most propositional ones, possibly using the rules for $\approx$ given in Table 1. To illustrate, we consider the formula

$$\langle\tfrac{\vec{s}}{\vec{x}}\rangle A \supset [\tfrac{\vec{s}}{\vec{x}}]A \tag{PF}$$

that corresponds to the *partial functionality* of $\mathcal{T}$:

$$\forall \mathbf{x} \in D_w, \mathbf{y} \in D_v, \mathbf{z} \in D_u (\mathbf{x}\mathcal{T}\mathbf{y} \wedge \mathbf{x}\mathcal{T}\mathbf{z} \supset \mathbf{z} = \mathbf{y})$$

This property is expressed by the following left coherent rule:

$$\frac{u \approx v, \{u : z = y \mid z \in \vec{z} \text{ and } y \in \vec{y}\}, (w, \vec{x})\mathscr{R}(v, \vec{y}), (w, \vec{x})\mathscr{R}(u, \vec{z}), \Gamma \Rightarrow \Delta}{(w, \vec{x})\mathscr{R}(v, \vec{y}), (w, \vec{x})\mathscr{R}(u, \vec{z}), \Gamma \Rightarrow \Delta}\ PFunc$$

and a straightforward proof-search shows that this rule make the sequent expressing (PF) derivable (we use two lemmas that will be proved in the next section and a double-line inference rule for many instances of the same rule):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{v : A[\vec{y}/\vec{x}], v : A[\vec{z}/\vec{x}], u \approx v, \{u : z = y \mid y \in \mathbf{y} \,\&\, z \in \mathbf{z}\}, (w, \vec{s})\mathscr{R}(u, \vec{z}), u : A[\vec{z}/\vec{x}], (w, \vec{s})\mathscr{R}(v, \vec{y}) \Rightarrow v : A[\vec{y}/\vec{x}]}\ Lem.\ 24}{v : A[\vec{z}/\vec{x}], u \approx v, \{u : z = y \mid y \in \mathbf{y} \,\&\, z \in \mathbf{z}\}, (w, \vec{s})\mathscr{R}(u, \vec{z}), u : A[\vec{z}/\vec{x}], (w, \vec{s})\mathscr{R}(v, \vec{y}) \Rightarrow v : A[\vec{y}/\vec{x}]}\ Lem.\ 31.3}{u \approx v, \{u : z = y \mid y \in \mathbf{y} \,\&\, z \in \mathbf{z}\}, (w, \vec{s})\mathscr{R}(u, \vec{z}), u : A[\vec{z}/\vec{x}], (w, \vec{s})\mathscr{R}(v, \vec{y}) \Rightarrow v : A[\vec{y}/\vec{x}]}\ Ref_{\approx}}{(w, \vec{s})\mathscr{R}(u, \vec{z}), u : A[\vec{z}/\vec{x}], (w, \vec{s})\mathscr{R}(v, \vec{y}) \Rightarrow v : A[\vec{y}/\vec{x}]}\ PFunc}{(w, \vec{s})\mathscr{R}(v, \vec{y}), w : \langle\tfrac{\vec{s}}{\vec{x}}\rangle A \Rightarrow v : A[\vec{y}/\vec{x}]}\ L\Diamond}{w : \langle\tfrac{\vec{s}}{\vec{x}}\rangle A \Rightarrow w : [\tfrac{\vec{s}}{\vec{x}}]A}\ R\Box$$

Nonetheless, there are many first-order properties in propositional correspondence theory that are not coherent—e.g., the finality of $\mathcal{T}$ considered in Example 1 where we have a universal quantifier in the scope of an existential one:

$$\forall \mathbf{x} \in D_w, \exists \mathbf{y} \in D_v (\mathbf{x}\mathcal{T}\mathbf{y} \supset \forall \mathbf{z} \in D_u (\mathbf{y}\mathcal{T}\mathbf{z} \supset \mathbf{y} = \mathbf{z})) \tag{Fin}$$

In order to make such cases amenable, we apply the method of *semidefinitional extensions* introduced in [12]. Roughly, the method allows to transform any first-order formula A into a coherent one by replacing non-coherent components of that formula with new atomic predicates behaving like the components they have replaced. This is done by applying the following algorithm:[26]

Algorithm 23 Let A be a first-order axiom (in a given signature).

1. We try to transform A into an equivalent formula A^* in (CNF);

[26] See [12] for a version of the algorithm based on the assumption that A is in prenex normal form with the propositional part in disjunctive normal form.

2. If A^* is in (CNF), the algorithm ends; else we identify the outermost non-coherent components of A^*, say $B_1(\vec{x}_1), \ldots, B_n(\vec{x}_n)$, where each $\vec{x}_i$ is the set of bound variables in B_i that are bound by quantifiers not occurring therein;
3. We extend the signature with new atomic predicates $Pred_1(\vec{x}_1),\ldots,Pred_n(\vec{x}_n)$ and we replace each B_i in A^* with $Pred_i$;
4. For each $Pred_i$ we add the axiom Ax_i: $\forall \vec{x}_i (Pred_i(\vec{x}_i) \supset B_i(\vec{x}_i))$;
5. We apply steps 1–4 to each Ax_i.

After a finite number of runs through the algorithm we have replaced the axiom A with a finite number of axioms in (CNF) which jointly imply A. We illustrate this algorithm with the case of (Fin). In this case step 2 identifies $\forall \mathbf{z} \in D_u(\mathbf{y}\mathcal{T}\mathbf{z} \supset \mathbf{y} = \mathbf{z})$ that in step 3 is replaced by the fresh predicate $Fin(\mathbf{y})$. Next, step 4 adds the axiom $\forall \mathbf{y}(Fin(\mathbf{y}) \supset \forall \mathbf{z} \in D_u(\mathbf{y}\mathcal{T}\mathbf{z} \supset \mathbf{y} = \mathbf{z}))$. Finally, this axiom is transformed into (CNF) by an instance of step 1. We have thus replaced the non-coherent axiom (Fin) with the following pair of axioms in (CNF):

$$\forall \mathbf{x} \in D_w(\top \supset \exists \mathbf{y} \in D_v(\mathbf{x}\mathcal{T}\mathbf{y} \;\&\; Fin(\mathbf{y}))) \qquad \forall \mathbf{y} \in D_v \forall \mathbf{z} \in D_u(Fin(\mathbf{y}) \;\&\; \mathbf{y}\mathcal{T}\mathbf{z} \supset \mathbf{y}=\mathbf{z})$$

which can be expressed by the following pair of left coherent rules:

$$\frac{(w, \vec{s})\mathscr{R}(v, \vec{y}),\ Fin(v, \vec{y}),\ \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle Fin_1,\ v, \vec{y}\ \text{fresh}}$$

$$\frac{v \approx u,\ \{v : s = r \mid s \in \vec{s} \;\&\; r \in \vec{r}\},\ Fin(v, \vec{s}),\ (v, \vec{s})\mathscr{R}(u, \vec{r}),\ \Gamma \Rightarrow \Delta}{Fin(v, \vec{s}),\ (v, \vec{s})\mathscr{R}(u, \vec{r}),\ \Gamma \Rightarrow \Delta}\ {\scriptstyle Fin_2}$$

By applying this method, we can transform each first-order definable correspondence result into a set of left coherent rules. As we are going to show in the next sections, this means that for each first-order definable IML we have a labelled calculus with good structural properties that is sound and complete.

To have more examples of left coherent rules expressing correspondence results, the reader can consult Table 2 where there are the left coherent rules for the properties of $\mathcal{T}$ given in Example 1 and in Theorems 8 and 9. In the next example we show that some of the formulas in Theorems 8 and 9 are derivable in calculi including the appropriate rule.

Example 2

1. $\mathbf{GIM.K} + \{Dec\} \vdash \Rightarrow w : [\vec{x}]\forall y A \supset \forall y[\vec{x}\ y]A$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{v : A[\vec{z}/\vec{x}, z/y],\ v : \forall y(A[\vec{z}/\vec{x}]),\ (w, \vec{x})\mathscr{R}(v, \vec{z}),\ (w, \vec{x}, y)\mathscr{R}(v, \vec{z}, z),\ w : [\vec{x}]\forall y A \Rightarrow v : A[\vec{z}/\vec{x}, z/y]}\ {\scriptstyle L.\,24}}{v : \forall y(A[\vec{z}/\vec{x}]),\ (w, \vec{x})\mathscr{R}(v, \vec{z}),\ (w, \vec{x}, y)\mathscr{R}(v, \vec{z}, z),\ w : [\vec{x}]\forall y A \Rightarrow v : A[\vec{z}/\vec{x}, z/y]}\ {\scriptstyle L\forall}}{(w, \vec{x})\mathscr{R}(v, \vec{z}),\ (w, \vec{x}, y)\mathscr{R}(v, \vec{z}, z),\ w : [\vec{x}]\forall y A \Rightarrow v : A[\vec{z}/\vec{x}, z/y]}\ {\scriptstyle L\Box}}{(w, \vec{x}, y)\mathscr{R}(v, \vec{z}, z),\ w : [\vec{x}]\forall y A \Rightarrow v : A[\vec{z}/\vec{x}, z/y]}\ {\scriptstyle Dec}}{w : [\vec{x}]\forall y A \Rightarrow w : [\vec{x}\ y]A}\ {\scriptstyle R\Box}}{w : [\vec{x}]\forall y A \Rightarrow w : \forall y[\vec{x}\ y]A}\ {\scriptstyle R\forall}$$

2. $\mathbf{GIM.K} + \{Func\} \vdash \Rightarrow w : x = y \supset [x\ y]x = y$

$$\dfrac{\dfrac{\dfrac{v : z_1 = z_2, (w, y, y)\mathscr{R}(v, z_1, z_2), (w, x, y)\mathscr{R}(v, z_1, z_2), w : x = y \Rightarrow v : z_1 = z_2}{(w, y, y)\mathscr{R}(v, z_1, z_2), (w, x, y)\mathscr{R}(v, z_1, z_2), w : x = y \Rightarrow v : z_1 = z_2}\ Func}{(w, x, y)\mathscr{R}(v, z_1, z_2), w : x = y \Rightarrow v : z_1 = z_2}\ Repl}{w : x = y \Rightarrow w : [x\ y]x = y}\ R\Box$$

3. $\mathbf{GIM.K} + \{Rig\} \vdash \Rightarrow w : [\vec{y}\,{}^{t}_{x}]A \supset [\vec{y}]A[t/x]$, where t is a closed term.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{v : A[t/x, \vec{z}/\vec{y}], (w, \vec{y}, t)\mathscr{R}(v, \vec{z}, t), (w, \vec{y})\mathscr{R}(v, \vec{z}), w : [\vec{y}\,{}^{t}_{x}]A \Rightarrow v : A[t/x, \vec{z}/\vec{y}]}\ L.\,24}{(w, \vec{y}, t)\mathscr{R}(v, \vec{z}, t), (w, \vec{y})\mathscr{R}(v, \vec{z}), w : [\vec{y}\,{}^{t}_{x}]A \Rightarrow v : A[t/x, \vec{z}/\vec{y}]}\ L\Box}{(w, \vec{y})\mathscr{R}(v, \vec{z}), w : [\vec{y}\,{}^{t}_{x}]A \Rightarrow v : A[t/x, \vec{z}/\vec{y}]}\ Rig}{w : [\vec{y}\,{}^{t}_{x}]A \Rightarrow w : [\vec{y}]A[t/x]}\ R\Box$$

8 Structural properties

We can now prove that each calculus **GIM.L** that is obtained by extending **GIM.K** with coherent rules expressing FO-definable properties of $\mathcal{T}$ has the good structural properties that are peculiar to **G3**-style calculi.

Lemma 24 *Sequents of shape $w : A, \Gamma \Rightarrow \Delta, w : A$ are* **GIM.L***-derivable.*

Proof By induction on the weight of $w : A$. For the inductive steps it is enough to apply, root first, the rules for the principal operator of $w : A$ and then the inductive hypothesis (IH). $\Box$

Lemma 25 (α-conversion). **GIM.L** $\vdash^n \Gamma \Rightarrow \Delta$ *entails* **GIM.L** $\vdash^n \Gamma' \Rightarrow \Delta'$, *where Γ' (Δ') is obtained from Γ (Δ) by renaming some bound variable (without capturing variables).*

Proof The proof, which is by induction on the height of the derivation of $\Gamma \Rightarrow \Delta$, is straightforward since α-conversion is built into the notion of substitution. $\Box$

Lemma 26 (Substitutions). *The following rules of substitution are height-preserving admissible in* **GIM.L***:*

$$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma[w/v] \Rightarrow \Delta[w/v]}\ S_l \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma[w : s/x] \Rightarrow \Delta[w : s/x]}\ S_t$$

Proof The proofs are by induction on the height of the derivation $\mathcal{D}$ of the premiss $\Gamma \Rightarrow \Delta$. Most cases are as in [36, Lemma 12.4].

We consider only some interesting cases. First, we consider the case of rule S_t with $\mathcal{D}$ ending as follows:

$$\dfrac{(u_1, \vec{t})\mathscr{R}(u_2, \vec{y}), \Gamma \Rightarrow \Delta, u_2 : A[\vec{y}/\vec{x}]}{\Gamma \Rightarrow \Delta, u_1 : [\,{}^{\vec{t}}_{\vec{x}}]A}\ R\Box,\ u_2, \vec{y}\text{ fresh}$$

If $w \equiv u_2$, there is nothing to prove, thus we assume that $w \not\equiv u_2$. If $w \not\equiv u_1$ the proof is immediate. If $w \equiv u_1$, we proceed as follows (for $\vec{z}$ new to $s, \mathcal{D}$):

$$\dfrac{\dfrac{\dfrac{\dfrac{(w,\vec{t})\mathscr{R}(u_2,\vec{y}),\Gamma \Rightarrow \Delta, u_2 : A[\vec{y}/\vec{x}]}{(w,\vec{t})\mathscr{R}(u_2,\vec{z}),\Gamma \Rightarrow \Delta, u_2 : A[\vec{z}/\vec{x}]}\ IH}{((w,\vec{t})\mathscr{R}(u_2,\vec{z}))[w:s/x],\Gamma[w:s/x] \Rightarrow \Delta[w:s/x],(u_2 : A[\vec{z}/\vec{x}])[w:s/x]}\ IH}{(w,\vec{t}[s/x])\mathscr{R}(u_2,\vec{z}),\Gamma[w:s/x] \Rightarrow \Delta[w:s/x], u_2 : A[\vec{z}/\vec{x}]}\ (\star)}{\dfrac{\Gamma[w:s/x] \Rightarrow \Delta[w:s/x], w : [\frac{\vec{t}[s/x]}{\vec{x}}]A}{\Gamma[w:s/x] \Rightarrow \Delta[w:s/x], (w : [\frac{\vec{t}}{\vec{x}}]A)[w:s/x]}\ (\star)}\ R\Box$$

where the steps marked by $(\star)$ are rewritings justified by the definition of label-sensitive substitution.

Next, we consider the case S_t with $\mathcal{D}$ ending by the following instance of $L\Box$:

$$\dfrac{u_2 : A[\vec{r}/\vec{x}],(u_1,\vec{t})\mathscr{R}(u_2,\vec{r}),u_1 : [\frac{\vec{t}}{\vec{x}}]A,\Gamma \Rightarrow \Delta}{(u_1,\vec{t})\mathscr{R}(u_2,\vec{r}),u_1 : [\frac{\vec{t}}{\vec{x}}]A,\Gamma \Rightarrow \Delta}\ L\Box$$

We have sub-cases according to whether w is in $\{u_1, u_2\}$. Suppose that $w \equiv u_2$ and $w \not\equiv u_1$. We transform $\mathcal{D}$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{u_2 : A[\vec{r}/\vec{x}],(u_1,\vec{t})\mathscr{R}(u_2,\vec{r}),u_1 : [\frac{\vec{t}}{\vec{x}}]A,\Gamma \Rightarrow \Delta}{(u_2 : A[\vec{r}/\vec{x}])[w:s/x],((u_1,\vec{t})\mathscr{R}(u_2,\vec{r}))[w:s/x],(u_1 : [\frac{\vec{t}}{\vec{x}}]A)[w:s/x],\Gamma[w:s/x] \Rightarrow \Delta[w:s/x]}\ IH}{u_2 : A[\vec{r}[s/x]/\vec{x}],(u_1,\vec{t})\mathscr{R}(u_2,\vec{r}[s/x]),(u_1 : [\frac{\vec{t}}{\vec{x}}]A)[w:s/x],\Gamma[w:s/x] \Rightarrow \Delta[w:s/x]}\ (\star)}{(u_1,\vec{t})\mathscr{R}(u_2,\vec{r}[s/x]),(u_1 : [\frac{\vec{t}}{\vec{x}}]A)[w:s/x],\Gamma[w:s/x] \Rightarrow \Delta[w:s/x]}\ L\Box}{((u_1,\vec{t})\mathscr{R}(u_2,\vec{r}))[w:s/x],(u_1 : [\frac{\vec{t}}{\vec{x}}]A)[w:s/x],\Gamma[w:s/x] \Rightarrow \Delta[w:s/x]}\ (\star)$$

The other sub-cases are similar.

Finally, we consider the case of rule S_l where the last step of $\mathcal{D}$ is as follows:

$$\dfrac{\Gamma \Rightarrow \Delta, u : A[z/x]}{\Gamma \Rightarrow \Delta, u : \forall x}\ R\forall, z \text{ fresh}$$

We proceed by applying the inductive hypothesis to the premiss and then an instance of $R\forall$. Observe that this is feasible only if the derivation of the premiss does not contain a formula with an explicit label-sensitive substitution of shape $[w : z/x]$ with $w \equiv u$ since by renaming w we could generate a clash with the eigenvariable of the final instance of $R\forall$. This assumption is not limitative since substitutions are a piece of notation and not primitive operators. □

Theorem 27 (Weakening). *The following rules are height-preserving admissible in* **GIM.L***:*

$$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta}\ LW \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \Delta'}\ RW, \ \Delta' \textit{ contains only labelled formulas}$$

Proof The proofs are by induction on the height of the derivation $\mathcal{D}$ of the premiss $\Gamma \Rightarrow \Delta$. We consider just the case of $\mathcal{D}$ ending with an instance of $R\Box$ and we refer the reader to [36, Thm. 12.5] for most other cases. Suppose $\mathcal{D}$ ends as follows:

$$\frac{(w,\vec{s})\mathscr{R}(u,\vec{y}),\Gamma \Rightarrow \Delta, u : A[\vec{y}/\vec{x}]}{\Gamma \Rightarrow \Delta, w : [\tfrac{\vec{s}}{\vec{x}}]A}\; R\Box,\, u,\vec{y}\text{ fresh}$$

The following transformation shows that LW is hp-admissible:

$$\cfrac{\cfrac{\cfrac{\cfrac{(w,\vec{s})\mathscr{R}(u,\vec{y}),\Gamma \Rightarrow \Delta, u : A[\vec{y}/\vec{x}]}{(w,\vec{s})\mathscr{R}(u,\vec{z}),\Gamma \Rightarrow \Delta, u : A[\vec{z}/\vec{x}]}\,S_t}{(w,\vec{s})\mathscr{R}(u',\vec{z}),\Gamma \Rightarrow \Delta, u' : A[\vec{z}/\vec{x}]}\,S_l}{(w,\vec{s})\mathscr{R}(u',\vec{z}),\Gamma',\Gamma \Rightarrow \Delta, u' : A[\vec{z}/\vec{x}]}\,IH}{\Gamma',\Gamma \Rightarrow \Delta, w : [\tfrac{\vec{s}}{\vec{x}}]A}\,R\Box$$

where u' and $\vec{z}$ are new to $\mathcal{D}$ and not occurring in Γ'. □

Lemma 28 (Invertibility). *Each rule of* **GIM.L** *is height-preserving invertible.*

Proof For each rule we proceed by induction on the height of the derivation $\mathcal{D}$ of a possible conclusion of an instance of that rule. The cases for the non-modal logical rules have been proved in [36]. Rules $L\Box$, $R\Diamond$ as well as all coherent rules are height-preserving invertible thanks to Theorem 27. Finally the height-preserving invertibility of rule $R\Box$ and $L\Diamond$ is straightforward. Suppose, e.g., that $\mathcal{D}$ is a derivation concluding $\Gamma \Rightarrow \Delta, w : [\frac{\vec{s}}{\vec{x}}]A$. If $w : [\frac{\vec{s}}{\vec{x}}]A$ is principal in the last rule applied in $\mathcal{D}$ then we already have a (shorter) derivation of $(w,\vec{s})\mathscr{R}(u,\vec{y}),\Gamma \Rightarrow \Delta, u : A[\vec{y}/\vec{x}]$, for some $u,\vec{y}$. Else the last rule applied in $\mathcal{D}$ has principal formula(s) in Γ,Δ. As usual, we apply (an instance of substitution if that rule has a variable condition and) the inductive hypothesis to the premiss and than an instance of the same rule to obtain a derivation—having same height of $\mathcal{D}$—of the desirered endsequent. □

Theorem 29 (Contraction). *The following rules are height-preserving admissible in* **GIM.L***:*

$$\frac{\Gamma',\Gamma',\Gamma \Rightarrow \Delta}{\Gamma',\Gamma \Rightarrow \Delta}\,LC \qquad \frac{\Gamma \Rightarrow \Delta,\Delta',\Delta'}{\Gamma \Rightarrow \Delta,\Delta'}\,RC$$

Proof The proof is handled by a simultaneous induction on the height of the derivations of the premisses of LC and RC. Without loss of generality, we assume the multiset we are contracting is made of only one formula E.

The base cases obviously hold, and the proof of the inductive cases depends on whether zero, one, or two instances of E are principal in the last step R of the derivation $\mathcal{D}$ of the premiss. If zero instances are principal in R, we apply IH to the premiss(es) of R and then an instance of rule R, and we are done.

If one instance is principal and R is by a propositional rule or by one of $R\forall$, $L\exists$, $R\Box$, and $L\Diamond$, we proceed by first applying invertibility to that rule, then we apply IH as many times as needed, and we conclude by applying an instance of that rule. If, instead, one instance is principal and R is by a rule with repetition of the principal formula(s) in the premiss, we apply directly IH—on LC and/or RC depending on the case we are considering—and an instance of R.

If two instances are principal, R is a coherent rule and we know that contraction is hp-admissible since R satisfies the closure condition given in Definition 22. ◻

Theorem 30 (Cut). *The following rule of Cut is admissible in* **GIM.L***:*

$$\frac{\Gamma \Rightarrow \Delta, w : A \quad w : A, \Gamma' \Rightarrow \Delta'}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ Cut$$

Proof The proof, which extends that of [36, Theorem 12.9], considers an uppermost instance of *Cut* which is handled by a principal induction on the weight of the cut-formula with a sub-induction on the sum of the heights of the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of the two premisses of cut (*cut-height*, for shortness). The proof is organised in three exhaustive cases: in **case 1** at least one of the two premisses is an initial sequent. In **case 2** the cut formula is not principal in at least one premiss. Finally, in **case 3** it is principal in both premisses.

Case 1 is treated in [36].

In **case 2** we permute the cut upwards in the derivation of one premiss where the cut formula is not principal, after having applied an instance of S_t and/or S_l if that last rule has a variable condition. To illustrate, suppose we have the following derivation:

$$\frac{\Gamma \Rightarrow \Delta, w : A \qquad \dfrac{(v, \vec{s})\mathcal{T}(u, \vec{y}), w : A, \Gamma' \Rightarrow \Delta'', u : B[\vec{y}/\vec{x}]}{w : A, \Gamma' \Rightarrow \Delta'', v : [\frac{\vec{s}}{\vec{x}}]B}\ R\Box}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'', v : [\frac{\vec{s}}{\vec{x}}]B}\ Cut$$

We transform it into the following derivation having an admissible cut of lesser cut-height:

$$\frac{\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad \dfrac{(v, \vec{s})\mathcal{T}(u, \vec{y}), w : A, \Gamma' \Rightarrow \Delta'', u : B[\vec{y}/\vec{x}]}{(v, \vec{s})\mathcal{T}(u', \vec{z}), w : A, \Gamma' \Rightarrow \Delta'', u' : B[\vec{z}/\vec{x}]}\ S_l{+}S_t}{(v, \vec{s})\mathcal{T}(u', \vec{z}), \Gamma', \Gamma \Rightarrow \Delta, \Delta'', u' : B[\vec{z}/\vec{x}]}\ IH}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'', v : [\frac{\vec{s}}{\vec{x}}]B}\ R\Box$$

where u' and $\vec{z}$ do not occur in the original derivation.

Finally, **case 3** has sub-cases according to the principal operator of the cut-formula. We consider only the case where the cut-formula is of shape $w : [\frac{\vec{s}}{\vec{x}}]B$. The procedure for $w : \langle\frac{\vec{s}}{\vec{x}}\rangle B$ is analogous, and the other cases have been considered in [36]. We transform the derivation:

$$
\dfrac{\dfrac{(w,\vec{s})\mathscr{R}(u,\vec{y}),\Gamma \Rightarrow \Delta, u : B[\vec{y}/\vec{x}]}{\Gamma \Rightarrow \Delta, w : [\genfrac{}{}{0pt}{}{\vec{s}}{\vec{x}}]B}\,{\scriptstyle R\Box} \qquad \dfrac{v : B[\vec{t}/\vec{x}], (w,\vec{s})\mathscr{R}(v,\vec{t}), w : [\genfrac{}{}{0pt}{}{\vec{s}}{\vec{x}}]B, \Gamma'' \Rightarrow \Delta'}{(w,\vec{s})\mathscr{R}(v,\vec{t}), w : [\genfrac{}{}{0pt}{}{\vec{s}}{\vec{x}}]B, \Gamma'' \Rightarrow \Delta'}\,{\scriptstyle L\Box}}{(w,\vec{s})\mathscr{R}(v,\vec{t}), \Gamma'', \Gamma \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}
$$

Into the following one:

$$
\dfrac{\dfrac{\dfrac{\dfrac{(w,\vec{s})\mathscr{R}(u,\vec{y}),\Gamma \Rightarrow \Delta, u : B[\vec{y}/\vec{x}]}{(w,\vec{s})\mathscr{R}(u,\vec{t}),\Gamma \Rightarrow \Delta, u : B[\vec{t}/\vec{x}]}\,{\scriptstyle S_t}}{(w,\vec{s})\mathscr{R}(v,\vec{t}),\Gamma \Rightarrow \Delta, v : B[\vec{t}/\vec{x}]}\,{\scriptstyle S_l} \qquad \dfrac{\Gamma \Rightarrow \Delta, w : [\genfrac{}{}{0pt}{}{\vec{s}}{\vec{x}}]B \quad w : [\genfrac{}{}{0pt}{}{\vec{s}}{\vec{x}}]B, v : B[\vec{t}/\vec{x}], (w,\vec{s})\mathscr{R}(v,\vec{t}), \Gamma'' \Rightarrow \Delta'}{v : B[\vec{t}/\vec{x}], (w,\vec{s})\mathscr{R}(v,\vec{t}), \Gamma'', \Gamma \Rightarrow \Delta, \Delta'}\,{\scriptstyle IH_2}}{(w,\vec{s})\mathscr{R}(v,\vec{t}), (w,\vec{s})\mathscr{R}(v,\vec{t}), \Gamma'', \Gamma', \Gamma \Rightarrow \Delta, \Delta', \Delta'}\,{\scriptstyle IH_1}}{(w,\vec{s})\mathscr{R}(v,\vec{t}), \Gamma'', \Gamma \Rightarrow \Delta, \Delta'}\,{\scriptstyle LC+RC}
$$

Where IH_1 refers to the principal inductive hypothesis and IH_2 to the secondary one. □

Lemma 31 (Properties of identity).

1. *Identity is an equivalence relation in* **GIM.L***;*
2. *The following sequents are* **GIM.L***-derivable:*

   (a) $\Rightarrow w : t = t$
   (b) $w : t = s, w : A[t/x] \Rightarrow w : A[s/x]$

3. *The following rule of replacement is admissible in* **GIM.L***:*

$$
\dfrac{w : A[s/x], w : t = s, w : A[t/x], \Gamma \Rightarrow \Delta}{w : s = t, w : A[t/x], \Gamma \Rightarrow \Delta}\,{\scriptstyle Repl_A}
$$

Proof The proofs of items 1 and 2(a) are left to the reader. We prove the following instance of 2(b): $w : s = t, w : [\genfrac{}{}{0pt}{}{\vec{r}[t/x]}{\vec{y}}]A \Rightarrow w : [\genfrac{}{}{0pt}{}{\vec{r}[s/x]}{\vec{y}}]A$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{v : A[\vec{z}/\vec{y}], (w,\vec{r}[t/x])\mathscr{R}(v,\vec{z}), w : s = t, (w,\vec{r}[s/x])\mathscr{R}(v,\vec{y}), w : t = s, w : [\genfrac{}{}{0pt}{}{\vec{r}[t/x]}{\vec{y}}]A \Rightarrow v : A[\vec{z}/\vec{y}]}\,{\scriptstyle Lem.\ 24}}{(w,\vec{r}[t/x])\mathscr{R}(v,\vec{z}), w : s = t, (w,\vec{r}[s/x])\mathscr{R}(v,\vec{y}), w : t = s, w : [\genfrac{}{}{0pt}{}{\vec{r}[t/x]}{\vec{y}}]A \Rightarrow v : A[\vec{z}/\vec{y}]}\,{\scriptstyle L\Box}}{w : s = t, (w,\vec{r}[s/x])\mathscr{R}(v,\vec{z}), w : t = s, w : [\genfrac{}{}{0pt}{}{\vec{r}[t/x]}{\vec{y}}]A \Rightarrow v : A[\vec{z}/\vec{y}]}\,{\scriptstyle Repl}}{(w,\vec{r}[s/x])\mathscr{R}(v,\vec{z}), w : t = s, w : [\genfrac{}{}{0pt}{}{\vec{r}[t/x]}{\vec{y}}]A \Rightarrow v : A[\vec{z}/\vec{y}]}\,{\scriptstyle Sim_=}}{w : t = s, w : [\genfrac{}{}{0pt}{}{\vec{r}[t/x]}{\vec{y}}]A \Rightarrow w : [\genfrac{}{}{0pt}{}{\vec{r}[s/x]}{\vec{y}}]A}\,{\scriptstyle R\Box}
$$

where rule $\dfrac{w : s = t, w : t = s, \Gamma \Rightarrow \Delta}{w : t = s, \Gamma \Rightarrow \Delta}\,{\scriptstyle Sim_=}$ is admissible.

Finally, item (3) is a corollary of the admissibility of contraction and cut:

$$
\dfrac{\dfrac{\dfrac{}{w : t = s, w : A[t/x] \Rightarrow w : A[s/x]}\,{\scriptstyle Lem.\ 31.2(b)} \qquad w : A[s/x], w : t = s, w : A[t/x], \Gamma \Rightarrow \Delta}{w : s = t, w : A[t/x], w : s = t, w : A[t/x], \Gamma \Rightarrow \Delta}\,{\scriptstyle Cut}}{w : s = t, w : A[t/x], \Gamma \Rightarrow \Delta}\,{\scriptstyle LC}
$$

□

9 Characterisation

9.1 Soundness

Definition 32 Given a model $\mathcal{M} = \langle \mathcal{W}, \mathcal{D}, \mathcal{T}, \mathcal{V} \rangle$, let $\alpha : LAB \longrightarrow \mathcal{W}$ be a function mapping labels to worlds of the model, and let $\beta : \mathcal{W} \longrightarrow (Var \longrightarrow \mathcal{D})$ be a function mapping worlds to assignments defined over that world. We say that:

$$\begin{array}{lll} \mathcal{M}, \alpha, \beta \text{ satisfies } w : A & \text{iff} & \beta(\alpha(w)) \models^{\mathcal{M}}_{\alpha(w)} A \\ \mathcal{M}, \alpha, \beta \text{ satisfies } \langle w, \sigma(\vec{t}) \rangle \mathcal{T} \langle v, \tau(\vec{s}) \rangle & \text{iff} & (\alpha(w), \beta(\alpha(w))(\vec{t})) \mathscr{R} (\alpha(v), \beta(\alpha(v))(\vec{s})) \end{array}$$

where, abusing notation, $\beta(\alpha(w))(t)$, for t not a variable, stands for $V_{\alpha(w)}(t)$.

Given a sequent $\Gamma \Rightarrow \Delta$ and a logic $\mathbf{L}$, we say that $\Gamma \Rightarrow \Delta$ is $\mathbf{L}$-*valid* iff for every triple $\mathcal{M}, \alpha, \beta$ where $\mathcal{M}$ is a model based on a skeleton for $\mathbf{L}$, if $\mathcal{M}, \alpha, \beta$ satisfies all formulas in Γ then it satisfies some formula in Δ.

Theorem 33 (Soundness). *If $\Gamma \Rightarrow \Delta$ is* **GIM.L**-*derivable, then it is* $\mathbf{L}$-*valid.*

Proof The proof is by induction on the height of the **GIM.L**-derivation of $\Gamma \Rightarrow \Delta$. The base case holds since either Γ and Δ have some atomic formula $w : P$ in common or Γ (Δ) contains an instance of $w : \bot$ ($w : \top$). It is also easy to see that the non-modal logical rules preserve validity on every model, see [36, Thm. 12.13].

Let us consider the following instance of $L\Box$:

$$\frac{v : A[\vec{s}/\vec{x}], (w, \vec{t}) \mathscr{R} (v, \vec{s}), w : [\tfrac{\vec{t}}{\vec{x}}]A, \Gamma \Rightarrow \Delta}{(w, \vec{t}) \mathscr{R} (v, \vec{s}), w : [\tfrac{\vec{t}}{\vec{x}}]A, \Gamma \Rightarrow \Delta} \; L\Box$$

and a triple $\mathcal{M}, \alpha, \beta$ satisfying all formulas in $(w, \vec{t}) \mathscr{R} (v, \vec{s}), w : [\tfrac{\vec{t}}{\vec{x}}]A, \Gamma$. This implies that $\mathcal{M}$ is such that $\langle \alpha(w), \beta(\alpha(w))(\vec{t}) \rangle \mathcal{T} \langle \alpha(v), \beta(\alpha(v))(\vec{s}) \rangle$ and, for each $u \in \mathcal{W}$ and each u-assignment τ, $\langle \alpha(w), \beta(\alpha(w))(\vec{t}) \rangle \mathcal{T} \langle u, \tau(\vec{x}) \rangle$ implies $\beta(\alpha(u)) \models_{\alpha(u)} A$. In particular, we have that $\langle \alpha(w), \beta(\alpha(w))(\vec{t}) \rangle \mathcal{T} \langle \alpha(v), \beta'(\alpha(v))(\vec{x}) \rangle$, where $\beta'(\alpha(v))(\vec{x}) = \beta(\alpha(v))(\vec{s})$ and β' behave like β for other variables. It follows that $\beta'(\alpha(v)) \models_{\alpha(v)} A$, which is equivalent to $\beta(\alpha(v)) \models_{\alpha(v)} A[\vec{s}/\vec{x}]$. Since $\mathcal{M}, \alpha, \beta$ satisfies all formulas in the antecedent of the premiss, we conclude that $\mathcal{M}, \alpha, \beta$ satisfies some formula in Δ.

Suppose, instead, that $\mathcal{D}$ ends by the following instance of $R\Box$:

$$\frac{(w, \vec{s}) \mathscr{R} (u, \vec{y}), \Gamma \Rightarrow \Delta, u : A[\vec{y}/\vec{x}]}{\Gamma \Rightarrow \Delta, w : [\tfrac{\vec{s}}{\vec{x}}]A} \; R\Box, u, \vec{y} \text{ fresh}$$

We consider a triple $\mathcal{M}, \alpha, \beta$ satisfying all formulas in Γ. If it satisfies also some formula in Δ we are done. If, instead, it satisfies no formula in Δ, we show it satisfies $w : [\tfrac{\vec{s}}{\vec{x}}]A$. If there is no tuple made of a world v and a sequence of objects $\vec{o}$ from its domain such that $\langle \alpha(w), \beta(\alpha(w))(\vec{s}) \rangle \mathcal{T} \langle v, \vec{o} \rangle$, we can already conclude that $\mathcal{M}, \alpha, \beta$ satisfies $w : [\tfrac{\vec{s}}{\vec{x}}]A$. Else, for one (generic) such $v, \vec{o}$ we consider α' and β' such that $\alpha'(u) = v$ and $\beta'(\alpha'(u))(\vec{y}) = \vec{o}$ and behaving like α and β for the other labels and variables. The triple $\mathcal{M}, \alpha', \beta'$ satisfies all formulas in $(w, \vec{s}) \mathscr{R} (u, \vec{y}), \Gamma$ and, by induction, it satisfies also some formula in $\Delta, u : A[\vec{y}/\vec{x}]$. Given that it cannot satisfies some formula in Δ (otherwise also $\mathcal{M}, \alpha, \beta$ would do so), it must satisfies $u : A[\vec{y}/\vec{x}]$. We can easily conclude that $\mathcal{M}, \alpha, \beta$ satisfies $w : [\frac{\vec{s}}{\vec{x}}]A$.

The proof that the rules for identity and l-identity, as well as rule *Perm*, preserve validity is standard and can, thus, be omitted. The proof that each non-logical rule preserves validity over the appropriate class of skeletons, is straightforward: each triple satisfying all formulas in the antecedent of the conclusion must satisfies all formulas in the antecedent of some premiss since $\mathcal{M}$ is based on a skeleton having the property expressed by that rule. We conclude that the given triple satisfies also some formula in the succedent of the conclusion of that rule instance. □

9.2 Completeness

We prove that each calculus is complete over the appropriate class of skeletons by the usual Tait-Schütte-Takeuti technique: we define a root first-procedure that outputs the informations needed to build a derivation for each derivable sequent and a countermodel (based on the appropriate class of skeletons) for each underivable one. This strategy is direct and has the advantage of being modular, see [34] for an overview of this and other strategies for proving completeness for modal logics. The proof is articulated in the following steps:

1. We define the notion of a *saturated branch* of a tree of sequents, where, intuitively, a branch is saturated if it contains no initial sequent nor instance of 0-premiss rules and, moreover, whenever one of its nodes contains the principal formulas of an instance of a rule, there is a node containing its active formulas.
2. We define a root-first proof-search procedure that either outputs the informations needed to build a derivation or a tree containing at least one saturated branch.
3. We define the model generated by a saturated branch.
4. We show that the model thus generated is a countermodel for the end-sequent of the proof-search and that it is based on a skeleton in the appropriate class.

Definition 34 (Saturated branch). Let $\mathcal{B}$ be a branch of a tree of sequents and let Γ (Δ) be the union of all formulas occurring in the antecedents (succedents, resp.) of its nodes. $\mathcal{B}$ is **L**-*saturated* if the following holds:

1. The branch contains no node that is an initial sequent or an instance of a 0-premiss rule. In particular:
   (a) No formula of shape $w : P$, for P atomic, occurs in $\Gamma \cap \Delta$;
   (b) No instance of $w : \bot$ ($w : \top$) occurs in Γ (Δ);
   (c) Γ does not contain the principal formulas of a coherent 0-ary rule of **GIM.L**.
2. If $\mathcal{B}$ contains the principal formulas of a propositional rule, it contains also the active formulas of one of its premisses—e.g., if Γ (Δ) contains $w : A \wedge B$ then Γ (Δ) contains $w : A$ and $w : B$ ($w : A$ or $w : B$, respectively).
3. If the branch contains the principal formula of a rule for the quantifiers, then it contains also its active formulas. For $\forall$ we have the following clauses:

(a) If $w : \forall x A$ occurs in Γ then Γ contains $w : A[t/x]$ for all terms t occurring in Γ in an expression of shape $w : t = t$ (or for a fresh term if no $w : t = t$ is in Γ);
(b) If $w : \forall x A$ occurs in Δ then $w : A[t/x]$ occurs in Δ for some term t.

4. If the branch contains the principal formula(s) of a rule for the indexed modalities, then it contains also its active formulas. For $[\cdot]$ we have the following clauses:

(a) If $w : [\frac{\vec{t}}{\vec{x}}]A$ and $(w, \vec{t})\mathscr{R}(v, \vec{s})$ occur in Γ then $v : A[\vec{s}/\vec{x}]$ occurs in Γ;
(b) If $w : [\frac{\vec{t}}{\vec{x}}]A$ occurs in Δ then, for some label u and some terms $\vec{s}$, $(w, \vec{t})\mathscr{R}(u, \vec{s})$ occurs in Γ and $u : A[\vec{s}/\vec{x}]$ occurs in Δ.

5. If $\mathcal{B}$ contains the principal formula(s) of a $n + 1$-ary coherent rule of **GIM**.**L** then it contains also its active formulas.[27]

Definition 35 (Proof-search tree). Given a sequent $\Gamma \Rightarrow \Delta$ and a calculus **GIM**.**L** we outline a procedure that applies root-first, all possible rules of inference.

At stage 0 we write the one-node tree $\Gamma \Rightarrow \Delta$.

At stage $n + 1$ we first check all leaves of the tree generated at stage n to see if they are all initial sequents or instances of a 0-ary rule of **GIM**.**L**. If they are so, the procedure ends. Else, we apply all possible (non 0-ary) rules of the calculus to each branch not satisfying the above condition. In particular, we have $20 + m$ substages, where m is the number of coherent rules of **GIM**.**L** (modulo the coherent rules of **GIM**.**K**), and at each sub-stage we apply in parallel all possible instances of a rule of **GIM**.**L**. To illustrate, at substage 13 we apply all possible instance of $R\Box$. If the leaf is

$$\Gamma \Rightarrow \Delta, w_1 : [\tfrac{\vec{t}_1}{\vec{x}_1}]A_1, \dots, w_k : [\tfrac{\vec{t}_k}{\vec{x}_k}]A_k$$

and no formula in Δ has $[\cdot]$ as principal operator, we extend the branch with the node

$$(w_1, \vec{t}_1)\mathscr{R}(u_1, \vec{y}_1), \dots, (w_k, \vec{t}_k)\mathscr{R}(u_k, \vec{y}_k), \Gamma \Rightarrow \Delta, u_1 A_1[\vec{y}_1/\vec{x}_1], \dots, u_k : A_k[\vec{y}_k/\vec{x}_k]$$

where $u_1, \dots, u_k$ are fresh labels and $\vec{y}_1, \dots, \vec{y}_k$ are fresh tuples of variables.

Two cases are possible: if the procedure ends, it outputs a finite tree containing the informations to write a **GIM**.**L**-derivation of $\Gamma \Rightarrow \Delta$. Else, it outputs an infinite tree which, by König's lemma, contains an infinite branch. It is immediate to notice that this infinite branch is **L**-saturated.

Definition 36 (Model $\mathcal{M}^{\mathcal{B}}$). let $\mathcal{B}$ be a **L**-saturated branch obtained by the procedure in Def. 35 and let Γ (Δ) be the union of all formulas in its antecedents (succedents). The model $\mathcal{M}^{\mathcal{B}}$ is thus defined:

$\mathcal{W}^{\mathcal{B}}$ is the set of all equivalence classes under l-identity of labels occurring in Γ;

[27] If the rule has no principal formula, then it is enough if it contains all instances of its active formulas containing (apart from eigenvariables) only labels w occurring in $\mathcal{B}$ and terms t occurring in Γ in an expression of shape $(w, \vec{s})$ or occurring in $\mathcal{B}$ in a formula of shape $w : A$ not inside the scope of an indexed modality or (only) of its lower index.

$\mathcal{D}^{\mathcal{B}}$ is the union of all sets D_w, for $w \in \mathcal{W}^{\mathcal{B}}$, such that: $t^w \in D_w$ iff t^w is the equivalence class of all terms s such that $w : t = s$ occurs in Γ;
$\mathcal{T}^{\mathcal{B}}$ is such that $\langle w, \vec{t}^w \rangle \mathcal{T}^{\mathcal{B}} \langle v, \vec{s}^v \rangle$ iff $(w, \vec{t}) \mathscr{R} (v, \vec{s})$ occurs in Γ;
$\mathcal{V}^{\mathcal{B}}$ is the union of world-bound valuations V_w, for all $w \in \mathcal{W}^{\mathcal{B}}$, such that $V_w(f(t_1, \dots, t_n)) = f^w(t_1^w, \dots, t_n^w)$ and
$V_w(P^n) = \{\langle t_1^w, \dots, t_n^w \rangle \ : \ w : P^n(t_1, \dots, t_n) \in \Gamma\}$.

Lemma 37 (Truth lemma). *Let $\mathcal{M}^{\mathcal{B}}$ be the model constructed from an $\mathbf{L}$-saturated branch $\mathcal{B}$ of a proof-search tree for the sequent $\Pi \Rightarrow \Sigma$ and let Γ (Δ) be the union of all formulas occurring in the antecedents (succedents) of $\mathcal{B}$. Let, moreover σ_v, for $v \in \mathcal{W}^{\mathcal{B}}$, be the v-assignment mapping each variable x to its equivalence class x^v. For each labelled formula $w : A$ occurring in $\mathcal{B}$ we have that:*

1. $\sigma_w \models_w^{\mathcal{M}^{\mathcal{B}}} A$ *if* $w : A$ *occurs in* Γ;
2. $\sigma_w \not\models_w^{\mathcal{M}^{\mathcal{B}}} A$ *if* $w : A$ *occurs in* Δ.

Proof The two claims are proved by simultaneous induction on the weight of $w : A$. For the non-modal cases we refer the reader to [36].

Assume that $w : A$ is $w : [\frac{\vec{t}}{\vec{x}}]B$ and that it occurs in Γ. Definition 34.4(a) ensures that, for each formula of shape $(w, \vec{t}) \mathscr{R} (v, \vec{s})$ occurring in Γ, $v : B[\vec{s}/\vec{x}]$ occurs in Γ. By induction we have that $\sigma_v \models_v B[\vec{s}/\vec{x}]$, which, by Lemma 6, is equivalent to $\sigma_v^{\vec{x} \triangleright \sigma_v(\vec{s})} \models_v B$. We easily conclude that $\sigma_w \models_w [\frac{\vec{t}}{\vec{x}}]B$.

Suppose, instead, that $w : [\frac{\vec{t}}{\vec{x}}]B$ occurs in Δ. By Definition 34.4(b) there are a label u and variables $\vec{y}$ such that $(w, \vec{t}) \mathscr{R} (u, \vec{y})$ occurs in Γ and $u : B[\vec{y}/\vec{x}]$ occurs in Δ. By induction, $\sigma_u \not\models_u B[\vec{y}/\vec{x}]$ and, since $\langle w, \vec{t}^w \rangle \mathcal{T}^{\mathcal{B}} \langle u, \vec{y}^u \rangle$, we conclude that $\sigma_w \not\models_w [\frac{\vec{t}}{\vec{x}}]B$. □

Theorem 38 (Completeness). *Each* $\mathbf{L}$*-valid sequent is derivable in* $\mathbf{GIM.L}$.

Proof Assume that $\mathbf{GIM.L} \nvdash \Gamma \Rightarrow \Delta$. Lemma 37 ensures that $\mathcal{M}^{\mathcal{B}}$ is a countermodel for $\Gamma \Rightarrow \Delta$—i.e., it satisfies all formulas in Γ and no formula in Δ. To prove the theorem we have only to show that $\mathcal{M}^{\mathcal{B}}$ is a model for $\mathbf{L}$. This follows immediately by clauses 1(c) and 5 of Definition 34: they ensure that whenever the principal formulas of a coherent rule expressing a property α of $\mathcal{T}^{\mathcal{B}}$ are in Γ, also (at least one disjunct of) the corresponding active formulas are in Γ. By construction of $\mathcal{T}^{\mathcal{B}}$ this means that it satisfies α. □

Corollary 39 *The rule of Cut is semantically admissible.*

Corollary 40 *Validity over FO-definable classed of Kripke frames can be characterised proof-theoretical by labelled sequent calculi.*

Proof An immediate consequence of Theorems 19 and 38. □

10 Conclusion

We have shown that IML provide a very expressive framework for QML: each formula where the FO-machinery interacts with modalities corresponds to a FO-expressible property of the transition relation. This is a big advantage with respect to Kripke semantics where there are formulas—like (CBF) and (GF)—that are valid when we adopt the classical theory of quantification and there are distinct formulas—like (NI) and (ND)—corresponding to the same semantical property.

We have also shown that labelled sequent calculi allow to give a proof-theoretic characterisation of an extremely wide class of QML: each indexed extension of a FO-definable PML has a cut-free and complete proof system. To our knowledge this completeness result is more general than existing ones which cover, at most, quantified extensions of canonical PML.[28] We avoided the problem of incomplete but completable axiomatic systems since all problematic interaction formulas are derivable. If, e.g., we consider the labelled calculus for the logic **Q.S4M**, an easy proof-search shows that the sequent expressing the axiom used in [11] to complete it is derivable in the corresponding labelled calculus. Analogously, the axiomatic system **Q.K2.BF** is incomplete since the formula

$$\Diamond(\forall x(A \supset \Box A) \wedge \Box\neg\forall x A) \ \wedge \ \Diamond\forall x(A \vee \Box A) \ \wedge \ \forall x(\Diamond A \supset \Box A) \qquad (Cr)$$

is **Q.K2.BF**-consistent but not satisfiable in the appropriate class of Kripke frames [10]. A (long but) simple proof-search shows that the sequent $\Rightarrow w : \neg Cr$ is derivable in the corresponding labelled calculus.

One question that remains open is whether the completeness result given here is optimal: the quantified extensions of second-order definable PML studied in the literature are not recursively axiomatisable. To our knowledge, there is no general result concerning the (non-)axiomatisability of quantified extensions of second-order definable PML.[29] Another direction for future research is the comparison of the present approach with those based on metaframes [18, 29, 39] and with those based on quantification over individual concepts [15, 19, 28].

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[28] If we exclude results using some kind of general-frame semantics, such as those in [23, 27, 28].

[29] At least for normal ones. See [38] for a recursively enumerable quantified extension of a second-order definable quasi-normal PML. We are grateful to Frank Wolter for making us aware of this paper.

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